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Homogenization and Electronic Polarization Effects in Dielectric Materials

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Abstract. This paper is devoted to the electronic polarization effect of dielectric materials in plasma physics. Using the two-scale convergence method introduced by G. Nguetseng and G. Allaire, we study the electronic polarization effect induced by two-scale homogenization. The microscopic properties of electrons in dielectric materials are characterized by Vlasov-Poisson system. The homogenized equations describing the mean behaviors of the microscopic equations are obtained. We also derive the modified Gauss law through the polarization effect. From the homogenized Poisson equation, the dielectric function is also acquired.

1. Introduction

The purpose of this paper is devoted to studying the electronic polarization effect by the theory of homogenization. The electronic polarization (or optical polarization) has been applied in physics or engineering like as optical polarization modulation, fiber optic polarization, semiconductor optical amplifier and so on. We will give a rigorous mathematical analysis of electronic polarization effect, which can be used as the foundations for the further improvement.

Electronic polarization can be thought of as the charge redistribution in a material caused by an external electric field. If an electric field is applied to a medium made up of a large number of atoms or molecules, which causes deformation or translation of the originally symmetrical distribution of the electron clouds of atoms or molecules. This is essentially the displacement of the outer electron clouds with respect to the inner positive atomic cores. From the macroscopic point of view, the charge density is changed and represents the polarization behavior.

In order to investigate this phenomenon, from the theory of plasma that states in [18, 19, 23] also to see [29, 30], we consider the perfect dielectric material such that inside

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the material no mobile charge carriers (electrons or ions) are present. We also suppose that the electrons are initial in a periodically modulated potential $V_{\text{in}}(\boldsymbol{x})$ in reference to dielectric materials, where $x \in \Omega$, and Ω is a periodic bounded domain in $\mathbb{R}^3_x$. This is owing to the potential that electrons see inside a crystal will be periodic in space since the atoms (or ions) are periodically arranged in space. When an external electric field is applied to the medium, the modulated potential $V_{\text{in}}(\boldsymbol{x})$ and the external electric field induce a highly oscillating distribution function of electrons inside a crystal in dielectric materials. And the averaging polarization effect hence occurs. We note that the topics are fundamentally important not only for dielectric materials, but also for electronic materials whose properties may be directly or indirectly associated with some of the dielectric phenomena. The topic can be characterized by the uniform and unmagnetized Vlasov-Poisson system

$$\partial_t F^\epsilon(\boldsymbol{x},t,\boldsymbol{v}) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}} F^\epsilon(\boldsymbol{x},t,\boldsymbol{v}) - \frac{e}{m}\left(E(\boldsymbol{x},t) + \nabla_{\boldsymbol{x}} V^\epsilon(\boldsymbol{x},t)\right)\cdot\nabla_{\boldsymbol{v}} F^\epsilon(\boldsymbol{x},t,\boldsymbol{v}) = 0, \tag{1.1}$$

$$\nabla_{\boldsymbol{x}}\cdot\left(E(\boldsymbol{x},t) + \nabla_{\boldsymbol{x}} V^\epsilon(\boldsymbol{x},t)\right) = -4\pi e\rho^\epsilon(\boldsymbol{x},t), \tag{1.2}$$

where $F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})$ is the velocity distribution function of electrons at location $\boldsymbol{x}$ and time t, traveling with velocity $\boldsymbol{v} = (v_1,v_2,v_3)$, with the initial $F_0(\boldsymbol{v})$ in the equilibrium. The macroscopic density $\rho^\epsilon(\boldsymbol{x},t)$ is defined by

$$\rho^\epsilon(\boldsymbol{x},t) = \int_{\mathbb{R}^3_v} F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v} - n_0$$

where

$$n_0 = \int_{\mathbb{R}^3_v} F_0(\boldsymbol{v})\,d\boldsymbol{v}$$

is the equilibrium number density of electrons which is the same as that of the ions. Note that

$$E(\boldsymbol{x},t) = -\nabla_{\boldsymbol{x}}\Phi(\boldsymbol{x},t)$$

is the external electric field corresponding to the electrostatic potential Φ. The supplementary of the initial conditions are given by $V^\epsilon(\boldsymbol{x},0) = V_{\text{in}}(\boldsymbol{x})$ and $\Phi(\boldsymbol{x},0)$ which are considered in $H^1(\Omega)$. Also the initial potential $V_{\text{in}}(\boldsymbol{x})$ is assumed to be harmonic, i.e., $\triangle_{\boldsymbol{x}} V_{\text{in}}(\boldsymbol{x}) = 0$, causing from the charges equilibrium in the dielectric material. The ions are assumed to be infinitely massive, that is, ion motion will be neglected. Thus the Vlasov-Poisson system (1.1)–(1.2) describes the nonlinear plasma waves on a uniform ion background. For other relative studies about Vlasov-Poisson system we refer to [5, 8, 13].

The homogenization theory studies the behavior of the associated solution sequence $\{F^\epsilon\}_\epsilon$ as $\epsilon \to 0$ and asks whether average behavior can be discerned from (1.1). We note that homogenization sometimes changes the type of the equation. Indeed, in some situations, the limit of a sequence of partial differential operators is not a partial differential operator. Tartar first investigated these problems [25–28], that there is an integral term (the nonlocal effect or memory effect) which appears in homogenization to arise from an equation with pure differential structure we also refer to [1, 3, 4, 10]. That is the homogenization process results in memory or nonlocal effects described by integro-differential equations we mention to [14–17].

To obtain a more accurate description of the limiting behavior of (1.1), it is more efficient to apply the two-scale convergence method introduced by G. Nguetseng [21, 22] and G. Allaire [2]. The basic idea is to consider the behavior of the homogenization process not only from the macroscopic point of view, but also from the microscopic one, by introducing an additional microscopic variable. The various asymptotic limits of solutions to the Vlasov-Poission equation in the presence of a strong external magnetic field is discussed in [11] also to see [9, 13].

The homogenization can be carried out formally by the method of asymptotic expansions, which provides us a great variety of models and equations posed in a periodic domain, given respectively by Bensoussan-Lions-Papanicolaou [6], E. Sanchez-Palencia [24] and references therein. The mathematical justification can be found in Section 3. The starting point is to look for a formal asymptotic expansion of F^ϵ, which is serving as a function of ϵ for $\epsilon \to 0$ and the heuristic device is to consider that F^ϵ in the equation (1.1) having two-scale expansions

$$F^\epsilon(\boldsymbol{x}, t, \boldsymbol{v}) = F_0(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) + \epsilon F_1(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) + \epsilon^2 F_2(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) + \cdots, \tag{1.3}$$

where $\boldsymbol{y} = \boldsymbol{x}/\epsilon$ and F_i, $i = 0, 1, 2, \ldots$, are Y-periodic functions of the fast variable $\boldsymbol{y}$. Furthermore, we also assume the zero mean conditions

$$\widetilde{F_i} = 0, \quad i = 1, 2, 3, \ldots,$$

where $\widetilde{F}$ denotes the average value of F over one period Y:

$$\widetilde{F} \equiv \frac{1}{|Y|} \int_Y F(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v})\, d\boldsymbol{y}.$$

Plugging (1.3) into the equations (1.1) and (1.2) gives

$$\begin{aligned} &\partial_t(F_0 + \epsilon F_1 + \cdots) + \boldsymbol{v} \cdot \left(\nabla_{\boldsymbol{x}} + \frac{1}{\epsilon}\nabla_{\boldsymbol{y}}\right)(F_0 + \epsilon F_1 + \cdots) \\ &- \frac{e}{m}\left(\left(\nabla_{\boldsymbol{x}} + \frac{1}{\epsilon}\nabla_{\boldsymbol{y}}\right)V(\boldsymbol{y}, t) + E(\boldsymbol{x}, t)\right) \cdot \nabla_{\boldsymbol{v}}(F_0 + \epsilon F_1 + \cdots) = 0 \end{aligned} \tag{1.4}$$

and

$$\begin{aligned} &\left(\nabla_{\boldsymbol{x}} + \frac{1}{\epsilon}\nabla_{\boldsymbol{y}}\right) \cdot \left(E(\boldsymbol{x}, t) + \left(\nabla_{\boldsymbol{x}} + \frac{1}{\epsilon}\nabla_{\boldsymbol{y}}\right)V(\boldsymbol{y}, t)\right) \\ &= -4\pi e \left(\int_{\mathbb{R}^3_v} (F_0 + \epsilon F_1 + \cdots)\, d\boldsymbol{v} - n_0\right). \end{aligned} \tag{1.5}$$

