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A Nonconforming Finite Element Method for Constrained Optimal Control Problems Governed by Parabolic Equations

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Abstract. In this paper, a nonconforming finite element method (NFEM) is proposed for the constrained optimal control problems (OCPs) governed by parabolic equations. The time discretization is based on the finite difference methods. The state and co-state variables are approximated by the nonconforming EQ_1^{rot} elements, and the control variable is approximated by the piecewise constant element, respectively. Some superclose properties are obtained for the above three variables. Moreover, for the state and co-state, the convergence and superconvergence results are achieved in L^2-norm and the broken energy norm, respectively.

1. Introduction

The state constrained OCPs play a crucial role in many science and engineering applications (cf. [2, 3]). But the exact solutions do not always exist or are difficult to be obtained for this kind of problems, so researching the corresponding numerical algorithms becomes very meaningful. [10] firstly gave the FEM for the elliptic OCPs and proved the convergence property. Later, [4] provided the optimality conditions and some important theoretical analyses for elliptic OCPs. Based on this research, more and more literatures appeared to study the FEMs for the OCPs governed by PDEs. For instance, [5] proved the convergence of FE approximations to OCPs governed by semi-linear elliptic equations with finitely many state constraints. Later, [6] extended these results to a less regular setting for the states, and gave the convergence analysis of FEM for semi-linear distributed and boundary control problems. On the other hand, [15] proposed a discretization concept which utilized for the discretization of the control variable the relation between adjoint state and control. The key feature is not to discretize the space of admissible control, but to implicitly utilize the first order optimality conditions and the discretization of the state and adjoint equations for the discretization of the control. Moreover, the linear FE was used to discretize the state equation and the error estimate of L^2-norm was obtained

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in [8]. Recently, a lot of studies are focusing on the conforming FEM for OCPs governed by elliptic equations, Stokes equations, convection-dominated diffusion equations, and so on (cf. [20,21,32]).

Some usual notations of Sobolev spaces and norms are presented in the following. Let $H^k(\Omega)$ be the standard Sobolev space of k-differential functions and less than k-differential functions in $L^2(\Omega)$ with norm $\|\cdot\|_k$ and semi-norm $|\cdot|_k$ respectively. $H_0^1(\Omega)$ denotes the closure of the space $C_0^\infty(\Omega)$ in $H^1(\Omega)$, $W^{k,\infty}(\Omega)$ denotes the Sobolev space of k-differential functions and less than k-differential functions are bounded. For Sobolev space Y, $L^p(m_1,m_2;Y)$ is the space of measurable Y-valued functions ϑ of $t\in(m_1,m_2)$ with the norm $\|\vartheta\|_{L^p(m_1,m_2;Y)}=\left(\int_{m_1}^{m_2}|\vartheta(\,\cdot\,,t)|_Y^p\right)^{1/p}$ if $1\le p<\infty$, or the norm $\|\vartheta\|_{L^\infty(m_1,m_2;Y)}=\operatorname{ess\,sup}_{m_1<t<m_2}|\vartheta(\,\cdot\,,t)|_Y<\infty$ if $p=\infty$.

In this paper, we will consider the following OCP with state constrained: find $(y,u)\in L^2(0,T;Y)\times L^2(0,T;U)$, such that

$$\min_{u\in L^2(0,T;K^*)}\frac{1}{2}\int_0^T\left(\|y-y_d\|_0^2+\alpha\,\|u\|_0^2\right)dt \tag{1.1}$$

subject to

$$\begin{cases} y_t-\Delta y=f+Bu & \text{in } \Omega,\\ y=0 & \text{on } \partial\Omega,\\ y(x,0)=y_0 & \text{in } \Omega, \end{cases} \tag{1.2}$$

where α is a positive constant parameter, Ω is a bounded polygon in R^2, $x=(x_1,x_2)\in\Omega$, $f\in L^2(0,T;L^2(\Omega))$ is a given function, $Y=H_0^1(\Omega)\cap H^3(\Omega)$, $U=H^1(\Omega)$, B is a linear continuous operator, and K^* is defined as

$$K^*=\{v\in U: v\le 0,\ \text{a.e. in } \Omega\}.$$

For the above parabolic OCPs, [17] investigated a FEM and obtained the $O(h^{3/2-\delta})$ convergence order. Later, by using the optimality conditions, [7] studied the fully discrete mixed FEM for semilinear parabolic OCPs, $O(h+\Delta t)$ error estimates are derived. [22] reported a posteriori error estimates for both the state and the control approximation. Recently, [31] derived the $O(h^2+\Delta t)$ order superconvergence results by using the linear FE approximations.

However, the findings mentioned above are mainly contributions to the conforming FEMs. In fact, NFEs have some advantages comparing with the conforming ones. For example, for the Crouzeix-Raviart type NFEs, such as Q_1^{rot} element [23], EQ_1^{rot} element [18], CNQ_1^{rot} element [16], P_1-nonconforming triangular element [1], etc. Since the unknowns of the elements are associated with the element edges or faces, each degree of freedom belongs to at most two elements, the use of the NFEs facilitates the exchange of information across each subdomain and provides spectral radius estimates for the iterative domain decomposition operator (cf. [9]). Especially, the nonconforming $EQ_1^{\rm rot}$ element has attracted scientists' more and more attentions to be applied to many problems. For example, [18] studied its superconvergence properties for the second order elliptic problems, [24] applied this element to solve diffusion-convection-reaction equation, [26] considered its superconvergence behaviors on anisotropic meshes, [29] used it to deal with the Signorini problem and obtained the global superconvergence results. Furthermore, this element was also employed to solve the Maxwell's equations [27], nonlinear Sobolev equation [28] and some other different problems [13, 25, 30]. Recently, we also researched the NFEM and mixed FEM for stationary OCPs, and obtained the superconvergence results and optimal order error estimates in [11, 12], respectively.

The aim of this paper is to derive the global superclose and superconvergence properties of NFEM for parabolic OCPs, in which the $EQ_1^{\rm rot}$ element is employed to approximate the state and co-state. The rest of this paper is organized as follows. In the next section, the discrete formulation will be presented for the OCPs. In Section 3, the superclose property will be established for the control variable. In Section 4, the superclose and superconvergence results will be derived both for the state and co-state variables.

2. The discrete formulation and some lemmas

By [19], we know that (1.1)–(1.2) has a unique solution (y, u) and that (y, u) is the solution of (1.1)–(1.2) if and only if there is a co-state $p \in L^2(0, T; Y)$ such that (y, p, u) satisfies the following optimality conditions:

$$
\begin{cases}
(y_t, v) + a(y, v) = (f + Bu, v) & \forall\, v \in H_0^1(\Omega), \\
y(x, 0) = y_0 & \text{in } \Omega, \\
-(p_t, v) + a(p, v) = (y - y_d, v) & \forall\, v \in H_0^1(\Omega), \\
p(x, T) = 0 & \text{in } \Omega, \\
(\alpha u + B^* p, v - u) \geq 0 & \forall\, v \in K^*,
\end{cases}
\tag{2.1}
$$

where $a(y, v) = \int_\Omega \nabla y \nabla v \, dx$, $(f + Bu, v) = \int_\Omega (f + Bu) v \, dx$.

