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Construction of Periodic Solutions for Nonlinear Wave Equations by a Para-differential Method

Bochao Chen, Yixian Gao* and Yong Li

Abstract. This paper is concerned with the existence of families of time-periodic solutions for the nonlinear wave equations with Hamiltonian perturbations on one-dimensional tori. We obtain the result by a new method: a para-differential conjugation together with a classical iteration scheme, which have been used for the nonlinear Schrödinger equation in [22]. Avoiding the use of KAM theorem and Nash-Moser iteration method, though a para-differential conjugation, an equivalent form of the investigated nonlinear wave equations can be obtained, while the frequencies are fixed in a Cantor-like set whose complement has small measure. Applying the non-resonant conditions on each finite-dimensional subspaces, solutions can be constructed to the block diagonal equation on the finite subspace by a classical iteration scheme.

1. Introduction

This paper is devoted to investigate the existence of the time-periodic solutions for the nonlinear wave equation. A quantity of research has been did concerning the periodic or quasi-periodic solutions of nonlinear wave equations. In [37], Pöschel considered the nonlinear wave equation with cubic, sine-Gordon or sinh-Gordon nonlinearities under Dirichlet boundary conditions

$$\begin{cases} u_{tt} - u_{xx} + mu + f(u) = 0, \\ u(t,0) = u(t,\pi) = 0, \quad x \in [0,\pi],\ t \in (-\infty,+\infty), \end{cases}$$

where $m > 0$, the nonlinearity

$$f(u) = au^3 + \sum_{k \geq 5} f_k u^k \quad \text{with } a \neq 0$$

is a real analytic, odd function of u. By an abstract KAM theorem, the results on the existence and stability of periodic and quasi-periodic solutions were obtained for the above

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*Corresponding author.

equation. For the periodic potential function wave equation under periodic boundary conditions

$$\begin{cases} u_{tt} - u_{xx} + V(x)u = f(u), \\ u(t,x) = u(t,x+2\pi), \\ u_t(t,x) = u_t(t,x+2\pi), \end{cases}$$

where the nonlinearity f is an analytic function vanishing together with derivative at $u = 0$, in [20], Chierchia and You arrived at the existence and linear stability of lower-dimensional tori (composing by quasi-periodic solutions) by an infinite-dimensional KAM theorem. In [12], using the Lyapunov-Schmidt technique, Bourgain proved the existence of nontrivial space and time-periodic solutions of the equation

$$u_{tt} - \Delta u + mu + \delta^2 u^3 = 0$$

in a neighborhood of the monochromatic wave on d-dimensional torus, where $\delta > 0$ is a small parameter, $m > 0$ is the given constant. For completely resonant nonlinear wave equations, Berti and Bolle [8] studied the existence of cantor families of periodic solutions of

$$\begin{cases} u_{tt} - u_{xx} + f(x,u) = 0, \\ u(t,0) = u(t,\pi) = 0, \end{cases}$$

where the nonlinearity

$$f(x,u) = a_p(x)u^p + O(u^{p+1}), \quad p \geq 2 \tag{1.1}$$

is analytic in u and H^1 in x. The proof relies on a Lyapunov-Schmidt decomposition and a Nash-Moser iteration.

In this paper, using the para-differential technique, we consider the forced oscillations equation in one-dimensional as [2, 38, 39], with the nonlinearities depending on the time and space variables:

$$u_{tt} - u_{xx} + mu = \epsilon \partial_u F(\omega t, x, u; \epsilon) + \epsilon f(\omega t, x), \quad x \in \mathbb{T} := \mathbb{R}/2\pi\mathbb{Z}. \tag{1.2}$$

In (1.2), $m > 0$, $\epsilon > 0$ is small enough and $\omega > 0$ is a frequency parameter. Moreover f is 2π-periodic in time and smooth on $\mathbb{R} \times \mathbb{T}$ with value in $\mathbb{R}$; the nonlinearity term F is 2π-periodic in time and satisfies

$$\partial_z^\alpha F(\,\cdot\,, \,\cdot\,, z; \epsilon)\big|_{z=0} \equiv 0 \quad \text{for } \alpha \leq 2. \tag{1.3}$$

If F is analytic function of u, and $f(\omega t, x)$ is vanishing in (1.2), we can consider the similar nonlinearities $\widetilde{f}(t,x,u) = a_p(t,x)u^p + O(u^{p+1})$, $p \geq 2$ as equation (1.1). The nonlinearity terms F are requested to be infinitely many times differentiable, i.e.,

$$F(t,x,z,\epsilon) \in C^\infty(\mathbb{R} \times \mathbb{T} \times \mathbb{R} \times [0,1]; \mathbb{R}). \tag{1.4}$$

Let us recall some known results about this type of problems. The problem of looking for time-periodic solutions to the nonlinear PDEs has been paid high attention since the pioneering paper of Rabinowitz [38, 39]. He rephrased the problem as a variational problem and proved the existence of periodic solutions whenever the time period T is a rational multiple of the length of spatial interval, and the nonlinearity f is monotonic in u. Subsequently, many authors, such as Bahri, Brézis, Corn, Nirenberg etc., had used and developed Rabinowitz's variational methods to prove both perturbative and global results, see [2, 16–18]. And some recent papers can be found in [1, 31, 32]. Most importantly, their time period had to be a rational multiple of their space period so that the wave operator $\partial_{tt} - \partial_{xx}$, acting on the corresponding space of x- and t-periodic functions, had discrete spectrum. For this reason, they also did not come in Cantor families. The case in which T is some irrational multiple of π (the space period) had also been investigated by Fečkan [25] and McKenna [36], where the frequencies are essentially the numbers whose continued fraction expansion is bounded. In other case, it appears the "small divisor" problem. The Kolmogorov-Arnold-Moser (KAM) method was an efficient tool to deal with this problem. In the later of 1980's, an approach via the KAM method was developed from the viewpoint of infinite-dimensional Hamiltonian partial differential equations by Kuksin [34], Eliasson [23] and Wayne [40]. This method allowed one to obtain solutions whose periods are irrational multiples of the length of the spatial interval, and it is also easily extended to construct quasi-periodic solutions for class of Hamiltonian PDEs, see [5, 7, 19, 20, 24, 26–29, 35, 41]. Later, in order to overcome some limitations of the KAM approach, in [11, 13, 15, 21], Craig, Wayne and Bourgain retrieved the Nash-Moser iteration method together with the Lyapunov-Schmidt reduction which involves the Green's function analysis and the control of the inverse of infinite matrices with small eigenvalues, successfully constructed the periodic and quasi-periodic solutions of PDEs with Dirichlet boundary conditions or periodic boundary conditions. The advantage of this approach is to require only the "first order Melnikov' non-resonance conditions, which are essentially the minimal assumptions. On the other hand, the main difficulty of this strategy lies in the inversion of the linearized operators obtained at each step of the iteration, and in achieving suitable estimates for their inverses in high (analytic) norms. Indeed these operators come from linear PDEs with non-constant coefficients and are small perturbations of a diagonal operator having arbitrarily small eigenvalues. Some recent results about Nash-Moser theorems can be found in [3, 4, 9, 10] and the references therein. There are actually a few results concerning existence of periodic solutions which do not appeal to Nash-Moser or KAM methods. Gentile and Procesi [30] verified the existence of Gevrey smooth periodic solutions. Their approach is based on a standard Lyapunov-Schmidt decomposition, which decomposes the original PDEs into two equations, traditionally called the P and Q equations, combined with renormalized expansions of Lindstedt series to handle the "small divisor problem". Bambusi and Paleari [6] constructed such solutions without making use of Nash-Moser or KAM methods, by a combination of the Lyapunov-Schmidt approach and an implicit function theorem, but only for a family of frequency parameters of measure zero (instead of a set of parameters whose complement has small measure). In the present paper, inspired by the technique of [22], avoiding the use of Nash-Moser theorems and KAM methods, by the para-differential method together with a standard iteration scheme, we establish an existence result about periodic solutions of (1.2) with nonlinearities $\partial_u F(\omega t, x, u, \epsilon)$, which is infinitely many times differentiable and can rely on space and time variables. In [22], Delort proposed that this method does not seem to be adapted to find periodic solutions of nonlinear wave equations in high-dimensional spaces, since the specific separation property does not hold. However, for the nonlinear wave equation on one-dimensional tori, we can obtain the separation property of the eigenvalues of $\sqrt{-\partial_{xx}+m}$. The properties of the operator in this paper is different from the Schördinger operator [22], and we will meet some similar difficulties in diagonalization of the equation. In a Nash-Moser iteration scheme, ones have to consider the treatment of losses of derivative coming from small divisors and the convergence of the sequence of approximations at the same time. Using para-differential approach, since such losses of derivative coming from small divisors will be compensated by the smoothing properties of the operator in the right-hand side of the equation, we don't worry about the convergence of the sequence of approximations of the solution when we treat small divisors. This approach allows one to separate on the one hand the treatment of losses of derivatives coming from small divisors, and on the other hand the question of convergence of the sequence of approximations.

This paper is organized as follows: in Section 2 we state the main theorem. We devote Section 3 to perform the first reduction of the equation applying the fixed point theorem with parameters. Then the equation on $\widetilde{\mathcal{H}}^\sigma$ is equivalent to the one on $\mathcal{H}^\sigma$, where $\widetilde{\mathcal{H}}^\sigma$, $\mathcal{H}^\sigma$ are, respectively, defined in (2.2), (2.5). The aim of Section 4 is to describe the para-linearization of the equation. We first define classes of convenient para-differential operators which can be used in the following; then we para-linearize the equation, and reduce it into

$$(\omega^2\partial_{tt}-\partial_{xx}+m+\epsilon V)u=\epsilon Ru+\epsilon f,$$

where V is a para-differential operator of order zero depending on u, ω, ϵ, self-adjoint, and R is a smoothing operator depending on u, ω, ϵ. The fifth section is the core of this paper. For a new unknown w, owing to a para-differential conjugation, we transform the equation on $\mathcal{H}^\sigma$ into a new form as follows:

$$(\omega^2\partial_{tt}-\partial_{xx}+m+\epsilon V_{\mathrm{D}})w=\epsilon Rw+\epsilon f,$$

where V_{D} depends on u, ω, ϵ. The operator V_{D} is block diagonal corresponding to an orthogonal decomposition of $L^2(\mathbb{T})$, which is in a sum of finite-dimensional subspaces introduced by Bourgain [14]. The operator R is still smoothing. In Section 6, our main goal is to construct the solution of the block diagonal equation by a standard iteration scheme. Combining with the non-resonant conditions (6.4), we show that $\omega^2\partial_{tt} - \partial_{xx} + m + \epsilon V_{\mathrm{D}}$ is invertible on each block when ω outside a subset. To guarantee that the measure of excluded ω remains small, we have to allow small divisors when inverting $\omega^2\partial_{tt} - \partial_{xx} + m + \epsilon V_{\mathrm{D}}$. While, such losses of derivatives coming from small divisors may be compensated by the smoothing operator R on the right-hand side of the equation. At the same time, we can construct an approximate sequence of the solution. We conclude the paper with some remarks in Section 7.

2. Main results

2.1. Statement of the main theorem

To fix ideas, we shall take ω inside a fixed compact sub-interval of $(0,\infty)$, such as $\omega \in [1,2]$ (in fact any compact interval $[a,b] \subset (0,\infty)$ is also true). After a time rescaling, equation (1.2) is changed into

$$(\omega^2\partial_{tt} - \partial_{xx} + m)u = \epsilon\partial_u F(t,x,u,\epsilon) + \epsilon f(t,x), \quad x \in \mathbb{T}. \tag{2.1}$$

In this paper, our goal is to present the existence of 2π-periodic solutions in time of (2.1) for small enough ϵ and for ω outside a subset of small measure. Denote by $\mathcal{D}'(\mathbb{T}\times\mathbb{T})$ the space of generalized functions on $\mathbb{T}\times\mathbb{T}$. Let us look for the solutions defined on $\mathbb{T}\times\mathbb{T}$ in the Sobolev space $\widetilde{\mathcal{H}}^\sigma$ with $\sigma \in \mathbb{R}$

$$\widetilde{\mathcal{H}}^\sigma := \widetilde{\mathcal{H}}^\sigma(\mathbb{T}\times\mathbb{T};\mathbb{R}) = \left\{u \in \mathcal{D}'(\mathbb{T}\times\mathbb{T}); \|u\|_{\widetilde{\mathcal{H}}^\sigma} < +\infty \text{ with } \overline{u}_{j,n} = u_{-j,-n}\right\}, \tag{2.2}$$

where

$$\|u\|^2_{\widetilde{\mathcal{H}}^\sigma} := \sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}}(1+j^2+n^2)^\sigma \left|u_{j,n}\right|^2, \quad u_{j,n} := \frac{1}{2\pi}\int_{\mathbb{T}\times\mathbb{T}} e^{-\mathrm{i}tj-\mathrm{i}xn}u(t,x)\,\mathrm{d}t\,\mathrm{d}x$$

for $(j,n) \in \mathbb{Z}\times\mathbb{Z}$.

We now state the main theorem as follows:

Theorem 2.1. *For $m > 0$, there exists a constant $B > 0$, a subset $\mathcal{O} \subset [1,2]\times(0,1]$ and a constant $\delta_0 \in (0,1]$ small enough such that for any $\delta \in (0,\delta_0]$, any $\epsilon \in (0,\delta^2]$, and any $\omega \in [1,2]$,*

- *when $(\omega,\epsilon) \notin \mathcal{O}$, equation (2.1) has a solution $u \in \widetilde{\mathcal{H}}^s(\mathbb{T}\times\mathbb{T};\mathbb{R})$ with $\|u\|_{\widetilde{\mathcal{H}}^s} \le B\epsilon\delta^{-1}$.*

- *the excluded measure of* ω *satisfies*

$$\text{meas}\,\{\omega \in [1,2];\ (\omega,\epsilon) \in \mathcal{O}\} \leq B\delta. \tag{2.3}$$

The proof of the above theorem are given at the end of Section 6.

2.2. Spaces of functions and notations

In this subsection, we first give some spaces and notations which will be used in the following. First, we consider the "separation property" of the spectrum. The spectrum of operator $\sqrt{-\partial_{xx}+m}$ is

$$\lambda_n = \sqrt{n^2+m}, \quad n \in \mathbb{Z}.$$

Then for any $n_1 \in \{-n, n\}$, $n_2 \in \{-n', n'\}$ with $n \neq n'$, $n, n' \in \mathbb{Z}$, we have

$$|\lambda_{n_1} - \lambda_{n_2}| = \left|\sqrt{n_1^2+m} - \sqrt{n_2^2+m}\right| = \frac{|(|n_1|+|n_2|)(|n_1|-|n_2|)|}{\sqrt{n_1^2+m}+\sqrt{n_2^2+m}},$$

where $\{-n, n\}$ denotes the two points set about n and $-n$. Obviously, it has

$$|\lambda_{n_1} - \lambda_{n_2}| > \begin{cases} \frac{1}{\sqrt{2}}\,||n_1|-|n_2|| \geq \frac{1}{\sqrt{2}} & \text{when } 0 < m \leq 1, \\ \frac{1}{\sqrt[n]{2m}}\,||n_1|-|n_2|| \geq \frac{1}{\sqrt{2m}} & \text{when } m > 1, \end{cases} \tag{2.4}$$

which shows that the eigenvalues of the wave equation on one-dimensional space have a nice separation property. This is similar to geometric properties of the spectrum of operator $-\Delta$ on $\mathbb{T}^d$ given by Bourgain in [14]. Considering t as a parameter, we denote by Π_n for any $n \in \mathbb{Z}$ the spectral projector

$$\Pi_n u = u_n(t)\frac{e^{\mathrm{i}xn}}{\sqrt{2\pi}} = \sum_{j\in\mathbb{Z}} u_{j,n}\frac{e^{\mathrm{i}tj+\mathrm{i}xn}}{2\pi}, \quad u(t,x) \in \mathcal{D}'(\mathbb{T}\times\mathbb{T}).$$

Let us set $\widetilde{\Pi}_0 = \Pi_0$, $\widetilde{\Pi}_n = \Pi_n + \Pi_{-n}$. Define a closed subspace $\mathcal{H}^\sigma$ of $\widetilde{\mathcal{H}}^\sigma$ by

$$\begin{aligned} \mathcal{H}^\sigma &:= \mathcal{H}^\sigma(\mathbb{T}\times\mathbb{T};\mathbb{R}) \\ &= \bigcap_{n\in\mathbb{N}} \Big\{ u \in \widetilde{\mathcal{H}}^\sigma(\mathbb{T}\times\mathbb{T};\mathbb{R}):\ u_{j,n'} = 0, \text{ for } n' \in \{-n,n\} \text{ with } n \in \mathbb{Z}, \\ &\qquad\qquad \forall\, j \text{ with } |j| > K_0\langle n\rangle \text{ or } |j| < K_0^{-1}\langle n\rangle \Big\}, \end{aligned} \tag{2.5}$$

where K_0 is large enough, $\langle n\rangle = (1+|n|^2)^{1/2}$ for $n \in \mathbb{N}$. In other words, for $n' \in \{-n, n\}$ with $n \in \mathbb{N}$, non-vanishing terms $u_{j,n'}$ have to satisfy $K_0^{-1}\langle n\rangle \leq |j| \leq K_0\langle n\rangle$ when $u \in \mathcal{H}^\sigma$. This implies that the restriction to $\mathcal{H}^\sigma$ of the $\widetilde{\mathcal{H}}^\sigma$-norm given by (2.2) is equivalent to the square root of

$$\sum_{j\in\mathbb{Z}}\sum_{n\in\mathbb{Z}} \langle n\rangle^{2\sigma} |u_{j,n}|^2, \tag{2.6}$$

and the the square root of

$$\sum_{n\in\mathbb{Z}} \langle n\rangle^{2\sigma} \|\Pi_n u\|^2_{L^2(\mathbb{T}\times\mathbb{T};\mathbb{R})}, \tag{2.7}$$

and the the square root of

$$\sum_{n\in\mathbb{N}} \langle n\rangle^{2\sigma} \left\|\widetilde{\Pi}_n u\right\|^2_{L^2(\mathbb{T}\times\mathbb{T};\mathbb{R})}. \tag{2.8}$$

Furthermore, we denote by $\mathcal{F}^\sigma$ the orthogonal complement of $\mathcal{H}^\sigma$ in $\widetilde{\mathcal{H}}^\sigma$. By (2.5), when $u \in \mathcal{F}^\sigma$, if for $n' \in \{-n, n\}$ with $n \in \mathbb{N}$, $u_{j,n'} \neq 0$, then it has $|j| > K_0 \langle n\rangle$ or $|j| < K_0^{-1} \langle n\rangle$. In addition, we have to fix some real number $\sigma_0 > 3/2$. For $\sigma \geq \sigma_0$, $\widetilde{\mathcal{H}}^\sigma$ is a Banach algebra with respect to multiplication of functions, i.e.,

$$u_1, u_2 \in \widetilde{\mathcal{H}}^\sigma \implies \|u_1u_2\|_{\widetilde{\mathcal{H}}^\sigma} \leq C \|u_1\|_{\widetilde{\mathcal{H}}^\sigma} \|u_2\|_{\widetilde{\mathcal{H}}^\sigma}.$$

In the remainder of this subsection, we set some new notations. Let us denote by $B_q(\mathcal{H}^\sigma)$ for $\sigma \in \mathbb{R}$, $q > 0$ the open ball with center 0, radius q in $\mathcal{H}^\sigma$, and denote by $\mathcal{L}(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2})$ for $\sigma_1 \in \mathbb{R}$, $\sigma_2 \in \mathbb{R}$ the space of continuous linear operators from $\mathcal{H}^{\sigma_1}$ to $\mathcal{H}^{\sigma_2}$. Specially, $\mathcal{L}(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_1})$ is written as $\mathcal{L}(\mathcal{H}^{\sigma_1})$. Moreover $\mathcal{L}_2(\mathcal{H}^{\sigma_1} \times \mathcal{H}^{\sigma_2}, \mathcal{H}^{\sigma_3})$ stands for the space of continuous bilinear operators from $\mathcal{H}^{\sigma_1} \times \mathcal{H}^{\sigma_2}$ to $\mathcal{H}^{\sigma_3}$ for $\sigma_1 \in \mathbb{R}$, $\sigma_2 \in \mathbb{R}$, $\sigma_3 \in \mathbb{R}$. If the operator $T \in \mathcal{L}(\mathcal{H}^{\sigma_1}, \mathcal{H}^{\sigma_2})$, then the transport operator ${}^tT \in \mathcal{L}(\mathcal{H}^{\sigma_2}, \mathcal{H}^{\sigma_1})$.