To gather the order ϵ, from the equation (1.4) and (1.5), we get the following equations respectively

$$\epsilon^{-1}: \quad \boldsymbol{v} \cdot \nabla_{\boldsymbol{y}} F_0(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) - \frac{e}{m} \nabla_{\boldsymbol{y}} V(\boldsymbol{y}, t) \cdot \nabla_{\boldsymbol{v}} F_0(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) = 0, \tag{1.6}$$

$$\begin{aligned} \epsilon^0: \quad & \partial_t F_0(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) + \boldsymbol{v} \cdot (\nabla_{\boldsymbol{x}} F_0(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) + \nabla_{\boldsymbol{y}} F_1(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v})) \\ & - \frac{e}{m} E(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{v}} F_0(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) - \frac{e}{m} \nabla_{\boldsymbol{y}} V(\boldsymbol{y}, t) \cdot \nabla_{\boldsymbol{v}} F_1(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) = 0, \end{aligned} \tag{1.7}$$

$$\epsilon^{-2}: \quad \triangle_{\boldsymbol{y}} V(\boldsymbol{y}, t) = 0, \tag{1.8}$$

$$\epsilon^{-1}: \quad \nabla_{\boldsymbol{y}} \cdot E(\boldsymbol{x}, t) + \nabla_{\boldsymbol{x}} \cdot \nabla_{\boldsymbol{y}} V(\boldsymbol{y}, t) = 0, \tag{1.9}$$

$$\epsilon^0: \quad \nabla_{\boldsymbol{x}} \cdot E(\boldsymbol{x}, t) = -4\pi e \left(\int_{\mathbb{R}^3_v} F_0(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v})\, d\boldsymbol{v} - n_0 \right). \tag{1.10}$$

Taking averaging in $\boldsymbol{y}$, the equations (1.7) and (1.10) express the homogenized system, with the restrained equations (1.6), (1.8) and (1.9), as follows:

$$\partial_t \widetilde{F}_0(\boldsymbol{x}, t, \boldsymbol{v}) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \widetilde{F}_0(\boldsymbol{x}, t, \boldsymbol{v}) - \frac{e}{m} E(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{v}} \widetilde{F}_0(\boldsymbol{x}, t, \boldsymbol{v}) = 0,$$

$$\nabla_{\boldsymbol{x}} \cdot E(\boldsymbol{x}, t) = -4\pi e \left(\int_{\mathbb{R}^3_v} \widetilde{F}_0(\boldsymbol{x}, t, \boldsymbol{v})\, d\boldsymbol{v} - n_0 \right).$$

Notice that if we define $\mathcal{P}$ such that

$$4\pi e \left(\int_{\mathbb{R}^3_v} \widetilde{F}_0(\boldsymbol{x}, t, \boldsymbol{v})\, d\boldsymbol{v} - n_0 \right) \equiv \nabla_{\boldsymbol{x}} \cdot \mathcal{P}(\boldsymbol{x}, t),$$

then the Gauss law becomes

$$\nabla_{\boldsymbol{x}} \cdot (E(\boldsymbol{x}, t) + \mathcal{P}(\boldsymbol{x}, t)) = 0$$

where $\mathcal{P}(\boldsymbol{x}, t)$ is the polarized electric field induced by the averaging effect of the interactions of the external electric field and the internal potential of the materials. From above sketchy expressions, we will derive the following theorems.

Theorem 1.1. *Let $F^\epsilon(\boldsymbol{x}, \boldsymbol{v}, 0) = F_0(\boldsymbol{v}) > 0$, $F^\epsilon(\boldsymbol{x}, t, \boldsymbol{v}) \to 0$ as $|\boldsymbol{v}| \to \infty$ and Ω be a bounded periodic domain in $\mathbb{R}^3_x$. Assume that the initial distribution function satisfies $F_0(\boldsymbol{v})$ and $|\boldsymbol{v}|^k F_0(\boldsymbol{v})$, $k \geq 3$, being an integer, are bounded in $L^1 \cap L^\infty(\mathbb{R}^3_v)$, and $\Phi(\boldsymbol{x}, 0) = \Phi_{\text{in}}(\boldsymbol{x}) \in H^1(\Omega)$, the sequence $\{(F^\epsilon, V^\epsilon)\}_\epsilon$ of solutions of (1.1)–(1.2) converges in the two-scale limit to $(\overline{F}, V)$ solution of the system*

$$\partial_t \overline{F}(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} \overline{F}(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) - \frac{e}{m} E(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{v}} \overline{F}(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) = 0,$$

$$\nabla_{\boldsymbol{x}} \cdot E(\boldsymbol{x}, t) = -4\pi e \left(\int_{\mathbb{R}^3_v} \overline{F}(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v})\, d\boldsymbol{v} - n_0 \right),$$

$$\triangle_{\boldsymbol{y}} V(\boldsymbol{y}, t) = 0,$$

$$\boldsymbol{v} \cdot \nabla_{\boldsymbol{y}} \overline{F}(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) - \frac{e}{m} \nabla_{\boldsymbol{y}} V(\boldsymbol{y}, t) \cdot \nabla_{\boldsymbol{v}} \overline{F}(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) = 0. \tag{1.11}$$

Theorem 1.2. *Under the same hypothesis of Theorem* 1.1*, there is a subsequence* $\{F^\epsilon(\boldsymbol{x}, t, \boldsymbol{v})\}_\epsilon$*, still denoted by* $\{F^\epsilon(\boldsymbol{x}, t, \boldsymbol{v})\}_\epsilon$*, of the solutions of the Vlasov-Poisson system* (1.1)–(1.2) *such that* $F^\epsilon(\boldsymbol{x}, t, \boldsymbol{v})$ *converges weakly* $*$ *in* $L^\infty(0, T; L^2(\Omega \times \mathbb{R}^3_v))$ *to the equilibrium distribution function* $F(\boldsymbol{x}, t, \boldsymbol{v})$ *solving the homogenized Vlasov-Poisson system*

$$\partial_t F(\boldsymbol{x}, t, \boldsymbol{v}) + \boldsymbol{v} \cdot \nabla_{\boldsymbol{x}} F(\boldsymbol{x}, t, \boldsymbol{v}) - \frac{e}{m} E(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{v}} F(\boldsymbol{x}, t, \boldsymbol{v}) = 0, \tag{1.12}$$

$$\nabla_{\boldsymbol{x}} \cdot E(\boldsymbol{x}, t) = -4\pi e \left(\int_{\mathbb{R}^3_v} F(\boldsymbol{x}, t, \boldsymbol{v})\, d\boldsymbol{v} - n_0 \right) \tag{1.13}$$

or

$$\nabla_{\boldsymbol{x}} \cdot (E(\boldsymbol{x}, t) + \mathcal{P}(\boldsymbol{x}, t)) = 0,$$

where

$$\overline{F}(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) = \overline{F}(\eta(\boldsymbol{y}, \boldsymbol{v}), \boldsymbol{x}, t), \quad F(\boldsymbol{x}, t, \boldsymbol{v}) = \int_Y \overline{F}(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v})\, d\boldsymbol{y}$$

and $\eta(\boldsymbol{y}, \boldsymbol{v})$ *is the characteristic curve of the equation* (1.11)*. The polarized electric field is given by*

$$\nabla_{\boldsymbol{x}} \cdot \mathcal{P}(\boldsymbol{x}, t) = 4\pi e \left(\int_{\mathbb{R}^3_v} F(\boldsymbol{x}, t, \boldsymbol{v})\, d\boldsymbol{v} - n_0 \right).$$

2. Basic a-priori estimates

The two-scale convergence was introduced by G. Nguetseng [21] and G. Allaire [2] as an efficient tool to study the homogenization problem. It is an alternative approach to the energy method of Tartar (see [7] and references therein). In particular, in applications there are homogenization problems where the solutions do not have classical limit and the weak limit cannot be viewed as a satisfactory approximation of the solution, the asymptotic behavior of the solution can be characterized by so-called *two-scale limit* (see [2,16,20,21] for detail and applications).

We denote by $C^\infty_\#(Y)$ the space of infinitely differentiable functions defined on $Y = [0, 1)^3$ and extended to $\mathbb{R}^3$ by Y-periodicity. For $p > 1$ and an open subset $\Omega \subset \mathbb{R}^3$, $L^p(\Omega; C^\infty_\#(Y))$ is the space of functions of $L^p(\Omega)$ with value in $C^\infty_\#(Y)$. A bounded sequence $\{u^\epsilon\}_\epsilon$ in $L^p(\Omega)$ is said to (weakly) two-scale converge to $u(\boldsymbol{x}, \boldsymbol{y}) \in L^p(\Omega \times Y)$ if and only if

$$\lim_{\epsilon \to 0} \int_\Omega u^\epsilon(\boldsymbol{x}) \psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\epsilon}\right) d\boldsymbol{x} = \int_\Omega \int_Y u(\boldsymbol{x}, \boldsymbol{y}) \psi(\boldsymbol{x}, \boldsymbol{y})\, d\boldsymbol{y} d\boldsymbol{x}$$

for any function $\psi(\boldsymbol{x}, \boldsymbol{y}) \in \mathcal{D}(\Omega; C^\infty_\#(Y))$ that is Y-periodic with respect to the second argument. This definition is justified by the following compactness theorem.

