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Nehari Type Ground State Solutions for Asymptotically Periodic Schrödinger-Poisson Systems

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Abstract. This paper is dedicated to studying the following Schrödinger-Poisson system

$$\begin{cases} -\Delta u + V(x)u + K(x)\phi(x)u = f(x,u), & x \in \mathbb{R}^3, \\ -\Delta\phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $V(x)$, $K(x)$ and $f(x,u)$ are periodic or asymptotically periodic in x. We use the non-Nehari manifold approach to establish the existence of the Nehari type ground state solutions in two cases: the periodic one and the asymptotically periodic case, by introducing weaker conditions $\lim_{|t|\to\infty}\left(\int_0^t f(x,s)\,\mathrm{d}s\right)/|t|^3 = \infty$ uniformly in $x \in \mathbb{R}^3$ and

$$\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\right]\operatorname{sign}(1-t) + \theta_0 V(x)\frac{|1-t^2|}{(t\tau)^2} \geq 0, \quad \forall\, x \in \mathbb{R}^3,\ t > 0,\ \tau \neq 0$$

with constant $\theta_0 \in (0,1)$, instead of $\lim_{|t|\to\infty}\left(\int_0^t f(x,s)\,\mathrm{d}s\right)/|t|^4 = \infty$ uniformly in $x \in \mathbb{R}^3$ and the usual Nehari-type monotonic condition on $f(x,t)/|t|^3$.

1. Introduction

In this paper we are concerned with the existence of ground state solutions for the nonlinear system

$$\text{(SP)} \qquad \begin{cases} -\triangle u + V(x)u + K(x)\phi(x)u = f(x,u), & x \in \mathbb{R}^3, \\ -\triangle\phi = K(x)u^2, & x \in \mathbb{R}^3, \end{cases}$$

where $V, K \colon \mathbb{R}^3 \to \mathbb{R}$ and $f \colon \mathbb{R}^3 \times \mathbb{R} \to \mathbb{R}$ satisfy the following basic assumptions, respectively

(V0) $V \in L^\infty(\mathbb{R}^3)$ and $\inf_{x\in\mathbb{R}^3} V(x) > 0$;

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(K0) $K \in L^{\infty}(\mathbb{R}^3)$, $0 \le K(x) \le K_{\infty}$, $\forall\, x \in \mathbb{R}^3$ and $K(x) \not\equiv 0$;

(F0) $f \in \mathcal{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $f(x,t) = o(|t|)$ as $t \to 0$, uniformly in $x \in \mathbb{R}^3$, and there exist constants $\mathcal{C}_0 > 0$ and $\kappa \in (2,6)$ such that

$$|f(x,t)| \le \mathcal{C}_0 \left(1 + |t|^{\kappa-1}\right), \quad \forall\, (x,t) \in \mathbb{R}^3 \times \mathbb{R}.$$

Under assumption (V0), the set

$$E = \left\{ u \in H^1(\mathbb{R}^3) : \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x < +\infty \right\},$$

is a Hilbert space equipped with the norm

$$\|u\| = \left(\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x \right)^{1/2}.$$

It is well known that the Poisson equation is solved by using Lax-Milgram theorem. Indeed, as we shall see in Section 2, for every $u \in E$, a unique $\phi_u \in D^{1,2}(\mathbb{R}^3)$ is obtained, such that $-\triangle \phi = K(x)u^2$ and so (SP) can be reduced to a single equation with a non-local term

$$-\triangle u + V(x)u + K(x)\phi_u(x)u = f(x,u). \tag{1.1}$$

Moreover, (1.1) is variational and its solutions are the critical points of the functional Φ defined on E by

$$\Phi(u) = \frac{1}{2}\int_{\mathbb{R}^3} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\, \mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\, \mathrm{d}x, \tag{1.2}$$

where $F(x,t) = \int_0^t f(x,s)\, \mathrm{d}s$. Define

$$\mathcal{N} := \left\{ u \in E : \left\langle \Phi'(u), u \right\rangle = 0, u \neq 0 \right\}, \tag{1.3}$$

which is the Nehari mainfold of Φ.

System (SP), also known as the nonlinear Schrödinger-Maxwell system, has a strong physical meaning because it appears in quantum mechanical models (see e.g., [5, 6, 15]) and in semiconductor theory [4, 19, 21]. For more details in the physical aspects, we refer the readers to [3, 4]. Note that when $\phi \equiv 0$, (SP) reduces to the well-known Schrödinger equation, which has been studied extensively in the last two decades, see for example [9–11, 18, 22, 23, 30–33] and the references therein.

In recent years, there have been enormous results on existence, nonexistence and multiplicity of solutions for systems like (SP) under various hypotheses on the potential and the nonlinearities. The greatest part of the literature focuses on the study of Problem (SP) with $V(x) \equiv 1$ or $V(x) = \overline{V}(|x|)$, and $f(x,u) = |u|^{p-1}u$ or $f(x,u) = a(x)\,|u|^{p-1}u$ with

$p \in (3,5)$, see e.g., [1,2,7,12,24,35]. Moreover, in [2,7,35], the existence of ground state solutions was obtained in several situations, where $V(x) \equiv 1$ or $\lim_{|x|\to\infty} V(x) = 1$, and $f(x,u) = |u|^{p-1} u$ with $p \in (3,5)$. We refer to [8,13,14,26] and the references therein for other cases.

When the potential and the nonlinearity are periodic, that is V and f satisfy

(V1) $V \in \mathcal{C}(\mathbb{R}^3, (0,\infty))$ and $V(x)$ is 1-periodic in x_1, x_2 and x_3;

(F1) $f(x,t)$ is 1-periodic in x_1, x_2 and x_3;

as far as we know, there are only two papers [27,35] dealing with the existence of ground state solutions to (SP) with $K \equiv 1$. Indeed, using the Nehari manifold approach, Zhao and Zhao [35] proved an existence theorem in the case when $f \in \mathcal{C}^1$ and f, f_u satisfy some suitable conditions. When only $f \in \mathcal{C}$, because $\mathcal{N}$ may not be a manifold, the arguments based on the Nehari manifold approach become invalid. Sun and Ma [27] adopted a technique developed in [28,29] to prove that (SP) has a ground solution if V and f satisfy (V1), (F0), (F1) and the following two assumptions:

(Ne) $f(x,t)/|t|^3$ is increasing in t on $\mathbb{R} \setminus \{0\}$ for every $x \in \mathbb{R}^3$;

(QF) $\lim_{|t|\to\infty} F(x,t)/|t|^4 = \infty$ uniformly in $x \in \mathbb{R}^3$.

We point out that assumption (Ne) is very crucial in [27]. In fact, the starting point of their approach is to show that for each $u \in E \setminus \{0\}$, the Nehari manifold $\mathcal{N}$ intersects E in exactly one point $\widehat{m}(u) = t_u u$ with $t_u > 0$. The uniqueness of $\widehat{m}(u)$ enables one to define a map $u \mapsto \widehat{m}(u)$, which is important in the remaining proof. If $t \mapsto f(x,t)/|t|^3$ *is not strictly increasing*, then $\widehat{m}(u)$ may not be unique and their arguments become invalid. This paper intends to address this problem caused by the dropping of this "strictly increasing" condition on f. Motivated by the works [27,35], we will use the non-Nehari manifold approach developed by Tang [32,33] to generalize and improve the results obtained in [27] by relaxing (Ne) and (QF) to the following assumptions:

(F2) there exists $\theta_0 \in (0,1)$ such that

(1.4)
$$\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\right] \operatorname{sign}(1-t) + \theta_0 V(x) \frac{\left|1-t^2\right|}{(t\tau)^2} \geq 0, \quad \forall\, x \in \mathbb{R}^3,\ t > 0,\ \tau \neq 0$$

and

(F3) $\lim_{|t|\to\infty} F(x,t)/|t|^3 = \infty$ uniformly in $x \in \mathbb{R}^3$,

respectively. Unlike the Nahari manifold method and the one used in [27], our approach lies on finding a minimizing Cerami sequence for Φ outside $\mathcal{N}$ by using the diagonal method, see Lemma 2.5.