3. An equivalent formulation on $\mathcal{H}^\sigma$

3.1. Functional setting

We now give some definitions of function space that will be used in the following. For brevity, let us denote by $\mathcal{H}_j^\sigma$, $j = 1, 2$ any one of the spaces $\mathcal{H}^\sigma$, $\mathcal{F}^\sigma$, $\widetilde{\mathcal{H}}^\sigma$.

Definition 3.1. For any $\sigma \geq \sigma_0$ and any open subset X of $\mathcal{H}_1^\sigma$, $k \in \mathbb{Z}$, denote the space of C^∞ maps $G\colon X \to \mathcal{H}_2^{\sigma-k}$ by $\Phi^{\infty,k}(X, \mathcal{H}_2^{\sigma-k})$, such that for any $u \in X \cap \mathcal{H}_1^s$ with $s \geq \sigma$, $G(u) \in \mathcal{H}_2^{s-k}$. Furthermore, the linear map $\mathrm{D}_u G(u) \in \mathcal{L}(\mathcal{H}_1^\sigma, \mathcal{H}_2^{\sigma-k})$ extends as an element of $\mathcal{L}(\mathcal{H}_1^{\sigma'}, \mathcal{H}_2^{\sigma'-k})$ for any $u \in X \cap \mathcal{H}_1^s$ with $s \geq \sigma$ and any $\sigma' \in [-s, s]$. Moreover, $u \to \mathrm{D}_u G(u)$ is smooth from $X \cap \mathcal{H}_1^s$ to the preceding space. In addition, for any $u \in X \cap \mathcal{H}_1^s$ with $s \geq \sigma$, the bilinear map $\mathrm{D}_u^2 G(u) \in \mathcal{L}_2(\mathcal{H}_1^\sigma \times \mathcal{H}_1^\sigma, \mathcal{H}_2^{\sigma-k})$ extends as an element of $\mathcal{L}_2(\mathcal{H}_1^{\sigma_1} \times \mathcal{H}_1^{\sigma_2}, \mathcal{H}_2^{-\sigma_3-k})$ for any $\{\sigma_1, \sigma_2, \sigma_3\} = \{\sigma', -\sigma', \max(\sigma_0, \sigma')\}$ with $\sigma' \in [0, s]$. In the same way, $u \to \mathrm{D}_u^2 G(u)$ is smooth from $X \cap \mathcal{H}_1^s$ to the preceding space.

Definition 3.2. For any $\sigma \geq \sigma_0$ and any open subset X of $\mathcal{H}_1^\sigma$, $k \in \mathbb{Z}$, let us denote the space of C^1 functions $\Phi\colon X \to \mathbb{R}$ by $C^{\infty,k}(X, \mathbb{R})$, such that for any $u \in X \cap \mathcal{H}_1^s$ with $s \geq \sigma$, $\nabla_u \Phi(u) \in \mathcal{H}_1^{s-k}$ and $u \to \nabla_u \Phi(u)$ belongs to $\Phi^{\infty,k}(X, \mathcal{H}_1^{\sigma-k})$.

Remark 3.3. For $n \in \mathbb{N}$, $\mathrm{D}_u^n G(u)$ denotes by the n-th order Frechet derivative of $G(u)$ with respect to u.

In the remainder of this paper, we shall consider elements $G(u,\omega,\epsilon)$, $\Phi(u,\omega,\epsilon)$ of the preceding spaces depending on (ω,ϵ), where (ω,ϵ) stays in a bounded domain of $\mathbb{R}^2$. If G, $\partial_\omega G$, $\partial_\epsilon G$ (resp. Φ, $\partial_\omega\Phi$, $\partial_\epsilon\Phi$) satisfy the conditions of Definition 3.1 (resp. Definition 3.2), we shall say that G, Φ are C^1 in (ω,ϵ).

The following two lemmas and a corollary are applied to analyze the properties of the functionals Φ_1, Φ_2 which are given by (3.7) and (3.8) respectively, and the proofs can be found in the appendix in [22].

Lemma 3.4. *If $s > 3/2$, then $\widetilde{\mathcal{H}}^s(\mathbb{T}\times\mathbb{T};\mathbb{C}) \subset L^\infty$. Furthermore, if F is a smooth function defined on $\mathbb{T}\times\mathbb{T}\times\mathbb{C}$ satisfying $F(t,x,0) \equiv 0$, there is some continuous function $\tau \to C(\tau)$, such that for any $u \in \widetilde{\mathcal{H}}^s$, $F(\,\cdot\,,u) \in \widetilde{\mathcal{H}}^s$ with $\|F(\,\cdot\,,u)\|_{\widetilde{\mathcal{H}}^s} \leq C(\|u\|_{\mathcal{L}^\infty})\,\|u\|_{\widetilde{\mathcal{H}}^s}$.*

Lemma 3.5. *If $s > 3/2$, when $u \in \widetilde{\mathcal{H}}^s$, $v \in \widetilde{\mathcal{H}}^{\sigma'}$, then $uv \in \widetilde{\mathcal{H}}^{\sigma'}$ with $\sigma' \in [-s,s]$. Moreover, for any $\sigma \in \mathbb{R}$, any $\sigma_0 > 3/2$, $\widetilde{\mathcal{H}}^\sigma \cdot \widetilde{\mathcal{H}}^{-\sigma} \subset \widetilde{\mathcal{H}}^{-\max(\sigma,\sigma_0)}$.*

Corollary 3.6. *If $F\colon \mathbb{T}\times\mathbb{T}\times\mathbb{C} \to \mathbb{C}$ is a smooth function with $F(t,x,0) \equiv 0$, then for any $\sigma > 3/2$, $u \to F(\,\cdot\,,u)$ is a smooth map from $\widetilde{\mathcal{H}}^\sigma$ to $\widetilde{\mathcal{H}}^\sigma$.*

Define the following map for all $\sigma \geq \sigma_0$, $\sigma' > 0$

$$\begin{aligned} G\colon \widetilde{H}^\sigma \cap \widetilde{H}^{\sigma'} &\to \quad \widetilde{H}^{\sigma'} \\ u \quad &\mapsto F(t,x,u,\epsilon), \end{aligned}$$

where F satisfies the conditions (1.3)–(1.4).

Lemma 3.7. *The map G is C^2 with respect to u and satisfies for all $h \in \widetilde{H}^\sigma \cap \widetilde{H}^{\sigma'}$*

$$\mathrm{D}_u\, G(u)[h] = \partial_u F(t,x,u,\epsilon)h, \quad \mathrm{D}_u^2\, G(u)[h,h] = \partial_u^2 F(t,x,u,\epsilon)h^2.$$

Proof. Corollary 3.6 implies that G is C^2 respect to u. Noting that by the continuity property of $u \mapsto \partial_u F(t,x,u,\epsilon)$, it has

$$\begin{aligned} &\|F(t,x,u+h,\epsilon) - F(t,x,u,\epsilon) - \partial_u F(t,x,u,\epsilon)h\|_{\widetilde{\mathcal{H}}^{\sigma'}} \\ &= \left\| h \int_0^1 \left(\partial_u F(t,x,u+\tau h,\epsilon) - \partial_u F(t,x,u,\epsilon)\right) \mathrm{d}\tau \right\|_{\widetilde{\mathcal{H}}^{\sigma'}} \\ &\leq C(\sigma')\,\|h\|_{\widetilde{\mathcal{H}}^{\max(\sigma,\sigma')}} \max_{\sigma\in[0,1]} \|\partial_u F(t,x,u+\tau h,\epsilon) - \partial_u F(t,x,u,\epsilon)\|_{\widetilde{\mathcal{H}}^{\max(\sigma,\sigma')}}\,. \end{aligned}$$

Therefore for all $h \in \widetilde{H}^\sigma \cap \widetilde{H}^{\sigma'}$, we have

$$\mathrm{D}_u\, G(u)[h] = \partial_u F(t,x,u,\epsilon)h$$

and $u \mapsto \mathrm{D}_u G(u)$ is continuous. Furthermore, it also has

$$\begin{aligned}&\partial_u F(t,x,u+\tau h,\epsilon)h - \partial_u F(t,x,u,\epsilon)h - \partial_u^2 F(t,x,u,\epsilon)h^2\\ &= h^2 \int_0^1 \big(\partial_u^2 F(t,x,u+\tau h,\epsilon) - \partial_u^2 F(t,x,u,\epsilon)\big)\,\mathrm{d}\tau.\end{aligned}$$

Similarly, we can obtain that G is twice differentiable with respect to u and $u \mapsto \mathrm{D}_u^2 G(u)$ is continuous. □

Lemma 3.8. *Let $\sigma \geq \sigma_0$, $k \in \mathbb{N}$, X and Y be the open subsets of $\mathcal{H}_1^\sigma$ and $\mathcal{H}_2^{\sigma+k}$ respectively. If $G \in \Phi^{\infty,-k}(X, \mathcal{H}_2^{\sigma+k})$, $\Phi \in C^{\infty,k}(Y,\mathbb{R})$, and $G(X) \subset Y$, then $\Phi \circ G \in C^{\infty,0}(X,\mathbb{R})$.*

Proof. We restrict our attention to $u \in X \cap \mathcal{H}_1^s$ with $s \geq \sigma$. This reads $G(u) \in Y \cap \mathcal{H}_2^{s+k}$. Definitions 3.1–3.2 indicate that

$$\mathrm{D}_u G(u) \in \mathcal{L}(\mathcal{H}_1^{\sigma'}, \mathcal{H}_2^{\sigma'+k}) \subset \mathcal{L}(\mathcal{H}_1^{\sigma'}, \mathcal{H}_2^{\sigma'}) \quad \text{for } |\sigma'| \leq s \tag{3.1}$$

and that $\nabla_u \Phi(G(u)) \in \mathcal{H}_2^s$ for $s \geq \sigma$. Consequently, we have for any σ' with $|\sigma'| \leq s$,

$$\mathrm{D}_u(\nabla_u \Phi(G(u))) \in \mathcal{L}(\mathcal{H}_2^{\sigma'+k}, \mathcal{H}_2^{\sigma'}). \tag{3.2}$$

Owing to formula (3.1) together with the fact that $\nabla_u(\Phi \circ G)(u)$ is equal to ${}^t\mathrm{D}_u G(u) \cdot (\nabla_u \Phi(G(u)))$, we deduce

$$\nabla_u(\Phi \circ G)(u) \in \mathcal{H}_1^s.$$

Let us check $\nabla(\Phi \circ G) \in \Phi^{\infty,0}(X, \mathcal{H}_1^\sigma)$. Write $\mathrm{D}_u(\nabla_u(\Phi \circ G)(u)) \cdot h$ as the sum of the following two terms

$${}^t\mathrm{D}_u G(u) \cdot ((\mathrm{D}_u \nabla_u \Phi)(G(u)) \cdot \mathrm{D}_u G(u) \cdot h), \tag{3.3a}$$

$$(\mathrm{D}_u({}^t\partial_u G)(u) \cdot h) \cdot \nabla_u \Phi(G(u)). \tag{3.3b}$$

Formulae (3.1) and (3.2) show that (3.3a) belongs to $\mathcal{H}_1^{\sigma'}$ with $\sigma' \in [0,s]$. According to integrating (3.3b) against $h' \in \mathcal{H}_1^{-\sigma'}$, it yields that

$$\int \big((\mathrm{D}_u({}^t\mathrm{D}_u G)(u) \cdot h) \cdot \nabla \Phi_u(G(u))\big)\, h' \,\mathrm{d}t\,\mathrm{d}x = \int \nabla_u \Phi(G(u)) \cdot \mathrm{D}_u^2 G(u) \cdot (h,h')\,\mathrm{d}t\,\mathrm{d}x. \tag{3.4}$$

Definition 3.1 gives that $\mathrm{D}_u^2 G(u) \cdot (h,h') \in \mathcal{H}_2^{-\max(\sigma_0,\sigma')+k}$. Combining this with the fact that $\nabla_u \Phi(G(u))$ is in $\mathcal{H}_2^s$ which is contained in $\mathcal{H}_2^{\max(\sigma_0,\sigma')}$, we get that the right-hand side of (3.4) is a continuous linear form with $h' \in \mathcal{H}_1^{-\sigma'}$.

Next, from integrating $\mathrm{D}_u^2 (\nabla_u(\Phi \circ G)(u)) \cdot (h_1,h_2)$ with $(h_1,h_2) \in \mathcal{H}_1^{\sigma_4} \times \mathcal{H}_1^{\sigma_5}$ against $h_3 \in \mathcal{H}_1^{\sigma_6}$, it follows that

$$\int \big(\mathrm{D}_u^2(\nabla_u(\Phi \circ G)(u)) \cdot (h_1,h_2)\big)\, h_3 \,\mathrm{d}t\,\mathrm{d}x = \mathrm{D}_u^2 \int (\nabla_u \Phi(G(u)))\,(\mathrm{D}_u G(u) \cdot h_3)\,\mathrm{d}t\,\mathrm{d}x, \tag{3.5}$$

where $\{\sigma_4, \sigma_5, \sigma_6\} = \{\sigma', -\sigma', \max(\sigma_0, \sigma')\}$ with $\sigma' \in [0, s]$. The right-hand side of (3.5) is the sum of the following four terms

$$\int (\nabla_u \Phi(G(u))) \left(\mathrm{D}_u^3 G(u) \cdot (h_1, h_2, h_3)\right) \mathrm{d}t \, \mathrm{d}x, \tag{3.6a}$$

$$\int (\mathrm{D}_u(\nabla_u \Phi(G(u))) \cdot h_1) \left(\mathrm{D}_u^2 G(u) \cdot (h_2, h_3)\right) \mathrm{d}t \, \mathrm{d}x, \tag{3.6b}$$

$$\int \left((\mathrm{D}_u \nabla_u \Phi)(G(u)) \cdot \mathrm{D}_u^2 G(u) \cdot (h_1, h_2)\right) (\mathrm{D}_u G(u) \cdot h_3) \, \mathrm{d}t \, \mathrm{d}x, \tag{3.6c}$$

$$\int \left((\mathrm{D}_u^2 \nabla \Phi)(G(u)) \cdot (\mathrm{D}_u G(u) \cdot h_1, \mathrm{D}_u G(u) \cdot h_2)\right) (\mathrm{D}_u G(u) \cdot h_3) \, \mathrm{d}t \, \mathrm{d}x, \tag{3.6d}$$

where $(h_1, h_2) \in \mathcal{H}_1^{\sigma_4} \times \mathcal{H}_1^{\sigma_5}$. We just consider $h_1 \in \mathcal{H}_1^{\sigma'}$, $h_2 \in \mathcal{H}_1^{-\sigma'}$ and $h_3 \in \mathcal{H}_1^{\max(\sigma_0, \sigma')}$ with $\sigma' \in [0, s]$. In (3.6a), since $u \to \mathrm{D}_u^2 G(u)$ is C^1 on $X \cap \mathcal{H}_1^{\max(\sigma_0, \sigma')}$ with values in $\mathcal{L}_2(\mathcal{H}_1^{\sigma'} \times \mathcal{H}_1^{-\sigma'}; \mathcal{H}_2^{-\max(\sigma_0, \sigma')+k})$, we obtain

$$\mathrm{D}_u^3 G(u) \cdot (h_1, h_2, h_3) \in \mathcal{H}_2^{-\max(\sigma_0, \sigma')+k}.$$

Combing this with $\nabla_u \Phi(G(u)) \in \mathcal{H}_2^s \subset \mathcal{H}_2^{\max(\sigma_0, \sigma')}$ for $s \geq \sigma' \geq 0$ and $s \geq \sigma$, the two factors in (3.6a) are integrable. In (3.6b), Definitions 3.1–3.2 verify

$$\mathrm{D}_u^2 G(u) \cdot (h_2, h_3) \in \mathcal{H}_2^{-\max(\sigma_0, \sigma')+k} \subset \mathcal{H}_2^{-\sigma'+k}, \quad \mathrm{D}_u(\nabla_u \Phi(G(u))) \cdot h_1 \in \mathcal{H}_2^{\sigma_1}.$$

Consequently, the two factors in (3.6b) are integrable. In (3.6c), formula (3.1) and Definitions 3.1–3.2 lead to

$$\mathrm{D}_u G(u) \cdot h_3 \in \mathcal{H}_2^{-\max(\sigma_0, \sigma')+k},$$

and

$$(\mathrm{D}_u \nabla_u \Phi)(G(u)) \cdot \mathrm{D}_u^2 G(u) \cdot (h_1, h_2) \in \mathcal{H}_2^{-\max(\sigma_0, \sigma')+k},$$

which implies that the two factors in (3.6c) are integrable. In (3.6d), from $\mathrm{D}_u G(u) \cdot h_1 \in \mathcal{H}_2^{\sigma'+k} \subset \mathcal{H}_2^{\sigma'}$ and $\mathrm{D}_u G(u) \cdot h_2 \in \mathcal{H}_2^{-\sigma'+k} \subset \mathcal{H}_2^{-\sigma'}$, it follows that

$$(\mathrm{D}_u^2 \nabla \Phi)(G(u)) \cdot (\mathrm{D}_u G(u) \cdot h_1, \mathrm{D}_u G(u) \cdot h_2) \in \mathcal{H}_2^{-\max(\sigma_0, \sigma')-k}.$$

As a result, the two factors in (3.6d) are integrable. This completes the proof of the lemma. □

3.2. An equivalent form

Since u is real-valued, we define the functionals $\Phi_1(u, f, \omega, \epsilon)$, $\Phi_2(u, \epsilon)$ by

$$\begin{aligned} \Phi_1(u, f, \omega, \epsilon) &:= \frac{1}{2} \int_{\mathbb{T}\times\mathbb{T}} (\widetilde{L}_\omega u(t,x)) u(t,x) \, \mathrm{d}t \, \mathrm{d}x \\ &\quad + \epsilon \int_{\mathbb{T}\times\mathbb{T}} f(t,x) u(t,x) \, \mathrm{d}t \, \mathrm{d}x, \quad u \in \widetilde{\mathcal{H}}^\sigma, \end{aligned} \tag{3.7}$$

$$\Phi_2(u, \epsilon) := \int_{\mathbb{T}\times\mathbb{T}} F(t, x, u(t,x), \epsilon) \, \mathrm{d}t \, \mathrm{d}x, \quad u \in \widetilde{\mathcal{H}}^\sigma,$$

where

$$\widetilde{L}_\omega = -(\omega^2 \partial_{tt} - \partial_{xx} + m), \tag{3.8}$$

Then

$$\nabla_u \Phi_1(u, f, \omega, \epsilon) = \widetilde{L}_\omega u + \epsilon f, \quad \nabla_u \Phi_2(u, \epsilon) = \partial_u F(u, \epsilon).$$

By the definition of $\widetilde{\mathcal{H}}^\sigma$, $\widetilde{L}_\omega$ is a bounded operator from $\widetilde{\mathcal{H}}^\sigma$ to $\widetilde{\mathcal{H}}^{\sigma-2}$. This shows $\Phi_1 \in C^{\infty,2}(\widetilde{\mathcal{H}}^\sigma, \mathbb{R})$ for $\sigma \geq \sigma_0$. Moreover, we also deduce $\Phi_2 \in C^{\infty,0}(\widetilde{\mathcal{H}}^\sigma, \mathbb{R})$ for $\sigma \geq \sigma_0$ from the condition (1.3) and Lemmas 3.4–3.5, Corollary 3.6. Then equation (2.1) may be written as

$$\nabla_u(\Phi_1(u, f, \omega, \epsilon) + \epsilon \Phi_2(u, \epsilon)) = 0. \tag{3.9}$$

Since $m > 0$, there exist some constants $c(m) > 0$ such that

$$n'^2 + m \geq c(m) \langle n \rangle^2, \quad \text{for } n' \in \{-n, n\} \text{ with } n \in \mathbb{N}.$$

By the definition of $\mathcal{F}^\sigma$ and $\omega \in [1, 2]$, if K_0 is chosen large enough, then there exists a constant $c > 0$, for $n' \in \{-n, n\}$ with $n \in \mathbb{N}$, such that the eigenvalues of $\widetilde{L}_\omega$ satisfy

$$\left|-\omega^2 j^2 + n'^2 + m\right| \geq c(|j|^2 + \langle n \rangle^2), \quad j \in \mathbb{Z}.$$

It is seen that for all $\omega \in [1, 2]$, the restriction of $\widetilde{L}_\omega$ on $\mathcal{F}^\sigma$ is an invertible operator from $\mathcal{F}^\sigma$ to $\mathcal{F}^{\sigma-2}$. Let us decompose any $u \in \widetilde{\mathcal{H}}^\sigma$ as $u = u_1 + u_2$, where $u_1 \in \mathcal{H}^\sigma$ and $u_2 \in \mathcal{F}^\sigma$. We decompose also any $f \in \widetilde{\mathcal{H}}^\sigma$ as $f = f_1 + f_2$, where $f_1 \in \mathcal{H}^\sigma$ and $f_2 \in \mathcal{F}^\sigma$. Then we will reduce (3.9) to an equivalent form on $\mathcal{H}^\sigma$.