Theorem 2.1. *Let* $\psi(\boldsymbol{x}, \boldsymbol{x}/\epsilon)$ *be measurable in* Ω *and* $\psi(\boldsymbol{x}, \boldsymbol{y}) \in L^p(\Omega; C^\infty_\#(Y))$*,* $1 < p < \infty$ *then for* $\epsilon > 0$ *we have*

$$\left\| \psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\epsilon}\right) \right\|_{L^p(\Omega)} \leq \|\psi(\boldsymbol{x}, \boldsymbol{y})\|_{L^p(\Omega; C^\infty_\#(Y))} \equiv \left[\int_\Omega \sup_{\boldsymbol{y} \in Y} |\psi(\boldsymbol{x}, \boldsymbol{y})|^p\, d\boldsymbol{x} \right]^{1/p}.$$

Moreover, if $\psi(\boldsymbol{x},\boldsymbol{y}) \in L^p(\Omega; C^\infty_\#(Y))$ *then*

$$\lim_{\epsilon\to 0}\int_\Omega \psi^p\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon}\right)d\boldsymbol{x} = \int_\Omega\int_Y \psi^p(\boldsymbol{x},\boldsymbol{y})\,d\boldsymbol{y}d\boldsymbol{x}$$

and $\psi(\boldsymbol{x},\boldsymbol{x}/\epsilon)$ *two-scale converges to* $\psi(\boldsymbol{x},\boldsymbol{y})$*.*

The proof is similar to the L^2 case as given by Allaire in [2] with modification (see also [9, 15]). Therefore, the proof is omitted. We now focus our attention to derive the priori estimates that are available for the Vlasov equation. First of all, we notice that its solution $F^\epsilon(\boldsymbol{x},\boldsymbol{v},t)$ satisfies the following estimate.

Lemma 2.2. *Under assumptions* (1.1)–(1.2)*, there exists a constant* C *independent of* ϵ *such that the solution* F^ϵ *of the Vlasov equation* (1.1) *satisfies*

$$\|F^\epsilon\|_{L^\infty(0,T;L^2(\Omega\times\mathbb{R}^3_v))} \leq C. \tag{2.1}$$

Proof. Multiplying the Vlasov equation (1.1) by F^ϵ and integrating over $\Omega\times\mathbb{R}^3_v$ we obtain the following equality

$$\begin{aligned}&\frac{1}{2}\iint_{\Omega\times\mathbb{R}^3_v}\partial_t(F^\epsilon(\boldsymbol{x},t,\boldsymbol{v}))^2\,d\boldsymbol{x}d\boldsymbol{v} + \frac{1}{2}\iint_{\Omega\times\mathbb{R}^3_v}\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}(F^\epsilon(\boldsymbol{x},t,\boldsymbol{v}))^2\,d\boldsymbol{x}d\boldsymbol{v}\\ &-\frac{1}{2}\frac{e}{m}\iint_{\Omega\times\mathbb{R}^3_v}(\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)+E(\boldsymbol{x},t))\cdot\nabla_{\boldsymbol{v}}(F^\epsilon(\boldsymbol{x},t,\boldsymbol{v}))^2\,d\boldsymbol{x}d\boldsymbol{v} = 0.\end{aligned}$$

The second and third integrals vanish after integration by part. Hence, we get

$$\frac{d}{dt}\int_{\Omega\times\mathbb{R}^3_v}(F^\epsilon(\boldsymbol{x},t,\boldsymbol{v}))^2\,d\boldsymbol{x}d\boldsymbol{v} = 0.$$

The L^2 norm of F^ϵ is conserved and (2.1) follows immediately because $F_0 \in L^2(\mathbb{R}^3_v)$. This completes the proof. □

The homogenization of the Vlasov-Poisson equation relies on the macroscopic averages such as the density and current. The first equation of fluid theory is the continuity equation. Integrating equation (1.1) over $\mathbb{R}^3_v$, we get

$$\begin{aligned}&\partial_t\int_{\mathbb{R}^3_v}F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v} + \nabla_{\boldsymbol{x}}\cdot\int_{\mathbb{R}^3_v}\boldsymbol{v}F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}\\ &-\int_{\mathbb{R}^3_v}\frac{e}{m}(\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)+E(\boldsymbol{x},t))\cdot\nabla_{\boldsymbol{v}}F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v} = 0.\end{aligned}$$

Hence we have obtained the charge continuity equation

$$\partial_t\rho^\epsilon(\boldsymbol{x},t) + \nabla_{\boldsymbol{x}}\cdot J^\epsilon(\boldsymbol{x},t) = 0, \tag{2.2}$$

where ρ^ϵ is the macroscopic density

$$\rho^\epsilon(\boldsymbol{x},t) = \int_{\mathbb{R}^3_v} F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v} - n_0$$

and J^ϵ is the macroscopic current density

$$J^\epsilon(\boldsymbol{x},t) = \int_{\mathbb{R}^3_v} \boldsymbol{v}F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}.$$

Employing (2.2) and integrating by part we obtain

$$\begin{aligned}
&\int_\Omega J^\epsilon(\boldsymbol{x},t)\cdot(\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)+E(\boldsymbol{x},t))\,d\boldsymbol{x}\\
&=\int_\Omega J^\epsilon(\boldsymbol{x},t)\cdot(\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)-\nabla_{\boldsymbol{x}}\Phi(\boldsymbol{x},t))\,d\boldsymbol{x}\\
&=-\int_\Omega (\nabla_{\boldsymbol{x}}\cdot J^\epsilon(\boldsymbol{x},t))\,(V^\epsilon(\boldsymbol{x},t)-\Phi(\boldsymbol{x},t))\,d\boldsymbol{x}\\
&=\int_\Omega \partial_t\rho^\epsilon(\boldsymbol{x},t)\,(V^\epsilon(\boldsymbol{x},t)-\Phi(\boldsymbol{x},t))\,d\boldsymbol{x}.
\end{aligned}\tag{2.3}$$

On the other hand, the Poisson equation (1.2) yields

$$\begin{aligned}
&\int_\Omega \partial_t\rho^\epsilon(\boldsymbol{x},t)\,(V^\epsilon(\boldsymbol{x},t)-\Phi(\boldsymbol{x},t))\,d\boldsymbol{x}\\
&=\frac{-1}{4\pi e}\int_\Omega (\partial_t\,(\nabla_{\boldsymbol{x}}\cdot\nabla_{\boldsymbol{x}}(V^\epsilon(\boldsymbol{x},t)-\Phi(\boldsymbol{x},t))))\,(V^\epsilon(\boldsymbol{x},t)-\Phi(\boldsymbol{x},t))\,d\boldsymbol{x}\\
&=\frac{1}{8\pi e}\partial_t\int_\Omega \left|\nabla_{\boldsymbol{x}}\,(V^\epsilon(\boldsymbol{x},t)-\Phi(\boldsymbol{x},t))\right|^2 d\boldsymbol{x}.
\end{aligned}\tag{2.4}$$

Moreover, we multiply the Vlasov equation (1.1) by $|\boldsymbol{v}|^2$ and integrate over $\mathbb{R}^3_v\times\Omega$ to obtain

$$\begin{aligned}
&\partial_t\iint_{\mathbb{R}^3_v\times\Omega}|\boldsymbol{v}|^2\,F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}d\boldsymbol{x}\\
&-\frac{e}{m}\iint_{\mathbb{R}^3_v\times\Omega}(\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)+E(\boldsymbol{x},t))\cdot|\boldsymbol{v}|^2\,\nabla_{\boldsymbol{v}}F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}d\boldsymbol{x}=0.
\end{aligned}\tag{2.5}$$

After integration by parts, (2.5) becomes

$$\begin{aligned}
&\partial_t\iint_{\mathbb{R}^3_v\times\Omega}|\boldsymbol{v}|^2\,F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}d\boldsymbol{x}\\
&+\frac{2e}{m}\iint_{\mathbb{R}^3_v\times\Omega}(\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)+E(x,t))\cdot\boldsymbol{v}F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}d\boldsymbol{x}=0,
\end{aligned}$$

or

$$\partial_t\iint_{\mathbb{R}^3_v\times\Omega}|\boldsymbol{v}|^2\,F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}d\boldsymbol{x}+\frac{2e}{m}\int_\Omega(\nabla_{\boldsymbol{x}}V^\epsilon(x,t)+E(x,t))\cdot J^\epsilon(x,t)\,d\boldsymbol{x}=0.\tag{2.6}$$

Using relation (2.3), the equation (2.6) can be further be rewritten as

$$\partial_t \iint_{\mathbb{R}_v^3\times\Omega} |\boldsymbol{v}|^2 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v} d\boldsymbol{x} + \frac{2e}{m}\int_\Omega \partial_t \rho^\epsilon(\boldsymbol{x},t)\left(V^\epsilon(\boldsymbol{x},t)-\Phi(\boldsymbol{x},t)\right) d\boldsymbol{x} = 0.$$

Also by means of the equation (2.4), we get

$$\partial_t \iint_{\mathbb{R}_v^3\times\Omega} |\boldsymbol{v}|^2 F^\epsilon(\boldsymbol{x},t)\, d\boldsymbol{v} d\boldsymbol{x} + \frac{1}{4\pi m}\int_\Omega \partial_t \left|\nabla_{\boldsymbol{x}}\left(V^\epsilon(\boldsymbol{x},t)-\Phi(\boldsymbol{x},t)\right)\right|^2 d\boldsymbol{x} = 0,$$

or

$$\frac{d}{dt}\left(\iint_{\mathbb{R}_v^3\times\Omega} |\boldsymbol{v}|^2 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v} d\boldsymbol{x} + \frac{1}{4\pi m}\int_\Omega \left|\nabla_{\boldsymbol{x}} V^\epsilon(\boldsymbol{x},t)+E(\boldsymbol{x},t)\right|^2 d\boldsymbol{x}\right) = 0.$$

Thus we have proven the following lemma.