Obviously, (1.4) is equivalent to

$$\begin{aligned}
&\begin{cases} \frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3} + \theta_0 V(x)\frac{1-t^2}{(t\tau)^2} \geq 0, & 0 < t \leq 1, \\ \frac{f(x,t\tau)}{(t\tau)^3} - \frac{f(x,\tau)}{\tau^3} + \theta_0 V(x)\frac{t^2-1}{(t\tau)^2} \geq 0, & t > 1, \end{cases} \quad \forall\, x \in \mathbb{R}^3,\ \tau \neq 0, \\
\iff &\begin{cases} \frac{f(x,\tau)}{|\tau|^3} - \frac{f(x,t\tau)}{|t\tau|^3} + \theta_0 V(x)\frac{\left|1-t^2\right|}{(t\tau)^2} \geq 0, & \tau \geq t\tau, \\ \frac{f(x,t\tau)}{|t\tau|^3} - \frac{f(x,\tau)}{|\tau|^3} + \theta_0 V(x)\frac{\left|1-t^2\right|}{(t\tau)^2} \geq 0, & \tau \leq t\tau, \end{cases} \quad \forall\, x \in \mathbb{R}^3,\ t > 0,\ \tau \neq 0.
\end{aligned} \tag{1.5}$$

Since $V(x) > 0$ for all $x \in \mathbb{R}^3$, it follows from (1.5) that (F2) is much weaker than (Ne). Furthermore, there are many functions satisfying (F1)–(F3), but not (Ne). We give the following example. For simplicity, we assume that $V(x) = 1$.

Example 1.1. $f(x,\tau) = b(x)\,|\tau|^3\,\tau + |\tau|\,\tau/2$ for all $(x,\tau) \in \mathbb{R}^3 \times \mathbb{R}$, where $b(x)$ is 1-periodic in x_1, x_2 and x_3 and $\inf_{\mathbb{R}^3} b \geq 1$.

It is easy to see that f satisfies (F1) and (F3), but not satisfy (Ne). Next, we show that f satisfies (F2). By elementary computations, one has

$$\begin{aligned}
&\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\right] \operatorname{sign}(1-t) + \theta_0 V(x)\frac{\left|1-t^2\right|}{(t\tau)^2} \\
&= b(x)\,|1-t|\,|\tau| - \frac{|1-t|}{2t\,|\tau|} + \theta_0 \frac{\left|1-t^2\right|}{(t\tau)^2} \\
&= \frac{|1-t|}{(t\tau)^2}\left[b(x)\,|\tau|^3\,t^2 - \frac{1}{2}\,|\tau|\,t + \theta_0(1+t)\right], \quad \forall\, x \in \mathbb{R}^3,\ t > 0,\ \tau \neq 0.
\end{aligned} \tag{1.6}$$

Note that

$$\begin{cases} |\tau|^3\,t^2 - \frac{1}{2}\,|\tau|\,t + \theta_0(1+t) \geq \left(\theta_0 - \frac{1}{2}\right)t, & |\tau| \leq 1,\ \forall\, t > 0, \\ |\tau|^3\,t^2 - \frac{1}{2}\,|\tau|\,t + \theta_0(1+t) \geq \left(t\,|\tau| - \frac{1}{4}\right)^2 + \theta_0 - \frac{1}{16}, & |\tau| > 1,\ \forall\, t > 0, \end{cases}$$

then (1.6) implies that f satisfies (1.4) with $\theta_0 = 1/2$.

In addition, (F3) is much weaker than (QF). Here, we give a nonlinear function satisfying assumptions (F1)–(F3), but they do not satisfy (Ne) and (QF). We assume that $V(x) = 1$.

Example 1.2. $f(x,\tau) = b(x)\tau^3 - |\tau|^{3/2}\,\tau + |\tau|\,\tau$ for all $(x,\tau) \in \mathbb{R}^3 \times \mathbb{R}$, where $b(x)$ is 1-periodic in x_1, x_2 and x_3 and $\inf_{\mathbb{R}^3} b > 0$.

Clearly, f satisfies (F1) and (F3), but does not satisfy (Ne) and (QF). Next, we show that f satisfies (F2). It is easy to check that

$$
\begin{aligned}
&\left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,t\tau)}{(t\tau)^3}\right]\operatorname{sign}(1-t) + \theta_0 V(x)\frac{|1-t^2|}{(t\tau)^2} \\
&= \frac{|1-t^{1/2}|}{|t\tau|^{1/2}} - \frac{|1-t|}{|t\tau|} + \theta_0\frac{|1-t^2|}{(t\tau)^2} \\
&= \frac{|1-t^{1/2}|}{(t\tau)^2}\left[|t\tau|^{3/2} - (1+t^{1/2})\,|t\tau| + \theta_0(1+t^{1/2})(1+t)\right] \\
&:= \frac{|1-t^{1/2}|}{(t\tau)^2} h(t,|\tau|), \quad \forall\, x\in\mathbb{R}^3,\ t>0,\ \tau\neq 0.
\end{aligned} \tag{1.7}
$$

By elementary computations, for any $t>0$, we have

$$h(t,|\tau|) \geq \min_{\tau\neq 0} h(t,|\tau|) = h(t,\tau_0) \quad \text{with } \tau_0 = \frac{4(1+t^{1/2})^2}{9t},$$

and so

$$
\begin{aligned}
h(t,|\tau|) &\geq -\frac{4}{27}(1+t^{1/2})^3 + \theta_0(1+t^{1/2})(1+t) \\
&= (1+t^{1/2})\left[\left(\theta_0 - \frac{4}{27}\right)t - \frac{8}{27}t^{1/2} + \left(\theta_0 - \frac{4}{27}\right)\right] \\
&= (1+t^{1/2})\frac{27\theta_0-4}{27}\left[\left(t^{1/2} - \frac{4}{27\theta_0-4}\right)^2 + 1 - \frac{16}{(27\theta_0-4)^2}\right].
\end{aligned} \tag{1.8}
$$

Hence, (1.7) and (1.8) imply that f satisfies (1.4) with $\theta_0 = 1/3$.

Before presenting our theorems, in addition to (V0), (V1), (K0) and (F0)–(F3), we introduce the following assumption:

(K1) $K\in\mathcal{C}(\mathbb{R}^3,\mathbb{R}^+)$ and $K(x)$ is 1-periodic in x_1, x_2 and x_3.

Now, we state the first result of this paper. In the periodic case, we establish the following theorem.

Theorem 1.3. *Assume that V, K and f satisfy* (V1), (K1) *and* (F0)–(F3). *Then Problem* (SP) *has a solution $u_0\in E$ such that $\Phi(u_0) = \inf_{\mathcal{N}}\Phi > 0$.*

Next, we assume that $V(x)$ is asymptotically periodic. In this case, the functional Φ loses the $\mathbb{Z}^3$-translation invariance. For this reason, many effective methods for periodic problems cannot be applied to asymptotically periodic ones. To the best of our knowledge, there are no results on the existence of ground state solutions for (SP) when $V(x)$ is asymptotically periodic. In this paper, we present new tricks to overcome the difficulties caused by the dropping of periodicity of $V(x)$.

Instead of (V1), (K1) and (F1), we make the following assumptions.

