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New Results for Second Order Discrete Hamiltonian Systems

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Abstract. In this paper, we deal with the second order discrete Hamiltonian system $\Delta[p(n)\Delta u(n-1)] - L(n)u(n) + \nabla W(n,u(n)) = 0$, where $L\colon \mathbb{Z} \to \mathbb{R}^{N\times N}$ is positive definite for sufficiently large $|n| \in \mathbb{Z}$ and $W(n,x)$ is indefinite sign. By using critical point theory, we establish some new criteria to guarantee that the above system has infinitely many nontrivial homoclinic solutions under the assumption that $W(n,x)$ is asymptotically quadratic and supquadratic, respectively. Our results generalize and improve some existing results in the literature.

1. Introduction

In this paper, we investigate the following second order discrete Hamiltonian system

$$\Delta[p(n)\Delta u(n-1)] - L(n)u(n) + \nabla W(n,u(n)) = 0, \quad \forall\, n \in \mathbb{Z}, \tag{1.1}$$

where $u \in \mathbb{R}^N$, $p, L\colon \mathbb{Z} \to \mathbb{R}^{N\times N}$, $\Delta u(n) = u(n+1) - u(n)$ is the forward difference, and $\nabla W(n,u)$ denotes the gradient of $W(n,u)$ with respect to u. As usual, we say that a solution $u(n)$ of system (1.1) is homoclinic (to 0) if $u(n) \to 0$ as $|n| \to \infty$. In addition, if $u(n) \not\equiv 0$ then $u(n)$ is called a nontrivial homoclinic solution. I_N denotes the $N \times N$ identity matrix.

Discrete Hamiltonian system can be applied in many different areas, such as chemistry, control theory, physics, and so on [11, 15]. The discrete Hamiltonian system has found a great deal of interest last years because not only it is important in applications but it provides a good model for developing mathematical methods. For more information on discrete Hamiltonian systems, we refer the reader to [1, 2].

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In the past ten years, many authors have studied the existence and multiplicity of homoclinic solutions for system (1.1) via variational methods, see [5–8, 12–14, 16–20, 23]. Most of them treated the superquadratic case [5, 6, 8, 12–14, 16, 19, 23]; the subquadratic case can be found in [17, 18, 23]; and except for [7], there are only a few papers that have studied the asymptotically quadratic case. Besides, among them, except for [6, 7, 16, 18, 23] all known results were obtained under the assumption that $L(n)$ is positive definite for all $n \in \mathbb{Z}$.

In this paper, motivated by the above papers, we study the existence of infinitely homoclinic solutions for system (1.1) under more relaxed assumptions on L and W, which generalize and improve some results in the references that we have mentioned above.

1.1. The asymptotically quadratic case

In 2014, by using the variant fountain theorem, Chen and He [7] obtained the following theorem.

Theorem 1.1. *Assume that the following conditions are satisfied.*

(F_1) $L(n)$ *is an* $N \times N$ *real symmetric matrix for all* $n \in \mathbb{Z}$ *and* $l(n) = \inf_{u \in \mathbb{R}^N, |u|=1} (L(n)u, u) \to +\infty$ *as* $|n| \to \infty$.

(F_2) $W \colon \mathbb{Z} \times \mathbb{R}^N \to \mathbb{R}$ *and* $W(n,u)$ *is continuously differentiable in* u *for every* $n \in \mathbb{Z}$.

(F_3) $W(n,u) = \frac{1}{2}\mu' |u|^2 + F(n,u)$, *where* $\mu' > 0$ *and* $\mu' \notin \sigma(A)$ *(*A *is defined in* (2.1)*, and* σ *denotes the spectrum).*

(F_4) $\widehat{a}(n) |u|^{\overline{\gamma}} \le (\nabla F(n,u), u)$, $|\nabla F(n,u)| \le \widehat{b}(n) |u|^{\overline{\gamma}-1} + \widehat{c}(n) |u|^{\overline{\nu}-1}$, $\forall\, (n,u) \in \mathbb{Z} \times \mathbb{R}^N$, *where* $\widehat{a}, \widehat{b}, \widehat{c} \colon \mathbb{Z} \to \mathbb{R}$ *are positive functions such that* $\widehat{a}, \widehat{b} \in l^{2/(2-\overline{\gamma})}(\mathbb{Z},\mathbb{R})$, $\widehat{c} \in l^{2/(2-\overline{\nu})}(\mathbb{Z},\mathbb{R})$ *and* $1 < \overline{\gamma} < 2$, $1 < \overline{\nu} < 2$ *are constants,* $F(n,0) = 0$, $F(n,u) = F(n,-u)$ *for all* $(n,u) \in \mathbb{Z} \times \mathbb{R}^N$.

Then system (1.1) *(with* $p = I_N$*) possesses infinitely many nontrivial homoclinic solutions.*

A natural question is whether system (1.1) possesses infinitely many nontrivial homoclinic solutions if $\mu' \in \sigma(A)$ or $\mu' < 0$. In order to solve the question, the main difficulties are: how to prove the boundedness of the (PS) (or (C)) sequence of the corresponding functional and how to check that the variational functional has mountain pass geometry. To overcome these difficulties, we have to find some new methods and techniques.

We will use the following conditions:

(R_1) $L(n)$ is a real symmetric matrix for all $n \in \mathbb{Z}$ and there exists $0 < \overline{\sigma} < 2$ such that $l(n) |n|^{-\overline{\sigma}} \to \infty$ as $|n| \to \infty$ where $l(n) = \inf_{u \in \mathbb{R}^N, |u|=1} (L(n)u, u)$.

(R_2) $\nabla W(n,u) = V(n)u + \nabla F(n,u)$, where $V\colon \mathbb{Z} \to \mathbb{R}^{N\times N}$ is bounded symmetric $N\times N$ matrix-valued function.

(R_3) $F(n,0) = 0$ and there exist $m_1', m_2' > 0$ and $\max\left\{\frac{1}{2}, \frac{1}{1+\sigma}\right\} < \nu_1 < \nu_2 < 1$ such that

$$|\nabla F(n,u)| \le m_1' |u|^{\nu_1} + m_2' |u|^{\nu_2}, \quad \forall (n,u) \in \mathbb{Z}\times\mathbb{R}^N.$$

(R_4) $W\colon \mathbb{Z}\times\mathbb{R}^N \to \mathbb{R}$ and $W(n,u)$ is continuously differentiable in u for every $n \in \mathbb{Z}$.

(R_5) $W(n,-u) = W(n,u)$, $\forall (n,u) \in \mathbb{Z}\times\mathbb{R}^N$.

(R_6) $\lim_{|u|\to 0} \frac{W(n,u)}{|u|^2} = +\infty$ uniformly in $n \in \mathbb{Z}$.

(R_7) $\widehat{W}(n,u) = 2W(n,u) - (\nabla W(n,u), u) \to +\infty$ as $|u| \to \infty$ uniformly in $n \in \mathbb{Z}$.

(R_8) $\widehat{W}(n,u) \ge 0$, $\forall (n,u) \in \mathbb{Z}\times\mathbb{R}^N$.

Up to now, we state our main result.