Lemma 2.3. *Suppose that $|\boldsymbol{v}|^2 F_0$ is bounded in $L^1(\mathbb{R}_v^3)$ and Φ_{in} is bounded in $H^1(\Omega)$ or equivalently E_{in} is bounded in $L^2(\Omega)$, then there is a constant C such that*

$$\left\| |\boldsymbol{v}|^2 F^\epsilon \right\|_{L^\infty(0,T;L^1(\Omega\times\mathbb{R}_v^3))} + \left\|\nabla_{\boldsymbol{x}} V^\epsilon(\boldsymbol{x},t)+E(\boldsymbol{x},t)\right\|_{L^\infty(0,T;L^2(\Omega))} \le C.$$

Using the conservation of mass and the conservation of energy, we deduce that the current $J^\epsilon(\boldsymbol{x},t)=\int_{\mathbb{R}_v^3} \boldsymbol{v}F^\epsilon\, d\boldsymbol{v}$ is bounded in $L^\infty(0,T;L^1(\Omega))$. Indeed, we have

$$\int_\Omega |J^\epsilon(t)|\, d\boldsymbol{x} \le \left(\iint_{\mathbb{R}_v^3\times\Omega} |\boldsymbol{v}|^2 F^\epsilon\, d\boldsymbol{v} d\boldsymbol{x}\right)^{1/2}\left(\iint_{\mathbb{R}_v^3\times\Omega} F^\epsilon\, d\boldsymbol{v} d\boldsymbol{x}\right)^{1/2} \le C.$$

In order to obtain the L^2 bound of ρ^ϵ, we need the boundness of the third moment. For this purpose, we first prove the following inequality.

Lemma 2.4. *There is a constant C such that*

$$\int_{\mathbb{R}_v^3} |\boldsymbol{v}|^2 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v} \le C\left[\int_{\mathbb{R}_v^3} |\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v}\right]^{5/6}.$$

Proof. Let $R>0$. We write

$$\begin{aligned}\int_{\mathbb{R}_v^3} |\boldsymbol{v}|^2 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v} &= \int_{|\boldsymbol{v}|\le R} |\boldsymbol{v}|^2 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v} + \int_{|\boldsymbol{v}|>R} |\boldsymbol{v}|^2 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v}\\ &\le \frac{4\pi}{3}R^5 \left\|F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\right\|_{L^\infty(\Omega\times\mathbb{R}_v^3)} + \frac{1}{R}\int_{\mathbb{R}_v^3} |\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v}.\end{aligned}$$

Choosing R such that

$$R^6 = \int_{\mathbb{R}_v^3} |\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v},$$

we obtain the inequality

$$\int_{\mathbb{R}_v^3} |\boldsymbol{v}|^2 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v} \le C\left(\int_{\mathbb{R}_v^3} |\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v}\right)^{5/6},$$

where the constant C depends on $\|F^\epsilon\|_{L^\infty(\Omega\times\mathbb{R}_v^3)}$. This completes the proof. □

Lemma 2.5. *If* $|\boldsymbol{v}|^3 F_0^\epsilon(\boldsymbol{v}) \in L^1(\mathbb{R}_v^3)$ *then* $|\boldsymbol{v}|^3 F^\epsilon$ *is bounded in* $L^\infty(0,T;L^1(\Omega\times\mathbb{R}_v^3))$. *Furthermore, from the third moment,*

$$\nabla_{\boldsymbol{x}}\cdot(E(\boldsymbol{x},t)+\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)) = -4\pi e\rho^\epsilon(\boldsymbol{x},t) = -4\pi e\left(\int_{\mathbb{R}_v^3} F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v} - n_0\right)$$

is bounded in $L^\infty(0,T;L^2(\Omega))$.

Proof. It suffices to show that $\int_{\mathbb{R}_v^3} F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}$ is bounded in $L^\infty(0,T;L^2(\Omega))$. Let $R>0$ then we may write

$$\begin{aligned}\int_{\mathbb{R}_v^3}|F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})|\,d\boldsymbol{v} &= \int_{|\boldsymbol{v}|\le R}|F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})|\,d\boldsymbol{v} + \int_{|\boldsymbol{v}|>R}|F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})|\,d\boldsymbol{v}\\ &\le CR^3\,\|F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\|_{L^\infty(\Omega\times\mathbb{R}_v^3)} + \frac{1}{R^3}\int_{\mathbb{R}_v^3}|\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}.\end{aligned}$$

Choosing R such that

$$R^6 = \int_{\mathbb{R}_v^3}|\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v},$$

we obtain the inequality

$$\int_{\mathbb{R}_v^3}|F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})|\,d\boldsymbol{v} \le C\left(\int_{\mathbb{R}_v^3}|\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}\right)^{1/2}$$

for some constant C depending on $\|F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\|_{L^\infty(\Omega\times\mathbb{R}_v^3)}$. Therefore, we obtain

$$\left\|\int_{\mathbb{R}_v^3}|F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})|\,d\boldsymbol{v}\right\|_{L^2(\Omega)} \le C_1\left(\int_{\mathbb{R}_v^3\times\Omega}|\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}d\boldsymbol{x}\right)^{1/2}. \tag{2.7}$$

Therefore we will have the boundedness of the charge density $\rho(\boldsymbol{x},t)$ in $L^\infty(0,T;L^2(\Omega))$, if we can prove the third moment of the right-hand side of the inequality (2.7) is bounded. For this aim, by way of Poisson equation (1.2) and the equation (2.7) we have the inequality

$$\|\nabla_{\boldsymbol{x}}\cdot(E(\boldsymbol{x},t)+\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t))\|_{L^2(\Omega)} \le C_2 + C_1\left(\int_{\mathbb{R}_v^3\times\Omega}|\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}d\boldsymbol{x}\right)^{1/2}$$

for some constants C_1 and C_2. Since the Sobolev space $H^1(\Omega)$ is a compact imbedding into $L^6(\Omega)$, the above equality implies

$$\|(E(\boldsymbol{x},t)+\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t))\|_{L^6(\Omega)} \le C_2 + C_1\left(\int_{\mathbb{R}_v^3\times\Omega}|\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}d\boldsymbol{x}\right)^{1/2}. \tag{2.8}$$

We now return to the Vlasov equation. Multiplying by $|\boldsymbol{v}|^3$ and integrating over $\mathcal{T}=\Omega\times\mathbb{R}_v^3$ we have

$$\partial_t\int_{\mathcal{T}}|\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{x}d\boldsymbol{v} - 3\int_{\mathcal{T}}|\boldsymbol{v}|\,\boldsymbol{v}\cdot(E(x,t)+\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t))\,F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{x}d\boldsymbol{v} = 0.$$

As $F_0 \geq 0$, using the positive preserving of the transport operator we have $F^\epsilon \geq 0$. Therefore

$$\partial_t \int_{\mathcal{T}} |\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{x}d\boldsymbol{v} \leq 3 \int_\Omega |E(\boldsymbol{x},t) + \nabla_{\boldsymbol{x}} V^\epsilon(\boldsymbol{x},t)| \left(\int_{\mathbb{R}^3_v} |\boldsymbol{v}|^2 F^\epsilon(x,t,\boldsymbol{v})\, d\boldsymbol{v} \right) d\boldsymbol{x}.$$

By the Hölder inequality and the equation (2.8), we finally get

$$\begin{aligned}
&\partial_t \int_{\mathcal{T}} |\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{x}d\boldsymbol{v} \\
&\leq 3 \left\| E(\boldsymbol{x},t) + \nabla_{\boldsymbol{x}} V^\epsilon(\boldsymbol{x},t) \right\|_{L^6(\Omega)} \left[\int_\Omega \left(\int_{\mathbb{R}^3_v} |\boldsymbol{v}|^2 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v} \right)^{6/5} d\boldsymbol{x} \right]^{5/6} \\
&\leq 3 \left(C_2 + C_1 \left(\int_{\mathcal{T}} |\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{v}d\boldsymbol{x} \right)^{1/2} \right) \left(\int_{\mathcal{T}} |\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\, d\boldsymbol{x}d\boldsymbol{v} \right)^{6/5}.
\end{aligned}$$

Therefore, $|\boldsymbol{v}|^3 F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})$ is bounded in $L^\infty(0,T;L^1(\Omega\times\mathbb{R}^3_v))$ by Gronwall's lemma. As the result, $\rho^\epsilon(\boldsymbol{x},t)$ is bounded in $L^\infty(0,T;L^2(\Omega))$ by (2.7). This completes the proof. □

3. Proofs of Theorem 1.1 and Theorem 1.2

In this section, we will prove Theorems 1.1 and 1.2. The basic ideas follow from the following compactness theorems of two-scale convergence. The proofs and further descriptions are referred to [2, 7, 21].