(V2) $V(x) = V_0(x) + V_1(x)$, $V_0, V_1 \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, $V_0(x)$ is 1-periodic in x_1, x_2 and x_3, and $-V_0(x) < V_1(x) \leq 0$ for $x \in \mathbb{R}^3$, $\lim_{|x|\to\infty} V_1(x) = 0$;

(K2) $K(x) = K_0(x) + K_1(x)$, $K_0, K_1 \in \mathcal{C}(\mathbb{R}^3, \mathbb{R})$, $K_0(x)$ is 1-periodic in x_1, x_2 and x_3, and $-K_0(x) \leq K_1(x) \leq 0$ for $x \in \mathbb{R}^3$, $\lim_{|x|\to\infty} K_1(x) = 0$;

(F1′) $f(x,t) = f_0(x,t) + f_1(x,t)$, $f_0 \in \mathcal{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$, $f_0(x,t)$ is 1-periodic in x_1, x_2 and x_3, and for any $x \in \mathbb{R}^3$, $t > 0$ and $\tau \neq 0$

$$\left[\frac{f_0(x,\tau)}{\tau^3} - \frac{f_0(x,t\tau)}{(t\tau)^3}\right] \operatorname{sign}(1-t) + V_0(x)\frac{|1-t^2|}{(t\tau)^2} \geq 0; \tag{1.9}$$

$f_1 \in \mathcal{C}(\mathbb{R}^3 \times \mathbb{R}, \mathbb{R})$ satisfies that

$$-V_1(x)t^2 + 2F_1(x,t) \geq 0, \quad |f_1(x,t)| \leq a(x)\left(|t| + |t|^{\kappa_0 - 1}\right),$$

where $F_1(x,t) = \int_0^t f_1(x,s)\,\mathrm{d}s$, $\kappa_0 \in (2,6)$ and $a \in \mathcal{C}(\mathbb{R}^3, \mathbb{R}^+)$ with $\lim_{|x|\to\infty} a(x) = 0$.

We are now in a position to state the second result of this paper.

Theorem 1.4. *Assume that V, K and f satisfy* (V2), (K2), (F0), (F1′), (F2) *and* (F3). *Then Problem* (SP) *has a solution $u_0 \in E$ such that $\Phi(u_0) = \inf_{\mathcal{N}} \Phi > 0$.*

The paper is organized as follows. In Section 2, we introduce some notations and preliminaries. We complete the proofs of Theorems 1.3 and 1.4 in Sections 3 and 4 respectively.

Throughout this paper, we denote the norm of $L^s(\mathbb{R}^3)$ by $\|u\|_s = \left(\int_{\mathbb{R}^3} |u|^s\,\mathrm{d}x\right)^{1/s}$ for $s \geq 2$, $B_r(x) = \{y \in \mathbb{R}^3 : |y-x| < r\}$, and positive constants possibly different in different places, by $C_1, C_2, \ldots$.

2. Notations and preliminaries

Hereafter, $H^1(\mathbb{R}^3)$ is the usual Sobolev space with the standard scalar product and norm

$$(u,v)_{H^1} = \int_{\mathbb{R}^3} (\nabla u \nabla v + uv)\,\mathrm{d}x, \quad \|u\|_{H^1}^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + u^2\right)\mathrm{d}x,$$

and

$$D^{1,2}(\mathbb{R}^3) = \{u \in L^6(\mathbb{R}^3) : \nabla u \in L^2(\mathbb{R}^3)\}$$

equipped with the norm defined by

$$\|u\|_{D^{1,2}}^2 = \int_{\mathbb{R}^3} |\nabla u|^2\,\mathrm{d}x.$$

It is easy to show that (SP) can be reduced to a single equation with a non-local term. Namely, for any $Ku^2 \in L^1_{\mathrm{loc}}(\mathbb{R}^3)$ such that

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{u^2(x)u^2(y)}{|x-y|}\,\mathrm{d}x\mathrm{d}y < \infty,$$

the distributional solution

$$\phi_u(x) = \int_{\mathbb{R}^3} \frac{K(y)u^2(y)}{|x-y|}\,\mathrm{d}y = \frac{1}{|x|} * Ku^2 \tag{2.1}$$

of the Poisson equation

$$-\triangle\phi = K(x)u^2, \quad x \in \mathbb{R}^3$$

belongs to $D^{1,2}(\mathbb{R}^3)$ and is the unique weak solution in $D^{1,2}(\mathbb{R}^3)$ (see e.g., [25] for more details), and

$$\int_{\mathbb{R}^3} \nabla\phi_u\nabla v\,\mathrm{d}x = \int_{\mathbb{R}^3} K(x)u^2 v\,\mathrm{d}x, \quad \forall\, v \in H^1(\mathbb{R}^3), \tag{2.2}$$

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K(x)K(y)}{|x-y|}u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y = \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,\mathrm{d}x. \tag{2.3}$$

Moreover, $\phi_u(x) > 0$ when $u \neq 0$, because K does (see (K0)). By using Hardy-Littlewood-Sobolev inequality (see [16] or [17, p. 98]), we have the following inequality:

$$\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)v(y)|}{|x-y|}\,\mathrm{d}x\mathrm{d}y \leq \frac{8\sqrt[3]{2}}{3\sqrt[3]{\pi}}\,\|u\|_{6/5}\,\|v\|_{6/5}\,, \quad u, v \in L^{6/5}(\mathbb{R}^3). \tag{2.4}$$

Formally, the solutions of (SP) are then the critical points of the reduced functional (1.2). Indeed, (V0), (K0), (F0) and (2.4) imply that Φ is a well-defined of class $\mathcal{C}^1$ functional, and that

$$\langle \Phi'(u), v\rangle = \int_{\mathbb{R}^3} (\nabla u\nabla v + V(x)uv)\,\mathrm{d}x + \int_{\mathbb{R}^3} [K(x)\phi_u(x)u - f(x,u)]\,v\,\mathrm{d}x. \tag{2.5}$$

Hence if $u \in E$ is a critical point of Φ , then the pair (u, ϕ_u), with ϕ_u as in (2.1), is a solution of (SP).

Lemma 2.1. *Under assumptions* (V0), (K0), (F0) *and* (F2)*,*

$$\Phi(u) \geq \Phi(tu) + \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2, \quad \forall\, u \in E,\ t \geq 0. \tag{2.6}$$

Proof. For any $x \in \mathbb{R}^3$, $t \geq 0$, $\tau \neq 0$, (F2) yields

$$\begin{aligned}
&\frac{1-t^4}{4}\tau f(x,\tau) + F(x,t\tau) - F(x,\tau) + \frac{\theta_0 V(x)}{4}(1-t^2)^2\tau^2 \\
&= \int_t^1 \left[\frac{f(x,\tau)}{\tau^3} - \frac{f(x,s\tau)}{(s\tau)^3} + \theta_0 V(x)\frac{(1-s^2)}{(s\tau)^2}\right] s^3\tau^4\,\mathrm{d}s \\
&\geq 0.
\end{aligned} \tag{2.7}$$

Note that

$$\Phi(u) = \frac{1}{2}\|u\|^2 + \frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,\mathrm{d}x - \int_{\mathbb{R}^3} F(x,u)\,\mathrm{d}x \tag{2.8}$$

and

$$\langle \Phi'(u), u\rangle = \|u\|^2 + \int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,\mathrm{d}x - \int_{\mathbb{R}^3} f(x,u)u\,\mathrm{d}x. \tag{2.9}$$

Thus, by (2.7), (2.8) and (2.9), one has

$$\begin{aligned}
\Phi(u) - \Phi(tu) &= \frac{1-t^2}{2}\|u\|^2 + \frac{1-t^4}{4}\int_{\mathbb{R}^3} K(x)\phi_u(x)u^2\,\mathrm{d}x + \int_{\mathbb{R}^3}[F(x,tu) - F(x,u)]\,\mathrm{d}x \\
&= \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-t^2)^2}{4}\|u\|^2 \\
&\quad + \int_{\mathbb{R}^3}\left[\frac{1-t^4}{4}f(x,u)u + F(x,tu) - F(x,u)\right]\mathrm{d}x \\
&\geq \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2 \\
&\quad + \int_{\mathbb{R}^3}\left[\frac{1-t^4}{4}f(x,u)u + F(x,tu) - F(x,u) + \frac{\theta_0 V(x)}{4}(1-t^2)^2u^2\right]\mathrm{d}x \\
&\geq \frac{1-t^4}{4}\langle \Phi'(u), u\rangle + \frac{(1-\theta_0)(1-t^2)^2}{4}\|u\|^2, \quad t \geq 0.
\end{aligned}$$