Theorem 1.2. *Assume that* (R_1)–(R_8) *hold. Then system* (1.1) *(with $p = I_N$) possesses infinitely many nontrivial homoclinic solutions.*

1.2. The supquadratic case

In 2006, by using the symmetric mountain pass theorem, Ma and Guo [13] obtained the following theorem.

Theorem 1.3. *Assume that the following conditions are satisfied.*

(W_1) $p(n) > 0$ *for all* $n \in \mathbb{Z}$.

(W_2) $L(n) > 0$ *for all* $n \in \mathbb{Z}$ *and* $\lim_{|n|\to\infty} L(n) = +\infty$.

(W_3) *There exists a constant* $\widetilde{\rho} > 2$ *such that*

$$0 < \widetilde{\rho} W(n,x) \le \nabla W(n,x)x, \quad \forall (n,x) \in \mathbb{Z}\times\mathbb{R}\setminus\{0\}.$$

(W_4) $|\nabla W(n,x)| = o(|x|)$ *as* $|x| \to 0$ *uniformly for* $n \in \mathbb{Z}$.

(W_5) $W(n,-x) = W(n,x)$, $\forall (n,x) \in \mathbb{Z}\times\mathbb{R}$.

Then system (1.1) *(with $N = 1$) possesses an unbounded sequence of homoclinic solutions.*

Remark 1.4. The condition (W_3) implies that $\lim_{|x|\to\infty} \frac{W(n,x)}{|x|^2} = +\infty$ uniformly for all $n \in \mathbb{Z}$, and

$$\frac{1}{2}\nabla W(n,x)x - W(n,x) \ge \left(\frac{\widetilde{\rho}}{2} - 1\right) W(n,x) \to +\infty \quad \text{as } |u| \to \infty.$$

In 2011, by using the symmetric mountain pass theorem, Lin and Tang [12] obtained the following theorem.

Theorem 1.5. *Assume that the following conditions are satisfied.*

(W_6) *$p(n)$ is a real symmetric positive definite matrix for all $n \in \mathbb{Z}$.*

(W_7) *$L(n)$ is a real symmetric positive definite matrix for all $n \in \mathbb{Z}$ and there exists a function $\widehat{l}\colon \mathbb{Z} \to (0,\infty)$ such that $\widehat{l}(n) \to \infty$ as $n \to \infty$ and*

$$(L(n)u,u) \geq \widehat{l}(n)\,|u|^2, \quad \forall\,(n,u) \in \mathbb{Z} \times \mathbb{R}^N.$$

(W_8) *$W(n,u) = W_1(n,u) - W_2(n,u)$ for every $n \in \mathbb{Z}$, W_1 and W_2 are continuously differentiable in u, and*

$$\frac{1}{\widehat{l}(n)}\,|\nabla W(n,u)| = o(|u|) \quad \textit{as } |u| \to 0$$

uniformly in $n \in \mathbb{Z}$.

(W_9) *$W(n,-u) = W(n,u)$, $\forall\,(n,u) \in \mathbb{Z} \times \mathbb{R}^N$.*

(W_{10}) *For any $\widehat{\tau} > 0$, there exist $\widehat{\tau}_1, \widehat{\tau}_2 > 0$ and $\widetilde{\nu}_1 < 2$ such that*

$$0 \leq \left(2 + \frac{1}{\widehat{\tau}_1 + \widehat{\tau}_2\,|u|^{\widetilde{\nu}_1}}\right) W(n,u) \leq (\nabla W(n,u),u),$$

$\forall\,(n,u) \in \mathbb{Z} \times \left\{u \in \mathbb{R}^N : |u| \geq \widehat{\tau}\right\}$.

(W_{11}) *For any $n \in \mathbb{Z}$, $\lim_{s\to+\infty} s^{-2} \min_{|u|=1} W(n,su) = +\infty$.*

Then system (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Remark 1.6. The conditions (W_{10}) and (W_{11}) imply that

$$\frac{1}{2}(\nabla W(n,u),u) - W(n,u) \geq \frac{1}{2\widehat{\tau}_1 + 2\widehat{\tau}_2\,|u|^{\widetilde{\nu}_1}} W(n,u) \to +\infty \quad \text{as } |u| \to \infty.$$

In 2013, by using fountain theorem, Chen and He [6] obtained the following theorems.

Theorem 1.7. *Assume that* (W_6)*,* (W_9) *and the following conditions hold.*

(B_1) *$L(n)$ is a real symmetric matrix for all $n \in \mathbb{Z}$ and $l(n) = \inf_{u\in\mathbb{R}^N,|u|=1}(L(n)u,u) \to +\infty$ as $|n| \to \infty$.*

(B_2) *$\lim_{|u|\to\infty} \frac{W(n,u)}{|u|^2} = +\infty$ uniformly for all $n \in \mathbb{Z}$.*

(B_3) *There exists* $\lambda > 0$ *such that* $W(n,u) \geq -\lambda |u|^2$ *for all* $(n,u) \in \mathbb{Z} \times \mathbb{R}^N$.

(B_4) $W(n,0) = 0$ *and there exist* $M_0 > 0$ *and* $\nu > 2$ *such that*

$$|\nabla W(n,u)| \leq M_0 \left(|u| + |u|^{\nu-1}\right), \quad \forall\, (n,u) \in \mathbb{Z} \times \mathbb{R}^N.$$

(B_5) *There exist* $\rho > 2$, $m_0 > 0$, $0 \leq m_1 < (\rho-2)/2$ *and* $0 < \theta < 2$ *such that*

$$(\nabla W(n,u), u) - \rho W(n,u) \geq -m_0 |u|^2 - m_1 (L(n)u, u) - m_2(n) |u|^\theta - m_3(n),$$

$\forall\, (n,u) \in \mathbb{Z} \times \mathbb{R}^N$, *where* $m_2 \in l^{2/(2-\theta)}(\mathbb{Z}, \mathbb{R}^+)$ *and* $m_3 \in l^1(\mathbb{Z}, \mathbb{R}^+)$.

Then system (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Theorem 1.8. *Assume that* (W_6), (W_9), (B_1)–(B_4) *and the following condition hold.*

(B_6) *There exist* $\widetilde{\nu}_2 \geq \gamma - 1$, $c > 0$ *and* $T_1 > 0$ *such that*

$$\begin{aligned} (\nabla W(n,u), u) - 2W(n,u) &\geq c\,|u|^{\widetilde{\nu}_2}, && \forall\, n \in \mathbb{Z},\ \forall\, |u| \geq T_1, \\ (\nabla W(n,u), u) &\geq 2W(n,u), && \forall\, n \in \mathbb{Z},\ \forall\, |u| \leq T_1. \end{aligned}$$

Then system (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Remark 1.9. The conditions (B_2) and (B_6) imply that

$$\frac{1}{2}(\nabla W(n,u), u) - W(n,u) \geq \frac{c}{2} |u|^{\widetilde{\nu}_2} \to +\infty \quad \text{as } |u| \to \infty.$$

In 2013, by using the symmetric mountain pass theorem, Tang and Chen [16] obtained the following theorems.