In (2.1), let $\Omega = \Omega^0 \cup \Omega^+$, where $\Omega^0 = \{x \in \Omega : u(x) = 0\}$ and $\Omega^+ = \{x \in \Omega : u(x) < 0\}$, then

$$
\begin{cases}
(\alpha u + B^* p, v - u) > 0 & \text{in } \Omega^0, \\
(\alpha u + B^* p, v - u) = 0 & \text{in } \Omega^+.
\end{cases}
$$

Let $0 = t_0 < t_1 < \cdots < t_{N-1} < t_N = T$ divide $[0, T]$ to N parts averagely, $0 < \Delta t = T/N < 1/2$, $u_n = u(x, t_n)$, $y_n = y(x, t_n)$, $p_n = p(x, t_n)$, $f_n = f(x, t_n)$. Then we have the following difference discrete scheme for the time variable t:

$$
(2.2)\qquad \begin{cases} (\frac{y_n-y_{n-1}}{\Delta t}, v) + a(y_n, v) = (f_n + Bu_n, v) & \forall\, v \in H_0^1(\Omega), \\ (\frac{p_{n-1}-p_n}{\Delta t}, v) + a(p_{n-1}, v) = (y_n - y_n^d, v) & \forall\, v \in H_0^1(\Omega), \\ p_N = 0 & \text{in } \Omega, \\ (\alpha u_n + B^* p_{n-1}, v - u_n) \geq 0 & \forall\, v \in K^*. \end{cases}
$$

On one hand, let T_{h_1} be a family of triangular subdivision of Ω with the mesh size $0 < h_1 < 1$. The corresponding FE space U_{h_1} contains the piecewise constants. $\Pi_{h_1}: L^2(\Omega) \rightarrow U_h$ is the interpolation operator on U_{h_1}. In the meantime, we denote K_{h_1} by the convex set associated with K^* in U_{h_1} as $K_{h_1} = K^* \cap U_{h_1}$.

On the other hand, let T_{h_2} be a family of rectangular subdivision of Ω with the mesh size $0 < h_2 < 1$. For any given $K \in T_{h_2}$, let $Z_1 = (x_0 - h_{x_1}, y_0 - h_{x_2})$, $Z_2 = (x_0 + h_{x_1}, y_0 - h_{x_2})$, $Z_3 = (x_0 + h_{x_1}, y_0 + h_{x_2})$ and $Z_4 = (x_0 - h_{x_1}, y_0 + h_{x_2})$ be the four vertices of K, $l_i = \overline{Z_i Z_{i+1}}$ ($i = 1, 2, 3, 4 \mod 4$). The corresponding EQ_1^{rot} FE space [26] is defined as

$$
V_{h_2} = \left\{ v \in L^2(\Omega),\, v|_K \in \operatorname{span}(1, x_1, x_2, x_1^2, x_2^2),\, \int_l [v]\, ds = 0,\, l \subset \partial K,\, \forall\, K \in T_{h_2} \right\},
$$

where $[v]$ denotes the jump of v across the edge F if F is an internal edge, and it is equal to v itself if l is a boundary edge. Let $\Pi_{h_2}: H^1(\Omega) \rightarrow V_{h_2}$ be the associated interpolation operator on V_{h_2}, $\Pi_K = \Pi_{h_2}|_K$ satisfies

$$
\begin{cases} \int_K (v - \Pi_K v)\, dx = 0, \\ \int_{l_i} (v - \Pi_K v)\, ds = 0, \quad i = 1, 2, 3, 4. \end{cases}
$$

Let y_n^h, p_n^h and u_n^h denote the FE approximation of y, p and u at t_n, respectively, then we have the following fully-discrete nonconforming FE scheme of (2.2) as follows:

$$
(2.3)\qquad \begin{cases} \left(\frac{y_n^h - y_{n-1}^h}{\Delta t}, v_h \right) + a_h(y_n^h, v_h) = (f_n + Bu_n^h, v_h) & \forall\, v_h \in V_{h_2}, \\ y_0^h = \Pi_{h_2} y_0, \\ \left(\frac{p_{n-1}^h - p_n^h}{\Delta t}, v_h \right) + a_h(p_{n-1}^h, v_h) = (y_n^h - y_n^d, v_h) & \forall\, v_h \in V_{h_2}, \\ p_N^h = 0, \\ (\alpha u_n^h + B^* p_{n-1}^h, v_h - u_n^h) \geq 0 & \forall\, v \in K_{h_1}, \end{cases}
$$

where $a_h(y_h, v_h) = \sum_{K \in T_{h_2}} \int_K \nabla y_h \nabla v_h\, dx$.

The following two lemmas have been proved in [26] and [28], respectively, which play an important role in our theoretical analysis.

Lemma 2.1. *Let $y \in Y$, then for all $v_h \in V_{h_2}$, we have*

$$\left| \sum_{K \in T_{h_2}} \int_{\partial K} \frac{\partial y}{\partial n} v_h \, ds \right| \le c h_2^2 \|y\|_3 \|v_h\|_h ,$$

here and later, c is a generic positive constant independent of h_1 or h_2, and $\|\cdot\|_h = \left(\sum_{K \in T_{h_2}} |\cdot|_{1,K}^2\right)^{1/2}$ is a broken energy norm on V_{h_2}.

Proof. For any $K \in T_{h_2}$, we define operators P_0 and P_{0i} as

$$P_0 v = \frac{1}{|K|} \int_K v \, dx, \quad P_{0i} \omega = \frac{1}{|l_i|} \int_{l_i} \omega \, ds,$$

respectively, where $|K|$ and $|l_i|$ denote the measures of K and l_i, respectively.

It can be checked that

$$\begin{aligned} \sum_{K \in T_{h_2}} \int_{\partial K} \frac{\partial y}{\partial n} v_h \, ds = \sum_{K \in T_{h_2}} & \left[- \int_{l_1} \frac{\partial y}{\partial x_2} (v_h - P_{01} v_h) \, dx_1 + \int_{l_2} \frac{\partial y}{\partial x_1} (v_h - P_{02} v_h) \, dx_2 \right. \\ & \left. + \int_{l_3} \frac{\partial y}{\partial x_2} (v_h - P_{03} v_h) \, dx - \int_{l_4} \frac{\partial y}{\partial x_1} (v_h - P_{04} v_h) \, dx_2 \right] \\ \doteq \sum_{K \in T_{h_2}} & \sum_{i=1}^{4} M_i. \end{aligned}$$

By the definition of P_{01}, we get

$$\begin{aligned} & \int_K (v_h(x_1, y_0 - h_{x_2}) - P_{01} v_h(x_1, y_0 - h_{x_2})) \, dx_1 \, dx_2 \\ = & \, 2h_{x_2} \int_{l_1} v_h(x_1, y_0 - h_{x_2}) \, dx_1 - \frac{4 h_{x_1} h_{x_2}}{|l_1|} \int_{l_1} v_h(x_1, y_0 - h_{x_2}) \, dx_1 = 0. \end{aligned}$$

Noticing that $(v_h - P_{01} v_h)|_{l_1}$ equals $(v_h - P_{03} v_h)|_{l_3}$ and $\frac{\partial v_h}{\partial x_1}$ is only dependent on x_1, we can derive that

$$\begin{aligned} M_1 + M_3 &= \int_{x_0 - h_{x_1}}^{x_0 + h_{x_1}} \left[\frac{\partial y}{\partial x_2}(x_1, y_0 + h_{x_2}) - \frac{\partial y}{\partial x_2}(x_1, y_0 - h_{x_2}) \right] (v_h - P_{01} v_h) \, dx_1 \\ &= \int_{x_0 - h_{x_1}}^{x_0 + h_{x_1}} \left[\int_{y_0 - h_{x_2}}^{y_0 + h_{x_2}} \frac{\partial^2 u}{\partial x_2^2}(x_1, x_2) \, dx_2 \right] (v_h - P_{01} v_h) \, dx_1 \\ &= \int_{x_0 - h_{x_1}}^{x_0 + h_{x_1}} \int_{y_0 - h_{x_2}}^{y_0 + h_{x_2}} \left(\frac{\partial^2 u}{\partial x_2^2} - P_0 \frac{\partial^2 u}{\partial x_2^2} \right) (v_h - P_{01} v_h) \, dx_2 \, dx_1 \\ &= \left\| \frac{\partial^2 u}{\partial x_2^2} - P_0 \frac{\partial^2 u}{\partial x_2^2} \right\|_{0,K} \|v_h - P_{01} v_h\|_{0,K} \\ &\le c h_2^2 |u|_{3,K} |v_h|_{1,K} . \end{aligned}$$