Proposition 3.9. *Set $\sigma \geq \sigma_0$, $q > 0$, $f_1 \in B_q(\mathcal{H}^\sigma)$, $W_q := B_q(\mathcal{H}^\sigma) \times B_q(\mathcal{F}^\sigma)$. There exist $\gamma_0 \in (0, 1]$ small enough, an element $(u_1, f_2) \to \Psi_2(u_1, f_2, \omega, \epsilon)$ of $C^{\infty,0}(W_q; \mathbb{R})$ and an element $(u_1, f_2) \to G(u_1, f_2, \omega, \epsilon)$ of $\Phi^{\infty,-2}(W_q; \mathcal{F}^{\sigma+2})$, are C^1 in $(\omega, \epsilon) \in [1, 2] \times [0, \gamma_0]$, such that for any given subset $\mathcal{A} \subset [1, 2] \times [0, \gamma_0]$, the following two conditions are equivalent, i.e.,*

(i) *For any $(\omega, \epsilon) \in \mathcal{A}$, the function $u = (u_1, G(u_1, f_2, \omega, \epsilon))$ satisfies*

$$\widetilde{L}_\omega u + \epsilon f + \epsilon \nabla_u \Phi_2(u, \epsilon) = 0; \tag{3.10}$$

(ii) *For any $(\omega, \epsilon) \in \mathcal{A}$, the function u_1 satisfies*

$$\widetilde{L}_\omega u_1 + \epsilon f_1 + \epsilon \nabla_{u_1} \psi_2(u_1, f_2, \omega, \epsilon) = 0. \tag{3.11}$$

Proof. Equation (3.10) may be written as the following system

$$\widetilde{L}_\omega u_1 + \epsilon f_1 + \epsilon \nabla_{u_1} \Phi_2(u_1, u_2, \epsilon) = 0, \tag{3.12a}$$

$$\widetilde{L}_\omega u_2 + \epsilon f_2 + \epsilon \nabla_{u_2} \Phi_2(u_1, u_2, \epsilon) = 0. \tag{3.12b}$$

Since the restriction of $\widetilde{L}_\omega$ on $\mathcal{F}^\sigma$ is an invertible operator, a solution of (3.12b) may be expressed in terms of the form $u_2 = -\epsilon \widetilde{L}_\omega^{-1} f_2 + \epsilon w_2$, where

$$w_2 = -\widetilde{L}_\omega^{-1} \nabla_{u_2} \Phi_2(u_1, -\epsilon \widetilde{L}_\omega^{-1} f_2 + \epsilon w_2, \epsilon). \tag{3.13}$$

For any $(u_1, h) \in B_q(\mathcal{H}^\sigma) \times B_q(\mathcal{F}^\sigma)$, any $(\omega, \epsilon) \in [1,2] \times [0,1]$, we have

$$\left\| \widetilde{L}_\omega^{-1} \nabla_{u_2} \Phi_2(u_1, h, \epsilon) \right\|_{\mathcal{F}^{\sigma+2}} \leq \frac{q_1}{2}$$

for some constant $q_1 > 0$. By means of the fixed point theorem with parameters, there exists $\gamma_0 \in (0,1]$, such that for any $(u_1, f_2) \in W_q$, any $\epsilon \in (0, \gamma_0]$, equation (3.13) has a unique solution $w_2 \in B_{q_1}(\mathcal{F}^{\sigma+2})$, which is denoted by $G(u_1, f_2, \omega, \epsilon)$. As a consequence

$$u_2 = -\epsilon \widetilde{L}_\omega^{-1} f_2 + \epsilon G. \tag{3.14}$$

Let us check that $G \in \Phi^{\infty,-2}(W_q; \mathcal{F}^{\sigma+2})$. Formula (3.13) indicates that G is a smooth function of u_1 with C^1 dependence on (ω, ϵ) and that G belongs to $\mathcal{F}^{s+2}$ for all $(u_1, f_2) \in W_q \cap \widetilde{\mathcal{H}}^s$ with $s \geq \sigma$. Furthermore

$$\begin{aligned}
\mathrm{D}_{u_1} G(u_1, f_2, \omega, \epsilon) &= -\widetilde{L}_\omega^{-1}(\mathrm{Id} - \epsilon M_2(u_1, f_2, \omega, \epsilon) \widetilde{L}_\omega^{-1})^{-1} M_1(u_1, f_2, \omega, \epsilon), \\
\mathrm{D}_{f_2} G(u_1, f_2, \omega, \epsilon) &= \epsilon \widetilde{L}_\omega^{-1}(\mathrm{Id} - \epsilon M_2(u_1, f_2, \omega, \epsilon) \widetilde{L}_\omega^{-1})^{-1} M_2(u_1, f_2, \omega, \epsilon) \widetilde{L}_\omega^{-1},
\end{aligned}$$

where

$$\begin{aligned}
M_1(u_1, f_2, \omega, \epsilon) &= (\mathrm{D}_{u_1} \nabla_{u_2} \Phi_2)(u_1, -\epsilon \widetilde{L}_\omega^{-1} f_2 + \epsilon G, \epsilon), \\
M_2(u_1, f_2, \omega, \epsilon) &= -(\mathrm{D}_{u_2} \nabla_{u_2} \Phi_2)(u_1, -\epsilon \widetilde{L}_\omega^{-1} f_2 + \epsilon G, \epsilon).
\end{aligned}$$

We restrict ourselves to $(u_1, f_2) \in W_q \cap \widetilde{\mathcal{H}}^s$ for $s \geq \sigma$. The fact of $\Phi_2 \in C^{\infty,0}(W_q, \mathbb{R})$ gives that $M_1(u_1, f_2, \omega, \epsilon)$ (resp. $M_2(u_1, f_2, \omega, \epsilon)$) sends $\mathcal{H}^{\sigma'}$ (resp. $\mathcal{F}^{\sigma'}$) to $\mathcal{F}^{\sigma'}$ for any $\sigma' \in [-s, s]$. We have to choose γ_0 small enough to ensure

$$\epsilon \left\| M_2(u_1, f_2, \omega, \epsilon) \widetilde{L}_\omega^{-1} \right\|_{\mathcal{L}(\mathcal{F}^\sigma, \mathcal{F}^\sigma)} \leq \frac{1}{2} \quad \text{for } \epsilon \in [0, \gamma_0].$$

This gives rise to

$$(\mathrm{Id} - \epsilon M_2 \widetilde{L}_\omega^{-1})^{-1} \in \mathcal{L}(\mathcal{F}^\sigma, \mathcal{F}^\sigma),$$

which then leads to that $\mathrm{D}_{u_1} G$ can be written as the sum of the following two terms

$$-\sum_{k=0}^{2N-1} \widetilde{L}_\omega^{-1} (\epsilon M_2 \widetilde{L}_\omega^{-1})^k M_1, \tag{3.15a}$$

$$-\widetilde{L}_\omega^{-1} (\epsilon M_2 \widetilde{L}_\omega^{-1})^N (\mathrm{Id} - \epsilon M_2 \widetilde{L}_\omega^{-1})^{-1} (\epsilon M_2 \widetilde{L}_\omega^{-1})^N M_1. \tag{3.15b}$$

If N is chosen large enough relatively to s, then $(\epsilon M_2 \widetilde{L}_\omega^{-1})^N M_1$ sends $\mathcal{H}^{\sigma'}$ to $\mathcal{F}^\sigma$ for any $\sigma' \in [-s, s]$. Then (3.15b) belongs to $\mathcal{F}^{s+2} \subset \mathcal{F}^{\sigma'+2}$. Moreover, (3.15a) is bounded from $\mathcal{H}^{\sigma'}$ to $\mathcal{F}^{\sigma'+2}$ for any $\sigma' \in [-s, s]$. Therefore $\mathrm{D}_{u_1} G$ extends as an element of $\mathcal{L}(\mathcal{H}^{\sigma'}, \mathcal{F}^{\sigma'+2})$ for any $\sigma' \in [-s, s]$. The discussion on $\mathrm{D}_{f_2} G$, $\mathrm{D}^2 G$ is similar to the one as above and so is omitted. Clearly, DG, D^2G are smooth with C^1 dependence on (ω, ϵ). Consequently, G is in $\Phi^{\infty,-2}(W_q; \mathcal{F}^{\sigma+2})$. Owing to (3.8) and (3.2), it follows that

$$\begin{aligned}\Phi_1(u_1, u_2, \omega, \epsilon) + \epsilon\Phi_2(u_1, u_2, \epsilon) = &\frac{1}{2}\int (\widetilde{L}_\omega u_1)u_1 \,\mathrm{d}t\,\mathrm{d}x + \epsilon \int f_1 u_1 \,\mathrm{d}t\,\mathrm{d}x \\ &+ \frac{1}{2}\int (\widetilde{L}_\omega u_2)u_2 \,\mathrm{d}t\,\mathrm{d}x + \epsilon \int f_2 u_2 \,\mathrm{d}t\,\mathrm{d}x + \epsilon\Phi_2(u_1, u_2, \epsilon).\end{aligned}$$

Substituting (3.14) into the above expression, we can get a new functional about $(u_1, f_2, \omega, \epsilon)$, which is denoted by $\Psi(u_1, f_2, \omega, \epsilon)$. A simple calculation yields

$$\begin{aligned}\Psi(u_1, f_2, \omega, \epsilon) = &\frac{1}{2}\int (\widetilde{L}_\omega u_1)u_1 \,\mathrm{d}t\,\mathrm{d}x + \epsilon \int f_1 u_1 \,\mathrm{d}t\,\mathrm{d}x \\ &- \frac{\epsilon^2}{2}\int (\widetilde{L}_\omega^{-1} f_2) f_2 \,\mathrm{d}t\,\mathrm{d}x + \epsilon\psi_2(u_1, f_2, \omega, \epsilon),\end{aligned}$$

where

$$\psi_2(u_1, f_2, \omega, \epsilon) = \frac{\epsilon}{2}\int G(\widetilde{L}_\omega G)\,\mathrm{d}t\,\mathrm{d}x + \Phi_2(u_1, -\epsilon\widetilde{L}_\omega^{-1} f_2 + \epsilon G, \epsilon). \tag{3.16}$$

The first term on the right-hand side of (3.16) belongs to $C^{\infty,2}(\mathcal{F}^{\sigma+2})$ thanks to that $\widetilde{L}_\omega$ is a bounded operator from $\mathcal{F}^{\sigma+2}$ to $\mathcal{F}^\sigma$. It follows from Lemma 3.8 that $\psi_2 \in C^{\infty,0}(W_q, \mathbb{R})$. Moreover

$$\begin{aligned}&\nabla_{u_1}\Psi(u_1, f_2, \omega, \epsilon)[h] \\ &= \nabla_{u_1}\Phi_0(u_1, u_2, \omega, \epsilon)[h] + {}^t[\mathrm{D}_{u_1} u_2(u_1, f_2, \omega, \epsilon)[h]] \cdot \nabla_{u_2}\Phi_0(u_1, u_2, \omega, \epsilon) \\ &= \nabla_{u_1}\Phi_0(u_1, u_2(u_1, f_2, \omega, \epsilon), \omega, \epsilon)[h] \\ &= \int (\widetilde{L}_\omega u_1 + \epsilon f_1 + \epsilon\nabla_{u_1}\psi_2(u_1, f_2, \omega, \epsilon))h \,\mathrm{d}t\,\mathrm{d}x,\end{aligned}$$

where $\Phi_0 := \Phi_1 + \epsilon\Phi_2$. Hence u_1 is a critical point of Ψ if and only if it is a solution of equation (3.11). □

4. Para-linearization of the equation

Proposition 3.9 shows that we just look for families of solutions $u_1 \in \mathcal{H}^\sigma$ to equation (3.11). To simplify this problem, we fix the force term $f = f_1 + f_2$, i.e., we take no account of ψ_2 (defined in (3.16)) depends in the f_2. We now turn to study equation

$$\widetilde{L}_\omega u + \epsilon f + \epsilon\nabla_u\psi_2(u, \omega, \epsilon) = 0, \tag{4.1}$$

where $u \in B_q(\mathcal{H}^\sigma)$, $f \in \mathcal{H}^s$, $\psi_2 \in C^{\infty,0}(B_q(\mathcal{H}^\sigma), \mathbb{R})$ for some $\sigma \in [\sigma_0, s]$, $q > 0$ and $\epsilon \in [0, \gamma_0]$ with $\gamma_0 \in (0,1]$ small enough. The goal of this section is to reduce (4.1) into a para-differential equation using the equivalent norm (2.6)–(2.8) in $\mathcal{H}^\sigma$. We first define classes of operators.

4.1. Spaces of operators

Define the spaces $\widetilde{\mathcal{H}}^\sigma_{\mathbb{C}} := \widetilde{\mathcal{H}}^\sigma(\mathbb{T}\times\mathbb{T};\mathbb{C})$, $\mathcal{H}^\sigma_{\mathbb{C}} := \mathcal{H}^\sigma(\mathbb{T}\times\mathbb{T};\mathbb{C})$ for complex valued functions. Other notations are defined in the similar way as in Section 2.2.

Definition 4.1. Let $m \in \mathbb{R}$, $q > 0$ with $u \in B_q(\mathcal{H}^\sigma_{\mathbb{C}})$, $N \in \mathbb{N}$, $\sigma \in \mathbb{R}$, with $\sigma \geq \sigma_0 + 2N + 2$. Denote the space of maps $u \to A(u)$ defined on $B_q(\mathcal{H}^\sigma_{\mathbb{C}})$ by $\Sigma^m(N,\sigma,q)$, with values in the space of linear maps from $C^\infty(\mathbb{T}\times\mathbb{T};\mathbb{C})$ to $\mathcal{D}'(\mathbb{T}\times\mathbb{T};\mathbb{C})$. And there exists a constant $C > 0$, such that for any $n, n' \in \mathbb{Z}$, $u \to \Pi_n A(u)\Pi_{n'}$ is smooth with values in $\mathcal{L}(\mathcal{H}^0_{\mathbb{C}})$. And for any $M \in \mathbb{N}$ with $2 \leq M \leq \sigma - \sigma_0 - 2N$, any $u \in B_q(\mathcal{H}^\sigma_{\mathbb{C}})$, any $j \in \mathbb{N}$, any $w_1, \ldots, w_j \in \mathcal{H}^\sigma_{\mathbb{C}}$, any $n, n' \in \mathbb{Z}$, the following holds:

(i) For $j \geq 1$, it has

$$\begin{aligned} &\left\|\Pi_n(\partial_u^j A(u)\cdot(w_1,\ldots,w_j))\Pi_{n'}\right\|_{\mathcal{L}(\mathcal{H}^0_{\mathbb{C}})} \\ &\leq C(1+|n|+\left|n'\right|)^m \left\langle n-n'\right\rangle^{-M} \times \mathbb{1}_{|n-n'|\leq\frac{1}{10}(|n|+|n'|)} \prod_{l=1}^{j} \|w_l\|_{\mathcal{H}^{\sigma_0+2N+M}_{\mathbb{C}}}. \end{aligned} \tag{4.2}$$

(ii) For $j = 0$, it has

$$\|\Pi_n A(u)\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0_{\mathbb{C}})} \leq C(1+|n|+\left|n'\right|)^m \left\langle n-n'\right\rangle^{-M} \mathbb{1}_{|n-n'|\leq\frac{1}{10}(|n|+|n'|)}.$$

Remark 4.2. In (4.2), the term $\langle n-n'\rangle^{-M}$ reflects the available x-smoothness of the symbol of a pseudo-differential operator, and the term $\mathbb{1}_{|n-n'|\leq\frac{1}{10}(|n|+|n'|)}$ reflects the cut-off.

Remark 4.3. By Definition 4.1, if $A \in \Sigma^m(N,\sigma,q)$, then $\partial_{tt}(A(u)) \in \Sigma^m(N+1,\sigma,q)$, where

$$\partial_{tt}A(u) = \partial_{uu}A(u)\cdot(\partial_t u)^2 + \partial_u A(u)\cdot\partial_{tt}u. \tag{4.3}$$

In fact, formulae (4.2) and (4.3) indicate for $j \geq 1$

$$\begin{aligned} &\left\|\Pi_n \partial_u^j(\partial_{tt}A(u))\cdot(w_1,\ldots,w_j)\Pi_{n'}\right\|_{\mathcal{L}(\mathcal{H}^0_{\mathbb{C}})} \\ &\leq C(1+|n|+\left|n'\right|)^m \left\langle n-n'\right\rangle^{-M} \mathbb{1}_{|n-n'|\leq\frac{1}{10}(|n|+|n'|)} \\ &\quad\times \left(\|\partial_t u\|^2_{\mathcal{H}^{\sigma_0+2N+M}_{\mathbb{C}}} + \|\partial_{tt}u\|_{\mathcal{H}^{\sigma_0+2N+M}_{\mathbb{C}}}\right) \prod_{l=1}^{j} \|w_l\|_{\mathcal{H}^{\sigma_0+2N+M}_{\mathbb{C}}} \end{aligned}$$

$$\leq C_1(1+|n|+|n'|)^m \langle n-n'\rangle^{-M} \mathbb{1}_{|n-n'|\leq\frac{1}{10}(|n|+|n'|)}(\|u\|^2_{\mathcal{H}^\sigma_{\mathbb{C}}}+\|u\|_{\mathcal{H}^\sigma_{\mathbb{C}}})\prod_{l=1}^{j}\|w_l\|_{\mathcal{H}^{\sigma_0+2N+M}_{\mathbb{C}}}$$

$$\leq C_2(1+|n|+|n'|)^m \langle n-n'\rangle^{-M} \mathbb{1}_{|n-n'|\leq\frac{1}{10}(|n|+|n'|)}\prod_{l=1}^{j}\|w_l\|_{\mathcal{H}^{\sigma_0+2(N+1)+M}_{\mathbb{C}}}$$

if we assume $M \leq \sigma - 2(N+1) - \sigma_0$ and $u \in B_q(\mathcal{H}^\sigma_{\mathbb{C}})$. In the same way, the case of $j=0$ is argued. This checks that $\partial_{tt}(A(u)) \in \Sigma^m(N+1,\sigma,q)$.

Lemma 4.4. *Let σ, m, N, q satisfy the conditions of Definition* 4.1. *Then for any $u \in B_q(\mathcal{H}^\sigma_{\mathbb{C}})$, any $s \in \mathbb{R}$, the operator $A(u)$ is bounded from $\mathcal{H}^s_{\mathbb{C}}$ to $\mathcal{H}^{s-m}_{\mathbb{C}}$. Moreover, $u \to A(u)$ is a smooth map from $B_q(\mathcal{H}^\sigma_{\mathbb{C}})$ to the space $\mathcal{L}(\mathcal{H}^s_{\mathbb{C}},\mathcal{H}^{s-m}_{\mathbb{C}})$. And for any $j \in \mathbb{N}$, any $u \in B_q(\mathcal{H}^\sigma_{\mathbb{C}})$, any $w_1,\dots,w_j \in \mathcal{H}^\sigma_{\mathbb{C}}$, we have*

$$\left\|\partial_u^j A(u)\cdot(w_1,\dots,w_j)\right\|_{\mathcal{L}(\mathcal{H}^s_{\mathbb{C}},\mathcal{H}^{s-m}_{\mathbb{C}})} \leq C\prod_{l=1}^{j}\|w_l\|_{\mathcal{H}^{\sigma_0+2N+2}_{\mathbb{C}}} \tag{4.4}$$

for some constant $C>0$.