Theorem 1.10. *Assume that* (W_6), (W_9), (B_1) *and the following conditions hold.*

(B_7) $W(n,u)$ *is continuously differentiable in* u *for every* $n \in \mathbb{Z}$, $W(n,0) = 0$, *and there exist constants* $c_0 > 0$ *and* $\widetilde{T}_0 > 0$ *such that*

$$|\nabla W(n,u)| \leq c_0 |u|, \quad \forall\, (n,u) \in \mathbb{Z} \times \mathbb{R}^N,\ |u| \leq \widetilde{T}_0.$$

(B_8) $\lim_{|u|\to\infty} \frac{W(n,u)}{|u|^2} = +\infty$ *for all* $n \in \mathbb{Z}$, *and*

$$W(n,u) \geq 0, \quad \textit{forall}\, (n,u) \in \mathbb{Z} \times \mathbb{R}^N,\ |u| \geq \widetilde{T}_0.$$

(B_9) $\widetilde{W}(n,u) := \frac{1}{2}(\nabla W(n,u), u) - W(n,u) \geq g(n)$, $\forall\, (n,u) \in \mathbb{Z} \times \mathbb{R}^N$, *where* $|g| \in l^1(\mathbb{Z}, \mathbb{R})$, *and there exists* $c_1 > 0$ *such that*

$$|W(n,u)| \leq c_1 |u|^2\, \widetilde{W}(n,u), \quad \forall\, (n,u) \in \mathbb{Z} \times \mathbb{R}^N,\ |u| \geq \widetilde{T}_0.$$

Then system (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Remark 1.11. The conditions (B_8) and (B_9) imply that

$$\widetilde{W}(n,u) \geq \frac{|W(n,u)|}{c_1 |u|^2} \to +\infty \quad \text{as } |u| \to \infty.$$

Theorem 1.12. *Assume that* (W_6), (W_9), (B_1), (B_7)–(B_8) *and the following condition hold.*

(B_{10}) *There exist* $\widehat{\rho}_1 > 2$ *and* $\widehat{\rho}_2 > 0$ *such that*

$$\widehat{\rho}_1 W(n,u) \leq (\nabla W(n,u), u) + \widehat{\rho}_2 |u|^2, \quad \forall (n,u) \in \mathbb{Z} \times \mathbb{R}^N.$$

Then system (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Remark 1.13. The conditions (B_8) and (B_{10}) imply that

$$\widetilde{W}(n,u) \to +\infty \quad \text{as } |u| \to \infty.$$

In this paper, using a different approach to that of [6, 12, 13, 16], we establish the existence of infinitely many nontrivial homoclinic solutions for system (1.1) under more general conditions, which generalize and improve the results mentioned above.

We will use the following conditions:

(H_1) $\lim_{|u|\to\infty} \frac{W(n,u)}{|u|^2} = +\infty$ uniformly for all $n \in \mathbb{Z}$, and there exists $M > 0$ such that

$$W(n,u) \geq 0, \quad \forall (n,u) \in \mathbb{Z} \times \mathbb{R}^N, \ |u| \geq M.$$

(H_2) $\widetilde{W}(n,u) := \frac{1}{2}(\nabla W(n,u), u) - W(n,u) \geq h(n)$, $\forall (n,u) \in \mathbb{Z} \times \mathbb{R}^N$, where $h \in l^1(\mathbb{Z}, \mathbb{R})$, $\lim_{|u|\to\infty} \widetilde{W}(n,u) = +\infty$ for all $n \in \mathbb{Z}$ and there exists $T_0 > 0$ such that

$$\widetilde{W}(n,u) \geq 0, \quad \forall (n,u) \in \mathbb{Z} \times \mathbb{R}^N, \ |u| \geq T_0.$$

(H_3) There exist $\eta > 2$, $k_0 > 0$, $0 \leq k_1 < (\eta - 2)/2$, $0 < \overline{\kappa} < 2$ and $T > 0$ such that

$$(\nabla W(n,u), u) - \eta W(n,u) \geq -k_0 |u|^2 - k_1 (L(n)u, u) - k_2(n) |u|^{\overline{\kappa}} - k_3(n),$$

$\forall (n,u) \in \mathbb{Z} \times \mathbb{R}^N$, $|u| \geq T$, where $k_2 \in l^{2/(2-\overline{\kappa})}(\mathbb{Z}, \mathbb{R}^+)$ and $k_3 \in l^1(\mathbb{Z}, \mathbb{R}^+)$.

Up to now, we state our main results.

Theorem 1.14. *Assume that* (W_6), (W_9), (B_1), (B_4) *and* (H_1)–(H_2) *hold. Then system* (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Remark 1.15. Theorem 1.14 generalizes and improves Theorems 1.3 and 1.5. First, it is easy to see that condition (B_1) is weaker than (W_2) in Theorem 1.3 and (W_7) in Theorem 1.5. Secondly, we remove (W_4) in Theorem 1.3 and (W_8) in Theorem 1.5. Lastly, (W_3) in Theorem 1.3 implies (H_2), as do (W_{10}) and (W_{11}) in Theorem 1.5, see Remarks 1.4 and 1.6.

Remark 1.16. Theorem 1.14 generalizes and improves Theorems 1.8, 1.10 and 1.12. In fact, as stated in Remarks 1.9, 1.11 and 1.13, we can deduce condition (H_2) from the supquadratic conditions of Theorems 1.8, 1.10 and 1.12.

Remark 1.17. There are many functions $W(n,u)$ satisfy our Theorem 1.14, but they do not satisfy Theorems 1.3, 1.5, 1.8, 1.10 and 1.12. For example, let $W(n,u) = |u|^2(|u|^2 + \sin|u|^2) + |u|^2(1 - 1/(1+|u|)) - 20|u|^2$.

Theorem 1.18. *Assume that* (W_6), (W_9), (B_1), (B_4), (H_1) *and* (H_3) *hold. Then system* (1.1) *possesses an unbounded sequence of homoclinic solutions.*

Remark 1.19. Theorem 1.18 generalizes and improves Theorems 1.3, 1.7 and 1.12. In fact, (W_3) in Theorem 1.3, (B_5) in Theorem 1.7 or (B_{10}) in Theorem 1.12 implies our condition (H_3). Furthermore, there are many functions $W(n,u)$ satisfy our Theorem 1.18, but they do not satisfy Theorems 1.3, 1.7 and 1.12. For example, let $W(n,u) = \frac{1}{1+n^2}|u|^6 - 2|u|^2(1 - 1/(1+|u|))$.

The remainder of this paper is organized as follows. In Section 2, we give the proof of Theorem 1.2. In Section 3, we give the proofs of Theorems 1.14 and 1.18.