This shows that (2.6) holds. □

Corollary 2.2. *Under assumptions* (V0), (K0), (F0) *and* (F2), *for* $u \in \mathcal{N}$

$$\Phi(u) = \max_{t\geq 0}\Phi(tu). \tag{2.10}$$

Unlike the super-cubic case (i.e., f satisfies (QF)), to show $\mathcal{N} \neq \emptyset$ in our situation, we have to overcome the competing effect of the nonlocal term. To this end, we define a set Λ as follows:

$$\Lambda = \left\{u \in E : \int_{\mathbb{R}^3}\left[V(x)u^2 + K(x)\phi_u u^2 - f(x,u)u\right]\mathrm{d}x < 0\right\}.$$

Lemma 2.3. *Under assumptions* (V0), (K0), (F0), (F2) *and* (F3), $\Lambda \neq \emptyset$ *and* $\mathcal{N} \subset \Lambda$. *Then, for any* $u \in \Lambda$, *there exists a unique* $t(u) > 0$ *such that* $t(u)u \in \mathcal{N}$.

Proof. First, we show that $\Lambda \neq \emptyset$. From (2.4) and Sobolev imbedding theorem, there exists $C_1 > 0$ such that $\int_{\mathbb{R}^3}\phi_u u^2\,\mathrm{d}x \leq C_1\|u\|^4$ for all $u \in E$. For any fixed $u \in E$ with $u \neq 0$, set $u_t(x) = u(tx)$ for $t > 0$. By (V0) and (K0), one has

$$\begin{aligned}
&\int_{\mathbb{R}^3}\left[V(x)(tu_t)^2 + K(x)\phi_{(tu_t)}(tu_t)^2 - f(x,tu_t)tu_t\right]\mathrm{d}x \\
&= t^{-1}\int_{\mathbb{R}^3} V(t^{-1}x)u^2\,\mathrm{d}x + t^{-1}\int_{\mathbb{R}^3} K(t^{-1}x)\phi_u u^2\,\mathrm{d}x - \int_{\mathbb{R}^3}\frac{f(t^{-1}x,tu)tu}{t^3}\,\mathrm{d}x \qquad (2.11) \\
&\leq V_\infty t^{-1}\|u\|_2^2 + C_1K_\infty t^{-1}\|u\|^4 - \int_{\mathbb{R}^3}\frac{f(t^{-1}x,tu)tu}{t^3}\,\mathrm{d}x,
\end{aligned}$$

where $V_\infty := \sup_{x\in\mathbb{R}^3} V(x)$. Note that for $u(x) \neq 0$, $F(t^{-1}x, tu)/|tu|^3 \to +\infty$ as $t \to +\infty$ uniformly in $x \in \mathbb{R}^3$ by (F2), and (2.7) with $t = 0$ yields

$$\frac{1}{4} f(x,\tau)\tau - F(x,\tau) + \frac{\theta_0 V(x)}{4}\tau^2 \geq 0, \quad \forall\, x \in \mathbb{R}^3,\ \tau \in \mathbb{R}, \tag{2.12}$$

then we have

$$\frac{f(t^{-1}x, tu)tu}{|tu|^3} \to +\infty \quad \text{as } t \to +\infty \text{ uniformly in } x \in \mathbb{R}^3. \tag{2.13}$$

Thus, it follows from (V0), (K0), (2.11) and (2.13) that

$$\int_{\mathbb{R}^3} \left[V(x)(tu_t)^2 + K(x)\phi_{(tu_t)}(tu_t)^2 - f(x, tu_t)tu_t\right] dx \to -\infty \quad \text{as } t \to +\infty.$$

Thus, taking $v = Tu_T$ for T large, we have $v \in \Lambda$. Hence, $\Lambda \neq \emptyset$. From (2.5), it is easy to see that $\mathcal{N} \subset \Lambda$.

Next, we prove the last part of lemma. Let $u \in \Lambda$ be fixed and define a function $g(t) := \langle \Phi'(tu), tu\rangle$ on $[0, \infty)$. By (F2), one has

$$f(x, t\tau)t\tau \geq f(x,\tau)\tau t^4 - \theta_0 V(x)(t^2 - 1)(t\tau)^2, \quad \forall\, x \in \mathbb{R}^3,\ t \geq 1,\ \tau \in \mathbb{R}, \tag{2.14}$$

which yields

$$\begin{aligned} &\int_{\mathbb{R}^3} \left[\theta_0 V(x)(t\tau)^2 + K(x)\phi_{t\tau}(t\tau)^2 - f(x, t\tau)t\tau\right] dx \\ &\leq t^4 \int_{\mathbb{R}^3} \left[\theta_0 V(x)\tau^2 + K(x)\phi_\tau \tau^2 - f(x,\tau)\tau\right] dx, \quad \forall\, t \geq 1,\ \tau \in \mathbb{R}. \end{aligned} \tag{2.15}$$

From (2.5) and (2.15) it follows that

$$\begin{aligned} g(t) \leq t^2 \|u\|^2 &+ t^4 \int_{\mathbb{R}^3} \left[V(x)u^2 + K(x)\phi_u u^2 - f(x,u)u\right] dx \\ &- \theta_0 t^2 \int_{\mathbb{R}^3} V(x)u^2\, dx, \quad \forall\, t \geq 1. \end{aligned} \tag{2.16}$$

Using (F0), (2.5) and (2.16), it is easy to verify that $g(0) = 0$, $g(t) > 0$ for $t > 0$ small and $g(t) < 0$ for t large due to $u \in \Lambda$. Therefore, there exist a $t_0 = t(u) > 0$ so that $g(t_0) = 0$ and $t(u)u \in \mathcal{N}$. We claim that $t(u)$ is unique for any $u \in \Lambda$. In fact, for any given $u \in \Lambda$, let $t_1, t_2 > 0$ such that $g(t_1) = g(t_2) = 0$. Jointly with (2.6), we have

$$\begin{aligned} \Phi(t_1 u) &\geq \Phi(t_2 u) + \frac{t_1^4 - t_2^4}{4t_1^4}\left\langle \Phi'(t_1 u), t_1 u\right\rangle + \frac{(1-\theta_0)(t_1^2 - t_2^2)^2}{4t_1^4}\|u\|^2 \\ &= \Phi(t_2 u) + \frac{(1-\theta_0)(t_1^2 - t_2^2)^2}{4t_1^4}\|u\|^2 \end{aligned} \tag{2.17}$$

and

$$\begin{aligned}\Phi(t_2u) &\geq \Phi(t_1u) + \frac{t_2^4 - t_1^4}{4t_2^4}\left\langle \Phi'(t_2u), t_2u\right\rangle + \frac{(1-\theta_0)(t_2^2-t_1^2)^2}{4t_2^4}\|u\|^2 \\ &= \Phi(t_1u) + \frac{(1-\theta_0)(t_2^2-t_1^2)^2}{4t_2^4}\|u\|^2 .\end{aligned} \tag{2.18}$$

(2.17) and (2.18) imply $t_1 = t_2$. Hence, $t(u) > 0$ is unique for any $u \in \Lambda$. □

Lemma 2.4. *Under assumptions* (V0), (K0), (F0), (F2) *and* (F3)*, then*

$$\inf_{u\in\mathcal{N}} \Phi(u) := c = \inf_{u\in\Lambda, u\neq 0} \max_{t\geq 0} \Phi(tu) > 0.$$

Proof. Both Corollary 2.2 and Lemma 2.3 imply that $c = \inf_{u\in\Lambda,u\neq 0}\max_{t\geq 0}\Phi(tu)$. Using Lemma 2.1, it is easy to see that $c > 0$. □