Similarly, $M_2 + M_4 \le c h_2^2 |u|_{3,K} |v_h|_{1,K}$. Thus the desired result follows. $\square$

Lemma 2.2. *Let $\varphi \in L^2(0,T;Y)$ and $\varphi_n^h \in V_{h_2}$ be the solutions of*

$$\begin{cases} (\varphi_t, v) + a(\varphi, v) = (f, v) & \forall\, v \in H_0^1(\Omega), \\ \varphi(x,0) = \varphi_0 & \text{in } \Omega \end{cases} \tag{2.4}$$

and

$$\begin{cases} \left(\frac{\varphi_n^h - \varphi_{n-1}^h}{\Delta t}, v_h\right) + a_h(\varphi_n^h, v_h) = (f_n, v_h) \quad \forall\, v_h \in V_{h_2}, \\ \varphi_0^h = \Pi_{h_2}\varphi_0, \end{cases} \tag{2.5}$$

respectively, then there holds the following optimal error estimate:

$$\max_{1\le n\le N} \left\|\varphi_n - \varphi_n^h\right\|_0 + \Delta t \sum_{n=1}^{N} \left\|\varphi_n - \varphi_n^h\right\|_h^2 \le c(h_2^2 + \Delta t),$$

in which $\varphi_n = \varphi(x, t_n)$.

Proof. Letting $t = t_n$ and $v = v_h$ in (2.4), we can get that

$$\left(\frac{\partial \varphi_n}{\partial t}, v_h\right) + a_h(\varphi_n, v_h) = (f_n, v_h), \tag{2.6}$$

where $\frac{\partial \varphi_n}{\partial t} = \frac{\partial \varphi}{\partial t}\big|_{t=t_n}$.

Subtracting (2.6) form (2.5), we have

$$\begin{aligned} &\left(\frac{\partial \varphi_n}{\partial t} - \frac{\varphi_n - \varphi_{n-1}}{\Delta t}, v_h\right) + \left(\frac{\varphi_n - \varphi_n^h}{\Delta t} - \frac{\varphi_{n-1} - \varphi_{n-1}^h}{\Delta t}, v_h\right) + a_h(\varphi_n - \varphi_n^h, v_h) \\ &= \sum_{K\in T_{h_2}} \int_{\partial K} \frac{\partial \varphi_n}{\partial n} v_h \, ds. \end{aligned} \tag{2.7}$$

We denote $\varphi_n - \varphi_n^h$ by θ_n, then letting $v_h = \theta_n$ in (2.7), there yields

$$\begin{aligned} &\|\theta_n\|_0^2 - \|\theta_{n-1}\|_0^2 + 2\Delta t \|\theta_n\|_h^2 \\ &\le -2\Delta t \left(\frac{\partial \varphi_n}{\partial t} - \frac{\varphi_n - \varphi_{n-1}}{\Delta t}, \theta_n\right) + 2\Delta t \sum_{K\in T_{h_2}} \int_{\partial K} \frac{\partial \varphi_n}{\partial n} \theta_n \, ds \\ &\le c(\Delta t)^2 \int_{t_{n-1}}^{t_n} \left\|\frac{\partial \varphi}{\partial t}\right\|_2^2 dt + \Delta t \|\theta_n\|_0^2 + c\Delta t h_2^2 \|\varphi_n\|_3^2 \|\theta_n\|_h, \end{aligned}$$

which means that

$$\begin{aligned} &(1 - \Delta t) \|\theta_n\|_0^2 - \|\theta_{n-1}\|_0^2 + \Delta t \|\theta_n\|_h^2 \\ &\le c(\Delta t)^2 \int_{t_{n-1}}^{t_n} \left\|\frac{\partial \varphi}{\partial t}\right\|_2^2 dt + \Delta t \|\theta_n\|_0^2 + c\Delta t h_2^4 \|\varphi_n\|_3^2. \end{aligned} \tag{2.8}$$

Summing (2.8) from 1 to n, we have

$$\|\theta_n\|_0^2 \le c(h_2^4 + (\Delta t)^2). \tag{2.9}$$

On the other hand, summing (2.8) from 1 to N, we can obtain that

$$\Delta t \sum_{n=1}^{N} \|\theta_n\|_h^2 \le c(h_2^2 + \Delta t). \tag{2.10}$$

Combining (2.9) and (2.10) gives the desired result. □

3. Superclose property of the control u

Lemma 3.1. *Let $(y_n(u_n^h), p_{n-1}(u_n^h))$ be the solution of*

$$\begin{cases} \left(\frac{y_n(u_n^h)-y_{n-1}(u_n^h)}{\Delta t}, v\right) + a(y_n(u_n^h), v) = (f_n + Bu_n^h, v) & \forall\, v \in H_0^1(\Omega), \\ y_0(u_n^h) = \Pi_h y_0, \\ \left(\frac{p_{n-1}(u_n^h)-p_n(u_n^h)}{\Delta t}, v\right) + a(p_{n-1}(u_n^h), v) = (y_n(u_n^h) - y_n^d, v) & \forall\, v \in H_0^1(\Omega), \\ p_N(u_n^h) = 0, \end{cases} \tag{3.1}$$

where we assumed that u_n^h $(n = 1, 2, \ldots, N)$ are given, then there holds

$$\max_{1\le n\le N} \left|p_{n-1}(u_n^h) - p_{n-1}\right|_1 \le \left(\sum_{n=1}^{N} \Delta t \left\|y_n(u_n^h) - y_n\right\|_0^2\right)^{1/2} \le c\left(\sum_{n=1}^{N} \Delta t \left\|u_n^h - u_n\right\|_0^2\right)^{1/2}.$$

Proof. From (2.2) and (3.1), we have the following error equation:

$$\frac{1}{\Delta t}(p_{n-1}(u_n^h) - p_n(u_n^h) - (p_{n-1} - p_n), v) + a(p_{n-1}(u_n^h) - p_{n-1}, v) = (y_n(u_n^h) - y_n, v). \tag{3.2}$$

Choosing $v = p_{n-1}(u_n^h) - p_n(u_n^h) - (p_{n-1} - p_n)$ in (3.2), we can get that

$$\begin{aligned} &\frac{1}{\Delta t}\left\|p_{n-1}(u_n^h) - p_n(u_n^h) - (p_{n-1} - p_n)\right\|_0^2 + \frac{1}{2}\left(\left|p_{n-1}(u_n^h) - p_{n-1}\right|_1^2 - \left|p_n(u_n^h) - p_n\right|_1^2\right) \\ &\le \left\|y_n(u_n^h) - y_n\right\|_0 \left\|p_{n-1}(u_n^h) - p_n(u_n^h) - (p_{n-1} - p_n)\right\|_0 \\ &\le \Delta t\left\|y_n(u_n^h) - y_n\right\|_0^2 + \frac{1}{\Delta t}\left\|p_{n-1}(u_n^h) - p_n(u_n^h) - (p_{n-1} - p_n)\right\|_0^2, \end{aligned}$$

which is

$$\left|p_{n-1}(u_n^h) - p_{n-1}\right|_1^2 - \left|p_n(u_n^h) - p_n\right|_1^2 \le \Delta t\left\|y_n(u_n^h) - y_n\right\|_0^2. \tag{3.3}$$

Summating (3.3) in time from n to N, we have

$$\max_{1\le n\le N} \left|p_{n-1}(u_n^h) - p_{n-1}\right|_1 \le \left(\sum_{n=1}^{N} \Delta t \left\|y_n(u_n^h) - y_n\right\|_0^2\right)^{1/2}. \tag{3.4}$$

