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Sums of Recursion Operators

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Abstract. Let $(M,\omega,\tau)_A$ be a $2n$-dimensional smooth manifold with a pair of symplectic forms ω and τ intertwined by a recursion operator $A \in \mathrm{End}(TM)$. We consider a codimension two submanifolds $Q \subset M$ with those restricted symplectic forms $(\omega|_Q, \tau|_Q)$. Assume that TQ is A-invariant. We call the tuple $(M,\omega,\tau,Q)_A$ symplectic-recursion data. In this paper, we consider the problem of fibre connected sum of such two symplectic-recursion data $(M_0,\omega_0,\tau_0,Q_0)_{A_0}$ and $(M_1,\omega_1,\tau_1,Q_1)_{A_1}$. It is interesting to consider potential applications of this result to integrable systems and mathematical string theory.

1. Introduction and main result

For a long time, the study of integrable systems occupies an important place in mathematics and in physics. For instance, we have now a complete picture of the geometry of completely integrable Hamiltonian systems, i.e., systems admitting a complete sequence of first integrals, which is called Arnold-Liouville integrability [1]. Since 1970's, much attention in the theory of integrability has been paid to systems admitting more than one Hamiltonian representation. The first examples belong to the class of infinite-dimensional systems, e.g., Lax representation of the the Korteweg-de Vries equation. Later it was realized that integrable systems of finite dimension also possess a bi-Hamiltonian representation, i.e., systems admitting two compatible Hamiltonian formulations. In mathematical terminology, phase spaces in classical mechanics are symplectic manifolds (M,ω), where M is an even-dimensional smooth manifold with a non-degenerate closed 2-form ω. A vector field X_H on M is called Hamiltonian if there exists a (Hamiltonian) function $H\colon M \to \mathbb{R}$ such that $\omega(\cdot\,, X_H) = dH(\cdot)$. Equivalently, a bi-Hamiltonian system on M can be described as a pair of symplectic forms ω and τ intertwined by the so-called recursion operator [16,17,20].

More precisely, given two closed and non-degenerate 2-forms ω and τ on the same manifold M, there exists a unique field of invertible endomorphisms A of the tangent bundle TM defined by $\omega(\cdot\,, X) = \tau(\cdot\,, AX)$. The field A is called a recursion operator and

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it is speculated that there should be a connection to topological recursion in integrable PDEs [7, 8]. In particular, the recursion operator $A = \pm \mathrm{Id}$ if and only if $\omega = \pm\tau$. Such recursion operators are called trivial. If the recursion operator A satisfies $A^2 = \mathrm{Id}$, it corresponds to the symplectic pair studied in [3]. And $A^2 = -\mathrm{Id}$ corresponds to holomorphic symplectic form [5]. The geometry of recursion operators was studied by Bande and Kotschick [4]. They obtained a simultaneous isotopy result for such a pair of symplectic forms which is somehow analogous to the Moser's stability theorem for symplectic manifolds. In [10], we obtained a certain neighborhood theorem for recursion operators.

On the other hand, symplectic connected sum is an important surgery on symplectic manifolds formulated by Gompf [9] and McCarthy-Wolfson [18]. Roughly speaking, connected sum of two symplectic manifolds produces a new symplectic manifold which preserves the most parts of those original two ones. From Donaldson's work for four-manifolds [6] we know that a pair of 2-forms may bring more non-trivial geometric structures. In this paper we consider similar surgery on manifolds which support a pair of symplectic forms intertwined by a recursion operator.

Rather than connected sum of general manifolds along their arbitrary isomorphic submanifolds, Gompf's fibre connected sum only works for two symplectic manifolds along their isomorphic codimension 2 symplectic submanifolds. Similarly, we will consider a codimension 2 compact submanifold $Q \subset M$ with those restricted symplectic forms $(\omega|_Q, \tau|_Q)$. Assume that TQ is A-invariant, then Q is called a (codimension 2 compact) symplectic-recursion submanifold of $(M, \omega, \tau)_A$. Generally, for symplectic-recursion submanifold Q of arbitrary codimension, the tuple $(M, \omega, \tau, Q)_A$ (and $(M, \omega, \tau)_A$, if without danger of confusion) is called symplectic-recursion data. We will study the construction of fibre connected sum of such two symplectic-recursion data $(M_0, \omega_0, \tau_0, Q_0)_{A_0}$ and $(M_1, \omega_1, \tau_1, Q_1)_{A_1}$. Our main result is given as follows:

Theorem 1.1. *Let $(M_0, \omega_0, \tau_0)_{A_0}$ and $(M_1, \omega_1, \tau_1)_{A_1}$ be two symplectic-recursion data each containing a symplectomorphic copy of a codimension two compact symplectic-recursion submanifold $(Q, \omega_Q, \tau_Q)_{A_Q}$. Suppose that their respective normal bundles have opposite Euler classes and that for a diffeomorphism $\widetilde{\Psi}$ (see the definition in (3.1)) between some neighborhoods of Q in M_0 and M_1, the associated recursion operators satisfy*

$$(d\widetilde{\Psi})^{-1} \circ A_1 \circ (d\widetilde{\Psi}) = A_0 = A$$

in a neighborhood of Q, and that in a possibly smaller neighborhood $\mathcal{N}$ of Q, we can find a pair of local 1-forms η_ω, $\eta_\tau \in \Omega^1(\mathcal{N})$ satisfying $\widetilde{\Psi}^*\omega_1 - \omega_0 = d\eta_\omega$ and $\widetilde{\Psi}^*\tau_1 - \tau_0 = d\eta_\tau$ with $\eta_\omega = \eta_\tau \circ A$. Then there exists a $(2n+2)$-dimensional symplectic-recursion data $(X, \Omega, \Gamma)_{\mathcal{A}}$ and a fibration $\lambda\colon X \to D$ over a small disk $D \subset \mathbb{C}$, such that for $\lambda \neq 0$, the fibers X_λ

are fibre connected sums which are smooth compact symplectic-recursion submanifolds of X*, while the central fiber* X_0 *is the singular symplectic-recursion manifold* $M_0 \cup_Q M_1$*.*