Proof. Applying (4.2) with $M=2$ and $\mathcal{H}^s_{\mathbb{C}}$-norm defined by (2.7), we may get the conclusion. □

Definition 4.5. Let $\sigma \in \mathbb{R}$ with $\sigma \geq \sigma_0+2N+2$, $N \in \mathbb{N}$, $\nu \in \mathbb{N}$, $q>0$, $r \geq 0$. One denotes by $\mathcal{R}^r_\nu(N,\sigma,q)$ the space of smooth maps $u \to R(u)$ defined on $B_q(\mathcal{H}^\sigma_{\mathbb{C}})$, with values in $\mathcal{L}(\mathcal{H}^s_{\mathbb{C}},\mathcal{H}^{s+r}_{\mathbb{C}})$ for any $s \geq \sigma_0+\nu$, satisfying for any j, any $s \geq \sigma_0+\nu$, any $u \in B_q(\mathcal{H}^\sigma_{\mathbb{C}})$, $w_1,\dots,w_j \in \mathcal{H}^\sigma_{\mathbb{C}}$

$$\left\|\partial_u^j R(u)\cdot(w_1,\dots,w_j)\right\|_{\mathcal{L}(\mathcal{H}^s_{\mathbb{C}},\mathcal{H}^{s+r}_{\mathbb{C}})} \leq C\prod_{l=1}^{j}\|w_l\|_{\mathcal{H}^{\sigma}_{\mathbb{C}}} \tag{4.5}$$

for some constant $C>0$. When $j=0$, we have $\|R(u)\|_{\mathcal{L}(\mathcal{H}^s_{\mathbb{C}},\mathcal{H}^{s+r}_{\mathbb{C}})} \leq C$.

Remark 4.6. Lemma 4.4 implies that $\Sigma^{-r}(N,\sigma,q) \subset \mathcal{R}^r_0(N,\sigma,q)$ for $r \geq 0$, $\sigma \geq \sigma_0+2N+2$.

Proposition 4.7. (i) *Let $\sigma \geq \sigma_0+2N+2$. If $A \in \Sigma^m(N,\sigma,q)$, then $A^* \in \Sigma^m(N,\sigma,q)$.*

(ii) *Let $m_1, m_2 \in \mathbb{R}$ and assume $\sigma \geq \sigma_0+2N+2+\max(m_1+m_2,0)$. Put*

$$r = \sigma-\sigma_0-2N-2-\max(m_1+m_2,0) \geq 0. \tag{4.6}$$

If $A \in \Sigma^{m_1}(N,\sigma,q)$, $B \in \Sigma^{m_2}(N,\sigma,q)$, then there exists $D \in \Sigma^{m_1+m_2}(N,\sigma,q)$ and $R \in \mathcal{R}^r_0(N,\sigma,q)$ such that

$$A(u)\circ B(u) = D(u)+R(u).$$

Proof. (i) It follows from Definition 4.1.

(ii) Define

$$D(u) = \sum_{n} \sum_{n'} \Pi_n (A(u) \circ B(u)) \Pi_{n'} \mathbb{1}_{|n-n'| \leq \frac{1}{10}(|n|+|n'|)},$$
$$R(u) = \sum_{n} \sum_{n'} \Pi_n (A(u) \circ B(u)) \Pi_{n'} \mathbb{1}_{|n-n'| > \frac{1}{10}(|n|+|n'|)}.$$

When $j = 0$, we get the upper bound

$$\|\Pi_n D(u) \Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0_{\mathbb{C}})} \overset{(4.2)}{\leq} C(1 + |n| + |n'|)^{(m_1+m_2)} \langle n - n' \rangle^{-M}.$$

Similarly, the estimates of $\left\|\Pi_n \partial_u^j D(u) \cdot (w_1, \ldots, w_j) \Pi_{n'}\right\|_{\mathcal{L}(\mathcal{H}^0)}$ for $j \geq 1$ are obtained. Formula (4.2) also infers for $j = 0$

$$\|\Pi_n R(u) \Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0_{\mathbb{C}})} \leq C(1 + |n| + |n'|)^{(m_1+m_2)} \sum_k \langle n - k \rangle^{-M} \langle k - n' \rangle^{-M}$$
$$\times \mathbb{1}_{|n-k| \leq \frac{1}{10}(|n|+|k|)} \mathbb{1}_{|k-n'| \leq \frac{1}{10}(|k|+|n'|)} \mathbb{1}_{|n-n'| > \frac{1}{10}(|n|+|n'|)}.$$

Clearly, either $|n - k| \geq \frac{1}{2}(|n - n'|)$ or $|n' - k| \geq \frac{1}{2}(|n - n'|)$ should be satisfied. This gives rise to

$$|n - n'| \leq \frac{1}{2}(|n| + |n'|).$$

According to the fact and taking $M = \sigma - \sigma_0 - 2N$, we have

$$\|\Pi_n R(u) \Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0_{\mathbb{C}})} \leq C(1 + |n| + |n'|)^{((m_1+m_2)-(M-2))} \langle n - n' \rangle^{-2} \mathbb{1}_{|n-n'| \leq \frac{1}{2}(|n|+|n'|)},$$

which gives that $R(u)$ sends $\mathcal{H}^s_{\mathbb{C}}$ to $\mathcal{H}^{s+r}_{\mathbb{C}}$ for any s, where r is given by (4.6). An argument similar to the one as above yields the estimates of $\left\|\partial_u^j R(u) \cdot (W_1, ..., W_j)\right\|_{\mathcal{L}(\mathcal{H}^s_{\mathbb{C}}, \mathcal{H}^{s+r}_{\mathbb{C}})}$ for $j \geq 1$. □

In the rest of this paper, we use those operators $A(u)$ (resp. $R(u)$) of $\Sigma^m(N, \sigma, q)$ (resp. $\mathcal{R}^r_\nu(N, \sigma, q)$) sending real valued functions to real valued functions, i.e., $\overline{A(u)} = A(u)$ (resp. $\overline{R(u)} = R(u)$). Furthermore, we shall consider operators $A(u, \omega, \epsilon)$, $R(u, \omega, \epsilon)$ depending on (ω, ϵ), where (ω, ϵ) stays in a bounded domain of $\mathbb{R}^2$. If $(\omega, \epsilon) \to \Pi_n A(u, \omega, \epsilon) \Pi_{n'}$ (resp. $(\omega, \epsilon) \to R(u, \omega, \epsilon)$) is C^1 in (ω, ϵ) with values in $\mathcal{L}(\mathcal{H}^0)$ (resp. $\mathcal{L}(\mathcal{H}^s, \mathcal{H}^{s+r})$) and if $\partial_\omega A$, $\partial_\epsilon A$ (resp. $\partial_\omega R$, $\partial_\epsilon R$) satisfy (4.2) (resp. (4.5)), then we shall say that operators $A(u, \omega, \epsilon)$ (resp. $R(u, \omega, \epsilon)$) are C^1 in (ω, ϵ).

4.2. Reduce to a para-differential equation

Denote by $\mathscr{R}((X^k)^\tau; k, \tau \in \mathbb{N}^{d_1})$ the space of polynomials in indeterminate X^k, which are the sum of those of monomials whose weights are equal. According to the fact, if

$(X^{k_1})^{\tau_1}\cdots(X^{k_l})^{\tau_l}$ is a monomial, then we define its weight as $\tau_1 k_1+\cdots+\tau_l k_l$. Let U be an open subset of $\mathcal{H}^{\sigma_0}$, ψ belong to $C^{\infty,0}(X,\mathbb{R})$. For any $u\in U\cap\mathcal{H}^{+\infty}$, $w_1,w_2\in\mathcal{H}^{+\infty}$, we set

$$L(u;w_1,w_2)=D_u^2\psi(u)\cdot(w_1,w_2). \tag{4.7}$$

This is a continuous bilinear form in $(w_1,w_2)\in\mathcal{H}^0\times\mathcal{H}^0$. By Riesz theorem, (4.7) can be written as

$$L(u;w_1,w_2)=\int_{\mathbb{T}\times\mathbb{T}}(W(u)w_1)w_2\,\mathrm{d}t\,\mathrm{d}x$$

for some symmetric $\mathcal{H}^0$-bounded operator $W(u)$. Definition 3.2 infers that $u\to D_u^2\psi(u)$ is a smooth map defined on U with values in the space of continuous bilinear forms on $\mathcal{H}^0\times\mathcal{H}^0$. This shows that $u\to W(u)$ is smooth with values in $\mathcal{L}(\mathcal{H}^0,\mathcal{H}^0)$, which then gives for any $u\in U\cap\mathcal{H}^{+\infty}$, $w_1,w_2\in\mathcal{H}^{+\infty}$

$$L(u;\partial_x w_1,w_2)+L(u;w_1,\partial_x w_2)=-(\partial_u L)(u;w_1,w_2)\cdot(\partial_x u). \tag{4.8}$$

Lemma 4.8. *Let $q>0$. For $l\in\mathbb{N}$, $N\in\mathbb{N}$, $N'\in\mathbb{N}$, there are polynomials $Q_N^l\in\mathscr{R}((X^k)^\tau;k,\tau\in\mathbb{N}^{d_1})$, of weight equal to N, a constant $C>0$, depending only on l,q,N', such that for any $u\in B_q(\mathcal{H}^{\sigma_0})\cap U\cap\mathcal{H}^{+\infty}$, any $h_1,\ldots,h_l\in\mathcal{H}^{+\infty}$, any $n,n'\in\mathbb{Z}$, the following holds:*

$$\begin{aligned}&\left\|\Pi_n\partial_u^l W(u)\cdot(h_1,\ldots,h_l)\Pi_{n'}\right\|_{\mathcal{L}(\mathcal{H}^0)}\\ &\le C\left\langle n-n'\right\rangle^{-N'}\sum_{N_0+\cdot+N_l=N'}Q_{N_0}^l((\|\partial_x^k u\|_{\mathcal{H}^{\sigma_0}})^\tau)\prod_{l'=1}^{l}\|h_{l'}\|_{\mathcal{H}^{\sigma_0+N_{l'}}},\end{aligned} \tag{4.9}$$

where $Q_{N_0}^l((\|\partial_x^k u\|_{\mathcal{H}^{\sigma_0}})^\tau)$ is the polynomial composed by these monomials like

$$(\|\partial^{k_1}v\|_{\mathcal{H}^{\sigma_0}})^{\tau_1}\cdots(\|\partial^{k_p}v\|_{\mathcal{H}^{\sigma_0}})^{\tau_l}$$

with $k_1\tau_1+\cdots+k_p\tau_p=N_0$.

Proof. Using ${}^t\Pi_n=\Pi_{-n}$ and (4.8), for $l=0$, we deduce for any $u\in B_q(\mathcal{H}^{\sigma_0})\cap U\cap\mathcal{H}^{+\infty}$, any $w_1,w_2\in\mathcal{H}^{+\infty}$

$$\begin{aligned}(n-n')\int(\Pi_n W(u)\Pi_{n'}w_1)\,w_2\,\mathrm{d}t\,\mathrm{d}x&=(n-n')\int(W(u)\Pi_{n'}w_1)\,\Pi_{-n}w_2\,\mathrm{d}t\,\mathrm{d}x\\ &=i(L(u;\partial_x\Pi_{n'}w_1,\Pi_{-n}w_2)+L(u;\Pi_{n'}w_1,\partial_x\Pi_{-n}w_2))\\ &=-\mathrm{i}(\partial_u L)(u;\Pi_{n'}w_1,\Pi_{-n}w_2)\cdot(\partial_x u).\end{aligned}$$

Iterating the above computation, it yields that

$$\left\langle n-n'\right\rangle^{N'}\left|\int(\Pi_n W(u)\Pi_{n'}w_1)w_2\,\mathrm{d}t\,\mathrm{d}x\right|$$

is bounded from above by finite sum of

$$\left|(\partial_u^j L)(u;\Pi_{n'}w_1,\Pi_{-n}w_2)(\partial_x^{\kappa_1}u,\dots,\partial_x^{\kappa_j}u)\right| \tag{4.10}$$

with $\kappa_1+\cdots+\kappa_j=N'$. According to the properties of the operator L, the term in (4.10) is bounded from above by

$$C\,\|\Pi_{n'}w_1\|_{\mathcal{H}^0}\,\|\Pi_{-n}w_2\|_{\mathcal{H}^0}\prod_{j'=1}^{j}\left\|\partial_x^{\kappa_{j'}}u\right\|_{\mathcal{H}^{\sigma_0}}.$$

The remainder of the discussion on $l\geq 1$ is analogous to the case of $l=0$, we have (4.9) for any $l\geq 1$. □

Put for $p\in\mathbb{N}$, $u\in\mathcal{H}^0$

$$\begin{aligned}\Delta_p u &= \sum_{\substack{n\in\mathbb{Z}\\ 2^{p-1}\leq|n|<2^p}}\Pi_n u, && p\geq 1,\quad \Delta_0 u=\Pi_0 u,\\ S_p u &= \sum_{p'=0}^{p-1}\Delta_{p'}u=\sum_{\substack{n\in\mathbb{Z}\\ |n|<2^{p-1}}}\Pi_n u, && p\geq 1,\quad S_0u=0.\end{aligned}\tag{4.11}$$

Lemma 4.9. *Let $q>0$, $\sigma\in\mathbb{R}$ with $\sigma\geq\sigma_0+2$, $\gamma_0\in(0,1]$ with γ_0 small enough. There exists a map $(u,\omega,\epsilon)\to W(u,\omega,\epsilon)$ on $B_q(\mathcal{H}^\sigma)\times[1,2]\times[0,\gamma_0]$ with values in $\mathcal{L}(\mathcal{H}^0)$, which is symmetric and is C^∞ in u with C^1 in (ω,ϵ), such that for any (u,ω,ϵ)*

$$\psi_2(u,\omega,\epsilon)=\int_{\mathbb{T}\times\mathbb{T}}(W(u,\omega,\epsilon)u)\,u\,\mathrm{d}t\,\mathrm{d}x \tag{4.12}$$

and satisfies the following estimate: for $l\in\mathbb{N}$, $N\in\mathbb{N}$, $N'\in\mathbb{N}$, there are polynomials $Q_N^l\in\mathscr{R}((X^k)^\tau;k,\tau\in\mathbb{N}^{d_1})$, of weight equal to N, and a constant C, depending on l, q, N', such that for any $u\in B_q(\mathcal{H}^\sigma)$, any $\epsilon\in[0,\gamma_0]$, any $\omega\in[1,2]$, any $(a_0,a_1)\in\mathbb{N}^2$ with $a_0+a_1\leq 1$, any $(h_1,\dots,h_l)\in(\mathcal{H}^\sigma)^l$, any $n,n'\in\mathbb{Z}$

$$\begin{aligned}&\left\|\Pi_n\partial_\omega^{a_0}\partial_\epsilon^{a_1}D_u^lW(u,\omega,\epsilon)\cdot(h_1,\dots,h_l)\Pi_{n'}\right\|_{\mathcal{L}(\mathcal{H}^0)}\\ &\leq C\,\langle n-n'\rangle^{-N'}\sum_{N_0+\cdots+N_l=N'}Q_{N_0}^l((\|\partial_x^\tau S(n,n')u\|_{\mathcal{H}^{\sigma_0}})^\tau)\prod_{l'=1}^{l}\|S(n,n')h_{l'}\|_{\mathcal{H}^{\sigma_0+N_{l'}}},\end{aligned}\tag{4.13}$$

where $S(n,n')=\sum_{|n''|\leq 2(1+\min(|n|,|n'|))}\Pi_{n''}$, $n,n'\in\mathbb{Z}$.

Proof. We just consider that W depends on u. By definition (3.16), we obtain that $\partial_u^\kappa\psi_2$ is continuous with $\kappa\leq 2$. Definition (4.11) gives rises to

$$S_pu\xrightarrow{\mathcal{H}^\sigma}u\quad\text{as } p\to+\infty.$$

This reads

$$\begin{aligned}\psi_2(v) &= \sum_{p_1=0}^{+\infty} (\psi_2(S_{p_1+1}v) - \psi_2(S_{p_1}v)) \\ &= \sum_{p_1=0}^{+\infty} \int_0^1 (D_v\psi_2)(S_{p_1}v + \theta_1\Delta_{p_1}v)\, d\theta_1 \cdot \Delta_{p_1}v \\ &= \sum_{p_1=0}^{+\infty}\sum_{p_2=0}^{+\infty} \int_0^1\int_0^1 (D_v^2\psi_2)\left(\Omega_{p_1,p_2}(\theta_1,\theta_2)v\right) d\theta_2 \cdot (\Delta_{p_2}(S_{p_1}+\theta_1\Delta_{p_1})v, \Delta_{p_1}v)\, \mathrm{d}\theta_1,\end{aligned}$$

where $\Omega_{p_1,p_2}(\theta_1,\theta_2) = \Pi_{l=1}^2(S_{p_l}+\theta_l\Delta_{p_l})$. According to the argument before Lemma 4.8, there exists a symmetric operator $\widetilde{W}(\Omega_{p_1,p_2}(\theta_1,\theta_2)u)$ satisfying (4.9), such that

$$D^2\psi_2(\Omega_{p_1,p_2}(\theta_1,\theta_2)u)\cdot(w_1,w_2) = \int \left(\widetilde{W}(\Omega_{p_1,p_2}(\theta_1,\theta_2)u)w_1\right) w_2\, \mathrm{d}t\, \mathrm{d}x.$$

Thus we can get (4.12), where

$$\begin{aligned}W(u) &= \frac{1}{2}\sum_{p_1}\sum_{p_2}\int_0^1\int_0^1 \Delta_{p_1}\left(\widetilde{W}(\Omega_{p_1,p_2}(\theta_1,\theta_2)u)\Delta_{p_2}(S_{p_1}+\theta_1\Delta_{p_1})\right) \mathrm{d}\theta_1\, \mathrm{d}\theta_2 \\ &\quad + \frac{1}{2}\sum_{p_1}\sum_{p_2}\int_0^1\int_0^1 \Delta_{p_2}(S_{p_1}+\theta_1\Delta_{p_1})\left(\widetilde{W}(\Omega_{p_1,p_2}(\theta_1,\theta_2)u)\Delta_{p_1}\right) \mathrm{d}\theta_1\, \mathrm{d}\theta_2,\end{aligned}$$

which is also a symmetric operator. Furthermore, the definition of $S(n,n')$ infers

$$\Pi_n W(u)\Pi_{n'} = \Pi_n W(S(n,n')u)\Pi_{n'}.$$

Combining this with (4.9), it leads to (4.13). If we choose $N' \geq 2$ to guarantee that $\sigma_0 + N' \leq \sigma$, then u, $h_{l'}$ are in $\mathcal{H}^\sigma$. Consequently, the right-hand side of (4.13) is bounded by $C\langle n-n'\rangle^{-N'}$, which implies that $W(u)$ is bounded from $\mathcal{H}^0$ to $\mathcal{H}^0$. □

Proposition 4.10. *Let $q>0$, $\sigma\in\mathbb{R}$ with $\sigma\geq\sigma_0+2$, $\gamma_0\in(0,1]$ with γ_0 small enough. Denote*

$$r = \sigma - \sigma_0 - 2. \tag{4.14}$$

There is a symmetric element $\widetilde{V}\in\Sigma^0(0,\sigma,q)$ and an element $\widetilde{R}\in\mathcal{R}_0^r(0,\sigma,q)$, are C^1 in (ω,ϵ), such that for any $u\in B_q(\mathcal{H}^\sigma)$, any $\epsilon\in[0,\gamma_0]$, any $\omega\in[1,2]$

$$\nabla_u\psi_2(u,\omega,\epsilon) = \widetilde{V}(u,\omega,\epsilon)u + \widetilde{R}(u,\omega,\epsilon)u.$$