Theorem 3.1. *For each bounded sequence $\{u^\epsilon\}_\epsilon$ in $L^p(\Omega)$, $1 < p \leq \infty$, there exists a subsequence still denoted by $\{u^\epsilon\}_\epsilon$ which two-scale converges to $u(\boldsymbol{x},\boldsymbol{y}) \in L^p(\Omega\times Y)$.*

Theorem 3.2. *Let u^ϵ and $\epsilon\nabla u^\epsilon$ be two bounded sequences in $L^2(\Omega)$ and $(L^2(\Omega))^3$. Then, there exists a function $u(\boldsymbol{x},\boldsymbol{y})$ in $L^2(\Omega;H^1_\#(Y))$ such that, up to a subsequence, u^ϵ and $\epsilon\nabla u^\epsilon$ two-scale converge to $u(x,y)$ and to $\nabla_{\boldsymbol{y}} u(\boldsymbol{x},\boldsymbol{y})$, respectively.*

We remark that Theorem 3.1 shows the well defined for the two-scale convergence, and which further generalizes the notion of weak convergence. Theorem 3.2 gives the properties of the derivatives, which points out the functions can be decomposed into the divergent free part and the gradient part with divergent free part zero. From Lemmas 2.2 and 2.3 we have the two-scale limiting of the Vlasov equation, and combining Lemma 2.5 with Theorem 3.2 we will obtain the two-scale limiting of the Poisson equation. The detail is given as follows.

Proofs of Theorems 1.1 *and* 1.2. The first thought, we want to go in search of the two-scale limit of Poisson equation. For this purpose, we multiply the Poisson equation (1.2) by the admissible function $\psi(\boldsymbol{x}, \boldsymbol{x}/\epsilon, t) = \psi(\boldsymbol{x}, \boldsymbol{y}, t)$ which is with compact support in $(\boldsymbol{x}, t)$ and periodic in $\boldsymbol{y}$, and we then obtain the equation

$$\begin{aligned}&\int_{\mathcal{T}} [\nabla_{\boldsymbol{x}} \cdot (E(\boldsymbol{x}, t) + \nabla_{\boldsymbol{x}} V^{\epsilon}(\boldsymbol{x}, t))]\, \psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\epsilon}, t\right) d\boldsymbol{x}dt \\ &= -4\pi e \int_{\mathcal{T}} \left(\int_{\mathbb{R}^3_v} F^{\epsilon}(\boldsymbol{x}, t, \boldsymbol{v})\, d\boldsymbol{v} - n_0\right) \psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\epsilon}, t\right) d\boldsymbol{x}dt.\end{aligned} \tag{3.1}$$

Integrating by parts we can rewrite (3.1) as

$$\begin{aligned}&-\int_{\mathcal{T}} (E(\boldsymbol{x}, t) + \nabla_{\boldsymbol{x}} V^{\epsilon}(\boldsymbol{x}, t)) \cdot \left(\nabla_{\boldsymbol{x}} + \frac{1}{\epsilon}\nabla_y\right) \psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\epsilon}, t\right) d\boldsymbol{x}dt \\ &= -4\pi e \int_{\mathcal{T}} \left(\int_{\mathbb{R}^3_v} F^{\epsilon}(\boldsymbol{x}, t, \boldsymbol{v})\, d\boldsymbol{v} - n_0\right) \psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\epsilon}, t\right) d\boldsymbol{x}dt.\end{aligned} \tag{3.2}$$

On the other hand, we have already derived the two-scale expansions in the Section 1 of the Vlasov-Poisson system, which predicts the likely order of the potential function V in variable $\boldsymbol{y}$ given by (1.8). Indeed, if we multiply the factor ϵ^2 on the both sides of the equation (3.2), we will get the equation

$$\begin{aligned}&-\int_{\mathcal{T}} (\epsilon E(\boldsymbol{x}, t) + \epsilon\nabla_{\boldsymbol{x}} V^{\epsilon}(\boldsymbol{x}, t))\, (\epsilon\nabla_{\boldsymbol{x}} + \nabla_{\boldsymbol{y}})\psi(\boldsymbol{x}, \boldsymbol{y}, t)\, d\boldsymbol{x}dt \\ &= -4\pi e \int_{\mathcal{T}} \epsilon^2 \left(\int_{\mathbb{R}^3_v} F^{\epsilon}(\boldsymbol{x}, t, \boldsymbol{v})\, dv - n_0\right) \psi\left(\boldsymbol{x}, \frac{\boldsymbol{x}}{\epsilon}, t\right) d\boldsymbol{x}dt.\end{aligned} \tag{3.3}$$

By Theorems 3.1 and 3.2, the two-scale limit of (3.3) takes the form

$$-\int_{Y\times\mathcal{T}} \nabla_{\boldsymbol{y}} V(\boldsymbol{y}, t) \cdot \nabla_{\boldsymbol{y}} \psi(\boldsymbol{x}, \boldsymbol{y}, t)\, d\boldsymbol{y}d\boldsymbol{x}dt = 0,$$

or

$$\int_{Y\times\mathcal{T}} \triangle_{\boldsymbol{y}} V(\boldsymbol{y}, t)\psi(\boldsymbol{x}, \boldsymbol{y}, t)\, d\boldsymbol{y}d\boldsymbol{x}dt = 0$$

after integration by parts. We therefore obtain the same restrained equation obtained by formal two scale expansion of the ϵ^{-2}-order related to the potential function $V(\boldsymbol{y}, t)$, which is harmonic in $\boldsymbol{y}$,

$$\triangle_{\boldsymbol{y}} V(\boldsymbol{y}, t) = 0. \tag{3.4}$$

In order to get the two-scale limit of Poisson equation, we need the restrained equation of ϵ^{-1}-order term which can be expressed as (1.9). Hence, we multiply (3.2) by the ϵ-order term to obtain

$$\begin{aligned}&-\int_{\mathcal{T}} \epsilon E(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{x}} \psi(\boldsymbol{x}, \boldsymbol{y}, t)\, d\boldsymbol{x}dt - \int_{\mathcal{T}} (\epsilon\nabla_{\boldsymbol{x}} V^{\epsilon}(\boldsymbol{x}, t)) \cdot \nabla_{\boldsymbol{x}} \psi(\boldsymbol{x}, \boldsymbol{y}, t)\, d\boldsymbol{x}dt \\ &-\int_{\mathcal{T}} E(\boldsymbol{x}, t) \cdot \nabla_{\boldsymbol{y}} \psi(\boldsymbol{x}, \boldsymbol{y}, t)\, d\boldsymbol{x}dt - \frac{1}{\epsilon}\int_{\mathcal{T}} (\epsilon\nabla_{\boldsymbol{x}} V^{\epsilon}(\boldsymbol{x}, t)) \cdot \nabla_{\boldsymbol{y}} \psi(\boldsymbol{x}, \boldsymbol{y}, t)\, d\boldsymbol{x}dt \\ &= -4\pi e\epsilon \int_{\mathcal{T}} \left(\int_{R^3_v} F^{\epsilon}(\boldsymbol{x}, t, \boldsymbol{v})\, d\boldsymbol{v} - n_0\right) \psi(\boldsymbol{x}, \boldsymbol{y}, t)\, d\boldsymbol{x}dt.\end{aligned} \tag{3.5}$$

Owing to the equation (3.4), the fourth term on the left-hand side of the equation (3.5) vanishes. Therefore, the two-scale limit of the equation (3.5) is given by

$$-\int_{Y\times\mathcal{T}} \nabla_{\boldsymbol{y}} V(\boldsymbol{y},t)\cdot\nabla_{\boldsymbol{x}}\psi(\boldsymbol{x},\boldsymbol{y},t)\,d\boldsymbol{y}d\boldsymbol{x}dt-\int_{Y\times\mathcal{T}} E(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{y}}\psi(\boldsymbol{x},\boldsymbol{y},t)\,d\boldsymbol{y}d\boldsymbol{x}dt=0.$$

That is,

$$\nabla_{\boldsymbol{x}}\cdot\nabla_{\boldsymbol{y}}V(\boldsymbol{y},t)+\nabla_{\boldsymbol{y}}\cdot E(\boldsymbol{x},t)=0. \tag{3.6}$$

Besides, the equation (3.2) can be separated as

$$\begin{aligned}
&-\int_{\mathcal{T}} E(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{x}}\psi(\boldsymbol{x},\boldsymbol{y},t)\,d\boldsymbol{x}dt\\
&-\frac{1}{\epsilon}\int_{\mathcal{T}}\left(E(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{y}}\psi(\boldsymbol{x},\boldsymbol{y},t)+(\epsilon\nabla_{\boldsymbol{x}}V^{\epsilon}(\boldsymbol{x},t))\cdot\nabla_{\boldsymbol{x}}\psi(\boldsymbol{x},\boldsymbol{y},t)\right)d\boldsymbol{x}dt\\
&-\frac{1}{\epsilon^2}\int_{\mathcal{T}}\epsilon\nabla_{\boldsymbol{x}}V^{\epsilon}(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{y}}\psi(\boldsymbol{x},\boldsymbol{y},t)\,d\boldsymbol{x}dt\\
=&-4\pi e\int_{\mathcal{T}}\left(\int_{\mathbb{R}^3_v}F^{\epsilon}(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}-n_0\right)\psi(\boldsymbol{x},\boldsymbol{y},t)\,d\boldsymbol{x}dt.
\end{aligned}$$