Remark 1.2. It is interesting to study the physical meaning of surgeries on symplectic manifolds and recursion operators. Since it is known that the symplectic sum and blow-up formulas of Gromov-Witten invariants have important applications in algebraic geometry and superstring theory [11, 12, 15], one suggests that there might be applications of our result in related fields.

In Section 2, we review some examples of recursion operators. As a preparation for proving the main result, we introduce some neighborhood theorems for submanifolds in Section 3. Then we prove Theorem 1.1 (i.e., Theorem 4.1) in Section 4. In Section 5, we discuss an application of the main result to the case of symplectic pairs.

2. Examples

We first show some examples of recursion operators and simpler example of connected sum.

Example 2.1 (No Darboux-type theorem for recursion operators)**.** Consider the standard symplectic space $(\mathbb{R}^{2n}, \omega_0)$ with the standard symplectic form

$$\omega_0 = \sum_{i=1}^{n} dy_i \wedge dx_i,$$

where $x_1, y_1, \ldots, x_n, y_n$ are coordinates on $\mathbb{R}^{2n}$, and an additional symplectic form τ_0 intertwined with ω_0 by a recursion operator A_0. Classical Darboux's theorem says that for any symplectic manifold (M, ω), there exists a locally diffeomorphism ϕ which maps an open neighborhood U of $0 \in \mathbb{R}^{2n}$ into a neighborhood $\mathcal{N}$ of a point x in M such that $\phi^*\omega = \omega_0$. For a recursion operator A on M, we consider the tuple $(M, \omega, \tau)_A$. If for a point $x \in \mathcal{N} \subset M$ there also exists a locally diffeomorphism $\phi\colon (U, \omega_0, \tau_0)_{A_0} \to (\mathcal{N}, \omega, \tau)_A$ such that $\phi(0) = x$, $\phi^*\omega = \omega_0$ and $\phi^*\tau = \tau_0$, it is easy to see that $A_0 = (d\phi)^{-1} \circ A \circ d\phi$. However, the operator A_0 depends on the point x. Thus, in general, there exists no such a standard symplectic-recursion space $(\mathbb{R}^{2n}, \omega_0, \tau_0)$ as a local model for arbitrary symplectic-recursion data.

Example 2.2 (A symplectic pair is an example of recursion operators)**.** Symplectic pairs are geometric objects on even-dimensional smooth manifolds introduced by Kotschick et al [3, 13, 14]. Given a $2n$-dimensional smooth manifold M, a pair of closed 2-forms (ξ, η) defined on M is called a symplectic pair, if they have constant and complementary ranks, and ξ restricts as symplectic form to the leaves of the kernel foliation $\mathcal{F}$ of η, η restricts as symplectic form to the leaves of the kernel foliation $\mathcal{G}$ of ξ. Note that $\mathcal{F}$ and $\mathcal{G}$ are complementary smooth foliations [19,21]. Given a symplectic pair (ξ, η) on a manifold M, then $(\xi \oplus \eta, \xi \oplus -\eta)$ is a pair of symplectic forms intertwined by a recursion operator A, which is defined by $\iota_X(\xi \oplus \eta) = \iota_{AX}(\xi \oplus -\eta)$.

Conversely, for a symplectic-recursion data $(M, \omega, \tau)_A$, if A is nontrivial and $A^2 = \mathrm{Id}$, then Bande and Kotschick proved that the recursion operator A gives rise to a symplectic pair

$$\xi = \frac{1}{2}(\omega + \tau), \quad \eta = \frac{1}{2}(\omega - \tau).$$

Every symplectic pair (ξ, η) can be given in this way [4, Theorem 3]. Some kinds of Gray-Moser stability theorems and neighborhood theorems for symplectic pairs are given in [2, 10].

Example 2.3 (Holomorphically symplectic forms and recursion operators)**.** A compact $2n_{\mathbb{C}}$-dimensional manifold X is called holomorphically symplectic if there is a holomorphic 2-form Ω over X such that Ω^n is a nowhere degenerate section of a canonical class of X. One sees that the holomorphically symplectic manifold has a trivial canonical bundle. A hyperkähler manifold is holomorphically symplectic. Recall that a hyperkähler manifold is a Riemannian manifold $(X, \langle \cdot\,, \cdot \rangle)$ endowed with three complex structures I, J and K, such that X is Kähler with respect to these three structures I, J and K, considered as endomorphisms of a real tangent bundle, satisfy the relation $I \circ J = -J \circ I = K$.