Lemma 2.5. *Under assumptions* (V0), (K0), (F0), (F2) *and* (F3)*, there exist a constant* $c_* \in (0, c]$ *and a sequence* $\{u_n\} \subset E$ *satisfying*

$$\Phi(u_n) \to c_*, \quad \left\|\Phi'(u_n)\right\|(1 + \|u_n\|) \to 0. \tag{2.19}$$

Proof. By (F0) and (1.2), we know that there exist $\delta_0 > 0$ and $\rho_0 > 0$ such that

$$\Phi(u) \geq \rho_0, \quad \|u\| = \delta_0. \tag{2.20}$$

In view of Lemmas 2.3 and 2.4, we may choose $v_k \in \mathcal{N} \subset \Lambda$ such that

$$c - \frac{1}{k} < \Phi(v_k) < c + \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.21}$$

Using Lemma 2.1 and (2.20), it is easy to check that $\Phi(tv_k) \geq \rho_0$ for small $t > 0$ and $\Phi(tv_k) < 0$ for large $t > 0$ due to $v_k \in \Lambda$. Since $\Phi(0) = 0$, then the Mountain pass Lemma implies that there exists a sequence $\{u_{k,n}\}_{n\in\mathbb{N}} \subset E$ satisfying

$$\Phi(u_{k,n}) \to c_k, \quad \left\|\Phi'(u_{k,n})\right\|(1 + \|u_{k,n}\|) \to 0, \quad k \in \mathbb{N}, \tag{2.22}$$

where $c_k \in \left[\rho_0, \sup_{t\geq 0}\Phi(tv_k)\right]$. By virtue of Corollary 2.2, one has $\Phi(v_k) = \sup_{t\geq 0}\Phi(tv_k)$. Hence, by (2.21) and (2.22), one has

$$\Phi(u_{k,n}) \to c_k \in \left[\rho_0, c + \frac{1}{k}\right), \quad \left\|\Phi'(u_{k,n})\right\|(1 + \|u_{k,n}\|) \to 0, \quad k \in \mathbb{N}. \tag{2.23}$$

Now, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\Phi(u_{k,n_k}) \in \left[\rho_0, c + \frac{1}{k}\right), \quad \left\|\Phi'(u_{k,n_k})\right\|(1 + \|u_{k,n_k}\|) < \frac{1}{k}, \quad k \in \mathbb{N}. \tag{2.24}$$

Let $u_k = u_{k,n_k}$, $k \in \mathbb{N}$. Then, going if necessary to a subsequence, we have

$$\Phi(u_n) \to c_* \in [\rho_0, c], \quad \left\|\Phi'(u_n)\right\|(1 + \|u_n\|) \to 0.$$ □

Lemma 2.6. *Under assumptions* (V0), (K0), (F0), (F2) *and* (F3), *any sequence* $\{u_n\} \subset E$ *satisfying* (2.19) *is bounded in* E.

Proof. By Lemma 2.1, one has

$$c_* + o(1) = \Phi(u_n) - \frac{1}{4}\left\langle \Phi'(u_n), u_n \right\rangle \geq \frac{1-\theta_0}{4}\left\| u_n \right\|^2.$$

This shows that sequence $\{u_n\}$ is bounded in E. □

Next, we prove the minimizer of the constrained problem is a critical point, which plays a crucial role in the asymptotically periodic case.

Lemma 2.7. *Under assumptions* (V0), (K0), (F0), (F2) *and* (F3), *if* $u_0 \in \mathcal{N}$ *and* $\Phi(u_0) = c$, *then* u_0 *is a critical point of* Φ.

Proof. Assume that $u_0 \in \mathcal{N}$, $\Phi(u_0) = c$ and $\Phi'(u_0) \neq 0$. Then there exist $\delta > 0$ and $\varrho > 0$ such that

$$\|u - u_0\| \leq 3\delta \implies \left\|\Phi'(u)\right\| \geq \varrho. \tag{2.25}$$

In view of Lemma 2.1, one has

$$\begin{aligned} \Phi(tu_0) &\leq \Phi(u_0) - \frac{(1-\theta_0)(1-t^2)^2}{4}\left\|u_0\right\|^2 \\ &= c - \frac{(1-\theta_0)(1-t^2)^2}{4}\left\|u_0\right\|^2, \quad \forall\, t \geq 0. \end{aligned} \tag{2.26}$$

For $\varepsilon := \min\left\{3(1-\theta_0)\left\|u_0\right\|^2/64, 1, \varrho\delta/8\right\}$, $S := B(u_0, \delta)$, [34, Lemma 2.3] yields a deformation $\eta \in \mathcal{C}([0,1] \times E, E)$ such that

(i) $\eta(1,u) = u$ if $\Phi(u) < c - 2\varepsilon$ or $\Phi(u) > c + 2\varepsilon$;

(ii) $\eta\left(1, \Phi^{c+\varepsilon} \cap B(u_0,\delta)\right) \subset \Phi^{c-\varepsilon}$;

(iii) $\Phi(\eta(1,u)) \leq \Phi(u)$, $\forall\, u \in E$;

(iv) $\eta(1,u)$ is a homeomorphism of E.

By Corollary 2.2, $\Phi(tu_0) \leq \Phi(u_0) = c$ for $t \geq 0$, then it follows from (ii) that

$$\Phi(\eta(1,tu_0)) \leq c - \varepsilon, \quad \forall\, t \geq 0,\ |t-1| < \delta/\left\|u_0\right\|. \tag{2.27}$$

On the other hand, by (iii) and (2.26), one has

$$\begin{aligned} \Phi(\eta(1,tu_0)) &\leq \Phi(tu_0) \\ &\leq c - \frac{(1-\theta_0)(1-t^2)^2}{4}\left\|u_0\right\|^2 \\ &\leq c - \frac{(1-\theta_0)\delta^2}{4}, \quad \forall\, t \geq 0,\ |t-1| \geq \delta/\left\|u_0\right\|. \end{aligned} \tag{2.28}$$

Combining (2.27) with (2.28), we have

$$\max_{t\in[1/2,\sqrt{7}/2]} \Phi(\eta(1,tu_0)) < c. \tag{2.29}$$

We prove that $\eta(1,tu_0)\cap\mathcal{N}\neq\emptyset$ for some $t\in[1/2,\sqrt{7}/2]$, contradicting to the definition of c. Define

$$\Psi_0(t) := \left\langle \Phi'(tu_0), tu_0 \right\rangle, \quad \Psi_1(t) := \left\langle \Phi'(\eta(1,tu_0)), \eta(1,tu_0) \right\rangle, \quad \forall\, t\geq 0.$$

Since $u_0\neq 0$, it follows from (iv) that $\eta(1,tu_0)\neq 0$ for all $t>0$. By Lemma 2.3 and the degree theory, one can derive that $\deg(\Psi_0,(1/2,\sqrt{7}/2),0)=1$. It follows from (2.26) and (i) that $\eta(1,tu_0) = tu_0$ for $t = 1/2$ and $t = \sqrt{7}/2$. Thus, $\deg(\Psi_1,(1/2,\sqrt{7}/2),0) = \deg(\Psi_0,(1/2,\sqrt{7}/2),0) = 1$. Hence, $\Psi_1(t_0) = 0$ for some $t_0\in(1/2,\sqrt{7}/2)$, that is $\eta(1,t_0u_0)\in\mathcal{N}$, which is a contradiction. □

3. The periodic case

In this section, we give the proof of Theorem 1.3.