Proof. For $h_1\in\mathcal{H}^{+\infty}$, it follows from Lemma 4.9 that

$$\mathrm{D}_u\,\psi_2(u,\omega,\epsilon)\cdot h_1 = 2\int (W(u,\omega,\epsilon)u)h_1\, \mathrm{d}t\, \mathrm{d}x + \int ((\mathrm{D}_u\, W(u,\omega,\epsilon)\cdot h_1)u)u\, \mathrm{d}t\, \mathrm{d}x. \tag{4.15}$$

Let us verify the first term on the right-hand side of (4.15). Define

$$\widetilde{V}(u,\omega,\epsilon)=2\sum_{n,n'}\mathbb{1}_{|n-n'|\leq\frac{1}{10}(|n|+|n'|)}\Pi_nW(u,\omega,\epsilon)\Pi_{n'},$$
$$\widetilde{R}'(u,\omega,\epsilon)=2\sum_{n,n'}\mathbb{1}_{|n-n'|>\frac{1}{10}(|n|+|n'|)}\Pi_nW(u,\omega,\epsilon)\Pi_{n'}.$$

In (4.13), if $|\tau|\leq N'\leq\sigma-\sigma_0$, then there exists a constant $C>0$ such that

$$\left\|\partial^\alpha S(n,n')u\right\|_{\mathcal{H}^{\sigma_0}}\leq C\left\|u\right\|_{\mathcal{H}^\sigma},\quad\left\|S(n,n')h_{l'}\right\|_{\mathcal{H}^{\sigma_0+N_{l'}}}\leq C\left\|h_{l'}\right\|_{\mathcal{H}^{\sigma_0+M}},$$

which shows that $\widetilde{V}$ satisfies (4.2). Then $\widetilde{V}\in\Sigma^0(0,\sigma,q)$. It is obvious to obtain that $\widetilde{R}'\in\mathcal{R}_0^r(0,\sigma,q)$ when $|\tau|\leq N'\leq\sigma-\sigma_0$. Furthermore, we have for some constant $C>0$

$$\left\|S(n,n')w\right\|_{\mathcal{H}^{\sigma_0+\beta}}\leq C(1+\inf(|n|,|n'|))^{\max(\beta+\sigma_0-\sigma,0)}\left\|w\right\|_{\mathcal{H}^\sigma}. \tag{4.16}$$

Formulae (4.13) and (4.16) derive that $\left\|\Pi_n\partial_\omega^{\eta_0}\partial_\epsilon^{\eta_1}\partial_u^l\widetilde{R}'(u,\omega,\epsilon)\cdot(h_1,\ldots,h_l)\Pi_{n'}\right\|_{\mathcal{L}(\mathcal{H}^0)}$ for $u\in B_q(\mathcal{H}^\sigma)$ is bounded from above by

$$C(1+|n|+|n'|)^{-N'}(1+\inf(|n|,|n'|))^{(N'+\sigma_0-\sigma)}\prod_{l'=1}^{l}\left\|h_{l'}\right\|_{\mathcal{H}^\sigma}$$

when $N'>\sigma-\sigma_0$. Then we get for any $s\geq\sigma_0$,

$$\left\|\partial_\omega^{\eta_0}\partial_\epsilon^{\eta_1}\partial_u^l\widetilde{R}'(u,\omega,\epsilon)\cdot(h_1,\ldots,h_l)\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s-m})}\leq C\prod_{l'=1}^{l}\left\|h_{l'}\right\|_{\mathcal{H}^\sigma}$$

if N' is given large enough, which shows $\widetilde{R}'\in\mathcal{R}_0^r(0,\sigma,q)$.

On the other hand, we study the second term on the right-hand side of (4.15). For any $h,w\in\mathcal{H}^{+\infty}$, assume there exists an operator $\widetilde{R}''(u,\omega,\epsilon)$ with

$$\int((\mathrm{D}_u W(u,\omega,\epsilon)\cdot h)u)\,w\,\mathrm{d}t\,\mathrm{d}x=\int\left(\widetilde{R}''(u,\omega,\epsilon)w\right)h\,\mathrm{d}t\,\mathrm{d}x.$$

Decomposing $u=\Sigma_{n'}\Pi_{n'}u$ and $w=\Sigma_n\Pi_nw$, the following estimate

$$\begin{aligned}&\left\|((\mathrm{D}_u W(u,\omega,\epsilon)\cdot h)u)\,w\right\|_{\mathcal{H}^0}\\&\leq\sum_n\sum_{n'}\left\|\Pi_n\,\mathrm{D}_u W(u,\omega,\epsilon)\cdot h\Pi_{n'}\right\|_{\mathcal{L}(\mathcal{H}^0)}\left\|\Pi_{n'}u\right\|_{\mathcal{H}^0}\left\|\Pi_nw\right\|_{\mathcal{H}^0}\end{aligned} \tag{4.17}$$

holds. For l^2-sequences $(c_n)_n$, $(c'_{n'})_{n'}$, we may obtain

$$\left\|\Pi_nw\right\|_{\mathcal{H}^0}\leq c_n\left\langle n\right\rangle^{-s}\left\|w\right\|_{\mathcal{H}^s},\quad\left\|\Pi_{n'}v\right\|_{\mathcal{H}^0}\leq c'_{n'}q\left\langle n'\right\rangle^{-\sigma}.$$

In addition, formula (4.13) gives for $l=1$ and $u\in B_q(\mathcal{H}^\sigma)$

$$Q_{N_0}^1((\|\partial^\tau S(n,n')u\|_{\mathcal{H}^{\sigma_0}})^\tau)\|S(n,n')h\|_{\mathcal{H}^{\sigma_0+N_1}}\leq C(1+\inf(|n|,|n'|))^{(N'+s+r+\sigma_0)}\left\|h\right\|_{\mathcal{H}^{-s-r}},$$

where $w \in \mathcal{H}^s$ and $h \in \mathcal{H}^{-s-r}$ for $s \geq \sigma_0$. Then

$$\begin{aligned}&\|\Pi_n \mathrm{D}_u W(u,\omega,\epsilon)\cdot h\Pi_{n'}\|_{\mathcal{L}(\mathcal{H}^0)} \|\Pi_{n'}u\|_{\mathcal{H}^0} \|\Pi_{n'}w\|_{\mathcal{H}^0}\\ &\leq C\left\langle n-n'\right\rangle^{-N'}\left((1+\inf(|n|,|n'|)\right)^{(N'+s+r+\sigma_0)}\langle n\rangle^{-s} c_n \left\langle n'\right\rangle^{-\sigma} c'_{n'}\|w\|_{\mathcal{H}^s}\|h\|_{\mathcal{H}^{-s-r}},\end{aligned}$$

where $s\geq\sigma_0$, $\sigma\geq\sigma_0+2$, r is given by (4.14). Taking $N'=2$, we may check that the sum in n, n' of (4.17) is convergent, which then leads to that $\widetilde{R}''\in\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+r})$. Similarly, the estimates of $\left\|\partial_\omega^{\eta_0}\partial_\epsilon^{\eta_1}\partial_u^l R_2(u,\omega,\epsilon)\cdot(h_1,\ldots,h_l)\right\|_{\mathcal{L}(\mathcal{H}^s,\mathcal{H}^{s+r})}$ for $l\geq1$ can be obtained. Therefore we get $\widetilde{R}''\in\mathcal{R}_0^r(0,\sigma,q)$. □

5. Diagonalization of the problem

5.1. Spaces of diagonal and non diagonal operators

It follows from Proposition 4.10 that we can decompose the nonlinearity in (4.1) as the sum of the action of the para-differential potential $\widetilde{V}(u,\omega,\epsilon)$ on u and of a remainder. Thus equation (4.1) can be reduced to

$$L_\omega u+\epsilon V(u,\omega,\epsilon)u=\epsilon\widetilde{R}(u,\omega,\epsilon)u+\epsilon f, \tag{5.1}$$

where $L_\omega:=-\widetilde{L}_\omega$, $V(u,\omega,\epsilon):=-\widetilde{V}(u,\omega,\epsilon)$. Furthermore, $V\in\Sigma^0(0,\sigma,q)$ is symmetric, and $\widetilde{R}\in\mathcal{R}_0^r(0,\sigma,q)$. Note the symmetric operator V is also self-adjoint.

Definition 5.1. Let $\sigma\in\mathbb{R}$, $N\in\mathbb{N}$, with $\sigma\geq\sigma_0+2N+2$, $m\in\mathbb{R}$, $q>0$.

(i) Denote by $\Sigma_{\mathrm{D}}^m(N,\sigma,q)$ the subspace of $\Sigma^m(N,\sigma,q)$ constituted by elements $A(u,\omega,\epsilon)$ satisfying $\widetilde{\Pi}_n A\widetilde{\Pi}_{n'}\equiv0$ for any $n,n'\in\mathbb{N}$ with $n\neq n'$.

(ii) Denote by $\Sigma_{\mathrm{ND}}^m(N,\sigma,q)$ the subspace of $\Sigma^m(N,\sigma,q)$ constituted by elements $A(u,\omega,\epsilon)$ satisfying $\widetilde{\Pi}_n A\widetilde{\Pi}_n\equiv0$ for any $n\in\mathbb{N}$.

It is straightforward to see that $\Sigma^m(N,\sigma,q)=\Sigma_{\mathrm{D}}^m(N,\sigma,q)\oplus\Sigma_{\mathrm{ND}}^m(N,\sigma,q)$.

Definition 5.2. Let $\rho'=1$, one denotes by $\mathcal{L}_{\rho'}^m(N,\sigma,q)$ the subspace of $\Sigma^m(N,\sigma,q)$ constituted by those elements $A(u,\omega,\epsilon)$ with

$$A(u,\omega,\epsilon)\in\Sigma^{m-\rho'}(N,\sigma,q). \tag{5.2}$$

Furthermore, one denotes by $\mathcal{L}'^m_{\rho'}(N,\sigma,q)$ the subspace of $\Sigma^m(N,\sigma,q)$ constituted by those elements $A(u,\omega,\epsilon)$ satisfying (5.2) and $A(u,\omega,\epsilon)^*=-A(u,\omega,\epsilon)$.

Remark 5.3. Assume $\sigma\geq\sigma_0+2N+2+\max(m_1+m_2-2\rho',0)$. By Proposition 4.7(ii), if $A\in\mathcal{L}_{\rho'}^{m_1}(N,\sigma,q)$, $B\in\mathcal{L}_{\rho'}^{m_2}(N,\sigma,q)$, then $A\circ B$ is the sum of an element of $\mathcal{L}_{\rho'}^{m_1+m_2-\rho'}(N,\sigma,q)$, and an element of $\mathcal{R}_0^r(N,\sigma,q)$ with $r=\sigma-\sigma_0-2N-2-(m_1+m_2-2\rho')$.

5.2. A class of sequences

Assume there exists a class of sequences $S_j(u,\omega,\epsilon)$, $0\le j\le N$ satisfying that S_j is written as $S_j=S_{1,j}+S_{2,j}$ with

$$\begin{aligned}&S_{1,j}\in\mathcal{L}'^{-j\rho'}_{\rho'}(j,\sigma,q), && [\partial_{xx},S_{1,j}]\in\Sigma^{-j\rho'}(j,\sigma,q), && j=0,\dots,N,\\ &S_{2,j}\in\mathcal{L}'^{-(j+2)\rho'}_{\rho'}(j,\sigma,q), && [\partial_{xx},S_{2,j}]\in\Sigma^{-(j+1)\rho'}(j,\sigma,q), && j=0,\dots,N-1,\\ &S_{2,N}=0.\end{aligned}\tag{5.3}$$

Let us check some properties of the class of sequences $S_j(u,\omega,\epsilon)$, $0\le j\le N$, satisfying (5.3).

Lemma 5.4. *Let r, σ, N satisfy $(N+1)\rho'\ge r+2$ and $\sigma\ge\sigma_0+2(N+1)+2+r$. Set*

$$S(u,\omega,\epsilon)=\Sigma_{j=0}^{N}S_j(u,\omega,\epsilon),$$

where $S_j=S_{1,j}+S_{2,j}$ and $S_{1,j}$, $S_{2,j}$ satisfy (5.3). The following two facts hold:

(i) *One may find, for $1\le j\le N$, $A_j\in\Sigma^{-j\rho'}(j-1,\sigma,q)$ depending only on S_l, $l\le j-1$ and satisfying $A_j^*=A_j$, one may find $R\in\mathcal{R}_2^r(N+1,\sigma,q)$, such that*

$$[S^*,L_\omega]S+S^*[L_\omega,S]=A^N+R,\tag{5.4}$$

where $A^N=\Sigma_{j=0}^{N}A_j$ with $A_0=0$, $[S^*,L_\omega]=S^*L_\omega-L_\omega S^*$.

(ii) *One may find, for $1\le j\le N$, A_j as above, $B_j\in\mathcal{L}^{-(j+1)\rho'}_{\rho'}(j,\sigma,q)$, $0\le j\le N-1$, satisfying $[\partial_{xx},B_j]\in\Sigma^{-(j+1)\rho'}(j,\sigma,q)$, B_j depending only on $S_{1,l}$, $l\le j$, $S_{2,l}$, $l\le j-1$ and $R\in\mathcal{R}_2^r(N+1,\sigma,q)$, such that*

$$S^*L_\omega S=A^N+(B^{N-1})^*L_\omega+L_\omega B^{N-1}+R,$$

where $B^{N-1}=\Sigma_{j=0}^{N-1}B_j$.

Proof. (i) Since $[L_\omega,S]=\omega^2[\partial_{tt},S]-[\partial_{xx},S]$, the left-hand side of (5.4) equals to

$$-S^*[\partial_{xx},S]-[S^*,\partial_{xx}]S+\omega^2S^*[\partial_{tt},S]+\omega^2[S^*,\partial_{tt}]S.$$

Define $\widehat{A}:=-S^*[\partial_{xx},S]-[S^*,\partial_{xx}]S$. It is clear to see that $\widehat{A}$ is self-adjoint. We write $\widehat{A}=\Sigma_{j=1}^{2N+1}\widehat{A}_j$, where

$$\widehat{A}_j:=-\sum_{\substack{j_1+j_2=j-1\\0\le j_1,j_2\le N}}\left([S^*_{j_1},\partial_{xx}]S_{j_2}+S^*_{j_2}[\partial_{xx},S_{j_1}]\right).\tag{5.5}$$

It follows from (5.3) and Proposition 4.7(ii) that for $1 \le j \le N$, $\widehat{A}_j$ may be written as the sum $A_j + R_j$, where

$$A_j \in \Sigma^{-j\rho'}(\min(N, j-1), \sigma, q), \quad R_j \in \mathcal{R}_0^{r_1}(\min(N, j-1), \sigma, q)$$

with $r_1 = \sigma - \sigma_0 - 2N - 2 + j\rho' \ge r$. The term in (5.5) implies A_j depends only on S_l, $l \le j-1$. Moreover, A_j is self-adjoint. For $j \ge N+1$, $A_j \in \Sigma^{-N\rho'}(N+1, \sigma, q)$, hence in $\mathcal{R}_0^r(N, \sigma, q)$ by $(N+1)\rho' \ge r$ and Remark 4.6. Define $\widehat{B} := \omega^2 S^*[\partial_{tt}, S] + \omega^2[S^*, \partial_{tt}]$. Obviously, it can be obtained that $\widehat{B}$ is self-adjoint. We write also $\widehat{B}$ as $\Sigma_{j=2}^{2N+2}\widehat{B}_j$, where

$$\widehat{B}_j = \omega^2 \sum_{\substack{j_1+j_2=j-2 \\ 0\le j_1,j_2\le N}} \left(S_{j_1}^*[\partial_{tt}, S_{j_2}] + [S_{j_2}^*, \partial_{tt}]S_{j_1}\right). \tag{5.6}$$

By (5.3), Remark 4.3 and Proposition 4.7(ii), $\widehat{B}_j$ for $1 \le j \le N$ can also be written as the sum $A_j + R_j$, where $A_j \in \Sigma^{-j\rho'}(\min(N+1, j-1), \sigma, q)$, $R_j \in \mathcal{R}_0^{r_1}(\min(N+1, j-1), \sigma, q)$. The term in (5.6) implies A_j depends only on S_l, $l \le j-2$. Furthermore, A_j is a self-adjoint operator. For $j \ge N+1$, $A_j \in \Sigma^{-(N+1)\rho'}(N+1, \sigma, q)$, hence in $\mathcal{R}_0^r(N+1, \sigma, q)$. Set $A^N = \Sigma_{j=0}^N A_j$ with $A_0 = 0$. This concludes the proof.

(ii) We express $S^*L_\omega S$ in terms of the sum of the following terms

$$\frac{1}{2}\left(S^*[L_\omega, S] + [S^*, L_\omega]S\right), \tag{5.7a}$$

$$\frac{1}{2}\left(S^*SL_\omega + L_\omega S^*S\right). \tag{5.7b}$$

By (i), the term in (5.7a) is written as $A^N + R$. In (5.7b), we write S^*S as the sum in j of

$$\sum_{\substack{j_1+j_2=j \\ 0\le j_1,j_2\le N}} {S_{1,j_1}}^* S_{1,j_2} + \sum_{\substack{j_1+j_2=j-1 \\ 0\le j_1,j_2\le N}} ({S_{1,j_1}}^* S_{2,j_2} + {S_{2,j_1}}^* S_{1,j_2}) + \sum_{\substack{j_1+j_2=j-2 \\ 0\le j_1,j_2\le N}} {S_{2,j_1}}^* S_{2,j_2}. \tag{5.8}$$

The condition (5.3) and Remark 5.3 shows that for $1 \le j \le N$, (5.8) may be written as $B_j + R_j$, where

$$B_j \in \mathcal{L}_{\rho'}^{-(j+1)\rho'}(\min(N, j), \sigma, q), \quad R_j \in \mathcal{R}_0^{r_2}(\min(N, j), \sigma, q)$$

with $r_2 = \sigma - \sigma_0 - 2N - 2 + (j+2)\rho' \ge r+2$. The expression in (5.8) indicates that B_j depends only on $S_{1,l}$, $l \le j$, $S_{2,l}$, $l \le j-1$. By construction, we also have

$$[\partial_{xx}, B_j] \in \Sigma^{-(j+1)\rho'}(\min(N, j), \sigma, q).$$

Furthermore, we have for $j \ge N+1$

$$B_j \in \Sigma^{-(N+1)\rho'}(N, \sigma, q),$$

hence in $\mathcal{R}_0^r(N+1, \sigma, q)$ by inequality $(N+1)\rho' \ge r+2$ and Remark 4.6. Set $B^{N-1} = \Sigma_{j=0}^{N-1} B_j$. Note that for $j \ge N+1$, B_jL_ω, $L_\omega B_j$ belong to $\mathcal{R}_2^r(N+1, \sigma, q)$. And R_jL_ω, $L_\omega R_j$ for $j \ge 0$ belongs to $\mathcal{R}_2^r(N+1, \sigma, q)$. □

Proposition 5.5. *Let r, σ, N, $S(u,\omega,\epsilon)$ satisfy the conditions of Lemma* 5.4.