Conversely, let X be a holomorphically symplectic Kähler manifold with the holomorphically symplectic form Ω, a Kähler class $[\omega] \in H^{1,1}(X)$ and a complex structure I. Then there is a unique hyperkähler structure $(I, J, K, \langle \cdot\,, \cdot \rangle)$ over X such that the cohomology class of the symplectic form $\omega_I = \langle \cdot\,, I \cdot \rangle$ is equal to $[\omega]$ and the canonical symplectic form $\omega_J + \mathrm{i}\omega_K$ is equal to Ω. The real part ω_J and imaginary part ω_K of Ω are two symplectic forms on X, intertwined by the complex structure I, which is the recursion operator such that $A^2 = -\mathrm{Id}$. Bande and Kotschick [3] show that for any symplectic-recursion data $(M, \omega, \tau)_A$ such that $A^2 = -\mathrm{Id}$, recursion operator A defines an integrable almost complex structure with a holomorphically symplectic form Ω whose real and imaginary parts are just ω and τ. Furthermore, every holomorphically symplectic form can arise in this way.

Example 2.4 (Sum for the case of trivial normal bundle)**.** We first assume that $Q_i \subset M_i$, $i = 0, 1$, has trivial normal bundle. This implies the isomorphism between normal bundles

$$\Psi \colon N_{Q_0} = Q_0 \times \mathbb{R}^2 \to N_{Q_1} = Q_1 \times \mathbb{R}^2$$

is a natural extension of ϕ. Suppose that the associated recursion operators satisfy

$$(d\widetilde{\Psi})^{-1} \circ A_1 \circ (d\widetilde{\Psi}) = A_0 = A$$

in a neighborhood of Q_0, and that in a possibly smaller neighborhood $\mathcal{N}$ of Q_0, we can find a pair of local 1-forms η_ω, $\eta_\tau \in \Omega^1(\mathcal{N})$ satisfying

$$\widetilde{\Psi}^*\omega_1 - \omega_0 = d\eta_\omega \quad \text{and} \quad \widetilde{\Psi}^*\tau_1 - \tau_0 = d\eta_\tau$$

with $\eta_\omega = \eta_\tau \circ A$. Then by Theorem 3.3, there exists a symplectomorphism $\Phi\colon \mathcal{N}(Q_0) \to \mathcal{N}(Q_1)$ of neighborhoods of Q_0 and Q_1 which is the extension of ϕ. Note that for each $i = 0, 1$, Q_i can be regarded as the image of an inclusion

$$\iota_i\colon \widetilde{Q} \to M_i,$$

where $(\widetilde{Q}, \widetilde{\omega}, \widetilde{\tau})_{\widetilde{A}}$ is a compact manifold of dimension $2n-2$ with a pair of symplectic forms $(\widetilde{\omega}, \widetilde{\tau})$ intertwined by a recursion operator $\widetilde{A}$ such that $\iota_i^*\omega_i = \widetilde{\omega}$, $\iota_i^*\tau_i = \widetilde{\tau}$ and $d\iota_i \circ \widetilde{A} = A_i \circ d\iota_i$. Combining with Theorem 3.2, this implies that there exist symplectic embeddings

$$f_i\colon \widetilde{Q} \times D^2(\epsilon) \to M_i, \quad f_i^*\omega_i = \widetilde{\omega} \times dx \wedge dy, \quad f_i^*\tau_i = \widetilde{\tau} \times \tau_\epsilon$$

such that $f_i(q, 0) = \iota_i(q)$ for $q \in \widetilde{Q}$, where $(D^2(\epsilon), dx \wedge dy, \tau_\epsilon)_{A_\epsilon}$ is a tuple consisting of radius ϵ disk in $\mathbb{R}^2$ with a pair of symplectic forms intertwined by recursion operator A_ϵ. Here we make an **additional assumption** that there exists such a 2-form τ_ϵ on $D^2(\epsilon)$ which is also preserved by the diffeomorphism $\mathcal{P}$ defined below.

Consider the annulus $A(\epsilon/2, \epsilon) = D^2(\epsilon) - D^2(\epsilon/2)$ and, with respect to the standard form $dx \wedge dy$, an orientation and area-preserving diffeomorphism

$$\mathcal{P}\colon A(\epsilon/2, \epsilon) \to A(\epsilon/2, \epsilon)$$

such that the two boundary components are interchanged. Under the additional assumption above, we see that $\mathcal{P}$ preserves the tuple $(A(\epsilon/2, \epsilon), dx \wedge dy, \tau_\epsilon)_{A_\epsilon}$ and turns inside out. We can glue together $M_0 - \mathcal{N}(Q_0)$ and $M_1 - \mathcal{N}(Q_1)$ along $\widetilde{Q} \times A(\epsilon/2, \epsilon)$ by the diffeomorphism $\phi \times \mathcal{P}$, where $\mathcal{N}(Q_i) = f_i(\widetilde{Q} \times D^2(\epsilon/2))$ satisfying $f_2(q, x) = f_1(q, \mathcal{P}(x))$ for $q \in \widetilde{Q}$ and $\epsilon/2 < |x| < \epsilon$.

Then the fibre connected sum is defined by

$$M_0 \#_{\widetilde{Q}} M_1 = (M_0 - \mathcal{N}(Q_0)) \cup_{\phi \times \mathcal{P}} (M_1 - \mathcal{N}(Q_1)).$$

The two pairs of symplectic forms (ω_0, τ_0) and (ω_1, τ_1) induce a pair of symplectic forms on $M_0 \#_{\widetilde{Q}} M_1$ intertwined by A whose restriction on M_i is A_i.