2. The asymptotically quadratic case

In the following, we will present some definitions and lemmas that will be used in the proof of our result. As usual, for $1 \le q < +\infty$, $j = 1$ or N, let

$$l^q(\mathbb{Z},\mathbb{R}^j) = \left\{ \{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^j, n \in \mathbb{Z}, \sum_{n\in\mathbb{Z}} |u(n)|^q < +\infty \right\}$$

and

$$l^\infty(\mathbb{Z},\mathbb{R}^j) = \left\{ \{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^j, n \in \mathbb{Z}, \sup_{n\in\mathbb{Z}} |u(n)| < +\infty \right\},$$

and their norms are defined by

$$\|u\|_q = \left(\sum_{n\in\mathbb{Z}} |u(n)|^q \right)^{1/q}, \ \forall\, u \in l^q(\mathbb{Z},\mathbb{R}^j); \quad \|u\|_\infty = \sup_{n\in\mathbb{Z}} |u(n)|, \ \forall\, u \in l^\infty(\mathbb{Z},\mathbb{R}^j),$$

respectively. It is well known that $l^q(\mathbb{Z},\mathbb{R}^N)$ is a Banach space for each $q \geq 1$ and $l^2(\mathbb{Z},\mathbb{R}^N)$ is a Hilbert space with the following inner product:

$$(u,v)_2 = \sum_{n\in\mathbb{Z}} (u(n), v(n)), \quad u,v \in l^2(\mathbb{Z},\mathbb{R}^N).$$

Now, we study the following difference operator:

$$\begin{aligned} A(u)(n) &= -\Delta^2 u(n-1) + L(n)u(n), \quad u = \{u(n)\}_{n\in\mathbb{Z}} \in D(A), \\ D(A) &= \left\{ u \in l^2(\mathbb{Z},\mathbb{R}^N) : Au \in l^2(\mathbb{Z},\mathbb{R}^N) \right\}. \end{aligned} \tag{2.1}$$

Lemma 2.1. [23] *Assume that L satisfies* (R_1). *Then, operator A is self-adjoint, has a pure discrete spectrum, and is bounded from below.*

Let $D(A)$ and $R(A)$ be the domain and range of K, respectively.

Lemma 2.2. [23] *Let A be a self-adjoint operator on a Hilbert space Y. Then, $A = U\,|A|$, where $|A|$ is nonnegative and self-adjoint and U is partially isometric operator on $\overline{R(|A|)}$. And, $|A|$ has one nonnegative and self-adjoint second root $|A|^{1/2}$. Further, $D(A) = D(|A|) \subset D(|A|^{1/2})$, $D(|A|)$ is the core of $|A|^{1/2}$, $|A| = UA = AU$, $|A|\,U = U\,|A|$, and $|A|^{1/2}\,U = U\,|A|^{1/2}$.*

If A is self-adjoint, then there is a spectral family E such that $A = \int \varsigma\, dE(\varsigma)$ (see, [21, Theorem 7.17]*). And, by the discussions on* [10, p. 358]*, one has*

$$|A| = \int_{-\infty}^{+\infty} |\mu|\, dE(\mu), \quad |A|^{1/2} = \int_{-\infty}^{+\infty} |\mu|^{1/2}\, dE(\mu), \quad U = I - E(0) - E(0-), \tag{2.2}$$

where I is the identity operator. By Lemma 2.1*, the distinct eigenvalues of A can be ordered as $\mu_1 < \mu_2 < \cdots \to \infty$. Let H_i be the eigenspace of A with respect to μ_i, P_i be the orthogonal projection from $l^2(\mathbb{Z},\mathbb{R}^N)$ to H_i. Hence, the spectral family of A is*

$$E(\mu) = \sum_{\mu_i \leq \mu} P_i. \tag{2.3}$$

Let $X := D(|A|^{1/2})$ be the domain of the self-adjoint operator $|A|^{1/2}$, which is a Hilbert space equipped with the inner product and norm given by

$$(u,v)_0 = (|A|^{1/2}\, u, |A|^{1/2}\, v)_2 + (u,v)_2, \quad \|u\|_0 = (u,u)_0^{1/2}$$

for $u, v \in X$.

Lemma 2.3. [23] *If* (R_1) *holds, then $(X, \|\cdot\|_0)$ is compactly embedded in $l^q(\mathbb{Z},\mathbb{R}^N)$ for each $1 < q \in (2/(1+\overline{\sigma}), +\infty]$.*

By Lemma 2.1, the spectrum $\sigma(A)$ consists of eigenvalues numbered by $\eta_1 \leq \eta_2 \leq \cdots \leq \eta_m \leq \cdots \to \infty$ (counted in their multiplicities) and a corresponding system of eigenfunctions $\{e_m\}$, $Ke_m = \eta_m e_m$ which forms an orthogonal basis in $l^2(\mathbb{Z}, \mathbb{R}^N)$.

Set

$$n^- = \sharp\{i \mid \eta_i < 0\}, \quad n^0 = \sharp\{i \mid \eta_i = 0\}, \quad n^+ = n^- + n^0, \tag{2.4}$$

and let

$$l^2(\mathbb{Z}, \mathbb{R}^N) = l^- \oplus l^0 \oplus l^+ \tag{2.5}$$

be the orthogonal decomposition in $l^2(\mathbb{Z}, \mathbb{R}^N)$ with

$$l^- = \operatorname{span}\{e_1, \ldots, e_{n^-}\}, \quad l^0 = \operatorname{span}\{e_{n^-+1}, \ldots, e_{n^0}\},$$
$$l^+ = (l^- \oplus l^0)^\perp = \overline{\operatorname{span}\{e_{n^++1}, \ldots\}}.$$

Now we introduce on X the following inner product

$$(u, v)_* = (|A|^{1/2} u, |A|^{1/2} v)_2 + (u^0, v^0)_2 \tag{2.6}$$

and the norm

$$\|u\| = (u, u)_*^{1/2}, \tag{2.7}$$

where $u = u^- + u^0 + u^+$ and $v = v^- + v^0 + v^+$ with respect to the decomposition (2.5). It is easy to verify that $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent, see [23]. Thus, there exist $M_1, M_2 > 0$ such that

$$M_1 \|u\| \leq \|u\|_0 \leq M_2 \|u\| \tag{2.8}$$

for any $u \in X$. By Lemma 2.3, there exists $N_q, C_q > 0$ such that

$$\|u\|_q \leq N_q \|u\|, \quad \|u\|_q \leq C_q \|u\|_0 \tag{2.9}$$

for any $u \in X$ and $1 < q \in (2/(1+\sigma), +\infty]$.

Remark 2.4. It is easy to check that X possesses the orthogonal decomposition

$$X = X^- \oplus X^0 \oplus X^+$$

with

$$X^- = l^-, \quad X^0 = l^0 \quad \text{and} \quad X^+ = X \cap l^+ = \overline{\operatorname{span}\{e_{n^++1}, \ldots\}},$$

where the closure is taken with respect to the norm $\|\cdot\|_0$. Evidently, the above decomposition is also orthogonal in $l^2(\mathbb{Z}, \mathbb{R}^N)$.