(i) *There are elements for $0 \le j \le N-1$*

$$B_j(u,\omega,\epsilon) \in \mathcal{L}_{\rho'}^{-(j+1)\rho'}(j,\sigma,q) \quad \text{with } [\partial_{xx}, B_j] \in \Sigma^{-(j+1)\rho'}(j,\sigma,q),$$

where B_j depends only on $S_{1,l}$, $l \le j$, $S_{2,l}$, $l \le j-1$;

(ii) *There are elements for $0 \le j \le N$*

$$V_j(u,\omega,\epsilon) \in \Sigma^{-j\rho'}(j,\sigma,q),$$

where $V_j^* = V_j$, and V_j depends only on S_l, $l \le j-1$;

(iii) *There is an element $R \in \mathcal{R}_2^r(N+1,\sigma,q)$, such that if we set*

$$V^N(u,\omega,\epsilon) = \sum_{j=0}^{N} V_j(u,\omega,\epsilon), \quad B^{N-1}(u,\omega,\epsilon) = \sum_{j=0}^{N-1} B_j(u,\omega,\epsilon), \quad S_i = \sum_{j=0}^{N} S_{i,j}, \quad i = 1,2,$$

then the following equality holds:

$$\begin{aligned}(\mathrm{Id} + \epsilon S)^*(L_\omega + \epsilon V)(\mathrm{Id} + \epsilon S) = {} & L_\omega + \epsilon V^N + \epsilon\left((B^{N-1})^* L_\omega + L_\omega (B^{N-1})\right) \\ & + \epsilon\left(S_1^*(-\partial_{xx} + m) + (-\partial_{xx} + m) S_1\right) \\ & + \epsilon(S_2^* L_\omega + L_\omega S_2) + \epsilon R.\end{aligned} \tag{5.9}$$

Proof. The left-hand side of (5.9) may be expressed in terms of the sum of the following terms

$$L_\omega + \epsilon V(u,\omega,\epsilon) + \epsilon^2 S^* L_\omega S, \tag{5.10a}$$

$$\epsilon(S_1^*(\omega^2 \partial_{tt}) + (\omega^2 \partial_{tt}) S_1), \tag{5.10b}$$

$$\epsilon^2(S^* V + V S) + \epsilon^3 S^* V S, \tag{5.10c}$$

$$\epsilon(S_1^*(-\partial_{xx} + m) + (-\partial_{xx} + m) S_1) + \epsilon(S_2^* L_\omega + L_\omega S_2). \tag{5.10d}$$

In (5.10a), the term V contributes to the V_0 component of V^N. Lemma 5.4 shows that the A_j component of A^N contributes to the V_j component of V^N and that the B_j, $0 \le j \le N-1$ satisfy the condition of Proposition 5.5. We write (5.10b) as the sum in j of ${S_{1,j-1}}^*(\omega^2 \partial_{tt}) + (\omega^2 \partial_{tt}) S_{1,j-1}$, which is self-adjoint. Remark 4.3 infers for $1 \le j \le N$

$${S_{1,j-1}}^*(\omega^2 \partial_{tt}) + (\omega^2 \partial_{tt}) S_{1,j-1} \in \Sigma^{-j\rho'}(j,\sigma,q).$$

Then we get a contribution to V_j for $1 \le j \le N$. Owing to Remark 4.6, it yields that

$${S_{1,N}}^*(\omega^2 \partial_{tt}) + (\omega^2 \partial_{tt}) S_{1,N} \in \mathcal{R}_0^r(N+1,\sigma,q).$$

We write (5.10c) as the sum in j of

$$
\begin{aligned}
&S_{1,j-1}{}^*V + VS_{1,j-1} + S_{2,j-2}{}^*V + VS_{2,j-2} + \epsilon \sum_{j_1+j_2=j-2} S_{1,j_1}{}^*VS_{1,j_2} \\
&+ \epsilon \sum_{j_1+j_2=j-3} (S_{2,j_1}{}^*VS_{1,j_2} + S_{1,j_1}{}^*VS_{2,j_2}) + \epsilon \sum_{j_1+j_2=j-4} S_{2,j_1}{}^*VS_{2,j_2}.
\end{aligned} \tag{5.11}
$$

Applying $S_{1,j} \in \Sigma^{-(j+1)\rho'}(j,\sigma,q)$, $S_{2,j} \in \Sigma^{-(j+2)\rho'}(j,\sigma,q)$, $V \in \Sigma^0(0,\sigma,q)$, we write the term for $1 \le j \le N$ in (5.11) as $V_j + R_j$, where $V_j \in \Sigma^{-j\rho'}(\min(N,j-1),\sigma,q)$ and $R_j \in \mathcal{R}_0^r(\min(N,j-1),\sigma,q)$. Moreover, V_j depends only on $S_{1,l}$, $l \le j-1$, $S_{2,l}$, $l \le j-2$. For $j \ge N+1$, we have $V_j \in \mathcal{R}_0^r(N+1,\sigma,q)$ by Remark 4.6. □

Proposition 5.6. *Let $A(u,\omega,\epsilon) \in \Sigma^m_{\mathrm{ND}}(N,\sigma,q)$ be self-adjoint. There is an element $B(u,\omega,\epsilon)$ of $\mathcal{L}'^m_{\rho'}(N,\sigma,q)$ such that*

$$B(u,\omega,\epsilon)^*(-\partial_{xx}+m) + (-\partial_{xx}+m)B(u,\omega,\epsilon) = A(u,\omega,\epsilon).$$

Moreover, $[\partial_{xx}, B] \in \Sigma^m(N,\sigma,q)$.

Proof. Let $A(u,\omega,\epsilon) \in \Sigma^m_{\mathrm{ND}}(N,\sigma,q)$ with $A^* = A$. Assume $B(u,\omega,\epsilon) \in \mathcal{L}^m_{\rho'}(N,\sigma,q)$ with $B^* = -B$ satisfying $B^*(-\partial_{xx}+m) + (-\partial_{xx}+m)B = A$, i.e., we have to solve the equation

$$[B, \partial_{xx}] = A, \tag{5.12}$$

which is equivalent to $(n^2 - n'^2)\Pi_n B \Pi_{n'} = \Pi_n A \Pi_{n'}$, i.e.,

$$\left(\sqrt{n^2+m} - \sqrt{n'^2+m}\right)\left(\sqrt{n^2+m} + \sqrt{n'^2+m}\right)\Pi_n B \Pi_{n'} = \Pi_n A \Pi_{n'}.$$

Owing to the separation property (2.4) together with the fact of $|n| \ne |n'|$ with $n, n' \in \mathbb{Z}$, we have for some $c(m) > 0$.

$$\left|\left(\sqrt{n^2+m} - \sqrt{n'^2+m}\right)\left(\sqrt{n^2+m} + \sqrt{n'^2+m}\right)\right| \ge \widetilde{c}(m)(1+|n|+|n'|),$$

where the constant $\widetilde{c}(m)$ depends the lower bound in (2.4). Applying $A \in \Sigma^m_{\mathrm{ND}}(N,\sigma,q)$, we define

$$
\begin{aligned}
B(u,\omega,\epsilon) &= \sum_{\substack{n_1,n_2\in\mathbb{N}\\ n_1\ne n_2}} \sum_{n\in\{-n_1,n_1\}} \sum_{n'\in\{-n_2,n_2\}} (n^2-n'^2)^{-1}\Pi_n A(u,\omega,\epsilon)\Pi_{n'} \\
&= \sum_{\substack{n,n'\in\mathbb{Z}\\ |n|\ne|n'|}} (n^2-n'^2)^{-1}\Pi_n A(u,\omega,\epsilon)\Pi_{n'}.
\end{aligned}
$$

Thus we have $B \in \Sigma^{m-\rho'}(N,\sigma,q)$, where $\rho' = 1$. Moreover, we can obtain $[\partial_{xx}, B] \in \Sigma^m(N,\sigma,q)$ by (5.12). □

5.3. Diagonalization theorem

The following proposition gives a reduction for operator $L_\omega + \epsilon V$ in (5.1). Through the para-differential conjugation, the para-differential potential $V(u,\omega,\epsilon)$ is replaced by $V_{\mathrm{D}}(u,\omega,\epsilon)$, where V_{D} is block diagonal relatively to an orthogonal decomposition of $L^2(\mathbb{T})$ in a sum of finite-dimensional subspaces.

Proposition 5.7. *Let r be a given positive number and N be an fixed integer satisfying $(N+1)\rho' \geq r+2$. Let $\sigma \in \mathbb{R}$ with*

$$\sigma \geq \sigma_0 + 2(N+1) + 2 + \frac{r}{\rho'}.$$

Setting $q > 0$, one can find elements $Q_j(u,\omega,\epsilon) \in \mathcal{L}_{\rho'}^{-j\rho'}(j,\sigma,q)$, $0 \leq j \leq N$, elements $V_{\mathrm{D},j}(u,\omega,\epsilon) \in \Sigma_{\mathrm{D}}^{-j\rho'}(j,\sigma,q)$, $0 \leq j \leq N$, an element $R_1(u,\omega,\epsilon) \in \mathcal{R}_2^r(N+1,\sigma,q)$, are C^1 in (ω,ϵ), such that for any $u \in B_q(\mathcal{H}^\sigma)$, this holds:

$$\begin{aligned}&(\mathrm{Id} + \epsilon Q(u,\omega,\epsilon))^*(L_\omega + \epsilon V(u,\omega,\epsilon))(\mathrm{Id} + \epsilon Q(u,\omega,\epsilon))\\ &= L_\omega + \epsilon V_{\mathrm{D}}(u,\omega,\epsilon) - \epsilon R_1(u,\omega,\epsilon),\end{aligned} \tag{5.13}$$

where

$$Q(u,\omega,\epsilon) = \sum_{j=0}^{N} Q_j(u,\omega,\epsilon), \quad V_{\mathrm{D}}(u,\omega,\epsilon) = \sum_{j=0}^{N} V_{\mathrm{D},j}(u,\omega,\epsilon). \tag{5.14}$$

Proof. Let us verify that the right-hand side of (5.9) may be written as the right-hand side of (5.13). Assume that $Q_0,\ldots,Q_{j-1}$ have been already determined, where Q_i, $0 \leq i \leq j-1$ may be written as the sum $Q_{1,i} + Q_{2,i}$ with $Q_{1,i}$, $Q_{2,i}$ satisfying (5.3), such that V_j depend only on Q_l, $l \leq j-1$ and the right-hand side of (5.9) can be written as

$$\begin{aligned}&L_\omega + \epsilon \sum_{j'=0}^{j-1} V_{\mathrm{D},j'} + \epsilon \sum_{j'=j}^{N-1} (B_{j'}^* L_\omega + L_\omega B_{j'}) + \epsilon \sum_{j'=j}^{N} ({Q_{1,j'}}^*(-\partial_{xx} + m) + (-\partial_{xx}+m)Q_{j'})\\ &+ \epsilon \sum_{j'=j}^{N-1} ({Q_{2,j'}}^* L_\omega + L_\omega Q_{2,j'}) + \epsilon \sum_{j'=j}^{N} V_{j'} + \epsilon R.\end{aligned} \tag{5.15}$$

It is straightforward to show that (5.15) with $j=0$ is the conclusion of Proposition 5.5. Since $V_j \in \Sigma^{-j\rho'}(j,\sigma,q)$ with $V_j^* = V_j$, depending on Q_l, $l \leq j-1$, we define

$$V_{\mathrm{D},j} = \sum_{n\in\mathbb{N}} \widetilde{\Pi}_n V_j \widetilde{\Pi}_{n'}, \quad V_{\mathrm{ND},j} = \sum_{\substack{n,n'\in\mathbb{N}\\ n\neq n'}} \widetilde{\Pi}_n V_j \widetilde{\Pi}_{n'}.$$

Then $V_{\mathrm{D},j} \in \Sigma_{\mathrm{D}}^{-j\rho'}(j,\sigma,q)$ with $(V_{\mathrm{D},j})^* = V_{\mathrm{D},j}$ and $V_{\mathrm{ND},j} \in \Sigma_{\mathrm{ND}}^{-j\rho'}(j,\sigma,q)$ with $(V_{\mathrm{ND},j})^* = V_{\mathrm{ND},j}$, where $V_{\mathrm{ND},j}$ depends only on Q_l, $l \leq j-1$. By Proposition 5.6, for $V_{\mathrm{ND},j}$, we may find $C_j \in \mathcal{L}'^{-j\rho'}_{\rho'}(j,\sigma,q)$ such that $C_j{}^*(-\partial_{xx}+m)+(-\partial_{xx}+m)C_j = V_{\mathrm{ND},j}$ with $[\partial_{xx}, C_j] \in \Sigma^{-j\rho'}(j,\sigma,q)$. Let $Q_{1,j} := -C_j$. This shows that we may eliminate the jth component of

$$\epsilon \sum_{j'=j}^{N} (Q_{1,j'}{}^*(-\partial_{xx}+m)+(-\partial_{xx}+m)Q_{1,j'})$$

and $\epsilon \sum_{j'=j}^{N} V_{j'}$. Moreover, $Q_{1,j}$ satisfies (5.3). Set $Q_{2,j} := -B_j$, $0 \le j \le N-1$. Then we may eliminate the jth component of $\epsilon \sum_{j'=j+1}^{N} (B_{j'}^* L_\omega + L_\omega B_{j'})$ and

$$\epsilon \sum_{j'=j+1}^{N} (Q_{2,j'}{}^* L_\omega + L_\omega Q_{2,j'}).$$

In addition, $Q_{2,j}$ satisfies (5.3). Therefore we may construct recursively $Q_{1,j}$, $0 \le j \le N$, $Q_{2,j}$, $0 \le j \le N-1$ satisfying (5.3), such that the equality in (5.13) holds. □

6. Iterative scheme

This section is concerned with the proof of Theorem 2.1. First, we investigate some properties about the restriction of the operator $L_\omega + \epsilon V_{\mathrm{D}}(u,\omega,\epsilon)$ to $\mathrm{Range}(\widetilde{\Pi}_n)$. Next, under the non-resonant conditions (6.3), we prove the restriction is invertible and the frequencies ω are in a Cantor-like set whose complement has small measure. Finally, we use a standard iterative scheme to construct the solutions.

6.1. Lower bounds for eigenvalues

Let $\gamma_0 \in (0,1]$, $\sigma \in \mathbb{R}$, $N \in \mathbb{N}$, $\zeta \in \mathbb{R}_+$ with $\sigma \ge \sigma_0 + 2(N+1) + 2 + \zeta/\rho'$. Define the space of functions by

$$\begin{aligned}\mathcal{E}^\sigma_\zeta &:= \mathcal{E}^\sigma_\zeta(\mathbb{T}\times\mathbb{T}\times[1,2]\times(0,\gamma_0];\mathbb{R}) \\ &= \Big\{u(t,x,\omega,\epsilon);\ u \in \mathcal{H}^\sigma,\ \partial_\omega u \in \mathcal{H}^{\sigma-\zeta-2},\ u(t,x,\omega,\epsilon),\ \partial_\omega u(t,x,\omega,\epsilon) \\ &\qquad \text{for any } \epsilon \in [0,\gamma_0] \text{ are continuous in } \omega,\ \|u\|_{\mathcal{E}^\sigma_\zeta} < +\infty\Big\}.\end{aligned}$$

where

$$\|u\|_{\mathcal{E}^\sigma_\zeta} := \sup_{(\omega,\epsilon)\in[1,2]\times[0,\gamma_0]} \|u(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^\sigma} + \sup_{(\omega,\epsilon)\in[1,2]\times[0,\gamma_0]} \|\partial_\omega u(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^{\sigma-\zeta-2}}.$$

Moreover, for fixed $n \in \mathbb{N}$, we define operator for any $u \in \mathcal{E}^\sigma_\zeta$, $\omega \in [1,2]$, $\epsilon \in (0,\gamma_0]$,

$$A_n(\omega;u,\epsilon) = \widetilde{\Pi}_n(L_\omega + \epsilon V_{\mathrm{D}}(u,\omega,\epsilon))\widetilde{\Pi}_n. \tag{6.1}$$

Set $F_n = \mathrm{Range}(\widetilde{\Pi}_n)$, $D_n = \dim F_n$. By (2.5), we have $D_n \le C_1 \langle n \rangle$ for some $C_1 > 0$. This implies that $A_n(\omega;u,\epsilon)$ is self-adjoint on a space of finite dimension. By means of (5.14), (4.2), $\partial_\omega u \in \mathcal{H}^{\sigma-\zeta-2}$ and the assumption on σ, we obtain that $A_n(\omega;u,\epsilon)$ is C^1 in ω.

Proposition 6.1. *Let $m > 0$, $q > 0$. There exists $\gamma_0 \in (0,1]$ small enough, $C_0 > 0$, and for any $u \in \mathcal{E}^\sigma(\zeta)$ with $\|u\|_{\mathcal{E}^\sigma(\zeta)} < q$, any $\epsilon \in [0,\gamma_0]$, any $n \in \mathbb{N}$, the eigenvalues of A_n form a finite family of C^1 real valued functions of ω, depending on (u,ϵ), i.e.,*

$$\omega \to \lambda_l^n(\omega; u, \epsilon), \quad 1 \leq l \leq D_n$$

satisfying the following properties:

(i) *For any $n \in \mathbb{N}$, any $u, u' \in \mathcal{H}^\sigma$ with $\|u\|_{\mathcal{H}^\sigma} < q$, $\|u'\|_{\mathcal{H}^\sigma} < q$, any $l \in \{1, \ldots, D_n\}$, any $\epsilon \in (0,\gamma_0]$, any $\omega \in [1,2]$, there is $l' \in \{1, \ldots, D_n\}$ such that*

$$\left|\lambda_l^n(\omega; u, \epsilon) - \lambda_{l'}^n(\omega; u', \epsilon)\right| \leq C_0 \epsilon \left\|u - u'\right\|_{\mathcal{H}^\sigma}. \tag{6.2}$$

(ii) *For any $n \in \mathbb{N}$, any $u \in \mathcal{E}^\sigma(\zeta)$ with $\|u\|_{\mathcal{E}^\sigma(\zeta)} < q$, any $\epsilon \in (0,\gamma_0]$, any $l \in \{1, \ldots, D_n\}$, any $\omega \in [1,2]$, this holds*

$$-4C_0 \langle n \rangle^2 \leq \partial_\omega \lambda_l^n(\omega; u, \epsilon) \leq -2C_0^{-1} \langle n \rangle^2. \tag{6.3}$$

(iii) *For any $n \in \mathbb{N}$, any $u \in \mathcal{E}_\zeta^\sigma$ with $\|u\|_{\mathcal{E}_\zeta^\sigma} < q$, $\delta \in (0,1]$, $\epsilon \in (0,\gamma_0]$, if we set*

$$I(n,u,\epsilon,\delta) = \Big\{\omega \in [1,2];\ \forall\, l \in \{1, \ldots, D_n\},\ |\lambda_l^n(\omega; u, \epsilon)| \geq \delta \langle n \rangle^{-\zeta}\Big\}, \tag{6.4}$$

then there is a constant E_0 depending only on the dimension, such that for any $\omega \in I(n,u,\epsilon,\delta)$, $A_n(\omega; u, \epsilon)$ is invertible and satisfies

$$\left\|A_n(\omega; u,\epsilon)^{-1}\right\|_{\mathcal{L}(\mathcal{H}^0)} \leq E_0 \delta^{-1} \langle n \rangle^{\zeta}, \quad \left\|\partial_\omega A_n(\omega; u,\epsilon)^{-1}\right\|_{\mathcal{L}(\mathcal{H}^0)} \leq E_0 \delta^{-2} \langle n \rangle^{2\zeta+2}. \tag{6.5}$$

Proof. (i) By the property of A_n, Theorem 6.8 in [33] shows that we may index eigenvalues $\lambda_l^n(\omega; u, \epsilon)$, $l \in \{1, \ldots, D_n\}$ of A_n such that they are C^1 functions of ω. On the other hand, for any eigenvalue $\lambda_l(B)$ of B, there is an eigenvalue $\lambda_{l'}(B')$ of B' with $l' \in \{1, \ldots, D_n\}$ such that $|\lambda_l(B) - \lambda_{l'}(B')| \leq \|B - B'\|$ when B B' are self-adjoint in the same dimension space. Moreover, $u \to A_n(\omega; u, \epsilon)$ is lipschitz with values in $\mathcal{L}(\mathcal{H}^0)$. Consequently formula (6.2) can be obtained with lipschitz constant $C_0\epsilon$.