On the other hand, from (2.2) and (3.1), we could get that

$$\frac{1}{\Delta t}(y_n(u_n^h) - y_{n-1}(u_n^h) - (y_n - y_{n-1}), v) + a(y_n(u_n^h) - y_n, v) = (B(u_n^h - u_n), v). \tag{3.5}$$

Choosing $v = y_n(u_n^h) - y_n$ in (3.5), there yields

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\left\|y_n(u_n^h) - y_n\right\|_0^2 - \left\|y_{n-1}(u_n^h) - y_{n-1}\right\|_0^2\right) + \left|y_n(u_n^h) - y_n\right|_1^2 \\
&\le c\left\|u_n^h - u_n\right\|_0 \left\|y_n(u_n^h) - y_n\right\|_0 \le c\left\|u_n^h - u_n\right\|_0^2 + \left\|y_n(u_n^h) - y_n\right\|_0^2.
\end{aligned}$$

So

$$(1 - 2\Delta t)\left\|y_n(u_n^h) - y_n\right\|_0^2 - \left\|y_{n-1}(u_n^h) - y_{n-1}\right\|_0^2 \le c\Delta t\left\|u_n^h - u_n\right\|_0^2, \tag{3.6}$$

then by summating (3.6) in time from 1 to n and noticing that $1 \ge (1 - 2\Delta t)^n \ge (1 - 2\Delta t)^N \ge e^{-2T}$, we find that

$$\max_{1\le n\le N}\left\|y_n(u_n^h) - y_n\right\|_0^2 \le ce^{2T}\Delta t\sum_{n=1}^{N}\left\|u_n^h - u_n\right\|_0^2. \tag{3.7}$$

Combining (3.4) and (3.7) completes the proof. □

Lemma 3.2. *Let $p_{n-1}(y_n^h)$ be the solution of*

$$\begin{cases}\left(\frac{p_{n-1}(y_n^h) - p_n(y_n^h)}{\Delta t}, v\right) + a(p_{n-1}(y_n^h), v) = (y_n^h - y_n^d, v), \quad \forall\, v \in H_0^1(\Omega), \\ p_N(u_n^h) = 0,\end{cases} \tag{3.8}$$

where we assumed that y_n^h $(n = 1, 2, \ldots, N)$ are given, then there holds

$$\max_{1\le n\le N}\left|p_{n-1}(u_n^h) - p_{n-1}^h\right|_0^2 \le c(h_2^4 + (\Delta t)^2).$$

Proof. By (3.1) and (3.8), we have the following error equation:

$$\begin{aligned}
&\frac{1}{\Delta t}(p_{n-1}(u_n^h) - p_n(u_n^h) - (p_{n-1}(y_n^h) - p_n(y_n^h), v) + a(p_{n-1}(u_n^h) - p_{n-1}(y_n^h), v) \\
&= (y_n(u_n^h) - y_n^h, v).
\end{aligned} \tag{3.9}$$

Choosing $v = p_{n-1}(u_n^h) - p_{n-1}(y_n^h)$ in (3.9), there yields

$$\begin{aligned}
&\frac{1}{2\Delta t}\left(\left\|p_{n-1}(u_n^h) - p_{n-1}(y_n^h)\right\|_0^2 - \left\|p_n(u_n^h) - p_n(y_n^h)\right\|_0^2\right) + \left|p_{n-1}(u_n^h) - p_{n-1}(y_n^h)\right|_1^2 \\
&\le \left\|y_n(u_n^h) - y_n^h\right\|_0^2 + \left\|p_{n-1}(u_n^h) - p_{n-1}(y_n^h)\right\|_0^2,
\end{aligned}$$

which is

$$(1-2\Delta t)\left\|p_{n-1}(u_n^h)-p_{n-1}(y_n^h)\right\|_0^2-\left\|p_n(u_n^h)-p_n(y_n^h)\right\|_0^2\leq 2\Delta t\left\|y_n(u_n^h)-y_n^h\right\|_0^2.$$

Summating the above equation in time from n to N and noticing that $p_N(u_n^h)=p_N(y_n^h)=0$, we can obtain that

$$(1-2\Delta t)^{N-n}\left\|p_{n-1}(u_n^h)-p_{n-1}(y_n^h)\right\|_0^2\leq 2\Delta t\sum_{n=1}^{N}\left\|y_n(u_n^h)-y_n^h\right\|_0^2.$$

So, using the fact that $1\geq(1-2\Delta t)^{N-n}\geq(1-2\Delta t)^N\geq e^{-2T}$, we find that

$$\max_{1\leq n\leq N}\left\|p_{n-1}(u_n^h)-p_{n-1}(y_n^h)\right\|_0^2\leq c\sum_{n=1}^{N}\Delta t\left\|y_n(u_n^h)-y_n^h\right\|_0^2. \tag{3.10}$$

Noticing that (y_n^h,p_{n-1}^h) and $y_n(u_n^h,p_{n-1}(y_n^h))$ are the solutions of (2.3) and (3.1), respectively, by using Lemma 2.2 directly, we can obtain that

$$\max_{1\leq n\leq N}\left\|y_n(u_n^h)-y_n^h\right\|_0^2\leq c(h_2^4+(\Delta t)^2) \tag{3.11}$$

and

$$\max_{1\leq n\leq N}\left\|p_{n-1}(y_n^h)-p_{n-1}^h\right\|_0^2\leq c(h_2^4+(\Delta t)^2). \tag{3.12}$$

Therefore, from (3.10)–(3.12) and the triangle inequality, the desired result follows. □

Theorem 3.3. *Let $u_n\in K^*$ and $u_n^h\in K_{h_1}$ be the solutions of* (2.2) *and* (2.3), *respectively, then there holds the following superclose result:*

$$\sum_{n=1}^{N}\Delta t\left\|\Pi_{h_1}u_n-u_n^h\right\|_0^2\leq c(h_1^3+h_2^4+(\Delta t)^2). \tag{3.13}$$

Proof. Noting the fact that u_n^h and $\Pi_{h_1}u_n$ are both in $K_{h_1}\subset K^*$, then substituting them into the inequalities (2.2) and (2.3), respectively, there yield

$$(\alpha u_n+B^*p_{n-1},u_n-u_n^h)\leq 0\quad\text{and}\quad(\alpha u_n^h+B^*p_{n-1}^h,u_n^h-\Pi_{h_1}u_n)\leq 0.$$

Then we can derive that

$$\begin{aligned}
&\alpha\left\|u_n^h-\Pi_{h_1}u_n\right\|_0^2\\
&=\alpha(u_n^h-\Pi_{h_1}u_n,u_n^h-\Pi_{h_1}u_n)\\
&\leq(-B^*p_{n-1}^h,u_n^h-\Pi_{h_1}u_n)-\alpha(\Pi_{h_1}u_n,u_n^h-\Pi_{h_1}u_n)\\
&=(B^*(p_{n-1}-p_{n-1}^h),u_n^h-\Pi_{h_1}u_n)-(B^*p_{n-1},u_n^h-u_n)\\
&\quad-(B^*p_{n-1},u_n-\Pi_{h_1}u_n)-\alpha(\Pi_{h_1}u_n,u_n^h-\Pi_{h_1}u_n)
\end{aligned}$$

$$
\begin{aligned}
&\le (B^*(p_{n-1}-p_{n-1}^h), u_n^h - \Pi_{h_1}u_n) + \alpha(u_n, u_n^h - u_n)\\
&\quad - (B^*p_{n-1}, u_n - \Pi_{h_1}u_n) - \alpha(\Pi_{h_1}u_n, u_n^h - \Pi_{h_1}u_n)\\
&= (\alpha u_n + B^*p_{n-1}, \Pi_{h_1}u_n - u_n) + (B^*(p_{n-1}-p_{n-1}(u_n^h)), u_n^h - \Pi_{h_1}u_n)\\
&\quad + (B^*(p_{n-1}(u_n^h)-p_{n-1}^h), u_n^h - \Pi_{h_1}u_n)\\
&= (\alpha u_n + B^*p_{n-1}, \Pi_{h_1}u_n - u_n) + (B^*(p_{n-1}-p_{n-1}(u_n^h)), u_n^h - u_n)\\
&\quad + (B^*(p_{n-1}-p_{n-1}(u_n^h)), u_n - \Pi_{h_1}u_n) + (B^*(p_{n-1}(u_n^h)-p_{n-1}^h), u_n^h - \Pi_{h_1}u_n)\\
&\circeq \sum_{i=1}^{4} M_i.
\end{aligned}
$$