By (2.2) and (2.3), we have

$$Uu = (I - E(0) - E(0-))u = u - (u^- + u^0) - u^- = u^+ - u^- \tag{2.10}$$

for any $u \in l^2(\mathbb{Z}, \mathbb{R}^N)$. For $g \in C^1(X, \mathbb{R})$, by $D_0 g(x)$ denote the Fréchet derivative of g at x in $(X, (\cdot\,,\cdot)_0)$. Introduce the following functional on $D(|K|^{1/2})$:

$$f(u) := \frac{1}{2}(|A|^{1/2}\, Uu, |A|^{1/2}\, u)_2, \quad \forall\, u \in D(|K|^{1/2}). \tag{2.11}$$

By (2.10), we obtain

$$\begin{aligned} f(u) &= \frac{1}{2}(|A|^{1/2}\,(u^+ - u^-), |A|^{1/2}\, u)_2 \\ &= \frac{1}{2}(|A|^{1/2}\,(u^+ - u^-), |A|^{1/2}\,(u^+ + u^-))_2 \\ &= \frac{1}{2}(\left\|u^+\right\|^2 - \left\|u^-\right\|^2), \quad \forall\, u \in D(|A|^{1/2}). \end{aligned} \tag{2.12}$$

Lemma 2.5. [23] *If* (R_1) *holds, then the functional* f *is* C^1 *in* $(X, (\cdot\,,\cdot)_0)$*, and*

$$D_0 f(u)v = ((I - (|A| + I)^{-1})Uu, v)_0 = (|A|^{1/2}\, Uu, |A|^{1/2}\, v)_2, \quad \forall\, u, v \in X. \tag{2.13}$$

Lemma 2.6. *Assume that* (R_1)–(R_4) *hold. Then the functional* $\Phi\colon X \to \mathbb{R}$ *defined by*

$$\begin{aligned} \Phi(u) &= f(u) - \sum_{n\in\mathbb{Z}} W(n, u(n)) \\ &= \frac{1}{2}(\left\|u^+\right\|^2 - \left\|u^-\right\|^2) - \sum_{n\in\mathbb{Z}} W(n, u(n)), \quad \forall\, u \in E \end{aligned} \tag{2.14}$$

is well defined and of class $C^1(X, \mathbb{R})$ *and*

$$D_0\Phi(u)v = (|A|^{1/2}\, Uu, |A|^{1/2}\, v)_2 - (\nabla W(\cdot\,, u), v)_2, \quad \forall\, u, v \in X. \tag{2.15}$$

Furthermore, the critical points of Ψ *in* X *are solutions of system* (1.1) *with* $u(\pm\infty) = 0$.

Proof. By using the same methods in [7, Lemma 2.6], we easily obtain the aforementioned result, and we omit it here. □

Lemma 2.7. [9] *Let* X *be an infinite-dimensional Banach space and* $\Phi \in C^1(X, \mathbb{R})$ *be even, satisfy the* (PS) *condition, and* $\Phi(0) = 0$. *If* $X = Y \oplus Z$*, where* Y *is finite-dimensional, and* Φ *satisfies*

- (G_1) Φ *is bounded from below on* Z*;*
- (G_2) *for each finite-dimensional subspace* $\widetilde{X} \subset X$*, there are positive constants* $\rho = \rho(\widetilde{X})$ *and* $\alpha = \alpha(\widetilde{X})$ *such that* $\Phi|_{B_\rho \cap \widetilde{X}} \le 0$ *and* $\Phi|_{\partial B_\rho \cap \widetilde{X}} \le -\alpha$*, where* $B_\rho = \{x \in X : \left\|x\right\|_0 \le \rho\}$.

Then Φ possesses infinitely many nontrivial critical points.

Remark 2.8. As shown in [3], a deformation lemma can be proved with (C) condition replacing (PS) condition, and it turns out that Lemma 2.7 holds true under (C) condition. We say that Φ satisfies (C) condition, i.e., for any $\{u_m\} \subset X$, $\{u_m\}$ has a convergent subsequence if $\Phi(u_m)$ is bounded and $(1 + \|u_m\|_0) \|D_0\Phi(u_m)\| \to 0$ as $m \to \infty$.

Define $X_j = \mathbb{R}e_j$,

$$Y_k = \bigoplus_{j=0}^{k} X_j, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}, \quad k \in \mathbb{N}. \tag{2.16}$$

Lemma 2.9. *Under assumption* (R_1), *for* $1 < \tau \in (2/(1+\overline{\sigma}), +\infty]$,

$$\beta_k(\tau) = \sup_{\substack{u \in Z_k \\ \|u\|_0 = 1}} \|u\|_\tau \to 0 \quad ask \to \infty. \tag{2.17}$$

Proof. It is clear that $0 < \beta_{k+1}(\tau) \leq \beta_k(\tau)$, so that $\beta_k(\tau) \to \overline{\beta}(\tau)$, $k \to \infty$. For every $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\|_0 = 1$ and $\|u_k\|_\tau > \beta_k/2$. For any $v \in X$, let $v = \sum_{i=1}^{\infty} \overline{b_i} e_i$, by the Cauchy-Schwartz inequality, one has

$$\begin{aligned} |(u_k, v)_0| &= \left| \left(u_k, \sum_{i=1}^{\infty} \overline{b_i} e_i \right)_0 \right| = \left| \left(u_k, \sum_{i=k}^{\infty} \overline{b_i} e_i \right)_0 \right| \\ &\leq \|u_k\|_0 \left\| \sum_{i=k}^{\infty} \overline{b_i} e_i \right\|_0 = \sum_{i=k}^{\infty} \left\| \overline{b_i} e_i \right\|_0 \to 0 \quad \text{as } k \to \infty, \end{aligned}$$

which implies that $u_k \rightharpoonup 0$. It follows from Lemma 2.3 that $u_k \to 0$ in $l^q(\mathbb{Z}, \mathbb{R}^N)$. Thus we have proved that $\overline{\beta}(\tau) = 0$.

By Lemma 2.9, we can choose a positive integer $k_0 \geq n^+ + 1$ such that

$$\|u\|_2^2 \leq \frac{1}{4m_0' M_2^2} \|u\|_0^2, \tag{2.18}$$

where $m_0' = \sup_{n \in \mathbb{Z}} \left[\sup_{x \in \mathbb{R}^N, |x|=1} (V(n)x, x) \right]$. □

Now we give the proof of Theorem 1.2.

Proof of Theorem 1.2. Let $Y = Y_{k_0}$, $Z = Z_{k_0}$. Obviously, $\Phi \in C^1(X, \mathbb{R})$ be even and $\Phi(0) = 0$. In the following, we will check that all conditions in Lemma 2.7 are satisfied.

First, we verify condition (G_1) in Lemma 2.7. By (R_2), (R_3), (2.8), (2.9) and (2.18), we have

$$
\begin{aligned}
\Phi(u) &= \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 - \sum_{n\in\mathbb{Z}} W(n,u(n)) \\
&= \frac{1}{2}\|u\|^2 - \sum_{n\in\mathbb{Z}} W(n,u(n)) \\
&\geq \frac{1}{2M_2^2}\|u\|_0^2 - m_0'\|u\|_2^2 - m_1'\|u\|_{\nu_1+1}^{\nu_1+1} - m_2'\|u\|_{\nu_2+1}^{\nu_2+1} \\
&\geq \frac{1}{4M_2^2}\|u\|_0^2 - m_1' C_{\nu_1+1}^{\nu_1+1}\|u\|_0^{\nu_1+1} - m_2' C_{\nu_2+1}^{\nu_2+1}\|u\|_0^{\nu_2+1} \to +\infty
\end{aligned} \tag{2.19}
$$

as $\|u\|_0 \to \infty$ and $u \in Z_{k_0}$.