Remark 2.5. Bande and Kotschick [3] have pointed out that the Gompf's fibre connected sum construction works for some special case of symplectic pairs (M_0, ξ_0, η_0) and

(M_1, ξ_1, η_1). For instance, they suppose that kernel foliations for ξ_0 and ξ_1 has codimension 2 and both symplectic pairs admit product structures in open neighborhoods of two closed leaves Σ_{η_0} and Σ_{η_1}, respectively. That means Σ_{η_i} has trivial normal bundle and locally looked as $\Sigma_{\eta_i} \times D^2$. Suppose further that there exists a diffeomorphism $\phi\colon (\Sigma_{\eta_0}, \xi_0) \to (\Sigma_{\eta_1}, \xi_1)$, then the Gompf type sum $M_0 \#_\phi M_1$ along submanifolds Σ_{η_i}, which glues together $M_0 \setminus \Sigma_{\eta_0}$ and $M_1 \setminus \Sigma_{\eta_1}$ along $\Sigma_{\eta_i} \times A(\epsilon/2, \epsilon)$, becomes the symplectic pair $(M_0 \#_\phi M_1, \xi, \eta)$. Recall from Example 2.3 that symplectic pairs are special cases of symplectic recursion operators. If for $i = 0, 1$, let $\omega_i = \xi_i \oplus \eta_i$, $\tau_i = \xi_i \oplus -\eta_i$, then $(M_i, \omega_i, \tau_i, \Sigma_{\eta_i})_{A_i}$ is symplectic-recursion data. Note that in this special case, the 2-form τ_ϵ defined on $D^2(\epsilon)$ is just $-dx \wedge dy$. Then the above additional assumption that τ_ϵ is preserved by the gluing map $\mathcal{P}$ will be automatically satisfied.

3. Neighborhood theorems for recursion operators

In [10, Theorem 5], we obtain the following neighborhood theorem for recursion operators.

Theorem 3.1. *Let M be a $2n$-dimensional smooth manifold and $Q \subset M$ be a compact submanifold. Suppose that (ω_0, τ_0) and (ω_1, τ_1) are two pairs of closed and non-degenerate 2-forms in a neighborhood of Q with associated recursion operators $A_0 = A_1 = A$, such that at each point $q \in Q$, the two pairs of symplectic forms (ω_0, τ_0) and (ω_1, τ_1) are equal on T_qM. Then there exist open neighborhoods $\mathcal{N}_0$ and $\mathcal{N}_1$ of Q and a diffeomorphism $\psi\colon \mathcal{N}_0 \to \mathcal{N}_1$ such that*

$$\psi|_Q = \mathrm{Id}, \quad \psi^*\omega_1 = \omega_0, \quad \psi^*\tau_1 = \tau_0$$

provided that there exists an open neighborhood $\mathcal{N}$ of Q, such that we can find a pair of local 1-forms η_ω, $\eta_\tau \in \Omega(\mathcal{N})$ satisfying $\omega_1 - \omega_0 = d\eta_\omega$ and $\tau_1 - \tau_0 = d\eta_\tau$ with $\eta_\omega = \eta_\tau \circ A$.

Proof. Here we rephrase the proof of Theorem 5 in [10]. Suppose that there exists an open neighborhood $\mathcal{N}$ of Q such that we can find a pair of local 1-forms $\eta_\omega, \eta_\tau \in \Omega(\mathcal{N})$ satisfying $\omega_1 - \omega_0 = d\eta_\omega$ and $\tau_1 - \tau_0 = d\eta_\tau$ with $\eta_\omega = \eta_\tau \circ A$. We define a pair of smooth families of closed 2-forms (ω_t, τ_t), $t \in [0, 1]$ where

$$\omega_t = \omega_0 + t(\omega_1 - \omega_0), \quad \tau_t = \tau_0 + t(\tau_1 - \tau_0).$$

Since the two pairs of symplectic forms (ω_0, τ_0) and (ω_1, τ_1) are equal on T_qM and hence both ω_t and τ_t are non-degenerate in a possible smaller neighborhood of Q, then there exist two vector fields X_t and Y_t in $\mathcal{N}$ defined by $\iota_{X_t}\omega_t = -\eta_\omega$ and $\iota_{Y_t}\tau_t = -\eta_\tau$. Note that $X_t|_Q = Y_t|_Q = 0$. For any vector field $Z \in T\mathcal{N}$, the following calculation

$$\iota_{X_t}\omega_t(Z) = -\eta_\omega(Z) = -\eta_\tau \circ A(Z) = \iota_{Y_t}\tau_t(AZ) = \iota_{AY_t}\tau_t(Z) = \iota_{Y_t}\omega_t(Z)$$

shows that $X_t = Y_t$ in $\mathcal{N}$. The equality $\iota_{Y_t}\tau_t(AZ) = \iota_{AY_t}\tau_t(Z)$ holds because for each t, τ_t and ω_t are intertwined by A, then for any two vector fields X and Y,

$$\tau_t(AX, Y) = \omega_t(X, Y) = -\omega_t(X, Y) = -\tau_t(AY, X) = \tau_t(X, AY).$$

Denote by ψ_t the flow generated by X_t. The compactness of Q and $X_t|_Q = 0$ implies that there exists a neighborhood $\mathcal{N}_0 \subset \mathcal{N}$ of Q such that ψ_t satisfies the following equation

$$\frac{d}{dt}\psi_t = X_t \circ \psi_t, \quad \psi_t|_Q = \mathrm{Id}, \quad \psi_0 = \mathrm{Id}$$

defined on $\mathcal{N}_0$ for all $t \in [0, 1]$. Then

$$\frac{d}{dt}\psi_t^*\omega_t = \psi_t^*\left(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t\right) = \psi_t^*(\omega_1 - \omega_0 + d\iota_{X_t}\omega_t) = \psi_t^*(\omega_1 - \omega_0 + d(-\eta_\omega)) = 0,$$

since ω_t is closed. For the same reason, $\frac{d}{dt}\psi_t^*\tau_t = 0$. Let $\psi = \psi_1$ and $\mathcal{N}_1 = \psi(\mathcal{N}_0)$, we have $\psi|_Q = \mathrm{Id}$, $\psi^*\omega_1 = \omega_0$ and $\psi^*\tau_1 = \tau_0$. Thus, we complete the proof. $\square$

In particular, we can consider compact symplectic submanifold Q which is even-dimensional submanifold of M such that $\omega|_Q$ and $\tau|_Q$ are also symplectic forms on Q. Moreover, we assume that TQ is A-invariant.