(ii) Set $L_\omega^n = \widetilde{\Pi}_n L_\omega \widetilde{\Pi}_n$. We denote by $\Lambda^0(n)$ the spectrum set of L_ω^n, where

$$\Lambda^0(n) = \big\{ -\omega^2 j^2 + n'^2 + m : n' \in \{-n, n\} \text{ with } n \in \mathbb{N}, \\ j \in \mathbb{Z} \text{ with } K_0^{-1} \langle n \rangle \leq |j| \leq K_0 \langle n \rangle \big\}.$$

Similarly, $\Lambda(n)$ stands for the spectrum of A_n. Let Γ be a contour in the complex plane turning once around $\Lambda^0(n)$, of length $O(\langle n \rangle^2)$, where Γ satisfies $\operatorname{dist}(\Gamma, \Lambda^0(n)) \geq c_0 \langle n \rangle^2$. If $\epsilon \in [0,\gamma_0]$ with γ_0 small enough, we also have $\operatorname{dist}(\Gamma, \Lambda(n)) \geq c_0 \langle n \rangle^2$. Moreover, we define the spectral projector $\Pi_n(\omega)$ (resp. $\Pi_n^0(\omega)$) associated to the eigenvalues $\Lambda(n)$ (resp. $\Lambda^0(n)$) of A_n (resp. L_ω^n) by

$$\Pi_n(\omega) = \frac{1}{2i\pi}\int_\Gamma (\zeta \operatorname{Id} - A_n)^{-1}\,\mathrm{d}\zeta, \quad \Pi_n^0 = \frac{1}{2i\pi}\int_\Gamma (\zeta \operatorname{Id} - L_\omega^n)^{-1}\,\mathrm{d}\zeta. \tag{6.6}$$

Then there exist some constant $C > 0$ such that $\|\Pi_n(\omega)\|_{\mathcal{L}(F_n)} \le C$, $\left\|\Pi_n^0\right\|_{\mathcal{L}(F_n)} \le C$. Note that Π_n^0 is just the orthogonal projector on

$$\operatorname{Vect}\left\{ e^{\mathrm{i}(jt+n'x)} : n' \in \{-n, n\} \text{ with } n \in \mathbb{N},\, j \in \mathbb{Z} \text{ with } K_0^{-1}\langle n\rangle \le |j| \le K_0\langle n\rangle \right\}.$$

This implies that Π_n^0 is independent of ω. Let us consider the upper bound of

$$\left\|\partial_\omega(\Pi_n(\omega)A_n\Pi_n(\omega) - \Pi_n^0 L_\omega^n \Pi_n^0)\right\|_{\mathcal{L}(F_n)}, \tag{6.7}$$

where

$$\begin{aligned}\Pi_n(\omega)A_n\Pi_n(\omega) - \Pi_n^0 L_\omega^n \Pi_n^0 = (\Pi_n(\omega) - \Pi_n^0)A_n\Pi_n(\omega) + \Pi_n^0(A_n - L_\omega^n)\Pi_n(\omega)\\ + \Pi_n^0 L_\omega^n(\Pi_n(\omega) - \Pi_n^0).\end{aligned}$$

Formula (6.6) indicates

$$\Pi_n(\omega) - \Pi_n^0 = \frac{1}{2\mathrm{i}\pi}\int_\Gamma (\zeta \operatorname{Id} - A_n)^{-1}(A_n - L_\omega^n)(\zeta \operatorname{Id} - L_\omega^n)^{-1}\,\mathrm{d}\zeta. \tag{6.8}$$

It follows from (6.1) that

$$\begin{aligned}\|A_n - L_\omega^n\|_{\mathcal{L}(F_n)} + \|\partial_\omega(A_n - L_\omega^n)\|_{\mathcal{L}(F_n)} \le C\epsilon,\\ \|\partial_\omega A_n\|_{\mathcal{L}(F_n)} + \|\partial_\omega L_\omega^n\|_{\mathcal{L}(F_n)} \le C\langle n\rangle^2.\end{aligned} \tag{6.9}$$

Then formulae (6.8) and (6.9) give rise to

$$\left\|\Pi_n(\omega) - \Pi_n^0\right\|_{\mathcal{L}(F_n)} \le C\epsilon\langle n\rangle^{-2}, \quad \|\partial_\omega\Pi_n(\omega)\|_{\mathcal{L}(F_n)} \le C\epsilon\langle n\rangle^{-2}.$$

Consequently, (6.7) is bounded from above by $C\epsilon$. Let $\mathcal{B}$ be a subinterval of $[1, 2]$. One of the eigenvalues $\lambda_l^n(\omega; u, \epsilon)$ of $\Pi_n(\omega)A_n\Pi_n(\omega)$ ($\omega \in \mathcal{B}$) has constant multiplicity m. Let us denote by $P(\omega)$ ($\omega \in \mathcal{B}$) the associated spectral projector, where $P(\omega)^2 = P(\omega)$ with C^1 dependence in $\omega \in \mathcal{B}$. Then we obtain

$$\lambda_l^n(\omega; u, \epsilon) = \frac{1}{m}\operatorname{tr}(P(\omega)\Pi_n(\omega)A_n\Pi_n(\omega)P(\omega)),$$

which then shows that

$$\partial_\omega\lambda_l^n(\omega; u, \epsilon) = \frac{1}{m}\operatorname{tr}(P(\omega)\partial_\omega(\Pi_n(\omega)A_n\Pi_n(\omega))P(\omega)).$$

Combining this with the fact that (6.7) has the upper bound $C\epsilon$, we get

$$\partial_\omega \lambda_l^n(\omega; u, \epsilon) = \frac{1}{m} \operatorname{tr}(P(\omega)\partial_\omega(\Pi_n^0 L_\omega^n \Pi_n^0)P(\omega)) + O(\epsilon).$$

Moreover, the definition of L_ω^n derives that $\Pi_n^0 L_\omega^n \Pi_n^0$ is diagonal with entries $-\omega^2 j^2 + n'^2 + m$ for $n' \in \{-n, n\}$ with $n \in \mathbb{N}$, $j \in \mathbb{Z}$ with $K_0^{-1}\langle n\rangle \le |j| \le K_0 \langle n\rangle$. This reads

$$-4K_0^2 \langle n\rangle^2 - C\epsilon \le \partial_\omega \lambda_l^n(\omega; u, \epsilon) \le -2K_0^{-2}\langle n\rangle^2 + C\epsilon.$$

Consequently, we get (6.3) when ϵ is in $(0, \gamma_0]$ with γ_0 small enough. □

6.2. Iterative scheme

In this subsection, our goal is to achieve the proof of Theorem 2.1. Fix indices s, σ, N, ζ, r, δ satisfying the following inequalities

$$\sigma \ge \sigma_0 + 2(N+1) + 2 + \frac{r}{\rho'},\ r = \zeta,\ (N+1)\rho' \ge r + 2,\ s \ge \sigma + \zeta + 2,\ \delta \in (0, \delta_0], \tag{6.10}$$

where $\delta_0 > 0$ is small enough. Let $m > 0$ and the force term f in (5.1) be given in $\mathcal{H}^{s+\zeta}$. First we study how to solve equation (5.1). Our main task is to construct a sequence $(G_k, \mathcal{O}_k, \psi_k, u_k, w_k)$, $k \ge 0$, where G_k, $\mathcal{O}_k$ will be subsets of $[1,2] \times [0, \delta^2]$, ψ_k will be a function of $(\omega, \epsilon) \in [1,2] \times [0, \delta^2]$, u_k, w_k will be functions of $(t, x, \omega, \epsilon) \in \mathbb{T} \times \mathbb{T} \times [1,2] \times [0, \delta^2]$. When $k = 0$, define

$$\begin{aligned}
&u_0 = w_0 = 0,\\
&\mathcal{O}_0 = \big\{(\omega, \epsilon) \in [1,2] \times [0, \gamma_0];\ \exists\, n \in \mathbb{Z} \text{ with } 1 \le \langle n\rangle < 3,\\
&\qquad\qquad\qquad\qquad \exists\, l \in \{1, \ldots, D_n\} \text{ with } |\lambda_l^n(\omega; 0, \epsilon)| < 2\delta\big\},\\
&G_0 = \left\{(\omega, \epsilon) \in [1,2] \times [0, \gamma_0];\ \operatorname{dist}(\omega, \mathbb{R} - \mathcal{O}_{0,\epsilon}) \ge \frac{\delta}{72C_0}\right\},
\end{aligned}$$

where C_0 is given in (6.3), $\mathcal{O}_{0,\epsilon}$ is the ϵ-section of $\mathcal{O}_0$ for any $\epsilon \in [0, \gamma_0]$. We also denote by $G_{0,\epsilon}$ the ϵ-section of G_0 for any $\epsilon \in [0, \gamma_0]$. Obviously, $G_{0,\epsilon}$ is a closed subset of $[1,2]$ for any $\epsilon \in [0, \gamma_0]$, contained in the open subset $\mathcal{O}_{0,\epsilon}$. By Urysohn's lemma, when ϵ is fixed, we may construct a C^1 function $\omega \to \psi_0(\omega, \epsilon)$, compactly supported in $\mathcal{O}_{0,\epsilon}$, equal to 1 on $G_{0,\epsilon}$, satisfying for any ω, ϵ

$$0 \le \psi_0(\omega, \epsilon) \le 1, \quad |\partial_\omega \psi_0(\omega, \epsilon)| \le C_1 \delta^{-1},$$

where C_1 is some uniform constant depending only on C_0. Set

$$\widetilde{S}_k = \sum_{\substack{n \in \mathbb{Z}\\ \langle n\rangle < 3^{k+1}}} \widetilde{\Pi}_n, \quad k \ge 0. \tag{6.11}$$

Proposition 6.2. *There are $\delta_0 \in (0, \sqrt{\gamma_0}]$ with γ_0 small enough, positive constants C_1, B_1, B_2, a* 5*-tuple $(G_k, \mathcal{O}_k, \psi_k, u_k, w_k)$ for any $k \geq 0$, any $\delta \in (0, \delta_0)$ satisfying the following conditions:*

$$
\begin{aligned}
\mathcal{O}_k = \Big\{(\omega,\epsilon) \in [1,2]\times[0,\delta^2];\ \exists\, n \in \mathbb{Z} \text{ with } 3^k \leq \langle n\rangle < 3^{k+1},\\
\exists\, l \in \{1,\dots,D_n\}\,,\ |\lambda_l^n(\omega; u_{k-1},\epsilon)| < 2\delta\, 3^{-k\zeta}\Big\},\\
G_k = \Big\{(\omega,\epsilon) \in [1,2]\times[0,\delta^2];\ \operatorname{dist}(\omega, \mathbb{R} - \mathcal{O}_{k,\epsilon}) \geq \frac{\delta}{72C_0} 3^{-k(\zeta+2)}\Big\},
\end{aligned} \tag{6.12}
$$

where C_0 is given in (6.3)*. And*

$$
\begin{aligned}
\psi_k\colon [1,2]\times[0,\delta^2] \to [0,1] \text{ is supported in } \mathcal{O}_k, \text{ equal to } 1 \text{ on } G_k,\\
C^1 \text{ in } \omega \text{ and for all } (\omega,\epsilon),\quad |\partial_\omega \psi_k(\omega,\epsilon)| \leq \frac{C_1}{\delta} 3^{k(\zeta+2)}.
\end{aligned} \tag{6.13}
$$

For any $\epsilon \in [0, \delta^2]$, it can be showed that

$$w_k \in \mathcal{H}^s, \quad \partial_\omega w_k \in \mathcal{H}^{s-\zeta-2},$$

and the functions $w_k(t,x,\omega,\epsilon)$, $\partial_\omega w_k(t,x,\omega,\epsilon)$ are continuous with respect to ω and satisfy

$$\|w_k(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^s} + \delta\, \|\partial_\omega w_k(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}} \leq B_1 \frac{\epsilon}{\delta}, \tag{6.14}$$

$$\|w_k - w_{k-1}\|_{\mathcal{H}^\sigma} \leq B_2 \frac{\epsilon}{\delta} 3^{-k\zeta} \tag{6.15}$$

uniformly in $\epsilon \in [0,\delta^2]$, $\omega \in [1,2]$, $\delta \in (0,\delta_0]$. Furthermore, for any $(\omega,\epsilon) \in [1,2]\times[0,\delta^2] - \bigcup_{k'=0}^{k} \mathcal{O}_{k'}$, w_k satisfies the equation

$$
\begin{aligned}
(L_\omega + \epsilon V_{\mathrm{D}}(u_{k-1},\omega,\epsilon)) w_k = \epsilon \widetilde{S}_k (\mathrm{Id} + \epsilon Q(u_{k-1},\omega,\epsilon))^* \widetilde{R}(u_{k-1},\omega,\epsilon) u_{k-1}\\
+ \epsilon \widetilde{S}_k (R_1(u_{k-1},\omega,\epsilon) w_{k-1})\\
+ \epsilon \widetilde{S}_k (\mathrm{Id} + \epsilon Q(u_{k-1},\omega,\epsilon))^* f,
\end{aligned} \tag{6.16}
$$

where $\widetilde{R}$ is defined in (5.1) *and Q, V_{D}, R_1 are defined in* (5.14) *and* (5.13)*. The function u_k is deduced from w_k by*

$$u_k(t,x,\omega,\epsilon) = (\mathrm{Id} + \epsilon Q(u_{k-1},\omega,\epsilon)) w_k \tag{6.17}$$

with

$$\|u_k(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^s} + \delta\, \|\partial_\omega u_k(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}} \leq B_2 \frac{\epsilon}{\delta}, \tag{6.18}$$

$$\|u_k - u_{k-1}\|_{\mathcal{H}^\sigma} \leq 2B_2 \frac{\epsilon}{\delta} 3^{-k\zeta} \tag{6.19}$$

uniformly for $\epsilon \in [0,\delta^2]$, $\omega \in [1,2]$, $\delta \in (0,\delta_0]$.

Remark 6.3. If we assume $\epsilon \le \delta^2$, then (6.18) implies for some constant $q > 0$

$$\|u_k\|_{\mathcal{E}^\sigma(\zeta)} < q. \tag{6.20}$$

Before the proof of Proposition 6.2, we need to introduce two lemmas.

Lemma 6.4. *There is $\delta_0 \in (0,1]$ small enough, depending only on the constants B_1, B_2, such that for any $k \ge 0$, any $k' \in \{0, \dots, k+1\}$, any $\epsilon \in [0, \delta^2]$, any $\delta \in (0, \delta_0]$, any $n \in \mathbb{N}$ with $3^{k'} \le \langle n \rangle < 3^{k'+1}$*

$$[1,2] - G_{k',\epsilon} \subset I(n, u_k, \epsilon, \delta),$$

where $I(\cdot)$ is defined by (6.4)*. When $k = 0$, we set $u_{-1} = 0$.*

Proof. We first consider $\omega \in [1,2] - \mathcal{O}_{k',\epsilon}$, $l \in \{1, \dots, D_n\}$. By Proposition 6.1(ii), (6.12) and (6.19), setting $(u, u') = (u_k, u_{k'-1})$, there exists $l' \in \{1, \dots, D_n\}$ such that

$$|\lambda_l^n(\omega; u_k, \epsilon)| \ge 2\delta 3^{-k'\zeta} - 2C_0 B_2 \frac{\epsilon^2}{\delta} \frac{3^{-k'\zeta}}{1 - 3^{-\zeta}} \ge \frac{3}{2}\delta 3^{-k'\zeta}, \tag{6.21}$$

when $\epsilon \le \delta^2$ if $\delta \in [0, \delta_0]$ with δ_0 small enough. Next, let $\omega \in \mathcal{O}_{k',\epsilon} - G_{k',\epsilon}$. The definition in (6.12) indicates that

$$|\omega - \widetilde{\omega}| < \frac{\delta}{72C_0} 3^{-k'(\zeta+2)},$$

where $\widetilde{\omega} \in [1,2] - \mathcal{O}_{k',\epsilon}$. Due to (6.3), we have that for any $u \in \mathcal{E}^\sigma(\zeta)$ with $\|u\|_{\mathcal{E}^\sigma(\zeta)} < q$, any $n \in \mathbb{N}$, any $l \in \{1, \dots, D_n\}$

$$\sup_{\omega \in [1,2]} \left|\partial_\omega \lambda_l^n(\omega'; u, \epsilon)\right| \le 4C_0 \langle n \rangle^2.$$

From (6.21) and $3^{k'} \le \langle n \rangle < 3^{k'+1}$, it yields that

$$|\lambda_l^n(\omega; u_k, \epsilon)| \ge |\lambda_l^n(\widetilde{\omega}; u_k, \epsilon)| - 4C_0 \langle n \rangle^2 |\omega - \widetilde{\omega}| \ge \delta \langle n \rangle^{-\zeta}. \qquad \square$$

In order to use the recurrence method, we shall also need to give the upper bound of the right-hand side of equation (6.16) at $k+1$-th step. Set

$$\begin{aligned} H_{k+1}(u_k, w_k) = {} & \widetilde{S}_{k+1}(\mathrm{Id} + \epsilon Q(u_k, \omega, \epsilon))^* \widetilde{R}(u_k, \omega, \epsilon) u_k \\ & + \widetilde{S}_{k+1}(R_1(u_k, \omega, \epsilon) w_k) + \epsilon \widetilde{S}_{k+1}(\mathrm{Id} + \epsilon Q(u_k, \omega, \epsilon))^* f. \end{aligned} \tag{6.22}$$

Lemma 6.5. *There exists a constant $C > 0$, depending on q in* (6.20) *but independent of k, such that for any $\omega \in [1,2]$, any $\epsilon \in [0, \delta^2]$, any $\delta \in [0, \delta_0]$, the following holds:*

$$\|H_{k+1}(u_k, w_k)\|_{\mathcal{H}^{s+\zeta}} \le C(\|u_k(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^s} + \|w_k(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^s}) + (1 + C\epsilon)\|f\|_{\mathcal{H}^{s+\zeta}}, \tag{6.23}$$

$$\begin{aligned} \|\partial_\omega H_{k+1}(u_k, w_k)\|_{\mathcal{H}^{s-2}} \le {} & C(\|u_k(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^s} + \|\partial_\omega u_k(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^{s-\zeta-2}} \\ & + \|w_k(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^s} + \|\partial_\omega w_k(\,\cdot\,, \omega, \epsilon)\|_{\mathcal{H}^{s-\zeta-2}} + \epsilon\|f\|_{\mathcal{H}^{s-2}}), \end{aligned} \tag{6.24}$$

and

$$
\begin{aligned}
&\|H_{k+1}(u_k,w_k)-H_k(u_{k-1},w_{k-1})\|_{\mathcal{H}^{\sigma+\zeta}}\\
&\le C(\|u_k-u_{k-1}\|_{\mathcal{H}^{\sigma}}+\|w_k-w_{k-1}\|_{\mathcal{H}^{\sigma}})\\
&\quad+3^{-k\zeta}\left(C(\|u_k\|_{\mathcal{H}^{\sigma+\zeta}}+\|w_k\|_{\mathcal{H}^{\sigma+\zeta}})+(1+C\epsilon)\,\|f\|_{\mathcal{H}^{\sigma+2\zeta}}\right).
\end{aligned}
\tag{6.25}
$$

Proof. Let u_k satisfy (6.20). It follows from Definition 4.5 and (6.10) that $\widetilde{R}$, R_1 are bounded from $\mathcal{H}^s$ to $\mathcal{H}^{s+\zeta}$ with $s\in\mathbb{R}$. Moreover, Lemma 4.4 shows that $Q(u_k,\omega,\epsilon)^*$ is bounded on space $\mathcal{H}^s$ with $s\in\mathbb{R}$, which yields (6.23).

The term in (6.22) implies that the upper bound of the following terms

$$\partial_\omega\left(Q(u_k,\omega,\epsilon)\right)=\partial_uQ(\,\cdot\,,\omega,\epsilon)\cdot(\partial_\omega u_k)+\partial_\omega Q(u_k,\omega,\epsilon), \tag{6.26a}$$

$$\partial_\omega\left(\widetilde{R}(u_k,\omega,\epsilon)\right)=\partial_u\widetilde{R}(\,\cdot\,,\omega,\epsilon)\cdot(\partial_\omega u_k)+\partial_\omega \widetilde{R}(u_k,\omega,\epsilon), \tag{6.26b}$$

$$\partial_\omega\left(R_1(u_k,\omega,\epsilon)\right)=\partial_uR_1(\,\cdot\,,\omega,\epsilon)\cdot(\partial_\omega u_k)+\partial_\omega R_1(u_k,\omega,\epsilon) \tag{6.26c}$$

has to be required. The assumption on s in (6.10) shows $\mathcal{H}^{s-\zeta-2}\subset\mathcal{H}^\sigma$. Formulae (4.4) and (6.20) read that (6.26a) is bounded on any space $\mathcal{H}^s$. Similarly, we see also that (6.26b), (6.26c) are bounded from $\mathcal{H}^s$ to $\mathcal{H}^{s+\zeta}$. This completes the proof of (6.24).