Secondly, we verify condition (G_2) in Lemma 2.7. Let $\widetilde{X} \subset X$ be any finite-dimensional subspace. Then there exists $T > 0$ such that

$$
\frac{1}{M_1^2}\|u\|_0^2 \leq T\|u\|_2^2, \quad \forall\, u \in \widetilde{X}. \tag{2.20}
$$

By virtue of (R_6), for T given above, there exists a constant $\delta > 0$

$$
W(n,u) \geq T|u|^2, \quad \forall\, n \in \mathbb{Z} \text{ and } |u| \leq \delta. \tag{2.21}
$$

By (2.9), for any $u \in \widetilde{X}$ with $\|u\|_0 \leq \delta/C_\infty$, there holds

$$
\|u\|_\infty \leq \delta. \tag{2.22}
$$

In view of (2.8) and (2.20)–(2.22), we have

$$
\begin{aligned}
\Phi(u) &= \frac{1}{2}\left\|u^+\right\|^2 - \frac{1}{2}\left\|u^-\right\|^2 + \sum_{n\in\mathbb{Z}} W(n,u(n)) \\
&\leq \frac{1}{2}\left\|u^+\right\|^2 - \sum_{n\in\mathbb{Z}} W(n,u(n)) \\
&\leq \frac{1}{2}\left\|u^+\right\|^2 - T\sum_{n\in\mathbb{Z}} |u(n)|^2 \\
&\leq -\frac{1}{2M_1^2}\|u\|_0^2
\end{aligned} \tag{2.23}
$$

for any $u = u^- + u^0 + u^+ \in \widetilde{X}$ with $\|u\|_0 \leq \delta/C_\infty$. Then there exist $\rho = \rho(\widetilde{X}) > 0$ and $\alpha = \alpha(\widetilde{X}) > 0$ such that

$$
\Phi(u) \leq 0,\ \forall\, u \in B_\rho \cap \widetilde{X}; \quad \Phi(u) \leq -\alpha,\ \forall\, u \in \partial B_\rho \cap \widetilde{X}.
$$

Finally, we will show that Φ satisfies (C) condition. Assume that $\{u_m\} \subset X$ is a (C) sequence of Φ, that is, $\{\Phi(u_m)\}$ is bounded and

$$
(1 + \|u_m\|_0)\,\|D_0\Phi(u_m)\| \to 0 \quad \text{as } m \to \infty, \tag{2.24}
$$

then there exists a constant $\widetilde{T}_0 > 0$ such that

$$|\Phi(u_m)| \leq \widetilde{T}_0, \quad (1 + \|u_m\|_0) \|D_0\Phi(u_m)\| \leq \widetilde{T}_0 \tag{2.25}$$

for every $m \in \mathbb{N}$. Since the norms $\|\cdot\|$ and $\|\cdot\|_0$ are equivalent, we will use the norm $\|\cdot\|$ in the following discussions for convenience. We choose $k \geq n^+ + 1$ large enough such that

$$\|u\|^2 \geq 2m_0' \|u\|_2^2, \quad \forall\, u \in Z_k, \tag{2.26}$$

where $m_0' = \sup_{n\in\mathbb{Z}} \left[\sup_{x\in\mathbb{R}^N,|x|=1} (V(n)x, x)\right]$. We now prove that $\{u_m\}$ is bounded in X. In fact, if not, we may assume by contradiction that $\|u_m\| \to \infty$ as $m \to \infty$. Let $u_m = \overline{w}_m + \overline{v}_m$, $z_m = u_m/\|u_m\|$, then $\|z_m\| = 1$, $z_m = w_m + v_m \in X$, where $w_m = \overline{w}_m/\|u_m\|$, $v_m = \overline{v}_m/\|u_m\|$, $\overline{w}_m \in Y_k$, $\overline{v}_m \in Z_k$. After passing a subsequence, we obtain $z_m \rightharpoonup z$, $w_m \to w$, and $\zeta = \lim_{m\to\infty} \|v_m\|$ exists.

Case 1: $\zeta = 0$. Since $\dim Y_k < \infty$, then we obtain that $\|w_m\| \to \|w\| = 1$. By virtue of (2.25) that

$$\begin{aligned} 3\widetilde{T}_0 &\geq 2\Phi(u_m) - D_0\Phi(u_m)u_m \\ &\geq \sum_{n\in\mathbb{Z}} \left(2W(n, u_m(n)) - (\nabla W(n, u_m(n)), u_m(n))\right). \end{aligned} \tag{2.27}$$

By (R$_7$), for $\widetilde{R}_1 > 3\widetilde{T}_0$, there exists $\widetilde{R}_2 > 0$ such that

$$\widehat{W}(n, u) = 2W(n, u) - (\nabla W(n, u), u) \geq \widetilde{R}_1, \quad \forall\, n \in \mathbb{Z},\ |u| \geq \widetilde{R}_2. \tag{2.28}$$

For any $\varepsilon > 0$, define $\Lambda_\varepsilon := \{n \in \mathbb{Z} : |w(n)| \geq \varepsilon\}$ and $\Lambda_{m\varepsilon} := \{n \in \mathbb{Z} : |v_m(n)| \geq \varepsilon/2\}$. First, we claim that there exists $\varepsilon_0 > 0$ such that

$$\{n \in \mathbb{Z} : |u(n)| \geq \varepsilon_0\} \neq \emptyset, \quad \forall\, u \in Y_k \text{ with } \|u\| = 1.$$

Arguing indirectly, for any positive integer i, there exists $u_i \in Y_k \setminus \{0\}$ such that

$$\Upsilon_1 = \left\{n \in \mathbb{Z} : |w_i(n)| \geq \frac{1}{i}\right\} = \emptyset, \tag{2.29}$$

where $w_i = u_i/\|u_i\|$. Passing to a subsequence if necessary, we may assume $w_i \to w_0$ in X for some $w_0 \in Y_k$ since $\dim Y_k < \infty$. Evidently, $\|w_0\| = 1$. By the equivalence of the norms on the finite-dimensional space Y_k, we have $w_i \to w_0$ in $l^2(\mathbb{Z}, \mathbb{R}^N)$, i.e.,

$$\sum_{n\in\mathbb{Z}} |w_i(n) - w_0(n)|^2 \to 0 \quad \text{as } i \to \infty. \tag{2.30}$$

Thus there exists $\varepsilon_1 > 0$ such that

$$\Upsilon_2 = \{n \in \mathbb{Z} : |w_0(n)| \geq \varepsilon_1\} \neq \emptyset. \tag{2.31}$$

In fact, if not, for all positive integers i, we have

$$\Upsilon_2 = \left\{ n \in \mathbb{Z} : |w_0(n)| \geq \frac{1}{i} \right\} = \emptyset. \tag{2.32}$$

This implies that

$$0 \leq \sum_{n\in\mathbb{Z}} |w_0(n)|^4 < \frac{1}{i^2} \|w_0\|_2^2 \to 0 \quad \text{as } i \to \infty.$$

Hence $w_0 = 0$ which contradicts that $\|w_0\|_0 = 1$. Thus, (2.31) holds.