Let's first consider the following Darboux type theorem. Consider a point $x \in M$ and fix a Riemannian metric g near x. We choose a basis $\{u_1, \ldots, u_n, v_1, \ldots, v_n\}$ of the tangent space (T_xM, ω_x, τ_x) so that

$$\omega(u_i, u_j) = \omega(v_i, v_j) = \omega(u_i, v_j) = 0$$

and

$$\omega(u_i, v_i) = 1 \quad \text{for } 1 \le i \le n.$$

Then we take an open neighborhood U of 0 in $\mathbb{R}^{2n}$, and define a local diffeomorphism $\Psi\colon U \to M$ by

$$\Psi(x_1, \ldots, x_n, y_1, \ldots, y_n) = \exp_x(x_1u_1 + \cdots + x_nu_n + y_1v_1 + \cdots + y_nv_n),$$

where $\exp_x$ is the exponential map

$$\exp_x\colon T_xM \to M, \quad \exp_x(\xi) = \gamma_\xi(1),$$

where $\gamma_\xi(t)$ is the geodesic with respect to g tangent to ξ at x.

Theorem 3.2. *Let $(M, \omega, \tau)_A$ be a $2n$-dimensional symplectic-recursion data, $(\mathbb{R}^{2n}, \omega_0, \tau_0)_{A_0}$ be a symplectic-recursion data on $\mathbb{R}^{2n}$ such that ω_0 is the standard symplectic form. Suppose that in a neighborhood of $0 \in \mathbb{R}^{2n}$ we have $A_0 = (d\Psi)^{-1} \circ A \circ d\Psi$ and*

$\Psi^*\tau = \tau_0|_{T_0\mathbb{R}^{2n}}$. Then there exist open neighborhoods U of $0 \in \mathbb{R}^{2n}$ and $\mathcal{N}$ of x and a diffeomorphism $\Phi\colon U \to \mathcal{N}$ such that

$$\Phi^*\omega = \omega_0 = \sum_{i=1}^{n} dx_i \wedge dy_i, \quad \Phi^*\tau = \tau_0,$$

provided that we can find a pair of local 1-forms $\eta_\omega, \eta_\tau \in \Omega^1(U)$ satisfying $\Psi^*\omega - \omega_0 = d\eta_\omega$ and $\Psi^*\tau - \tau_0 = d\eta_\tau$ with $\eta_\omega = \eta_\tau \circ A_0$.

Proof. Given the neighborhood U and the local diffeomorphism Ψ as above, (U, Ψ) is a chart around $x \in M$. Since $\{u_1, \dots, u_n, v_1, \dots, v_n\}$ is a basis of the tangent space and $d(\exp_x)|_{\{0\}} = \mathrm{Id}$, it follows that $\Psi^*\omega = \omega_0|_{T_0\mathbb{R}^{2n}}$. Note the condition that $\Psi^*\tau = \tau_0|_{T_0\mathbb{R}^{2n}}$. Then applying Theorem 3.1 to the special case that Q is one point $0 \in \mathbb{R}^{2n}$, there exists a local diffeomorphism ψ in a neighborhood of 0 such that $\psi(0) = 0$, $\psi^*(\Psi^*\omega) = \omega_0$ and $\psi^*(\Psi^*\tau) = \tau_0$. Let $\Phi = \Psi \circ \psi$, which is the local diffeomorphism such that the conclusion of the theorem holds. $\square$

More generally, let $(M_0, \omega_0, \tau_0)_{A_0}$ and $(M_1, \omega_1, \tau_1)_{A_1}$ be two such manifolds of dimension $2n$ as above with $2k$-dimensional compact symplectic submanifolds S_0 and S_1, respectively. Suppose that there exists a diffeomorphism $\phi\colon S_0 \to S_1$ such that $\phi^*(\omega_1|_{S_1}) = \omega_0|_{S_0}$ and $\phi^*(\tau_1|_{S_1}) = \tau_0|_{S_0}$, and an isomorphism

$$\Psi\colon (N_{S_0}, \omega_0, \tau_0) \to (N_{S_1}, \omega_1, \tau_1)$$

between symplectic normal bundles which covers ϕ. For any Riemann metric g_i on M_i, we can identify N_{S_i} with the orthogonal complement TS_i^{T} with respect to this metric g_i. The exponential map induces a map $f_i\colon N_{S_i} \to M_i$. We can define

$$\widetilde{\Psi} := f_1 \circ \Psi \circ f_0^{-1} \tag{3.1}$$

which is a diffeomorphism between a neighborhood of S_0 and a neighborhood of S_1. As an application of Theorem 3.1, we have