Notice that the second and third terms vanish because of (3.6) and (3.4). Hence, taking two-scale limit we obtain

$$\begin{aligned}
&-\int_{Y\times\mathcal{T}} E(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{x}}\psi(\boldsymbol{x},\boldsymbol{y},t)\,d\boldsymbol{y}d\boldsymbol{x}dt\\
=&-4\pi e\int_{Y\times\mathcal{T}}\left(\int_{\mathbb{R}^3_v}\overline{F}(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\,d\boldsymbol{v}-n_0\right)\psi(\boldsymbol{x},\boldsymbol{y},t)\,d\boldsymbol{y}d\boldsymbol{x}dt.
\end{aligned}$$

The two-scale limiting Poisson equation will be

$$\nabla_x\cdot E(\boldsymbol{x},t)=-4\pi e\left(\int_{\mathbb{R}^3_v}\overline{F}(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\,d\boldsymbol{v}-n_0\right). \tag{3.7}$$

We are now in a position to investigate the two-scale limits of the Vlasov equation. Multiplying the Vlasov equation (1.1) by the admissible test function, we have

$$\begin{aligned}
&\int_{\mathcal{O}}\partial_t F^{\epsilon}(\boldsymbol{x},t,\boldsymbol{v})\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,v\right)d\boldsymbol{x}dtd\boldsymbol{v}\\
&+\int_{\mathcal{O}}\boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}F^{\epsilon}(\boldsymbol{x},t,\boldsymbol{v})\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,v\right)d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\frac{e}{m}\int_{\mathcal{O}}\left(\nabla_{\boldsymbol{x}}V^{\epsilon}(\boldsymbol{x},t)+E(\boldsymbol{x},t)\right)\cdot\nabla_{\boldsymbol{v}}F^{\epsilon}(\boldsymbol{x},t,\boldsymbol{v})\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right)d\boldsymbol{x}dtd\boldsymbol{v}=0,
\end{aligned} \tag{3.8}$$

where $\mathcal{O} = \mathcal{T} \times \mathbb{R}^3_v$. After integrating by parts, (3.8) can be rewritten as

$$
\begin{aligned}
&-\int_{\mathcal{O}} F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\partial_t\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right)d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\int_{\mathcal{O}} \boldsymbol{v}\cdot F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\nabla_{\boldsymbol{x}}\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right)d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\int_{\mathcal{O}} \frac{1}{\epsilon}\boldsymbol{v}\cdot F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\nabla_{\boldsymbol{y}}\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right)d\boldsymbol{x}dtd\boldsymbol{v}\\
&+\frac{e}{m}\int_{\mathcal{O}} \left(\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)+E(\boldsymbol{x},t)\right)\cdot F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\nabla_{\boldsymbol{v}}\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right)d\boldsymbol{x}dtd\boldsymbol{v}=0.
\end{aligned}
\tag{3.9}
$$

The two-scale expansions in Section 1 reveals the order ϵ^{-1} about the restrained equation (1.6). To see this, we multiply (3.9) by the factor ϵ and obtain

$$
\begin{aligned}
&-\int_{\mathcal{O}} \epsilon F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\partial_t\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right)d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\int_{\mathcal{O}} \epsilon\boldsymbol{v}\cdot F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\nabla_{\boldsymbol{x}}\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right)d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\int_{\mathcal{O}} \boldsymbol{v}\cdot F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\nabla_{\boldsymbol{y}}\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right)d\boldsymbol{x}dtd\boldsymbol{v}\\
&+\frac{e}{m}\int_{\mathcal{O}} \epsilon\left(\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)+E(\boldsymbol{x},t)\right)\cdot F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\nabla_{\boldsymbol{v}}\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right)d\boldsymbol{x}dtd\boldsymbol{v}=0.
\end{aligned}
\tag{3.10}
$$

Applying Lemmas 2.2 and 2.3, we obtain the two-scale limit of (3.10)

$$
\begin{aligned}
&-\int_{\mathcal{O}}\int_Y \boldsymbol{v}\cdot\overline{F}(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v}))\nabla_{\boldsymbol{y}}\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\,d\boldsymbol{y}d\boldsymbol{x}dtd\boldsymbol{v}\\
&+\frac{e}{m}\int_{\mathcal{O}}\int_Y \nabla_{\boldsymbol{y}}V(\boldsymbol{y},t)\overline{F}(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\cdot\nabla_{\boldsymbol{v}}\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\,d\boldsymbol{y}d\boldsymbol{x}dtd\boldsymbol{v}=0.
\end{aligned}
\tag{3.11}
$$

Integrating by parts again, (3.11) can be further represented as

$$
\begin{aligned}
&\int_{\mathcal{O}}\int_Y \boldsymbol{v}\cdot\nabla_{\boldsymbol{y}}\overline{F}(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v}))\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\,d\boldsymbol{y}d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\frac{e}{m}\int_{\mathcal{O}}\int_Y \nabla_{\boldsymbol{y}}V(\boldsymbol{y},t)\cdot\nabla_{\boldsymbol{v}}\overline{F}(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v}))\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\,d\boldsymbol{y}d\boldsymbol{x}dtd\boldsymbol{v}=0.
\end{aligned}
$$

Therefore, we have the restrained equation

$$
\boldsymbol{v}\cdot\nabla_{\boldsymbol{y}}\overline{F}(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})-\frac{e}{m}\nabla_{\boldsymbol{y}}V(\boldsymbol{y},t)\cdot\nabla_{\boldsymbol{v}}\overline{F}(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})=0.
\tag{3.12}
$$

We note that the equation (3.12) can be solved explicitly by the method of characteristics:

$$
\overline{F}(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})=\overline{F}(\eta(\boldsymbol{y},\boldsymbol{v}),\boldsymbol{x},t),
$$

where $\eta(\boldsymbol{y},\boldsymbol{v})$ is the characteristic curve of (3.12).

We now want to view the two-scale limiting equation of the Vlasov equation (1.1). Similarly, multiplying the equation (1.1) by the admissible function we have

$$
\begin{aligned}
&\int_{\mathcal{O}} \partial_t F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,v\right) d\boldsymbol{x}dtd\boldsymbol{v}\\
&+\int_{\mathcal{O}} \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right) d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\frac{e}{m}\int_{\mathcal{O}} E(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{v}}F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right) d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\frac{e}{m}\int_{\mathcal{O}} \nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{v}}F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\psi\left(\boldsymbol{x},\frac{\boldsymbol{x}}{\epsilon},t,\boldsymbol{v}\right) d\boldsymbol{x}dtd\boldsymbol{v}=0.
\end{aligned}
$$

After integration by parts, we obtain

$$
\begin{aligned}
&-\int_{\mathcal{O}} F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\partial_t\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\, d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\int_{\mathcal{O}} \boldsymbol{v}\cdot F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\left(\nabla_{\boldsymbol{x}}+\frac{1}{\epsilon}\nabla_{\boldsymbol{y}}\right)\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\, d\boldsymbol{x}dtd\boldsymbol{v}\\
&+\frac{e}{m}\int_{\mathcal{O}} E(\boldsymbol{x},t)\cdot F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\nabla_{\boldsymbol{v}}\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\, d\boldsymbol{x}dtd\boldsymbol{v}\\
&+\frac{e}{m}\int_{\mathcal{O}} \nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\cdot\nabla_{\boldsymbol{v}}\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\, d\boldsymbol{x}dtd\boldsymbol{v}=0,
\end{aligned}
$$

or

$$
\begin{aligned}
&-\int_{\mathcal{O}} F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\partial_t\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\, d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\int_{\mathcal{O}} \boldsymbol{v}\cdot F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\cdot\nabla_{\boldsymbol{x}}\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\, d\boldsymbol{x}dtd\boldsymbol{v}\\
&+\frac{e}{m}\int_{\mathcal{O}} E(\boldsymbol{x},t)\cdot F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\nabla_{\boldsymbol{v}}\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\, d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\frac{1}{\epsilon}\int_{\mathcal{O}} \Big[\boldsymbol{v}F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\cdot\nabla_{\boldsymbol{y}}\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\\
&\qquad -\frac{e}{m}\left(\epsilon\nabla_{\boldsymbol{x}}V^\epsilon(\boldsymbol{x},t)\right)F^\epsilon(\boldsymbol{x},t,\boldsymbol{v})\cdot\nabla_{\boldsymbol{v}}\psi(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})\Big]d\boldsymbol{x}dtd\boldsymbol{v}=0.
\end{aligned}
\tag{3.13}
$$