Now let

$$\Omega_0 = \{n \in \mathbb{Z} : |w_0(n)| \geq \varepsilon_1\}, \quad \Omega_i = \left\{ n \in \mathbb{Z} : |w_i(n)| < \frac{1}{i} \right\}.$$

By (2.29) and (2.31), we have

$$\Omega_0 \cap \Omega_i \neq \emptyset$$

for all positive integers i. Let i be large enough such that $\frac{1}{2}\varepsilon_1 - \frac{1}{i} > 0$. Then we have

$$\begin{aligned}\sum_{n\in\mathbb{Z}} |w_i(n) - w_0(n)|^2 &\geq \sum_{n\in\Omega_0\cap\Omega_i} |w_i(n) - w_0(n)|^2 \\ &\geq \frac{1}{2} \sum_{n\in\Omega_0\cap\Omega_i} |w_0(n)|^2 - \sum_{n\in\Omega_0\cap\Omega_i} |w_i(n)|^2 \\ &\geq \sum_{n\in\Omega_0\cap\Omega_i} \left(\frac{1}{2}\varepsilon_1 - \frac{1}{i} \right) > 0\end{aligned}$$

for all large i, which is a contradiction to (2.30). Thus, the claim above is true. Hence, there exists $\varepsilon > 0$ such that $\Lambda_\varepsilon := \{n \in \mathbb{Z} : |w(n)| \geq \varepsilon\} \neq \emptyset$.

By (2.9), we obtain

$$\sharp\Lambda_{m\varepsilon} \leq \frac{4}{\varepsilon^2} \sum_{n\in\mathbb{Z}} |v_m(n)|^2 \leq \frac{4N_2^2}{\varepsilon^2} \|v_m\|^2 \to 0 \quad \text{as } m \to \infty.$$

Then we have $\sharp(\Lambda_\varepsilon \setminus \Lambda_{m\varepsilon}) \to \sharp\Lambda_\varepsilon$ as $m \to \infty$. Therefore, there exists $N_0' > 0$ such that $|z_m(n)| \geq \varepsilon/3$, $\forall\, n \in \Lambda_\varepsilon \setminus \Lambda_{m\varepsilon}$ and $m \geq N_0'$, then we have $|u_m(n)| \geq \frac{\varepsilon}{3} \|u_m\|$, $\forall\, n \in \Lambda_\varepsilon \setminus \Lambda_{m\varepsilon}$ and $m \geq N_0'$. By (R_8), (2.27) and (2.28), we get

$$3\widetilde{T}_0 \geq \sum_{n\in\mathbb{Z}} \widehat{W}(n, u_m(n)) \geq \sum_{n\in\Lambda_\varepsilon\setminus\Lambda_{m\varepsilon}} \widetilde{R}_1 \geq \widetilde{R}_1 \quad \text{as } m \to \infty,$$

which gives a contradiction, since $\widetilde{R}_1 > 3\widetilde{T}_0$.

Case 2: $\zeta > 0$. By (R_2), (R_3), (2.26), (2.9) and Hölder's inequality, we get

$$\begin{aligned}T_0 \geq D_0\Phi(u_m)\overline{v}_m &= \|\widetilde{v}_m\|^2 - \sum_{n\in\mathbb{Z}} (\nabla W(n, u_m(n)), \overline{v}_m(n)) \\ &\geq \|\overline{v}_m\|^2 - \sum_{n\in\mathbb{Z}} \left[(V(n)u_m(n), \overline{v}_m(n)) + \left(m_1' |u_m(n)|^{\nu_1} + m_2' |u_m(n)|^{\nu_2} \right) |\overline{v}_m(n)| \right] \\ &\geq \|\overline{v}_m\|^2 - m_0' \|\overline{v}_m\|_2^2 - m_1' \|u_m\|_{2\nu_1}^{\nu_1} \|\overline{v}_m\|_2 - m_2' \|u_m\|_{2\nu_2}^{\nu_2} \|\overline{v}_m\|_2 \\ &\geq \frac{1}{2} \|\overline{v}_m\|^2 - m_1' N_2 N_{2\nu_1}^{\nu_1} \|u_m\|^{\nu_1+1} - m_2' N_2 N_{2\nu_2}^{\nu_2} \|u_m\|^{\nu_2+1}.\end{aligned} \tag{2.33}$$

Divided by $\|u_m\|^2$ on both sides of (2.33), we obtain

$$0 \geq \frac{\zeta^2}{2} > 0,$$

which gives a contradiction.

Therefore, $\{u_m\}$ is bounded in $\|\cdot\|$. Consequently, $\{u_m\}$ is bounded in $\|\cdot\|_0$. Next, we show that $\{u_m\}$ has a convergent subsequence in $\|\cdot\|_0$. In view of the boundness of $\{u_m\}$, without loss of generality, we may assume that

$$u_m \rightharpoonup u,\ u_m^+ \rightharpoonup u^+,\ u_m^- \to u^-,\ u_m^0 \to u^0 \quad \text{in } \|\cdot\|_0. \tag{2.34}$$

Now, we show that $u_m \to u$ in $\|\cdot\|$. By (2.15), we easily obtain that

$$\begin{aligned} \left\|u_m^+ - u^+\right\|^2 &= (D_0\Phi(u_m) - D_0\Phi(u))(u_m^+ - u^+) \\ &\quad + (\nabla W(\cdot\,, u_m) - \nabla W(\cdot\,, u), u_m^+ - u^+)_2. \end{aligned} \tag{2.35}$$

It is clear that

$$(D_0\Phi(u_m) - D_0\Phi(u))(u_m^+ - u^+) \to 0 \quad \text{as } m \to \infty. \tag{2.36}$$

By (2.9), (2.34), Lemma 2.3 and the Hölder's inequality, we have

$$\begin{aligned} &(\nabla W(\cdot\,, u_m) - \nabla W(\cdot\,, u), u_m^+ - u^+)_2 \\ &= \sum_{n\in\mathbb{Z}} (\nabla W(n, u_m(n)) - \nabla W(n, u(n)), u_m^+(n) - u^+(n)) \\ &\leq \left(m_0' \|u_m\|_2 + m_1' \|u_m\|_{2\nu_1}^{\nu_1} + m_2' \|u_m\|_{2\nu_2}^{\nu_2}\right) \left\|u_m^+ - u^+\right\|_2 \\ &\quad + \left(m_0' \|u\|_2 + m_1' \|u\|_{2\nu_1}^{\nu_1} + m_2' \|u\|_{2\nu_2}^{\nu_2}\right) \left\|u_m^+ - u^+\right\|_2 \\ &\leq \left(m_0' N_2 \|u_m\| + m_1' N_{2\nu_1}^{\nu_1} \|u_m\|^{\nu_1} + m_2' N_{2\nu_2}^{\nu_2} \|u_m\|^{\nu_2}\right) \left\|u_m^+ - u^+\right\|_2 \\ &\quad + \left(m_0' N_2 \|u\| + m_1' N_{2\nu_1}^{\nu_1} \|u\|^{\nu_1} + m_0' N_{2\nu_2}^{\nu_2} \|u\|^{\nu_2}\right) \left\|u_m^+ - u^+\right\|_2 \to 0 \end{aligned} \tag{2.37}$$

as $m \to \infty$. Therefore, by (2.35)–(2.37), we get $\|u_m^+ - u^+\| \to 0$ as $m \to \infty$, which, together with the fact that $\dim(X^- \oplus X^0) < \infty$, yields that $u_m \to u$ in $\|\cdot\|$. Consequently, $u_m \to u$ in $\|\cdot\|_0$. Hence, Φ satisfies (C) condition.