Theorem 3.3. *Let $(M_0, \omega_0, \tau_0, S_0)_{A_0}$ and $(M_1, \omega_1, \tau_1, S_1)_{A_1}$ be two data. Maps ϕ, Ψ and $\widetilde{\Psi}$ are defined as above. Suppose that the associated recursion operators satisfy*

$$(d\widetilde{\Psi})^{-1} \circ A_1 \circ (d\widetilde{\Psi}) = A_0 = A$$

in a neighborhood of S_0. Then there exist neighborhoods $\mathcal{N}_0$ and $\mathcal{N}_1$ of S_0 and S_1 respectively, such that ϕ extends to a diffeomorphism $\Phi\colon \mathcal{N}_0 \to \mathcal{N}_1$ satisfying $d\Phi = \Psi$ on N_{S_0} and

$$\Phi^*\omega_1 = \omega_0, \quad \Phi^*\tau_1 = \tau_0,$$

provided that in a possibly smaller neighborhood $\mathcal{N}$ of S_0, we can find a pair of local 1-forms $\eta_\omega, \eta_\tau \in \Omega^1(\mathcal{N})$ satisfying $\widetilde{\Psi}^*\omega_1 - \omega_0 = d\eta_\omega$ and $\widetilde{\Psi}^*\tau_1 - \tau_0 = d\eta_\tau$ with $\eta_\omega = \eta_\tau \circ A$.

Proof. It is easy to see that the $\widetilde{\Psi}$ satisfies

$$\widetilde{\Psi}^*\omega_1 = \omega_0 \quad \text{and} \quad \widetilde{\Psi}^*\tau_1 = \tau_0$$

on T_xM_0 for any $x \in S_0$. Since $(d\widetilde{\Psi})^{-1} \circ A_1 \circ (d\widetilde{\Psi}) = A_0 = A$, then the theorem follows by applying Theorem 3.1 to pairs (ω_0, τ_0) and $(\widetilde{\Psi}^*\omega_1, \widetilde{\Psi}^*\tau_1)$ around the compact submanifold S_0 to obtain a diffeomorphism ψ around S_0 and defining $\Phi = \widetilde{\Psi} \circ \psi$. □

4. Constructing fibre connected sum

To construct the fibre connected sum, we suppose that each of the two tuples $(M_0, \omega_0, \tau_0)_{A_0}$ and $(M_1, \omega_1, \tau_1)_{A_1}$ contains a symplectomorphic copy of a codimension two compact symplectic submanifold $(Q, \omega_Q, \tau_Q)_{A_Q}$. We now just assume their respective normal bundles have opposite Euler classes, i.e.,

$$e(N_{M_0}Q) + e(N_{M_1}Q) = 0.$$

Then there exists a natural orientation-preserving isomorphism between normal bundles

$$\Psi\colon N_{M_0}Q \to (N_{M_1}Q)^*. \tag{4.1}$$

We fix this isomorphism once and for all. The diffeomorphism $\widetilde{\Psi}$ is defined similarly as (3.1). Suppose that the associated recursion operators satisfy

$$(d\widetilde{\Psi})^{-1} \circ A_1 \circ (d\widetilde{\Psi}) = A_0 = A \tag{4.2}$$

in a neighborhood of Q, and that in a possibly smaller neighborhood $\mathcal{N}$ of Q, we can find a pair of 1-forms $\eta_\omega, \eta_\tau \in \Omega^1(\mathcal{N})$ satisfying $\widetilde{\Psi}^*\omega_1 - \omega_0 = d\eta_\omega$ and $\widetilde{\Psi}^*\tau_1 - \tau_0 = d\eta_\tau$ with $\eta_\omega = \eta_\tau \circ A$.

Then we can construct a new $(2n+2)$-dimensional symplectic-recursion data $(X, \Omega, \Gamma)_{\mathcal{A}}$ as a family of fibre connected sums $X_\lambda = M_0 \#_{Q,\lambda} M_1$ parameterized by small $\lambda \in \mathbb{C}$. Actually, this family fits together to form a smooth manifold X which is a fibration over a disk. Under the conditions above, the following is the restatement of the Theorem 1.1.

Theorem 4.1. *For data constructed as above, there are a $(2n+2)$-dimensional symplectic-recursion data $(X, \Omega, \Gamma)_{\mathcal{A}}$ and a fibration $\lambda\colon X \to D$ over a small disk $D \subset \mathbb{C}$, such that for $\lambda \neq 0$, the fibers X_λ are fibre connected sums which are smooth compact symplectic-recursion submanifolds of X, while the central fiber X_0 is the singular symplectic-recursion manifold $M_0 \cup_Q M_1$.*

Proof. We mimic the symplectic connected sum constructions from [9,12] to show X_λ as a deformation in the symplectic category of the singular space $M_0 \cup_Q M_1$. The normal bundle

$N_{M_0}Q$, denoted simply by L, can be considered as a complex line bundle $\pi\colon L \to Q$. Then $N_{M_1}Q$ is the dual line bundle L^*. Fix a Hermitian metric on L and choose a compatible connection on L. The connection defines a real-valued 1-form α on $L \setminus \{0\text{-section}\}$ and the curvature defines a 2-form F on Q such that $\pi^* F = d\alpha$. Let $\rho(x) = |x|^2/2$ for $x \in L$, then the two forms

$$\omega_L = \pi^*\omega_Q + \rho\pi^* F + d\rho \wedge \alpha = \pi^*\omega_Q + d(\rho\alpha)$$

and

$$\tau_L = \pi^*\tau_Q + \rho\pi^* F + d\rho \wedge \alpha = \pi^*\tau_Q + d(\rho\alpha)$$

are closed and non-degenerate for small ρ. For L^* and its dual metric, we have 1-form α^* with $d\alpha^* = -\pi^* F$. Let $\rho^*(y) = |y|^2/2$, we can similarly obtain such pair of 2-forms $(\omega_{L^*}, \tau_{L^*})$. By Theorem 3.3, we can identify a neighborhood of Q in M_0 with a disk bundle in L of radius ϵ small enough, and a neighborhood of Q in M_1 with a ϵ-disk bundle in L^*. So up to these identifications, we say $(\omega_0, \tau_0) \sim (\omega_L, \tau_L)$ and $(\omega_1, \tau_1) \sim (\omega_{L^*}, \tau_{L^*})$. Then for ϵ small enough, ω_L and τ_L are also intertwined by recursion operator A_0. This implies $d(\rho\alpha)$ is A_0-invariant. Similarly, $d(\rho^*\alpha^*)$ is A_1-invariant.