Since the admissible test function satisfied the restrained equation (3.12) then by change of variables, the last term on the left-hand side of (3.13) vanishes, thus the two-scale limits of the (3.13) will be

$$
\begin{aligned}
&-\int_{\mathcal{O}\times Y} \overline{F}(\boldsymbol{x},\ ,t,\eta(\boldsymbol{y},\boldsymbol{v}))\partial_t\psi(\boldsymbol{x},t,\eta(\boldsymbol{y},\boldsymbol{v}))\, d\boldsymbol{y}d\boldsymbol{x}dtd\boldsymbol{v}\\
&-\int_{\mathcal{O}\times Y} \boldsymbol{v}\cdot\overline{F}(\boldsymbol{x},\ ,t,\eta(\boldsymbol{y},\boldsymbol{v}))\cdot\nabla_{\boldsymbol{x}}\psi(\boldsymbol{x},t,\eta(\boldsymbol{y},\boldsymbol{v}))\, d\boldsymbol{y}d\boldsymbol{x}dtd\boldsymbol{v}\\
&+\frac{e}{m}\int_{\mathcal{O}\times Y} E(\boldsymbol{x},t)\cdot\overline{F}(\boldsymbol{x},\ ,t,\eta(\boldsymbol{y},\boldsymbol{v}))\nabla_{\boldsymbol{v}}\psi(\boldsymbol{x},t,\eta(\boldsymbol{y},\boldsymbol{v}))\, d\boldsymbol{y}d\boldsymbol{x}dtd\boldsymbol{v}=0.
\end{aligned}
\tag{3.14}
$$

Integrating by parts again, (3.14) becomes

$$\int_{\mathcal{O}\times Y} \partial_t \overline{F}(\boldsymbol{x}, , t, \eta(\boldsymbol{y},\boldsymbol{v}))\psi(\boldsymbol{x}, t, \eta(\boldsymbol{y},\boldsymbol{v}))\, d\boldsymbol{x}d\boldsymbol{y}dtd\boldsymbol{v}$$

$$+ \int_{\mathcal{O}\times Y} \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\overline{F}(\boldsymbol{x}, , t, \eta(\boldsymbol{y},\boldsymbol{v}))\cdot\psi(\boldsymbol{x}, t, \eta(\boldsymbol{y},\boldsymbol{v}))\, d\boldsymbol{x}d\boldsymbol{y}dtd\boldsymbol{v}$$

$$- \frac{e}{m}\int_{\mathcal{O}\times Y} E(\boldsymbol{x}, t)\cdot\nabla_{\boldsymbol{v}}\overline{F}(\boldsymbol{x}, , t, \eta(\boldsymbol{y},\boldsymbol{v}))\psi(\boldsymbol{x}, t, \eta(\boldsymbol{y},\boldsymbol{v}))\, d\boldsymbol{x}d\boldsymbol{y}dtd\boldsymbol{v} = 0.$$

This means

$$\text{(3.15)}\quad \partial_t\overline{F}(\boldsymbol{x}, , t, \eta(\boldsymbol{y},\boldsymbol{v})) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}\overline{F}(\boldsymbol{x}, , t, \eta(\boldsymbol{y},\boldsymbol{v})) - \frac{e}{m}E(\boldsymbol{x}, t)\cdot\nabla_{\boldsymbol{v}}\overline{F}(\boldsymbol{x}, , t, \eta(\boldsymbol{y},\boldsymbol{v})) = 0.$$

Combining (3.4), (3.7), (3.12) and (3.15) together we have completed the proof of Theorem 1.1. Moreover, averaging in Y on the equations (3.7) and (3.15), we obtain

$$\partial_t F(\boldsymbol{x}, t, \boldsymbol{v}) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}F(\boldsymbol{x}, t, \boldsymbol{v}) - \frac{e}{m}E(\boldsymbol{x}, t)\cdot\nabla_{\boldsymbol{v}}F(\boldsymbol{x}, t, \boldsymbol{v}) = 0,$$

$$\nabla_{\boldsymbol{x}}\cdot E(\boldsymbol{x}, t) = -4\pi e\left(\int_{\mathbb{R}^3_v} F(\boldsymbol{x}, t, \boldsymbol{v})\, d\boldsymbol{v} - n_0\right),$$

where $\overline{F}(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) = \overline{F}(\eta(\boldsymbol{y},\boldsymbol{v}), \boldsymbol{x}, t)$ and $F(\boldsymbol{x}, t, \boldsymbol{v}) = \int_Y \overline{F}(\boldsymbol{x}, , t, \eta(\boldsymbol{y},\boldsymbol{v}))\, d\boldsymbol{y}$. This yields Theorem 1.2. □

4. Dielectric function and the dispersion relation

From the theory of electric polarization, the electric field strength changes with the time; then the polarization need not be in the equilibrium with the field. The motions of the microscopic particles required to reach a certain value of the polarization have characteristic times. When the electric field varies appreciably within a period of the same order as the characteristic time, the motions of the microscopic particles will not be sufficiently rapid to build up the equilibrium polarization, and the actual value of the polarization will, as it was, lags behind the changing electric field. Therefore, it gives rise to dielectric loss. To view this and the corresponding dispersion relation, we consider an unbounded uniform electron plasma with a fixed neutralizing ion background and under equilibrium conditions, which is given slightly displaced from the equilibrium positions. Since we are dealing with small deviations from the equilibrium, the equations can be linearized. To describe small deviations from the equilibrium, we assume the density F^ϵ in (1.1) and (1.2) can be separated into

$$F^\epsilon(\boldsymbol{x}, t, \boldsymbol{v}) = F_0(v) + F_1^\epsilon(\boldsymbol{x}, t, \boldsymbol{v}),$$

where $v = |\boldsymbol{v}|$ and the two-scale limiting function will be

$$\overline{F}(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}) = F_0(v) + \overline{F}_1(\boldsymbol{x}, \boldsymbol{y}, t, \boldsymbol{v}).$$

We note that the homogenized function becomes

$$F(\boldsymbol{x},t,\boldsymbol{v}) = F_0(v) + F_1(\boldsymbol{x},t,\boldsymbol{v}). \tag{4.1}$$

The restrained equation (3.12) can be rewritten as

$$\boldsymbol{v}\cdot\nabla_{\boldsymbol{y}}\overline{F}_1(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v}) - \frac{e}{m}\nabla_{\boldsymbol{y}}V(\boldsymbol{y},t)\cdot\nabla_{\boldsymbol{v}}\overline{F}_1(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v}) = \frac{e}{m}\nabla_{\boldsymbol{y}}V(\boldsymbol{y},t)\cdot\nabla_{\boldsymbol{v}}F_0(v). \tag{4.2}$$

By characteristics, F_1 can be represented as $\overline{F}_1(\boldsymbol{x},\boldsymbol{y},\boldsymbol{v},t) = \overline{F}_1(\eta(\boldsymbol{y},\boldsymbol{v}),\boldsymbol{x},t)$, where η is the characteristic curve of the equation (4.2), and also $F_1(\boldsymbol{x},t,\boldsymbol{v}) = \int_Y \overline{F}_1(\boldsymbol{x},\boldsymbol{y},t,\boldsymbol{v})d\boldsymbol{y}$. We now focus on deducing dielectric function. To this end, we plug (4.1) into the equations (1.12) and (1.13), and ignore the second order term. Thus, the linearized Vlasov-Poisson system becomes

$$\partial_t F_1(\boldsymbol{x},t,\boldsymbol{v}) + \boldsymbol{v}\cdot\nabla_{\boldsymbol{x}}F_1(\boldsymbol{x},t,\boldsymbol{v}) = \frac{e}{m}E(\boldsymbol{x},t)\cdot\nabla_{\boldsymbol{v}}F_0(v) \tag{4.3}$$

and

$$\nabla_{\boldsymbol{x}}\cdot E(\boldsymbol{x},t) = -4\pi e\int_{\mathbb{R}^3_v} F_1(\boldsymbol{x},t,\boldsymbol{v})\,d\boldsymbol{v}. \tag{4.4}$$

For the mathematical treatment, however, without losing the essential of the plasma behavior under consideration, we might assume

$$F_1(\boldsymbol{x},t,\boldsymbol{v}) = \widehat{F}_1(\omega,\boldsymbol{k},\boldsymbol{v})e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}, \quad E(\boldsymbol{x},t) = \widehat{E}(\omega,\boldsymbol{k})e^{i\boldsymbol{k}\cdot\boldsymbol{x}-i\omega t}. \tag{4.5}$$

Substituting (4.5) into (4.3), we get

$$-i\omega\widehat{F}_1(\omega,\boldsymbol{k},\boldsymbol{v}) + i\boldsymbol{v}\cdot\boldsymbol{k}\widehat{F}_1(\omega,\boldsymbol{k},\boldsymbol{v}) = \frac{e}{m}\widehat{E}(\omega,\boldsymbol{k})\cdot\nabla_{\boldsymbol{v}}F_0(v),$$

whose solution is

$$\widehat{F}_1(\omega,\boldsymbol{k},\boldsymbol{v}) = \frac{e}{m}\widehat{E}(\omega,\boldsymbol{k})\cdot\frac{\nabla_{\boldsymbol{v}}F_0(v)}{-i\omega+i\boldsymbol{k}\cdot\boldsymbol{v}}. \tag{4.6}$$