Now we start to estimate each M_i $(i=1,2,3,4)$. Noticing that $(\alpha u_n + B^*p_{n-1})|_{\Omega^+} = 0$ and $(\Pi_{h_1}u_n - u_n)|_{\Omega^0} = 0$, it is easy to know that $M_1 = 0$.

By (2.2) and (3.1), we have

$$
\begin{aligned}
M_2 &= (p_{n-1}-p_{n-1}(u_n^h), B(u_n^h - u_n))\\
&= a(y_n(u_n^h) - y_n, p_{n-1}-p_{n-1}(u_n^h))\\
&\quad + \frac{1}{\Delta t}(y_n(u_n^h) - y_{n-1}(u_n^h) - (y_n - y_{n-1}), p_{n-1}-p_{n-1}(u_n^h))\\
&= (y_n(u_n^h) - y_n, y_n - y_n(u_n^h)) + \frac{1}{\Delta t}(p_n - p_{n-1} - (p_n(u_n^h) - p_{n-1}(u_n^h)), y_n(u_n^h) - y_n)\\
&\quad + \frac{1}{\Delta t}(y_n(u_n^h) - y_{n-1}(u_n^h) - (y_n - y_{n-1}), p_{n-1}-p_{n-1}(u_n^h))\\
&= \frac{1}{\Delta t}(p_n - p_n(u_n^h), y_n(u_n^h) - y_n) - \frac{1}{\Delta t}(p_{n-1} - p_{n-1}(u_n^h), y_{n-1}(u_n^h) - y_{n-1})\\
&\quad - \left\|y_n(u_n^h) - y_n\right\|_0^2.
\end{aligned}
$$

By the property of interpolation operator Π_{h_1}, we can derive that

$$
\begin{aligned}
|M_3| &= \left|(B^*(p_{n-1}-p_{n-1}(u_n^h)) - \Pi_{h_1}(B^*(p_{n-1}-p_{n-1}(u_n^h))), u_n - \Pi_{h_1}u_n)\right|\\
&\le ch_1^2\left|p_{n-1}-p_{n-1}(u_n^h))\right|_1 |u_n|_1.
\end{aligned}
$$

By Cauchy inequality, we have

$$
|M_4| \le c\left|p_{n-1}(u_n^h) - p_{n-1}^h\right|_0^2 + \frac{\alpha}{2}\left\|u_n^h - \Pi_{h_1}u_n\right\|_0^2.
$$

Thus,

$$
\begin{aligned}
&\alpha\left\|u_n^h - \Pi_{h_1}u_n\right\|_0^2\\
&\le \frac{1}{\Delta t}\left[(p_n - p_n(u_n^h), y_n(u_n^h) - y_n) - (p_{n-1} - p_{n-1}(u_n^h), y_{n-1}(u_n^h) - y_{n-1})\right]\\
&\quad + ch_1^2\left|p_{n-1}-p_{n-1}(u_n^h)\right|_1 + c\left|p_{n-1}(u_n^h) - p_{n-1}^h\right|_0^2.
\end{aligned}
$$

Noticing that $p_N - p_N(u_n^h) = 0$, $y_0(u_n^h) - y_0 = 0$ and using Lemmas 3.1–3.2, there yields

$$\begin{aligned}
&\sum_{n=1}^{N} \Delta t \left\| u_n^h - \Pi_{h_1} u_n \right\|_0^2 \\
&\le ch_1^2 \sum_{n=1}^{N} \Delta t \left| p_{n-1} - p_{n-1}(u_n^h) \right|_1 + c \sum_{n=1}^{N} \Delta t \left| p_{n-1}(u_n^h) - p_{n-1}^h \right|_0^2 \\
&\le ch_1^2 \left(\sum_{n=1}^{N} \Delta t \left\| u_n^h - \Pi_{h_1} u_n \right\|_0^2 \right)^{1/2} + ch_1^2 \left(\sum_{n=1}^{N} \Delta t \left\| \Pi_{h_1} u_n - u_n \right\|_0^2 \right)^{1/2} + c(h_2^4 + (\Delta t)^2) \\
&\le c(h_1^4 + h_1^3 + h_2^4 + (\Delta t)^2) + \frac{1}{2} \sum_{n=1}^{N} \Delta t \left\| u_n^h - \Pi_{h_1} u_n \right\|_0^2,
\end{aligned}$$

which completes the proof. □

Theorem 3.4. *Under the assumptions of Theorem* 3.3*, we have*

$$\sum_{n=1}^{N} \Delta t \left\| u_n - u_n^h \right\|_0^2 \le c(h_1^2 + h_2^4 + (\Delta t)^2).$$

Proof. Noting that

$$\left\| u_n - u_n^h \right\|_0^2 \le c \left(\left\| u_n - \Pi_{h_1} u_n \right\|_0^2 + \left\| \Pi_{h_1} u_n - u_n^h \right\|_0^2 \right)$$

and

$$\left\| u_n - \Pi_{h_1} u_n \right\|_0 \le ch \left| u_n \right|_1,$$

the desired result can be obtained by Theorem 3.3 and the triangle inequality. □

Remark 3.5. The above error estimate is optimal, but which is one half order lower than Theorem 3.3. So we consider to derive the negative norm error estimate to obtain the superclose and superconvergence results for y and p.

Theorem 3.6. *Under the assumptions of Theorem* 3.3*, we have*

$$\sum_{n=1}^{N} \Delta t \left\| u_n - u_n^h \right\|_{-1}^2 \le c(h_1^3 + h_2^4 + (\Delta t)^2).$$

Proof. For any $\Phi \in H^1(\Omega)$, we have

$$\begin{aligned}
(u_n - u_n^h, \Phi - \Pi_{h_1}\Phi) &\le \left\| u_n - u_n^h \right\|_0 \left\| \Phi - \Pi_{h_1}\Phi \right\|_0 \\
&\le \left(\left\| u_n - \Pi_{h_1} u_n \right\|_0 + \left\| \Pi_{h_1} u_n - u_n^h \right\|_0 \right) \left\| \Phi - \Pi_{h_1}\Phi \right\|_0
\end{aligned} \tag{3.14}$$

and

$$(u_n - u_n^h, \Pi_{h_1}\Phi) = (\Pi_{h_1} u_n - u_n^h, \Pi_{h_1}\Phi) = (\Pi_{h_1} u_n - u_n^h, \Phi). \tag{3.15}$$