Consider the sum $\pi\colon L \oplus L^* \to Q$, similarly, we have two closed and non-degenerate 2-forms

$$\begin{aligned}\mathcal{W} &= \pi^*\omega_Q + (\rho - \rho^*)\pi^* F + d\rho \wedge \alpha + d\rho^* \wedge \alpha^* \\ &= \pi^*\omega_Q + d(\rho\alpha) + d(\rho^*\alpha^*)\end{aligned}$$

and

$$\begin{aligned}\mathcal{T} &= \pi^*\tau_Q + (\rho - \rho^*)\pi^* F + d\rho \wedge \alpha + d\rho^* \wedge \alpha^* \\ &= \pi^*\tau_Q + d(\rho\alpha) + d(\rho^*\alpha^*).\end{aligned}$$

Denote by

$$O_\epsilon = \{(q, x, y) \in L \oplus L^* \mid |x| < \epsilon, |y| < \epsilon\}$$

an open set in $L \oplus L^*$. Using isomorphism (4.1) and (4.2) and identifications above, we see that in O_ϵ the two 2-forms $\mathcal{W}$ and $\mathcal{T}$ are also intertwined by recursion operator A.

Let $D = D^2(\delta) \subset \mathbb{R}^2$ be a radius $\delta < \epsilon/2$ disk with symplectic form $\omega_D = r\, dr d\theta$. Then using the identification, we define gluing diffeomorphisms

$$\begin{aligned}\mathcal{G}_{M_0}\colon O_\epsilon \setminus L^* &\to (M_0 \setminus Q) \times D, \quad (q, x, y) \mapsto ((q, x), \lambda = xy), \\ \mathcal{G}_{M_1}\colon O_\epsilon \setminus L &\to (M_1 \setminus Q) \times D, \quad (q, x, y) \mapsto ((q, y), \lambda = xy).\end{aligned}$$

The space X is constructed by gluing these three open sets, so X is a smooth manifold. The function $\lambda\colon L \oplus L^* \to \mathbb{C}$ can be extended to $(M_i \setminus Q) \times D$ so that we get a projection

$\lambda\colon X \to D$ whose fibers are smooth submanifolds X_λ for small $\lambda \neq 0$. Applying the similar argument of smooth deformation in [9,12], note that under the identifications above $d(\rho\alpha)$ and $d(\rho^*\alpha^*)$ are A-invariant, we can simultaneously extend the forms $\mathcal{W}$ and $\mathcal{T}$ in O_ϵ over $(M_0 \setminus Q) \times D$ (resp. $(M_1 \setminus Q) \times D$) as symplectic forms whose restrictions to the part of X_λ with $|x| \geq 1$ (resp. $|y| \geq 1$) is the original symplectic forms ω_0 and τ_0 (resp. ω_1 and τ_1) on $M_0 \times \{\lambda\}$ (resp. $M_1 \times \{\lambda\}$). Thus, we obtain global symplectic forms Ω and Γ on X which are intertwined by a recursion operator $\mathcal{A}$ such that $\mathcal{A}|_{M_i \times \{\lambda\}} = A_i$, $\mathcal{A}|_D = \mathrm{Id}$. Furthermore, along $O_\epsilon \cap X_\lambda$, Ω and Γ restrict to

$$\Omega_\lambda = \pi^*\omega_Q + (\rho - \rho^*)\pi^*F + d(\rho - \rho^*) \wedge \alpha$$

and

$$\Gamma_\lambda = \pi^*\tau_Q + (\rho - \rho^*)\pi^*F + d(\rho - \rho^*) \wedge \alpha,$$

respectively. So for $\lambda \neq 0$, $(X_\lambda, \Omega_\lambda, \Gamma_\lambda)_{\mathcal{A}}$ is a symplectic-recursion data which is the connected sum we desired. For $\lambda = 0$, the central fiber X_0 is the singular symplectic-recursion data. □

5. Application to symplectic pairs

In this section, we apply the main result to a certain case of sum of two symplectic pairs. To study the sum of symplectic pairs, let us recall more notions and facts in addition to Example 2.2.

In general, for a $2n$-dimensional manifold M with symplectic pair (ξ, η), we know that (ξ, η) have constant and complementary ranks, and ξ restricts as symplectic form to the leaves of the kernel foliation $\mathcal{F}$ of η, η restricts as symplectic form to the leaves of the kernel foliation $\mathcal{G}$ of ξ, where $\mathcal{F}$ and $\mathcal{G}$ are complementary smooth foliations. We say (M, ξ, η) are of rank $(2d, 2n-2d)$ if the dimension of leave of $\mathcal{F}$ is $2d$. Given a $2k$-dimensional submanifold S of a $2n$-dimensional manifold M with symplectic pair (ξ, η), S is called a $(2l, 2k-2l)$-labelled symplectic submanifold of the symplectic pair (ξ, η) if the restrictions $(\xi|_S, \eta|_S)$ is also a symplectic pair on S of rank $(2l, 2k-2l)$. We can define the symplectic normal bundle N_S of a $(2l, 2k-2l)$-labelled symplectic submanifold to be a vector bundle over S whose each fiber carries a symplectic pair of rank $(2d-2l, 2n-2k-2d-2l)$.