Proof of Theorem 1.3. Lemma 2.5 implies the existence of a sequence $\{u_n\}\subset E$ satisfying (2.19), then

$$\Phi(u_n)\to c_*>0, \quad \left\langle \Phi'(u_n), u_n \right\rangle \to 0. \tag{3.1}$$

By Lemma 2.6, $\{u_n\}$ is bounded in E. If

$$\delta := \limsup_{n\to\infty}\sup_{y\in\mathbb{R}^3}\int_{B_1(y)} |u_n|^2\,\mathrm{d}x = 0,$$

then by Lion's concentration compactness principle [20] or [34, Lemma 1.21], $u_n\to 0$ in $L^s(\mathbb{R}^3)$ for $2<s<6$. Moreover, there exists $C_2>0$ such that $\|u_n\|_2\leq C_2$. By (F0), for $\varepsilon = c_*/2C_2^2$, there exists $C_\varepsilon>0$ such that

$$\limsup_{n\to\infty}\int_{\mathbb{R}^3}\left|\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right|\mathrm{d}x \leq \frac{3}{2}\varepsilon C_2^2 + C_\varepsilon \lim_{n\to\infty}\|u_n\|_\kappa^\kappa = \frac{3c_*}{4}. \tag{3.2}$$

By (K0), (2.3) and (2.4), we have

$$\begin{aligned}\limsup_{n\to\infty}\int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)u_n^2\,\mathrm{d}x &= \limsup_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K(x)K(y)}{|x-y|}u_n^2(x)u_n^2(y)\,\mathrm{d}x\mathrm{d}y\\ &\leq K_\infty^2\limsup_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{u_n^2(x)u_n^2(y)}{|x-y|}\,\mathrm{d}x\mathrm{d}y\\ &\leq \mathcal{C}_1K_\infty^2\limsup_{n\to\infty}\|u_n\|_{12/5}^4\\ &=0,\end{aligned} \tag{3.3}$$

where and in the sequel, $\mathcal{C}_1 = 8\sqrt[3]{2}/3\sqrt[3]{\pi}$. From (1.2), (2.5), (3.1), (3.2) and (3.3), one has

$$\begin{aligned} c_* &= \Phi(u_n) - \frac{1}{2}\left\langle \Phi'(u_n), u_n \right\rangle + o(1) \\ &= -\frac{1}{4}\int_{\mathbb{R}^3} K(x)\phi_{u_n}(x)u_n^2\,\mathrm{d}x + \int_{\mathbb{R}^3}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]\mathrm{d}x + o(1) \\ &\le \frac{3c_*}{4} + o(1). \end{aligned}$$

This contradiction shows $\delta > 0$.

Going if necessary to a subsequence, we may assume the existence of $k_n \in \mathbb{Z}^3$ such that

$$\int_{B_2(k_n)} |u_n|^2\,\mathrm{d}x > \frac{\delta}{2}. \tag{3.4}$$

Let $v_n(x) = u_n(x + k_n)$. Then

$$\int_{B_2(0)} |v_n|^2\,\mathrm{d}x > \frac{\delta}{2}. \tag{3.5}$$

Since $V(x)$, $K(x)$ and $f(x,u)$ are periodic on x, we have

$$\Phi(v_n) \to c_* \in (0, c], \quad \left\|\Phi'(v_n)\right\|(1 + \|v_n\|) \to 0. \tag{3.6}$$

Passing to a subsequence, we have $v_n \rightharpoonup \overline{v}$ in E, $v_n \to \overline{v}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$, $2 \le s < 6$ and $v_n(x) \to \overline{v}(x)$ a.e. on $\mathbb{R}^3$. Thus, (3.5) implies that $\overline{v} \neq 0$. For every $\phi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\left\langle \Phi'(\overline{v}), \phi \right\rangle = \lim_{n\to\infty} \left\langle \Phi'(v_n), \phi \right\rangle = 0.$$

Hence $\Phi'(\overline{v}) = 0$. This shows that $\overline{v} \in \mathcal{N}$ is a nontrivial solution of Problem (SP) and $\Phi(\overline{v}) \ge c$. It follows from (F2), (3.6) and Fatou's lemma that

$$\begin{aligned} c \ge c_* &= \lim_{n\to\infty}\left[\Phi(v_n) - \frac{1}{4}\left\langle \Phi'(v_n), v_n \right\rangle\right] \\ &= \lim_{n\to\infty}\left\{\frac{1-\theta_0}{4}\|v_n\|^2 + \frac{\theta_0}{4}\|v_n\|^2_{D^{1,2}} + \int_{\mathbb{R}^3}\left[\frac{1}{4}f(x,v_n)v_n - F(x,v_n) + \frac{\theta_0 V(x)}{4}v_n^2\right]\mathrm{d}x\right\} \\ &\ge \frac{1}{4}\liminf_{n\to\infty}\left[(1-\theta_0)\|v_n\|^2 + \theta_0\|v_n\|^2_{D^{1,2}}\right] \\ &\quad + \liminf_{n\to\infty}\int_{\mathbb{R}^3}\left[\frac{1}{4}f(x,v_n)v_n - F(x,v_n) + \frac{\theta_0 V(x)}{4}v_n^2\right]\mathrm{d}x \\ &\ge \frac{1}{4}\|\overline{v}\|^2 + \int_{\mathbb{R}^3}\left[\frac{1}{4}f(x,\overline{v})\overline{v} - F(x,\overline{v})\right]\mathrm{d}x \\ &= \Phi(\overline{v}) - \frac{1}{4}\left\langle \Phi'(\overline{v}), \overline{v} \right\rangle = \Phi(\overline{v}). \end{aligned}$$

This shows that $\Phi(\overline{v}) \le c$ and so $\Phi(\overline{v}) = c = \inf_{\mathcal{N}} \Phi > 0$. □

4. The asymptotically periodic case

In this section, we have $V(x) = V_0(x) + V_1(x)$, $K(x) = K_0(x) + K_1(x)$ and $f(x,u) = f_0(x,u) + f_1(x,u)$. Define functional Φ_0 as follows:

$$\begin{aligned}\Phi_0(u) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|\nabla u|^2 + V_0(x)u^2\right)\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K_0(x)K_0(y)}{|x-y|}u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y \\ - \int_{\mathbb{R}^3} F_0(x,u)\,\mathrm{d}x, \quad u \in E,\end{aligned} \tag{4.1}$$

where $F_0(x,u) := \int_0^u f_0(x,s)\,ds$. Then (V2), (K2), (F0) and (F1′) imply that $\Phi_0 \in \mathcal{C}^1(E,\mathbb{R})$ and

$$\begin{aligned}\left\langle \Phi_0'(u), v\right\rangle = \int_{\mathbb{R}^3}(\nabla u\nabla v + V_0(x)uv)\,\mathrm{d}x + \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K_0(x)K_0(y)}{|x-y|}u^2(x)u^2(y)\,\mathrm{d}x\mathrm{d}y \\ - \int_{\mathbb{R}^3} f_0(x,u)v\,\mathrm{d}x, \quad u,v \in E.\end{aligned} \tag{4.2}$$

Lemma 4.1. *Under assumptions* (V0), (V2), (K0), (K2), (F0) *and* (F1′), *if* $u_n \rightharpoonup 0$ *in* E, *then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} V_1(x)u_n^2\,\mathrm{d}x = 0, \quad \lim_{n\to\infty}\int_{\mathbb{R}^3} V_1(x)u_n v\,\mathrm{d}x = 0, \quad \forall\, v \in E, \tag{4.3}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} F_1(x,u_n)\,\mathrm{d}x = 0, \quad \lim_{n\to\infty}\int_{\mathbb{R}^3} f_1(x,u_n)v\,\mathrm{d}x = 0, \quad \forall\, v \in E, \tag{4.4}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3} K_1(x)\phi_{u_n}(x)u_n^2\,\mathrm{d}x = 0, \quad \lim_{n\to\infty}\int_{\mathbb{R}^3} K_1(x)\phi_{u_n}(x)u_n v\,\mathrm{d}x = 0, \quad \forall\, v \in E, \tag{4.5}$$

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K_1(x)K_1(y)}{|x-y|}u_n^2(x)u_n^2(y)\,\mathrm{d}x\mathrm{d}y = 0 \tag{4.6}$$

and

$$\lim_{n\to\infty}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K_1(x)K_1(y)}{|x-y|}u_n(x)v(x)u_n^2(y)\,\mathrm{d}x\mathrm{d}y = 0, \quad \forall\, v \in E. \tag{4.7}$$