So, from (3.13), (3.14) and (3.15), we can get that

$$\begin{aligned}\sum_{n=1}^{N}\Delta t\left\|u_n-u_n^h\right\|_{-1}^2&=\sum_{n=1}^{N}\Delta t\sup_{\Phi\in H^1(\Omega)}\frac{\left|(u_n-u_n^h,\Phi)\right|^2}{\|\Phi\|_1^2}\\&\le c\sup_{\Phi\in H^1(\Omega)}\sum_{n=1}^{N}\Delta t\frac{\left|(u_n-u_n^h,\Phi-\Pi_{h_1}\Phi)\right|^2+\left|(u_n-u_n^h,\Pi_{h_1}\Phi)\right|^2}{\|\Phi\|_1^2}\\&\le ch_1^4+c\sum_{n=1}^{N}\Delta t\left\|\Pi_{h_1}u_n-u_n^h\right\|_0^2\\&\le c(h_1^3+h_2^4+(\Delta t)^2),\end{aligned}$$

which is the desired result. □

4. Superconvergence analysis of y and p

The following two properties are essential to our analysis, which can be found in [28] and [29], respectively: for $y\in H^1(\Omega)$, there hold

$$a_h(y-\Pi_{h_2}y,v_h)=0,\quad\forall\,v_h\in V_{h_2}$$

and

$$\|v_h\|_0\le c\,\|v_h\|_h\,,\quad\forall\,v_h\in V_{h_2}.$$

Theorem 4.1. *Let $y_n\in Y$ and $y_n^h\in V_{h_2}$ be the solutions of* (2.2) *and* (2.3)*, respectively, then there hold the following error estimates for y_n:*

$$\max_{1\le n\le N}\left\|y_n^h-\Pi_{h_2}y_n\right\|_0\le c(h_1^{3/2}+h_2^2+\Delta t)\tag{4.1}$$

and

$$\sum_{n=1}^{N}\Delta t\left\|y_n^h-\Pi_{h_2}y_n\right\|_h^2\le c(h_1^3+h_2^4+(\Delta t)^2).\tag{4.2}$$

Proof. For convenience, we denote $\xi_n=y_n^h-\Pi_{h_2}y_n$ and $\eta_n=\Pi_{h_2}y_n-y_n$. By (2.2) and (2.3), letting the test function $v_h=\xi_n$, we get the error equation as follows:

$$\begin{aligned}\frac{1}{\Delta t}(\xi_n-\xi_{n-1},\xi_n)+a_h(\xi_n,\xi_n)&=(B(u_n^h-u_n),\xi_n)-\frac{1}{\Delta t}(\eta_n-\eta_{n-1},\xi_n)\\&\quad+\sum_{K\in T_{h_2}}\int_{\partial K}\frac{\partial y_n}{\partial n}\xi_n\,ds.\end{aligned}\tag{4.3}$$

By the Cauchy inequality, the left-hand terms of (4.3) can be estimated as

$$\frac{1}{\Delta t}(\xi_n-\xi_{n-1},\xi_n)+a_h(\xi_n,\xi_n)\ge\frac{1}{2\Delta t}(\|\xi_n\|_0^2-\|\xi_{n-1}\|_0^2)+\|\xi_n\|_h^2\,.\tag{4.4}$$

Now we only need to estimate the right-hand terms of (4.3). First, we have

$$\left|(B(u_n^h - u_n), \xi_n)\right| \le c \left\|u_n^h - u_n\right\|_{-1} \|\xi_n\|_h \le c \left\|u_n^h - u_n\right\|_{-1}^2 + \frac{1}{6} \|\xi_n\|_h^2 . \tag{4.5}$$

Secondly,

$$\begin{aligned} \left|\frac{1}{\Delta t}(\eta_n - \eta_{n-1}, \xi_n)\right| &\le \frac{c}{\Delta t} \|\eta_n - \eta_{n-1}\|_0^2 + \frac{1}{6} \|\xi_n\|_h^2 \le \frac{c}{\Delta t} \int_{t_{n-1}}^{t_n} \left\|\frac{\partial \eta}{\partial t}\right\|_0^2 dt + \frac{1}{6} \|\xi_n\|_h^2 \\ &\le \frac{ch_2^4}{\Delta t} \int_{t_{n-1}}^{t_n} \left|\frac{\partial y}{\partial t}\right|_2^2 dt + \frac{1}{6} \|\xi_n\|_h^2 . \end{aligned} \tag{4.6}$$

Lastly, by Lemma 2.1, we get

$$\sum_{K \in T_{h_2}} \int_{\partial K} \frac{\partial y_n}{\partial n} \xi_n \, ds \le ch_2^2 \|y_n\|_3 \|\xi_n\|_h \le ch_2^4 \|y_n\|_3^2 + \frac{1}{6} \|\xi_n\|_h^2 . \tag{4.7}$$

Substitute (4.4)–(4.7) into (4.3), there yields

$$(\|\xi_n\|_0^2 - \|\xi_{n-1}\|_0^2) + \Delta t \|\xi_n\|_h^2 \le c\Delta t \left\|u_n^h - u_n\right\|_{-1}^2 + ch_2^4 \int_{t_{n-1}}^{t_n} \left|\frac{\partial y}{\partial t}\right|_2^2 dt + ch_2^4 \Delta t \|y_n\|_3^2 . \tag{4.8}$$

So, by using Theorem 3.6 and summating (4.8) in time from 1 to n, there yields (4.1).

Similarly, by summating (4.8) from 1 to N, we can get (4.2). The proof is thus completed. □

Theorem 4.2. *Let $p_n \in Y$ and $p_n^h \in V_{h_2}$ be the solutions of* (2.2) *and* (2.3), *respectively, there holds the following error estimates for p_n:*

$$\max_{0 \le n \le N-1} \left\|p_n^h - \Pi_{h_2} p_n\right\|_0 \le c(h_1^{3/2} + h_2^2 + \Delta t) \tag{4.9}$$

and

$$\sum_{n=0}^{N-1} \Delta t \left\|p_n^h - \Pi_{h_2} p_n\right\|_h^2 \le c(h_1^3 + h_2^4 + (\Delta t)^2). \tag{4.10}$$

Proof. Let $\theta_n = p_n^h - \Pi_{h_2} p_n$, $\zeta_n = \Pi_{h_2} p_n - p_n$. By (2.2) and (2.3), letting the test function $v_h = \theta_{n-1}$, we get the error equation as follows:

$$\begin{aligned} &\frac{1}{\Delta t}(\theta_{n-1} - \theta_n, \theta_{n-1}) + a_h(\theta_{n-1}, \theta_{n-1}) \\ &= (y_n^h - y_n, \theta_{n-1}) - \frac{1}{\Delta t}(\zeta_{n-1} - \zeta_n, \theta_{n-1}) + \sum_{K \in T_{h_2}} \int_{\partial K} \frac{\partial p_{n-1}}{\partial n} \theta_{n-1} \, ds. \end{aligned} \tag{4.11}$$

By Cauchy inequality, the left-hand terms of (4.11) can be estimated as

$$\frac{1}{\Delta t}(\theta_{n-1} - \theta_n, \theta_{n-1}) + a_h(\theta_{n-1}, \theta_{n-1}) \ge \frac{1}{2\Delta t}(\|\theta_{n-1}\|_0^2 - \|\theta_n\|_0^2) + \|\theta_{n-1}\|_h^2 . \tag{4.12}$$

Now we only need to estimate the right-hand terms of (4.11). First, by Theorem 4.1, we have

$$\begin{aligned}\left|(y_n^h - y_n, \theta_{n-1})\right| &= \left|(y_n^h - \Pi_{h_1} y_n, \theta_{n-1})\right| \le c\left\|y_n^h - \Pi_{h_1} y_n\right\|_0^2 + \frac{1}{6}\left\|\theta_{n-1}\right\|_h^2 \\ &\le c(h_1^3 + h_2^4 + (\Delta t)^2) + \frac{1}{6}\left\|\theta_{n-1}\right\|_h^2.\end{aligned} \tag{4.13}$$