Let us write the difference of $H_{k+1}(u_k,w_k)-H_k(u_{k-1},w_{k-1})$ as the sum of the following three parts:

$$
\begin{cases}
(\widetilde{S}_{k+1}-\widetilde{S}_k)(\mathrm{Id}+\epsilon Q(u_k,\omega,\epsilon))^*\widetilde{R}(u_k,\omega,\epsilon)u_k,\\
(\widetilde{S}_{k+1}-\widetilde{S}_k)R_1(u_k,\omega,\epsilon)w_k,\\
(\widetilde{S}_{k+1}-\widetilde{S}_k)(\mathrm{Id}+\epsilon Q(u_k,\omega,\epsilon))^*f,
\end{cases}
\tag{6.27}
$$

$$
\begin{cases}
\epsilon\widetilde{S}_k(Q(u_k,\omega,\epsilon)^*-Q(u_{k-1},\omega,\epsilon)^*)\widetilde{R}(u_k,\omega,\epsilon)u_k,\\
\widetilde{S}_k(\mathrm{Id}+\epsilon Q(u_{k-1},\omega,\epsilon))^*(\widetilde{R}(u_k,\omega,\epsilon)-\widetilde{R}(u_{k-1},\omega,\epsilon))u_k,\\
\widetilde{S}_k(R_1(u_k,\omega,\epsilon)-R_1(u_{k-1},\omega,\epsilon))w_k,\\
\epsilon\widetilde{S}_k(Q(u_k,\omega,\epsilon)^*-Q(u_{k-1},\omega,\epsilon)^*)f,
\end{cases}
\tag{6.28}
$$

and

$$
\begin{cases}
\widetilde{S}_k(\mathrm{Id}+\epsilon Q(u_{k-1},\omega,\epsilon))^*\widetilde{R}(u_{k-1},\omega,\epsilon)(u_k-u_{k-1}),\\
\widetilde{S}_kR_1(u_k,\omega,\epsilon)(w_k-w_{k-1}).
\end{cases}
\tag{6.29}
$$

Formulae (6.14) and (6.19) lead to that u_k, w_k are in a bounded subset of $\mathcal{H}^\sigma$. This establishes that $\widetilde{R}$, R_1 are bounded operators from $\mathcal{H}^{\sigma+\zeta}$ to $\mathcal{H}^{\sigma+2\zeta}$ with $\sigma\in\mathbb{R}$. Owing to (6.11), we have that the $\mathcal{H}^{\sigma+\zeta}$-norm of (6.27) is bounded from above by

$$3^{-k\zeta}(C(\|u_k\|_{\mathcal{H}^{\sigma+\zeta}}+\|w_k\|_{\mathcal{H}^{\sigma+\zeta}}+(1+C\epsilon)\,\|f\|_{\mathcal{H}^{\sigma+2\zeta}}).$$

It follows from (4.5) and (4.4) that there exists a constant C such that

$$\begin{aligned} \left\|\widetilde{R}(u_k,\omega,\epsilon)-\widetilde{R}(u_{k-1},\omega,\epsilon)\right\|_{\mathcal{L}(\mathcal{H}^\sigma,\mathcal{H}^{\sigma+\zeta})} &\le C\left\|u_k-u_{k-1}\right\|_{\mathcal{H}^\sigma}, \\ \left\|R_1(u_k,\omega,\epsilon)-R_1(u_{k-1},\omega,\epsilon)\right\|_{\mathcal{L}(\mathcal{H}^\sigma,\mathcal{H}^{\sigma+\zeta})} &\le C\left\|u_k-u_{k-1}\right\|_{\mathcal{H}^\sigma}, \\ \left\|Q(u_k,\omega,\epsilon)^*-Q(u_{k-1},\omega,\epsilon)^*\right\|_{\mathcal{L}(\mathcal{H}^{\sigma+\zeta},\mathcal{H}^{\sigma+\zeta})} &\le C\left\|u_k-u_{k-1}\right\|_{\mathcal{H}^\sigma}. \end{aligned}$$

Since $Q(u_k,\omega,\epsilon)^*$ is bounded on any space $\mathcal{H}^\sigma$ with $\sigma\in\mathbb{R}$, the $\mathcal{H}^{\sigma+\zeta}$-norm of (6.28) is bounded from above by $C\left\|u_k-u_{k-1}\right\|_{\mathcal{H}^\sigma}$. Similarly, $\mathcal{H}^{\sigma+\zeta}$-norm of (6.29) is bounded from above by $C(\left\|u_k-u_{k-1}\right\|_{\mathcal{H}^\sigma}+\left\|w_k-w_{k-1}\right\|_{\mathcal{H}^\sigma})$. Thus we get (6.25). □

Let us complete the proof of Proposition 6.2.

Proof of Proposition 6.2. We apply a recursive argument to Proposition 6.2. We have already defined $(G_0,\mathcal{O}_0,\psi_0,u_0,w_0)$ satisfying (6.12)–(6.19). Suppose that $(G_k,\mathcal{O}_k,\psi_k,u_k,w_k)$ have been constructed satisfying (6.12)–(6.19). Now let us construct these data at $k+1$-th step and verify that these data at $k+1$-th step still satisfy (6.12)–(6.19). When u_k is given, the sets $\mathcal{O}_{k+1}$, G_{k+1} are defined by (6.12) at $k+1$-th step. Fixing ϵ, $G_{k+1,\epsilon}$ is a compact subset of the open set $\mathcal{O}_{k+1,\epsilon}$, where $G_{k+1,\epsilon}$ has to satisfy the distance between $G_{k+1,\epsilon}$ and the complement of $\mathcal{O}_{k+1,\epsilon}$ is bounded from below $\frac{\delta}{72C_0}3^{-(k+1)(\zeta+2)}$. It is easy to construct a function ψ_{k+1} satisfying (6.13) at the $k+1$-th step applying Urysohn's lemma.

For $(\omega,\epsilon)\in[1,2]\times[0,\delta^2]-\bigcup_{k'=0}^{k+1}G_{k'}$, let us construct w_{k+1}. By construction, the operator $V_{\mathrm{D}}(u_k,\omega,\epsilon)$ is a block-diagonal operator, which implies

$$\widetilde{\Pi}_n(L_\omega+\epsilon V_{\mathrm{D}}(u_k,\omega,\epsilon))w_{k+1}=(L_\omega+\epsilon V_{\mathrm{D}}(u_k,\omega,\epsilon))\widetilde{\Pi}_n w_{k+1}.$$

Then equation (6.16) at the $k+1$-th step can be written as for any $n\in\mathbb{N}$

$$(L_\omega+\epsilon V_{\mathrm{D}}(u_k,\omega,\epsilon))\widetilde{\Pi}_n w_{k+1}=\epsilon\widetilde{\Pi}_n H_{k+1}(u_k,w_k). \tag{6.30}$$

Notice the right-hand side of (6.30) vanishes when $\langle n\rangle\ge 3^{k+2}$ by (6.11). Let $k'\in\{0,\dots,k+1\}$, $n\in\mathbb{N}$ with $3^{k'}\le\langle n\rangle<3^{k'+1}$, $\omega\in[1,2]-G_{k',\epsilon}$. Moreover, by Lemma 6.4, Proposition 6.1(iii), equation (6.30) may be simplified as

$$\widetilde{\Pi}_n w_{k+1}=\epsilon A_n(\omega;u_k,\epsilon)^{-1}\widetilde{\Pi}_n H_{k+1}(u_k,w_k). \tag{6.31}$$

Then we define $w_{k+1}(t,x,\omega,\epsilon)$ as

$$w_{k+1}(t,x,\omega,\epsilon):=\sum_{k'=0}^{k+1}\sum_{\substack{n\in\mathbb{Z}\\ 3^{k'}\le\langle n\rangle<3^{k'+1}}}(1-\psi_{k'}(\omega,\epsilon))\widetilde{\Pi}_n w_{k+1}(t,x,\omega,\epsilon) \tag{6.32}$$

for any $(\omega, \epsilon) \in [1,2] \times [0,\delta^2]$. Let us first verify that (6.14) holds at the $k+1$-th step. Formulae (6.5) and (6.31) deduce that for any $k' \in \{0,\dots,k+1\}$, any $n \in \mathbb{N}$ with $3^{k'} \le \langle n \rangle < 3^{k'+1}$, any $(\omega,\epsilon) \in [1,2]\times[0,\delta^2] - G_{k'}$

$$\left\|\widetilde{\Pi}_n w_{k+1}(\,\cdot\,,\omega,\epsilon)\right\|_{\mathcal{H}^s} \le E_0 \frac{\epsilon}{\delta} \left\|\widetilde{\Pi}_n H_{k+1}(u_k,w_k)(\,\cdot\,,\omega,\epsilon)\right\|_{\mathcal{H}^{s+\zeta}} \tag{6.33}$$

and

$$\begin{aligned}\left\|\widetilde{\Pi}_n \partial_\omega w_{k+1}(\,\cdot\,,\omega,\epsilon)\right\|_{\mathcal{H}^{s-\zeta-2}} &\le E_0 \frac{\epsilon}{\delta} \left\|\widetilde{\Pi}_n \partial_\omega H_{k+1}(u_k,w_k)(\,\cdot\,,\omega,\epsilon)\right\|_{\mathcal{H}^{s-2}} \\ &\quad + E_0 \frac{\epsilon}{\delta^2} \left\|\widetilde{\Pi}_n H_{k+1}(u_k,w_k)(\,\cdot\,,\omega,\epsilon)\right\|_{\mathcal{H}^{s+\zeta}}.\end{aligned} \tag{6.34}$$

Furthermore formula (6.13) gives

$$\left\|\partial_\omega \psi_{k'} \widetilde{\Pi}_n w_{k+1}\right\|_{\mathcal{H}^{s-\zeta-2}} \le \frac{C_1}{\delta} \left\|\widetilde{\Pi}_n w_{k+1}\right\|_{\mathcal{H}^s}. \tag{6.35}$$

From (6.14), (6.19), (6.23), (6.24), and (6.32)–(6.35), it follows that

$$\begin{aligned}\|w_{k+1}(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^s} &\le E_0\frac{\epsilon}{\delta}\left(C(B_1+B_2)\frac{\epsilon}{\delta} + (1+C\epsilon)\|f\|_{\mathcal{H}^{s+\zeta}}\right),\\ \|\partial_\omega w_{k+1}(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}} &\le E_0\frac{\epsilon}{\delta}\left(C\frac{\epsilon}{\delta^2}(B_1+B_2) + C\epsilon\|f\|_{\mathcal{H}^{s-2}}\right)\\ &\quad + E_0\frac{\epsilon}{\delta^2}\left(C\frac{\epsilon}{\delta}(B_1+B_2) + (1+C\epsilon)\|f\|_{\mathcal{H}^{s+\zeta}}\right)\\ &\quad + E_0 C_1\frac{\epsilon}{\delta^2}\left(\frac{\epsilon}{\delta}C(B_1+B_2) + (1+C\epsilon)\|f\|_{\mathcal{H}^{s+\zeta}}\right).\end{aligned}$$

Notice C depends only on q, E_0, C_1, where q is given by (6.20), E_0, C_1 are uniform constants. If $\epsilon \le \delta^2 \le \delta_0^2$ with δ_0 small enough, when B_1 is taken large enough corresponding to E_0, C_1 and $\|f\|_{\mathcal{H}^{s+\zeta}}$, then we have that (6.14) still holds at the $k+1$-th step. Furthermore, using that $Q(u_k,\omega,\epsilon)$ is bounded on any space $\mathcal{H}^s$ with $s \in \mathbb{R}$, we derive

$$\begin{aligned}&\|u_{k+1}(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^s} + \delta\|\partial_\omega u_{k+1}(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}}\\ &\le (1+C\epsilon+C\epsilon\delta)\left(\|w_k\|_{\mathcal{H}^s} + \delta\|\partial_\omega w_{k+1}\|_{\mathcal{H}^{s-\zeta-2}}\right).\end{aligned}$$

When δ_0 is small enough, if we take $B_2 = 2B_1$, then (6.18) holds at the $k+1$-th step.

Next, we check (6.15) still holds at the $k+1$-th step. By (6.31), for $k' \in \{0,\dots,k\}$, $(\omega,\epsilon) \in [1,2]\times[0,\delta^2] - G_{k'}$, $n \in \mathbb{N}$ with $3^{k'} \le \langle n \rangle < 3^{k'+1}$, we can get the upper bound

$$\begin{aligned}\left\|\widetilde{\Pi}_n(w_{k+1}-w_k)\right\|_{\mathcal{H}^\sigma} \overset{(6.5)}{\le} E_0\frac{\epsilon}{\delta}\Big(&\left\|\widetilde{\Pi}_n(V_{\mathrm{D}}(u_{k-1},\omega,\epsilon) - V_{\mathrm{D}}(u_k,\omega,\epsilon))w_k\right\|_{\mathcal{H}^{\sigma+\zeta}}\\ &+ \left\|\widetilde{\Pi}_n(H_{k+1}(u_k,w_k) - H_k(u_{k-1},w_{k-1}))\right\|_{\mathcal{H}^{\sigma+\zeta}}\Big).\end{aligned} \tag{6.36}$$

Furthermore, formula (4.4) infers for $s \ge \sigma+\zeta$

$$\|(V_{\mathrm{D}}(u_{k-1},\omega,\epsilon) - V_{\mathrm{D}}(u_k,\omega,\epsilon))w_k\|_{\mathcal{H}^{\sigma+\zeta}} \le C\|u_k - u_{k-1}\|_{\mathcal{H}^\sigma}\|w_k\|_{\mathcal{H}^s}.$$

Applying (6.10) and (6.14), there exist some universal constants C_3 such that

$$\left\| \sum_{\substack{n\in\mathbb{N} \\ 3^{k+1}\leq\langle n\rangle<3^{k+2}}} (1-\varphi_{k+1})\Pi_n w_{k+1} \right\|_{\mathcal{H}^\sigma} \leq C_3 3^{-k(s-\sigma)} \|w_{k+1}\|_{\mathcal{H}^s} \leq C_3 B_1 \frac{\epsilon}{\delta} 3^{-k(s-\sigma)}. \tag{6.37}$$

Owing to (6.32), (6.15), (6.25), (6.36)–(6.37), it yields that for $s\geq\sigma+\zeta$

$$\begin{aligned}
&\|w_{k+1}-w_k\|_{\mathcal{H}^\sigma} \\
&\leq E_0\frac{\epsilon}{\delta}\left(2CB_1B_2\frac{\epsilon^2}{\delta^2}3^{-k\zeta}+3CB_2\frac{\epsilon}{\delta}3^{-k\zeta}+C\frac{\epsilon}{\delta}(B_1+B_2)3^{-k\zeta}+(1+C\epsilon)\|f\|_{\mathcal{H}^{\sigma+2\zeta}}3^{-k\zeta}\right) \\
&\quad + C_2B_1\frac{\epsilon}{\delta}3^{-k(s-\sigma)}.
\end{aligned}$$

We have $\|f\|_{\mathcal{H}^{\sigma+2\zeta}}\leq\|f\|_{\mathcal{H}^{s+\zeta}}$ from $s\geq\sigma+\zeta$. If $0\leq\epsilon\leq\delta^2\leq\delta_0^2$ with δ_0 small enough, when B_1 is chosen large enough relatively to E_0, $\|f\|_{\mathcal{H}^{s+\zeta}}$, and B_2 is taken large enough corresponding to B_1, C_2, then (6.15) is obtained for $s\geq\sigma+\zeta$ holds at the $k+1$-th step. It is clear to verify that (6.19) still holds at the $k+1$-th step by the definition (6.17). This concludes the proof of Proposition 6.2. □

Our aim is to construct the solution of (5.1). Therefore we consider the equation about u_k. According to (6.17), Proposition 5.5, (6.20) and (6.16), it follows that for any $(\omega,\epsilon)\in[1,2]\times[0,\delta^2]-\bigcup_{k'=0}^{k}\mathcal{O}_{k'}$, $\delta\in(0,\delta_0]$

$$\begin{aligned}
&(L_\omega+\epsilon V(u_{k-1},\omega,\epsilon))u_k \\
&=\epsilon((\mathrm{Id}+\epsilon Q(u_{k-1},\omega,\epsilon))^*)^{-1}\Big(\widetilde{S}_k(\mathrm{Id}+\epsilon Q(u_{k-1},\omega,\epsilon))^*\widetilde{R}(u_{k-1},\omega,\epsilon)u_{k-1} \\
&\quad+\widetilde{S}_k(R_1(u_{k-1},\omega,\epsilon)w_{k-1})+(\widetilde{S}_k(\mathrm{Id}+\epsilon Q(u_{k-1},\omega,\epsilon))^*f+R_1(u_{k-1},\omega,\epsilon)w_k)\Big).
\end{aligned}\tag{6.38}$$

Finally, let us complete the proof of Theorem 2.1.

Proof of Theorem 2.1. Formulae (6.17) and (6.19) indicate that the sequence u_k is well defined and converges to u in $\mathcal{H}^\sigma$ with

$$\|u(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^s}+\delta\|\partial_\omega u(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}}\leq B_2\frac{\epsilon}{\delta}.$$

Moreover, by (6.14), (6.15), the sequence w_k converges in $\mathcal{H}^\sigma$ to w, which satisfies

$$\|w(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^s}+\delta\|\partial_\omega w(\,\cdot\,,\omega,\epsilon)\|_{\mathcal{H}^{s-\zeta-2}}\leq B_1\frac{\epsilon}{\delta}.$$

If (ω,ϵ) is in $[1,2]\times[0,\delta^2]-\bigcup_{k'=0}^{+\infty}G_{k'}$, $\delta\in(0,\delta_0]$ with δ_0 small enough, then equation (6.38) is satisfied for any $k\in\mathbb{N}$. Therefore u satisfies

$$(L_\omega+\epsilon V(u,\omega,\epsilon))u=\epsilon\widetilde{R}(u,\omega,\epsilon)u+\epsilon f$$

as $k \to +\infty$. This shows that u is a solution of equation (5.1). By Proposition 4.10, equation (5.1) is equivalent to equation (4.1) which is also equivalent to (3.9) by Proposition 3.9. Thus we may get a solution satisfying the conditions of Theorem 2.1. Let $\mathcal{O} = \bigcup_{k'=0}^{+\infty} \mathcal{O}_{k'}$. For $\omega, \omega' \in \mathcal{O}_{k',\epsilon}$, using (6.3) and (6.12), we may obtain the bound

$$\left|\omega - \omega'\right| \overset{\theta \in (0,1)}{=} \frac{\left|\lambda_n^l(\omega; u_{k'}, \epsilon) - \lambda_n^l(\omega'; u_{k'}, \epsilon)\right|}{\left|\partial_\omega \lambda_n^l(\theta\omega' + (1-\theta)\omega; u_{k'}, \epsilon)\right|} \leq C 3^{-(2+\zeta)k'} \delta.$$

Moreover, we deduce $D_n \leq C_1 3^{k'+1}$ with $n \in \mathbb{N}$ from $\langle n \rangle < 3^{k'+1}$ and definition of $\widetilde{\Pi}_n$. Thus the upper bound of ω-measure of the ϵ-section of $\mathcal{O}$ is

$$C\delta \sum_{k'=0}^{+\infty} 3^{-(2+\zeta)k' + (k'+1) + (k'+1)}.$$

The series converges if we take $\zeta > 0$. This implies that we obtain the bound $O(\delta)$, which gives the proof of (2.3). □

7. Concluding remarks

In this paper, we have investigated the existence of time-periodic solutions of nonlinear wave equation with general nonlinear terms on the one-dimensional torus. Without applying the use of Nash-Moser or KAM methods, through para-differential conjugation, the equation under study are reduced to an equivalent form for which periodic solutions can be constructed for a large set of frequencies by a classical iteration scheme. This approach allows ones to separate on the one hand the treatment of losses of derivatives coming from small divisors, and on the other hand the question of convergence of the sequence of approximations. In [22], Delort proposed that this method does not seem to be adapted to find periodic solutions of nonlinear wave equations in high-dimensional spaces, since the specific separation property does not hold. However, for the nonlinear wave equation on one-dimensional tori, we can obtain the separation property of the eigenvalues of $\sqrt{-\partial_{xx} + m}$. One direction for this research is the construction of quasi-periodic solutions of the nonlinear PDEs by the para-differential method. This is our ongoing work and will be reported elsewhere.

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Bochao Chen, Yixian Gao and Yong Li
School of Mathematics and Statistics, Center for Mathematics and Interdisciplinary Sciences, Northeast Normal University, Changchun, Jilin 130024, P. R. China
E-mail address: chenbc758@nenu.edu.cn, gaoyx643@nenu.edu.cn, yongli@nenu.edu.cn