Proof. It follows from the fact $u_n \rightharpoonup 0$ in E that $\{\|u_n\|\}$ is bounded, $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$, $2 \le s < 6$ and $u_n(x) \to 0$ a.e. on $\mathbb{R}^3$. For any $\varepsilon > 0$, by (V2), there exists $R_\varepsilon > 0$ such that $|V_1(x)| \le \varepsilon$ for $|x| \ge R_\varepsilon$. Hence,

$$\begin{aligned}\int_{\mathbb{R}^3}|V_1(x)|\,u_n^2\,\mathrm{d}x &= \int_{B_{R_\varepsilon}(0)}|V_1(x)|\,u_n^2\,\mathrm{d}x + \int_{\mathbb{R}^3\setminus B_{R_\varepsilon}(0)}|V_1(x)|\,u_n^2\,\mathrm{d}x \\ &\le \sup_{x\in\mathbb{R}^3}|V_1(x)|\int_{B_{R_\varepsilon}(0)}u_n^2\,\mathrm{d}x + \varepsilon\int_{\mathbb{R}^3\setminus B_{R_\varepsilon}(0)}u_n^2\,\mathrm{d}x \\ &\le o(1) + \varepsilon\,\|u_n\|_2^2 \\ &\le o(1) + C_1\varepsilon.\end{aligned} \tag{4.8}$$

For any $v \in E$, it follows that

$$\int_{\mathbb{R}^3} |V_1(x)| |u_n v| \, \mathrm{d}x \le \left[\int_{\mathbb{R}^3} |V_1(x)| \, u_n^2 \, \mathrm{d}x \int_{\mathbb{R}^3} |V_1(x)| \, v^2 \, \mathrm{d}x \right]^{1/2} \le o(1) + C_2 \varepsilon. \tag{4.9}$$

Since $\varepsilon > 0$ is arbitrary, then (4.8) and (4.9) imply that (4.3) holds. Similarly, by (F1′), one can prove that (4.4) holds also. From (K0), (2.1) and (2.4), we have

$$\begin{aligned} \int_{\mathbb{R}^3} |K_1(x)| \, \phi_{u_n}(x) u_n^2 \, \mathrm{d}x &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|K_1(x)| \, K(y)}{|x-y|} u_n^2(x) u_n^2(y) \, \mathrm{d}x\mathrm{d}y \\ &\le K_\infty \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|K_1(x)| \, u_n^2(x) u_n^2(y)}{|x-y|} \, \mathrm{d}x\mathrm{d}y \\ &\le \mathcal{C}_1 K_\infty \left[\int_{\mathbb{R}^3} |K_1(x)|^{6/5} \, |u_n(x)|^{12/5} \, \mathrm{d}x \right]^{5/6} \|u_n\|_{12/5}^2 , \end{aligned} \tag{4.10}$$

where $\mathcal{C}_1 = 8\sqrt[3]{2}/3\sqrt[3]{\pi}$, and for any $v \in E$

$$\begin{aligned} \int_{\mathbb{R}^3} |K_1(x) \phi_{u_n}(x) u_n v| \, \mathrm{d}x &= \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|K_1(x)| \, K(y)}{|x-y|} \, |u_n(x) v(x)| \, u_n^2(y) \, \mathrm{d}x\mathrm{d}y \\ &\le K_\infty \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|K_1(x) u_n(x) v(x)| \, u_n^2(y)}{|x-y|} \, \mathrm{d}x\mathrm{d}y \\ &\le \mathcal{C}_1 K_\infty \left[\int_{\mathbb{R}^3} |K_1(x) u_n(x) v(x)|^{6/5} \, \mathrm{d}x \right]^{5/6} \|u_n\|_{12/5}^2 \\ &\le \mathcal{C}_1 K_\infty \left[\int_{\mathbb{R}^3} |K_1(x) u_n(x)|^{12/5} \, \mathrm{d}x \right]^{5/12} \|v\|_{12/5} \, \|u_n\|_{12/5}^2 . \end{aligned} \tag{4.11}$$

Since $\lim_{|x|\to\infty} |K_1(x)| = 0$, similar to the proof of (4.8), it follows from (4.10) and (4.11) that (4.5) holds. Similarly, by (2.4), we have

$$\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|K_1(x) K_1(y)|}{|x-y|} u_n^2(x) u_n^2(y) \, \mathrm{d}x\mathrm{d}y \le \mathcal{C}_1 \left[\int_{\mathbb{R}^3} |K_1(x)|^{6/5} \, |u_n(x)|^{12/5} \, \mathrm{d}x \right]^{5/3} \tag{4.12}$$

and for any $v \in E$

$$\begin{aligned} &\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|K_1(x) K_1(y)|}{|x-y|} \, |u_n(x) v(x)| \, u_n^2(y) \, \mathrm{d}x\mathrm{d}y \\ &\le \mathcal{C}_1 \left[\int_{\mathbb{R}^3} |K_1(x) u_n(x) v(x)|^{6/5} \, \mathrm{d}x \right]^{5/6} \left[\int_{\mathbb{R}^3} |K_1(x)|^{6/5} \, |u_n(x)|^{12/5} \, \mathrm{d}x \right]^{5/6} \\ &\le \mathcal{C}_1 \left[\int_{\mathbb{R}^3} |K_1(x) u_n(x)|^{12/5} \, \mathrm{d}x \right]^{5/12} \left[\int_{\mathbb{R}^3} |K_1(x)|^{6/5} \, |u_n(x)|^{12/5} \, \mathrm{d}x \right]^{5/6} \|v\|_{12/5} . \end{aligned} \tag{4.13}$$

Since $\lim_{|x|\to\infty} |K_1(x)| = 0$, similar to the proof of (4.8), it follows from (4.12) and (4.13) that (4.6) and (4.7) hold. □

Remark 4.2. If the functions $V_1(x)$, $K_1(x)$ and $tf_1(x,t)$ are sign-changing, the conclusions in Lemma 4.1 still hold.

Proof of Theorem 1.4. Lemma 2.5 implies the existence of a sequence $\{u_n\} \subset E$ satisfying (2.19), then

$$\Phi(u_n) \to c_*, \quad \langle \Phi'(u_n), u_n \rangle \to 0.$$

By Lemma 2.6, $\{u_n\}$ is bounded in E. Passing to a subsequence, we have $u_n \rightharpoonup \overline{u}$ in E and $u_n(x) \to \overline{u}(x)$ a.e. on $\mathbb{R}^3$. There are two possible cases: (i) $\overline{u} = 0$; (ii) $\overline{u} \neq 0$.

Case (i): $\overline{u} = 0$. Then $u_n \rightharpoonup 0$ in E, and so $u_n \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$, $2 \le s < 6$ and $u_n(x) \to 0$ a.e. on $\mathbb{R}^3$. Note that

$$\|u\|^2 = \int_{\mathbb{R}^3} \left(|\nabla u|^2 + V_0(x)u^2\right) \mathrm{d}x + \int_{\mathbb{R}^3} V_1(x)u^2 \,\mathrm{d}x, \quad u \in E, \tag{4.14}$$

$$\begin{aligned}\Phi_0(u) = \Phi(u) &- \frac{1}{2}\int_{\mathbb{R}^3} V_1(x)u^2 \,\mathrm{d}x - \frac{1}{2}\int_{\mathbb{R}^3} K_1(x)\phi_u(x)u^2 \,\mathrm{d}x \\ &+ \frac{1}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_1(x)K_1(y)}{|x-y|} u^2(x)u^2(y) \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} F_1(x,u)\,\mathrm{d}x\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\langle \Phi_0'(u), v\rangle = \langle \Phi'(u), v\rangle &- \int_{\mathbb{R}^3} V_1(x)uv \,\mathrm{d}x - 2\int_{\mathbb{R}^3} K_1(x)\phi_u(x)uv \,\mathrm{d}x \\ &+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{K_1(x)K_1(y)}{|x-y|} u(x)v(x)u^2(y) \,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3} f_1(x,u)v \,\mathrm{d}x.\end{aligned} \tag{4.16}$$