Secondly,

$$\begin{aligned}&\left|\frac{1}{\Delta t}(\zeta_n - \zeta_{n-1}, \theta_{n-1})\right| \\ &\le \frac{c}{\Delta t}\left\|\zeta_n - \zeta_{n-1}\right\|_0^2 + \frac{1}{6}\left\|\theta_{n-1}\right\|_h^2 \le \frac{c}{\Delta t}\int_{t_{n-1}}^{t_n}\left\|\frac{\partial \zeta}{\partial t}\right\|_0^2 dt + \frac{1}{6}\left\|\theta_{n-1}\right\|_h^2 \\ &\le \frac{ch_2^4}{\Delta t}\int_{t_{n-1}}^{t_n}\left|\frac{\partial p}{\partial t}\right|_2^2 dt + \frac{1}{6}\left\|\theta_{n-1}\right\|_h^2.\end{aligned} \tag{4.14}$$

Lastly, by Lemma 2.1, we get

$$\sum_{K\in T_{h_2}}\int_{\partial K}\frac{\partial p_{n-1}}{\partial n}\theta_{n-1}\,ds \le ch_2^2\left\|p_{n-1}\right\|_3\left\|\theta_{n-1}\right\|_h \le ch_2^4\left\|p_{n-1}\right\|_3^2 + \frac{1}{6}\left\|\theta_{n-1}\right\|_h^2. \tag{4.15}$$

Substitute (4.12)–(4.15) into (4.11), there yields

$$\begin{aligned}&(\left\|\theta_{n-1}\right\|_0^2 - \left\|\theta_n\right\|_0^2) + \Delta t\left\|\theta_{n-1}\right\|_h^2 \\ &\le c(h_1^3 + h_2^4 + (\Delta t)^2) + ch_2^4\int_{t_{n-1}}^{t_n}\left|\frac{\partial p}{\partial t}\right|_2^2 dt + ch_2^4\Delta t\left\|y_n\right\|_3^2.\end{aligned} \tag{4.16}$$

Noticing the fact that $\|\theta_N\|_0 = 0$, then summating (4.16) from $n+1$ to N yields (4.9).

Similarly, by summating (4.16) from 1 to N, we can get (4.10). The proof is thus completed. □

The following theorem reports the convergence results for y and p in L^2-norm:

Theorem 4.3. *Under the assumptions of Theorems* 4.1–4.2*, we have the following error estimates for y and p, respectively:*

$$\max_{1\le n\le N}\left\|y_n - y_n^h\right\|_0 \le c(h_1^{3/2} + h_2^2 + \Delta t)$$

and

$$\max_{0\le n\le N-1}\left\|p_n - p_n^h\right\|_0 \le c(h_1^{3/2} + h_2^2 + \Delta t). \tag{4.17}$$

Proof. By (4.1) and the interpolation theory, there holds

$$\max_{1\le n\le N}\left\|y_n - y_n^h\right\|_0 \le \max_{1\le n\le N}\left\|y_n - \Pi_{h_2} y_n\right\|_0 + \max_{1\le n\le N}\left\|\Pi_{h_2} y_n - y_n^h\right\|_0 \le c(h_1^{3/2} + h_2^2 + \Delta t).$$

Similarly, by (4.9) and the interpolation theory, we can get (4.17). The proof is thus completed. □

In order to obtain the global superconvergence results for y and p in the broken energy norm, we combine four neighbouring elements $K_1, K_2, K_3, K_4 \in T_{h_2}$ into a big rectangular element K_0 (see Figure 4.1). T_{2h_2} presents the corresponding new partition (cf. [18]). We construct the interpolated postprocessing operator as follows:

$$\begin{cases} \Pi_{2h_2} w|_{K_0} \in P_2(K_0) & \forall\, K_0 \in T_{2h_2}, \\ \int_{L_i} (\Pi_{2h_2} w - w)\, ds = 0 & i = 1,2,3,4, \\ \int_{K_1 \cup K_3} (\Pi_{2h_2} w - w)\, dx = 0, \quad \int_{K_2 \cup K_4} (\Pi_{2h_2} w - w)\, dx = 0 & \forall\, K_0 \in T_{2h_2}, \end{cases}$$

in which L_i $(i = 1,2,3,4)$ are the four edges of K_0, P_2 denotes the set of polynomials of degree 2.

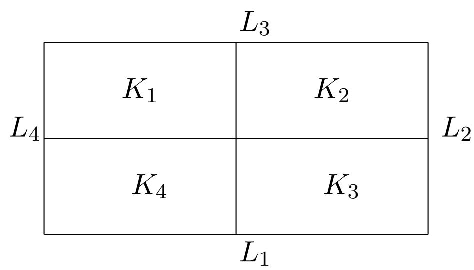

Figure 4.1: Big element K_0

The following properties for Π_{2h_2} have been validated in [18]:

$$\begin{cases} \Pi_{2h_2} \Pi_{h_2} w = \Pi_{2h_2} w & \forall\, w \in H^2(\Omega), \\ \|\Pi_{2h_2} w - w\|_h \le c h_2^r \,|w|_{r+1} & \forall\, w \in H^{r+1}(\Omega),\ 0 \le r \le 2, \\ \|\Pi_{2h_2} v_h\|_h \le c\,\|v_h\|_h & \forall\, v_h \in V_{h_2}. \end{cases} \tag{4.18}$$

Theorem 4.4. *Under the assumptions of Theorems* 4.1–4.2*, we have the following superconvergence results for* y *and* p*, respectively:*

$$\sum_{n=1}^{N} \Delta t \left\| y_n - \Pi_{2h_2} y_n^h \right\|_h^2 \le c(h_1^3 + h_2^4 + (\Delta t)^2)$$

and

$$\sum_{n=0}^{N-1} \Delta t \left\| p_n - \Pi_{2h_2} p_n^h \right\|_h^2 \le c(h_1^3 + h_2^4 + (\Delta t)^2). \tag{4.19}$$

Proof. By (4.18) and Theorem 4.1, there hold

$$\begin{aligned} \sum_{n=1}^{N} \Delta t \left\| \Pi_{2h_2} \Pi_{h_2} y_n - \Pi_{2h_2} y_n^h \right\|_h^2 &= \sum_{n=1}^{N} \Delta t \left\| \Pi_{2h_2} (\Pi_{h_2} y_n - y_n^h) \right\|_h^2 \\ &\le c \sum_{n=1}^{N} \Delta t \left\| \Pi_{h_2} y_n - y_n^h \right\|_h^2 \le c(h_1^3 + h_2^4 + (\Delta t)^2) \end{aligned}$$

and

$$\|y_n - \Pi_{2h_2}\Pi_{h_2}y_n\|_h^2 = \|y_n - \Pi_{2h_2}y_n\|_h^2 \le ch_2^4\,|y_n|_3^2\,.$$

So we have

$$\begin{aligned}\sum_{n=1}^{N}\Delta t\left\|y_n - \Pi_{2h_2}y_n^h\right\|_h^2 &= \sum_{n=1}^{N}\Delta t\left\|y_n - \Pi_{2h_2}\Pi_{h_2}y_n + \Pi_{2h_2}\Pi_{h_2}y_n - \Pi_{2h_2}y_n^h\right\|_h^2\\ &\le \sum_{n=1}^{N}\Delta t\,\|y_n - \Pi_{2h_2}\Pi_{h_2}y_n\|_h^2 + \sum_{n=1}^{N}\Delta t\left\|\Pi_{2h_2}\Pi_{h_2}y_n - \Pi_{2h_2}y_n^h\right\|_h^2\\ &\le c(h_1^3 + h_2^4 + (\Delta t)^2).\end{aligned}$$

Similarly, by (4.18) and Theorem 4.2, we can get (4.19). The proof is thus completed. □