For definiteness it is convenient to consider the direction of propagation of the waves as being the direction of the first component. Therefore, $\boldsymbol{k}\cdot\boldsymbol{v} = kv_1$, and (4.6) becomes

$$\widehat{F}_1(\omega,\boldsymbol{k},\boldsymbol{v}) = \frac{e}{m}\widehat{E}(\omega,\boldsymbol{k})\cdot\frac{\nabla_{\boldsymbol{v}}F_0(v)}{-i\omega+ikv_1}.$$

We apply the useful identity

$$\nabla_{\boldsymbol{v}}F_0(v) = \frac{\boldsymbol{v}}{v}\frac{dF_0(v)}{dv},$$

to reducing the Poisson equation (4.4) to

$$ik\widehat{E}_1(\omega,\boldsymbol{k}) = -\frac{4\pi e^2}{m}\widehat{E}(\omega,\boldsymbol{k})\cdot\int_{\mathbb{R}^3_v}\frac{\boldsymbol{v}}{v}\frac{dF_0(v)}{dv}\frac{1}{-i\omega+ikv_1}\,d\boldsymbol{v}. \tag{4.7}$$

Observe that

$$-\frac{4\pi e^2}{m}\widehat{E}_j(\omega,\boldsymbol{k})\int_{\mathbb{R}^3_v}\frac{v_j}{v}\frac{dF_0(v)}{dv}\frac{1}{k\omega-k^2v_1}\,d\boldsymbol{v}=0,\quad j=2,3$$

since the integrand is an odd function of v_j. Consequently, the only contribution comes from the term $\widehat{E}(\omega,\boldsymbol{k})\cdot\nabla_{\boldsymbol{v}}F_0(v)$ to the first component of $\widehat{E}(\omega,\boldsymbol{k})$. Explicitly, it is $\widehat{E}_1(\omega,\boldsymbol{k})\frac{\partial F_0(v)}{\partial v_1}$, so that the equation (4.7) can be written as

$$\widehat{E}_1(\omega,\boldsymbol{k})=-\frac{4\pi e^2}{m}\widehat{E}_1(\omega,\boldsymbol{k})\int_{\mathbb{R}^3_v}\frac{\partial F_0(v)/\partial v_1}{k\omega-k^2v_1}\,d\boldsymbol{v}. \tag{4.8}$$

Thus the dipole moment is given by

$$\begin{aligned}\widehat{\mathcal{P}}(\omega,\boldsymbol{k})&=\left(\frac{4\pi e^2}{m}\int_{\mathbb{R}^3_v}\frac{\partial F_0(v)/\partial v_1}{k\omega-k^2v_1}\,d\boldsymbol{v}\right)\widehat{E}_1(\omega,\boldsymbol{k})\\&=4\pi\chi(\omega,\boldsymbol{k})\widehat{E}_1(\omega,\boldsymbol{k}),\end{aligned}$$

where χ is the electric susceptibility, and the dielectric function is defined as

$$\mathcal{E}(\omega,\boldsymbol{k})=1+4\pi\chi=1+\frac{4\pi e^2}{m}\int_{\mathbb{R}^3_v}\frac{\partial F_0(v)/\partial v_1}{k\omega-k^2v_1}\,d\boldsymbol{v}.$$

Dividing this equation by $\widehat{E}_1(\omega,\boldsymbol{k})\neq 0$, then using (4.8), we have the dispersion relation

$$1=-\omega_e^2\int_{\mathbb{R}^3_v}\frac{\partial f_0(v)/\partial v_1}{k\omega-k^2v_1}\,d\boldsymbol{v}, \tag{4.9}$$

where $\omega_e^2=4\pi n_0e^2/m$ is the natural plasma frequency and $F_0=n_0f_0$. The equation (4.9) has a singularity at $v_1=\omega/k$. In order to calculate the integral we need the Plemelj formula

$$\lim_{\tau\to 0}\int_{-\infty}^{\infty}\frac{\phi(t)}{t-t_0-i\tau}\,dt=\mathrm{Pr}\int_{-\infty}^{\infty}\frac{\phi(t)}{t-t_0}\,dt+\pi i\phi(t_0), \tag{4.10}$$

where Pr denotes the principal value, t_0 is a point on the real axis and $\phi(t)$ is a continuous function of t. Applying the Plemelj formula (4.10) to (4.9) we have the dispersion relation

$$1=-\omega_e^2\int_{-\infty}^{\infty}\frac{\partial\widehat{f}_0(v_1)/\partial v_1}{k\omega-k^2v_1}\,dv_1=\frac{-\omega_e^2}{k\omega}\left[\mathrm{Pr}\int_{-\infty}^{\infty}\frac{\partial\widehat{f}_0(v_1)/\partial v_1}{1-kv_1/\omega}\,dv_1+\pi i\frac{\partial\widehat{f}_0(\omega/k)}{\partial v_1}\right] \tag{4.11}$$

and the dielectric function is given by

$$\mathcal{E}(\omega,\boldsymbol{k})=1+\omega_e^2\int_{-\infty}^{\infty}\frac{\partial\widehat{f}_0(v_1)/\partial v_1}{k\omega-k^2v_1}\,dv_1=\frac{-\omega_e^2}{k\omega}\left[\mathrm{Pr}\int_{-\infty}^{\infty}\frac{\partial\widehat{f}_0(v_1)/\partial v_1}{1-kv_1/\omega}\,dv_1+\pi i\frac{\partial\widehat{f}_0(\omega/k)}{\partial v_1}\right] \tag{4.12}$$

where

$$\widehat{f}_0(v_1)=\int_{\mathbb{R}^2_{v_2,v_3}}f_0(v_1,v_2,v_3)\,dv_2dv_3.$$

We note that the complex part of the equation (4.12) is the dielectric loss due to the polarization delay. The result is in virtue of the particle and wave interaction, when the velocity v_1 is closed to ω/k.

Another important result can be immediately obtained from the dispersion relation (4.11), for the limiting case in which the wave phase velocity ω/k is very large compared to the velocity of almost all of the electrons. In this high phase velocity limit with $kv/\omega \ll 1$, that is the finite frequency and long wavelength (small k) limit, it is reasonable to expand by power series. The principle value can be approximated by

$$\mathrm{Pr}\int_{-\infty}^{\infty} \frac{\partial \widehat{f}_0(v_1)/\partial v_1}{1 - kv_1/\omega}\, dv_1 = \int_{-\infty}^{\infty} \frac{\partial \widehat{f}_0(v_1)}{\partial v_1}\left(1 + \frac{kv_1}{\omega} + \left(\frac{kv_1}{\omega}\right)^2\right) dv_1 = -\frac{k}{\omega}. \tag{4.13}$$

Therefore, the dispersion relation of the equation (4.11) alters to

$$1 = -\omega_e^2\left[\frac{-1}{\omega^2} + i\frac{\pi}{k\omega}\frac{\partial \widehat{f}_0}{\partial v_1}\left(\frac{\omega}{k}\right)\right] = \frac{\omega_e^2}{\omega^2} - i\frac{\pi\omega_e^2}{k\omega}\frac{\partial \widehat{f}_0}{\partial v_1}\left(\frac{\omega}{k}\right),$$

here we assume $f_0(v) = \frac{1}{(\sqrt{2\pi})^3}e^{-v^2/2}$. By (4.13), the dielectric function of the equation (4.12) can be written as

$$\mathcal{E} = 1 - \frac{\omega_e^2}{\omega^2} + i\frac{\pi\omega_e^2}{k\omega}\frac{\partial \widehat{f}_0}{\partial v_1}\left(\frac{\omega}{k}\right).$$

Integrating by parts, the right-hand side of (4.9) becomes

$$1 = \omega_e^2\int_{-\infty}^{\infty} \frac{\widehat{f}_0(v_1)}{(\omega - kv_1)^2}\, dv_1. \tag{4.14}$$

Expanding the denominator of the integrand up to and including second order terms in v_1k/ω, the equation (4.14) leads to the approximation

$$1 = \frac{\omega_e^2}{\omega^2}\int_{-\infty}^{\infty} \widehat{f}_0(v_1)\left(1 + \frac{2kv_1}{\omega} + \frac{3k^2v_1^2}{\omega^2}\right) dv_1 = \frac{\omega_e^2}{\omega^2} + \frac{3k^2\omega_e^2}{\omega^4}.$$

For the the long wavelength limit ($k \to 0$), the dielectric function becomes

$$\mathcal{E} = 1 - \frac{\omega_e^2}{\omega^2}$$

which means that we have the real-valued dispersion relation ω and $\omega \sim \omega_e$. This result is also called the cold plasma approximation. Using the approximation $\omega^2 \sim \omega_e^2$, we derived the so-called *Langmuir wave dispersion relation*

$$\omega^2(k) = \omega_e^2 + 3k^2.$$

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