Let (M_0, ξ_0, η_0) and (M_1, ξ_1, η_1) be two $2n$-dimensional symplectic pairs of same ranks $(2d, 2n-2d)$ containing a copy of a $(2l, 2n-2-2l)$-labelled $(2n-2)$-dimensional compact submanifold $Q := Q_0 \cong Q_1$. Note that in this codimension 2 special case one has either the relation $l = d$ or $l = d-1$. In the sequel, we just need to consider the case of $l = d-1$. The argument for the other case is similar.

We assume that $\xi_0|_Q \cong \xi_1|_Q$ and $\eta_0|_Q \cong \eta_1|_Q$, and that their respective normal bundles have opposite Euler classes, that is $e(N_{M_0}Q) + e(N_{M_1}Q) = 0$. In precise, we assume that there exist a diffeomorphism $\phi\colon Q_0 \to Q_1$ such that $\phi^*(\xi_1|_{Q_1}) = \xi_0|_{Q_0}$ and $\phi^*(\eta_1|_{Q_1}) = \eta_0|_{Q_0}$, and an isomorphism

$$\Psi\colon N_{Q_0} \to N_{Q_1}$$

between normal bundles which covers ϕ. We can identify N_{Q_i} with the orthogonal complement TQ_i^{T}. Then the exponential map induces a map $f_i\colon N_{Q_i} \to M_i$. Define

$$\widetilde{\Psi} := f_1 \circ \Psi \circ f_0^{-1}$$

which is a diffeomorphism between a neighborhood of Q_0 and a neighborhood of Q_1. Assume that in a possibly smaller neighborhood $\mathcal{N}$ of Q_0, we can find a pair of 1-forms $\alpha, \beta \in \Omega^1(\mathcal{N})$, such that for $\widetilde{\Psi}$ the symplectic pairs (ξ_i, η_i), $i = 0, 1$, satisfy

$$\widetilde{\Psi}^*\xi_1 - \xi_0 = d\alpha, \quad \widetilde{\Psi}^*\eta_1 - \eta_0 = d\beta.$$

We can see that

$$\widetilde{\Psi}^*(\xi_1 \oplus \pm\eta_1) - (\xi_0 \oplus \pm\eta_0) = d(\alpha \pm \beta).$$

It is clear that $(\xi_i \oplus \eta_i, \xi_i \oplus -\eta_i)$ is a pair of symplectic forms intertwined by a recursion operator A_i defined by $\iota_X(\xi_i \oplus \eta_i) = \iota_{A_iX}(\xi_i \oplus -\eta_i)$. To apply Theorem 1.1, we set $\omega_i = \xi_i \oplus \eta_i$, $\tau_i = \xi_i \oplus -\eta_i$, $i = 0, 1$. Then $\omega_Q := \omega_i|_Q$ and $\tau_Q := \tau_i|_Q$ are a pair of symplectic forms on Q intertwined by a recursion operator $A_Q := A_i|_Q$. We see that $(M_0, \omega_0, \tau_0)_{A_0}$ and $(M_1, \omega_1, \tau_1)_{A_1}$ are two symplectic-recursion data containing a codimension two symplectic-recursion submanifold $(Q, \omega_Q, \tau_Q)_{A_Q}$.

Let $\eta_\omega = \alpha + \beta$, $\eta_\tau = \alpha - \beta$. So we have $\widetilde{\Psi}^*\omega_1 - \omega_0 = d\eta_\omega$ and $\widetilde{\Psi}^*\tau_1 - \tau_0 = d\eta_\tau$. If we further assume that $\alpha + \beta = (\alpha - \beta) \circ A_0$, then Theorem 1.1 implies that there exist a $(2n+2)$-dimensional manofold X, a fibration $\lambda\colon X \to D$ over a small disk $D \subset \mathbb{C}$, carrying a pair of symplectic forms Ω and Γ such that they are intertwined by a recursion operator $\mathcal{A}$ satisfying $\mathcal{A}|_{M_i \times \{\lambda\}} = A_i$, $\mathcal{A}|_D = \mathrm{Id}$. Moreover, along $O_\epsilon \cap X_\lambda$, Ω and Γ restrict to Ω_λ and Γ_λ, respectively. When $\lambda = 0$, $\Omega_0|_{M_i} = \omega_i$ and $\Gamma_0|_{M_i} = \tau_i$. When $\lambda \neq 0$, the fibers X_λ are fibre connected sums which are smooth compact codimension 2 submanifolds of X, while the central fiber X_0 is the singular manifold $M_0 \cup_Q M_1$ with symplectic pair (ξ^*, η^*) such that $\xi^*|_{M_i} = \xi_i$, $\eta^*|_{M_i} = \eta_i$, $i = 0, 1$.

Since we are considering the special case that the rank of $\eta_i|_Q$ is still $2n - 2d$, it is direct that we have symplectic pair $(\Xi, \mathcal{E})$ on the sum $\pi\colon L \oplus L^* \to Q$ such that

$$\Xi = \pi^*\xi_Q + d(\rho\alpha) + d(\rho^*\alpha^*) \quad \text{and} \quad \mathcal{E} = \pi^*\eta_Q,$$

where ρ and α are the same as ones appeared in the proof of Theorem 4.1.

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