From (2.19), (4.3)–(4.7), (4.15) and (4.16), one has

$$\Phi_0(u_n) \to c_*, \quad \|\Phi_0'(u_n)\|(1 + \|u_n\|) \to 0. \tag{4.17}$$

Similar to the proof of (3.4), we may prove that there exists $k_n \in \mathbb{Z}^3$, going if necessary to a subsequence, such that

$$\int_{B_2(k_n)} |u_n|^2 \,\mathrm{d}x > \frac{\delta}{2} > 0.$$

Let us define $v_n(x) = u_n(x + k_n)$ so that

$$\int_{B_2(0)} |v_n|^2 \,\mathrm{d}x > \frac{\delta}{2}. \tag{4.18}$$

Since $V_0(x)$, $K_0(x)$ and $f_0(x,u)$ are periodic on x, we have

$$\Phi_0(v_n) \to c_* \in (0, c], \quad \|\Phi_0'(v_n)\|(1 + \|v_n\|) \to 0. \tag{4.19}$$

Passing to a subsequence, we have $v_n \rightharpoonup \overline{v}$ in E, $v_n \to \overline{v}$ in $L^s_{\mathrm{loc}}(\mathbb{R}^3)$, $2 \le s < 6$ and $v_n(x) \to \overline{v}(x)$ a.e. on $\mathbb{R}^3$. Thus, (4.18) implies that $\overline{v} \neq 0$. For every $\phi \in C_0^\infty(\mathbb{R}^3)$, we have

$$\langle \Phi_0'(\overline{v}), \phi\rangle = \lim_{n\to\infty} \langle \Phi_0'(v_n), \phi\rangle = 0.$$

Hence $\Phi_0'(\overline{v}) = 0$. For any $x \in \mathbb{R}^3$, $t \geq 0$, $\tau \neq 0$, (1.9) yields

$$\begin{aligned} &\frac{1-t^4}{4}\tau f_0(x,\tau) + F_0(x,t\tau) - F_0(x,\tau) + \frac{V_0(x)}{4}(1-t^2)^2\tau^2 \\ &= \int_t^1 \left[\frac{f_0(x,\tau)}{\tau^3} - \frac{f_0(x,s\tau)}{(s\tau)^3} + V_0(x)\frac{(1-s^2)}{(s\tau)^2}\right] s^3\tau^4 \, \mathrm{d}s \geq 0. \end{aligned} \tag{4.20}$$

Observe that (4.20) with $t = 0$ yields

$$\frac{1}{4}f_0(x,\tau)\tau - F_0(x,\tau) + \frac{V_0(x)}{4}\tau^2 \geq 0, \quad \forall\, x \in \mathbb{R}^3,\ \tau \in \mathbb{R}, \tag{4.21}$$

then it follows from (F1′), (4.21) and Fatou's lemma that

$$\begin{aligned} c \geq c_* &= \lim_{n\to\infty}\left[\Phi_0(v_n) - \frac{1}{4}\langle \Phi_0'(v_n), v_n\rangle\right] \\ &= \lim_{n\to\infty}\left\{\frac{1}{4}\|v_n\|_{D^{1,2}}^2 + \int_{\mathbb{R}^3}\left[\frac{1}{4}f_0(x,v_n)v_n - F_0(x,v_n) + \frac{V_0(x)}{4}v_n^2\right]\mathrm{d}x\right\} \\ &\geq \frac{1}{4}\liminf_{n\to\infty}\|v_n\|_{D^{1,2}}^2 + \liminf_{n\to\infty}\int_{\mathbb{R}^3}\left[\frac{1}{4}f_0(x,v_n)v_n - F_0(x,v_n) + \frac{V_0(x)}{4}v_n^2\right]\mathrm{d}x \\ &\geq \frac{1}{4}\int_{\mathbb{R}^3}\left(|\nabla \overline{v}|^2 + V_0(x)\overline{v}^2\right)\mathrm{d}x + \int_{\mathbb{R}^3}\left[\frac{1}{4}f_0(x,\overline{v})\overline{v} - F_0(x,\overline{v})\right]\mathrm{d}x \\ &= \Phi_0(\overline{v}) - \frac{1}{4}\langle \Phi_0'(\overline{v}), \overline{v}\rangle \\ &= \Phi_0(\overline{v}). \end{aligned}$$

This shows that $\Phi_0(\overline{v}) \leq c$. Since $\overline{v} \neq 0$, it follows from Lemma 2.3 that there exists $t_0 = t(\overline{v}) > 0$ such that $t_0\overline{v} \in \mathcal{N}$, and so $\Phi(t_0\overline{v}) \geq c$. Now, we prove that $\Phi(t_0\overline{v}) = c$. Arguing by indirectly, we assume that $\Phi(t_0\overline{v}) > c$. Then from (4.1), (4.2), (4.20), (V2), (K0), (K2) and (F1′), we have

$$\begin{aligned} c \geq \Phi_0(\overline{v}) &= \Phi_0(t_0\overline{v}) + \frac{1-t_0^4}{4}\langle \Phi_0'(\overline{v}), \overline{v}\rangle + \frac{(1-t_0^2)^2}{4}\int_{\mathbb{R}^3}|\nabla\overline{v}|^2\,\mathrm{d}x \\ &\quad + \int_{\mathbb{R}^3}\left[\frac{1-t_0^4}{4}f_0(x,\overline{v})\overline{v} + F_0(x,t_0\overline{v}) - F_0(x,\overline{v}) + \frac{V_0(x)}{4}(1-t_0^2)^2\overline{v}^2\right]\mathrm{d}x \\ &\geq \Phi_0(t_0\overline{v}) = \Phi(t_0\overline{v}) - \frac{t_0^2}{2}\int_{\mathbb{R}^3}V_1(x)\overline{v}^2\,\mathrm{d}x - \frac{t_0^4}{2}\int_{\mathbb{R}^3}K_1(x)\phi_{\overline{v}}(x)\overline{v}^2\,\mathrm{d}x \\ &\qquad + \frac{t_0^4}{4}\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{K_1(x)K_1(y)}{|x-y|}\overline{v}^2(x)\overline{v}^2(y)\,\mathrm{d}x\mathrm{d}y + \int_{\mathbb{R}^3}F_1(x,t_0\overline{v})\,\mathrm{d}x \\ &\geq \Phi(t_0\overline{v}) - \frac{t_0^2}{2}\int_{\mathbb{R}^3}V_1(x)\overline{v}^2\,\mathrm{d}x - \frac{t_0^4}{2}\int_{\mathbb{R}^3}K_1(x)\phi_{\overline{v}}(x)\overline{v}^2\,\mathrm{d}x + \int_{\mathbb{R}^3}F_1(x,t_0\overline{v})\,\mathrm{d}x \\ &\geq \Phi(t_0\overline{v}) > c. \end{aligned}$$

This contradiction shows that $\Phi(t_0\overline{v}) = c$.

Let $u_0 = t_0\overline{v}$. Then $u_0 \in \mathcal{N}$ and $\Phi(u_0) = c$. In view of Lemma 2.7, $\Phi'(u_0) = 0$. This shows that $u_0 \in E$ is a solution for (SP) with $\Phi(u_0) = \inf_{\mathcal{N}} \Phi > 0$.

Case (ii): $\overline{u} \neq 0$. By the same fashion as the last part of the proof of Theorem 1.3, we can prove that $\Phi'(\overline{u}) = 0$ and $\Phi(\overline{u}) = c = \inf_{\mathcal{N}} \Phi$. This shows that $\overline{u} \in E$ is a solution for System (SP) with $\Phi(\overline{u}) = \inf_{\mathcal{N}} \Phi > 0$. □

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