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References

[1] T. Apel, S. Nicaise and J. Schöberl, *Crouzeix-Raviart type finite elements on anisotropic meshes*, Numer. Math. **89** (2001), no. 2, 193–223. https://doi.org/10.1007/pl00005466

[2] R. Becker, H. Kapp and R. Rannacher, *Adaptive finite element methods for optimal control of partial differential equations: basic concept*, SIAM J. Control. Optim. **39** (2000), no. 1, 113–132. https://doi.org/10.1137/s0363012999351097

[3] J. F. Bonnans and H. Zidani, *Optimal control problems with partially polyhedric constraints*, SIAM J. Control. Optim. **37** (1999), no. 6, 1726–1741. https://doi.org/10.1137/s0363012998333724

[4] E. Casas, *Control of an elliptic problem with pointwise state constraints*, SIAM J. Control. Optim. **24** (1986), no. 6, 1309–1318. https://doi.org/10.1137/0324078

[5] ______, *Error estimates for the numerical approximation of semilinear elliptic control problems with finitely many state constraints*, ESAIM Control. Optim. Calc. Var. **8** (2002), 345–374. https://doi.org/10.1051/cocv:2002049

[6] E. Casas and J.-P. Raymond, *Error estimates for the numerical approximation of Dirichlet boundary control for semilinear elliptic equations*, SIAM J. Control. Optim. **45** (2006), no. 5, 1586–1611. https://doi.org/10.1137/050626600

[7] Y.-P. Chen and Z.-L. Lu, *Error estimates of fully discrete mixed finite element methods for semilinear quadratic parabolic optimal control problem*, Comput. Methods Appl. Mech. Engrg. **199** (2010), no. 23-24, 1415–1423. https://doi.org/10.1016/j.cma.2009.11.009

[8] K. Deckelnick and M. Hinze, *Convergence of a finite element approximation to a state-constrained elliptic control problem*, SIAM J. Numer. Anal. **45** (2007), no. 5, 1937–1953. https://doi.org/10.1137/060652361

[9] J. Douglas Jr., J. E. Santos, D. Sheen and X. Ye, *Nonconforming Galerkin methods based on quadrilateral elements for second order elliptic problems*, M2AN Math. Model. Numer. Anal. **33** (1999), no. 4, 747–770. https://doi.org/10.1051/m2an:1999161

[10] R. S. Falk, *Approximation of a class of optimal control problems with order of convergence estimates*, J. Math. Anal. Appl. **44** (1973), no. 1, 28–47. https://doi.org/10.1016/0022-247x(73)90022-x

[11] H.-B. Guan and D.-Y. Shi, *A high accuracy NFEM for constrained optimal control problems governed by elliptic equations*, Appl. Math. Comput. **245** (2014), 382–390. https://doi.org/10.1016/j.amc.2014.07.077

[12] ———, *A stable nonconforming mixed finite element scheme for elliptic optimal control problems*, Comput. Math. Appl. **70** (2015), no. 3, 236–243. https://doi.org/10.1016/j.camwa.2015.04.005

[13] ———, *P_1-nonconforming triangular FEM for elliptic and parabolic interface problems*, Appl. Math. Mech. **36** (2015), no. 9, 1197–1212. https://doi.org/10.1007/s10483-015-1974-6

[14] H.-B. Guan, D.-Y. Shi and X.-F. Guan, *High accuracy analysis of nonconforming MFEM for constrained optimal control problems governed by Stokes equations*, Appl. Math. Lett. **53** (2016), 17–24. https://doi.org/10.1016/j.aml.2015.09.016

[15] M. Hinze, *A variational discretization concept in control constrained optimization: the linear-quadratic case*, Comput. Optim. Appl. **30** (2005), no. 1, 45–61. https://doi.org/10.1007/s10589-005-4559-5

[16] J. Hu, H.-Y. Man and A.-H. Zhou, *Constrained nonconforming rotated Q_1 element for stokes flow and planar elasticity*, Math. Numer. Sin. **27** (2005), no. 3, 311–324.

[17] G. Knowles, *Finite element approximation of parabolic time optimal control problems*, SIAM J. Control Optim. **20** (1982), no. 3, 414–427. https://doi.org/10.1137/0320032

[18] Q. Lin, L. Tobiska and A.-H. Zhou, *Superconvergence and extrapolation of nonconforming low order finite elements applied to the Poisson equation*, IMA J. Numer. Anal. **25** (2005), no. 1, 160–181. https://doi.org/10.1093/imanum/drh008

[19] J.-L. Lions, *Optimal Control of Systems Governed by Partial Differential Equations*, Springer-Verlag, Berlin, 1971. https://doi.org/10.1007/978-3-642-65024-6

[20] W.-B. Liu, W. Gong and N.-N. Yan, *A new finite element approximation of a state-constrained optimal control problem*, J. Comput. Math. **27** (2009), no. 1, 97–114.

[21] H.-P. Liu and N.-N. Yan, *Global superconvergence for optimal control problems governed by Stokes equations*, Int. J. Numer. Anal. Model. **3** (2006), no. 3, 283–302.

[22] W.-B. Liu and N.-N. Yan, *A posteriori error estimates for optimal control problems governed by parabolic equations*, Numer. Math. **93** (2003), no. 3, 497–521. https://doi.org/10.1007/s002110100380

[23] P.-B. Ming and Z.-C. Shi, *Nonconforming rotated $\mathcal{Q}_1$ element for Reissner-Mindlin plate*, Math. Models Methods Appl. Sci. **11** (2001), no. 8, 1311–1342. https://doi.org/10.1142/s0218202501001343

[24] U. Risch, *Superconvergence of a nonconforming low order finite element*, Appl. Numer. Math. **54** (2005), no. 3-4, 324–338. https://doi.org/10.1016/j.apnum.2004.09.006

[25] D.-Y. Shi, H.-B. Guan and W. Gong, *High accuracy analysis of the characteristic-nonconforming FEM for a convection-dominated transport problem*, Math. Methods Appl. Sci. **37** (2014), no. 8, 1130–1136. https://doi.org/10.1002/mma.2873

[26] D.-Y. Shi, S.-P. Mao and S.-C. Chen, *An anisotropic nonconforming finite element with some superconvergence results*, J. Comput. Math. **23** (2005), no. 3, 261–274.

[27] D.-Y. Shi and L.-F. Pei, *Low order Crouzeix-Raviart type nonconforming finite element methods for approximating Maxwell's equations*, Int. J. Numer. Anal. Model. **5** (2008), no. 3, 373–385.

[28] D.-Y. Shi, H.-H. Wang and Y.-P. Du, *An anisotropic nonconforming finite element method for approximating a class of nonlinear Sobolev equations*, J. Comput. Math. **27** (2009), no. 2-3, 299–314.

[29] D.-Y. Shi and C. Xu, EQ_1^{rot} *nonconforming finite element approximation to Signorini problem*, Sci. China Math. **56** (2013), no. 6, 1301–1311. https://doi.org/10.1007/s11425-013-4615-z

[30] D.-Y. Shi, C. Xu and J.-H. Chen, *Anisotropic nonconforming EQ_1^{rot} quadrilateral finite element approximation to second order elliptic problems*, J. Sci. Comput. **56** (2013), no. 3, 637–653. https://doi.org/10.1007/s10915-013-9690-3

[31] Y.-L. Tang and Y.-C. Hua, *Superconvergence analysis for parabolic optimal control problems*, Calcolo **51** (2014), no. 3, 381–392. https://doi.org/10.1007/s10092-013-0091-7

[32] N.-N. Yan and Z.-J. Zhou, *A priori and a posteriori error analysis of edge stabilization Galerkin method for the optimal control problem governed by convection-dominated diffusion equation*, J. Comput. Appl. Math. **223** (2009), no. 1, 198–217. https://doi.org/10.1016/j.cam.2008.01.006

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