By Lemma 2.7, we get that Φ possesses infinitely many nontrivial critical points, that is, system (1.1) possesses infinitely many nontrivial homoclinic solutions. $\square$

3. The supquadratic case

Before establishing the variational setting for system (1.1), we have the following:

Remark 3.1. It follows from (B_1) that there exist $a > 0$ such that $\overline{L}(n) := L(n) + aI_N$ are real symmetric positive definite matrices for all $n \in \mathbb{Z}$.

In what follows, we always assume that $p(n)$ and $\overline{L}(n)$ are real symmetric positive definite matrices for all $n \in \mathbb{Z}$. Let $\bar{l}(n) = \inf_{u\in\mathbb{R}^N,|u|=1}(\overline{L}(n)u,u)$,

$$H = \left\{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^N, n \in \mathbb{Z}\right\},$$

$$X = \left\{u \in H : \sum_{n\in\mathbb{Z}} \left[(p(n+1)\Delta u(n), \Delta u(n)) + (\overline{L}(n)u(n), u(n))\right] < +\infty\right\}$$

and for $u, v \in X$, let

$$\langle u, v\rangle = \sum_{n\in\mathbb{Z}} \left[(p(n+1)\Delta u(n), \Delta v(n)) + (\overline{L}(n)u(n), v(n))\right],$$

and the corresponding norm is

$$\|u\| = \left\{\sum_{n\in\mathbb{Z}} \left[(p(n+1)\Delta u(n), \Delta u(n)) + (\overline{L}(n)u(n), u(n))\right]\right\}^{1/2}, \quad \forall\, u \in X.$$

Then X is a Hilbert space with the above inner product. As usual, for $1 \le q < +\infty$, $j = 1$ or N, let

$$l^q(\mathbb{Z}, \mathbb{R}^j) = \left\{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^j, n \in \mathbb{Z}, \sum_{n\in\mathbb{Z}} |u(n)|^q < +\infty\right\}$$

and

$$l^\infty(\mathbb{Z}, \mathbb{R}^j) = \left\{\{u(n)\}_{n\in\mathbb{Z}} : u(n) \in \mathbb{R}^j, n \in \mathbb{Z}, \sup_{n\in\mathbb{Z}} |u(n)| < +\infty\right\},$$

and their norms are defined by

$$\|u\|_q = \left(\sum_{n\in\mathbb{Z}} |u(n)|^q\right)^{1/q}, \ \forall\, u \in l^q(\mathbb{Z}, \mathbb{R}^j); \quad \|u\|_\infty = \sup_{n\in\mathbb{Z}} |u(n)|, \ \forall\, u \in l^\infty(\mathbb{Z}, \mathbb{R}^j),$$

respectively.

Lemma 3.2. [17] *For $u \in X$,*

$$\|u\|_\infty \le \frac{1}{\sqrt[4]{(\varrho_2 + 4\varrho_1)\varrho_2}} \|u\|, \tag{3.1}$$

where $\varrho_1 = \inf\left\{(p(n)x, x) : n \in \mathbb{Z}, x \in \mathbb{R}^N, |x| = 1\right\}$ and $\varrho_2 = \inf\left\{\bar{l}(n) : n \in \mathbb{Z}\right\}$.

Lemma 3.3. [18] *Assume that L satisfies* (B_1). *Then X is compactly embedded in $l^q(\mathbb{Z}, \mathbb{R}^N)$ for any $2 \le q < \infty$, and*

$$\|u\|_q^q \le \varrho_2^{-1}[(\varrho_2 + 4\varrho_1)\varrho_2]^{(2-q)/4} \|u\|^q, \qquad \forall\, u \in X, \tag{3.2}$$

$$\sum_{|n|>N} |u(n)|^q \le \frac{[(\varrho_2 + 4\varrho_1)\varrho_2]^{(2-q)/4}}{\min_{|s|\ge N_0} \bar{l}(s)} \|u\|^q, \qquad \forall\, u \in X,\ N_0 \ge 1. \tag{3.3}$$

Define the functional Φ on X by

$$\Phi(u) = \frac{1}{2}\|u\|^2 - \frac{a}{2}\|u\|_2^2 - \sum_{n\in\mathbb{Z}} W(n,u(n)). \tag{3.4}$$

Then $\Phi \in C^1(X,\mathbb{R})$ and one can easily check that

$$\langle \Phi'(u), v\rangle = \sum_{n\in\mathbb{Z}} [(p(n+1)\Delta u(n), \Delta v(n)) + (L(n)u(n), v(n)) - (\nabla W(n,u(n)), v(n))] \tag{3.5}$$

for any $u, v \in X$. Furthermore, the critical points of Φ in X are the solutions of system (1.1) with $u(\pm\infty) = 0$, see [7,18].

In this section, the following fountain theorem will be needed in our argument. Assume that X is a Banach with the norm $\|\cdot\|$ and $X = \overline{\bigoplus_{j\in\mathbb{N}} X_j}$, where X_j are finite-dimensional subspaces of X. For each $k \in \mathbb{N}$, let $Y_k = \bigoplus_{j=0}^{k} X_j$, $Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j}$. The functional Φ is said to satisfy (PS) condition if any sequence $\{u_i\}$ such that $\{\Phi(u_i)\}$ is bounded and $\Phi'(u_i) \to 0$ as $i \to \infty$ has a convergent subsequence.

Theorem 3.4. [4,22] *Suppose that the functional $\Phi \in C^1(X,\mathbb{R})$ is even. If, for every $k \in \mathbb{N}$, there exist $\rho_k > r_k > 0$ such that*

(A$_1$) $a_k := \max_{u\in Y_k, \|u\|=\rho_k} \Phi(u) \le 0$;

(A$_2$) $b_k := \inf_{u\in Z_k, \|u\|=r_k} \Phi(u) \to +\infty$ *as* $k \to \infty$;

(A$_3$) Φ *satisfies* (PS) *condition.*

Then Φ possesses an unbounded sequence of critical values.

Remark 3.5. As shown in [3], a deformation lemma can be proved with (C) condition replacing (PS) condition, and it turns out that Theorem 3.4 holds true under (C) condition. We say that Φ satisfies (C) condition, i.e., for any $\{u_m\} \subset X$, $\{u_m\}$ has a convergent subsequence if $\Phi(u_m)$ is bounded and $(1 + \|u_m\|)\|\Phi'(u_m)\| \to 0$ as $m \to \infty$.

Now we give the proof of Theorem 1.14.

Proof of Theorem 1.14. We choose a completely orthonormal basis $\{e_j\}$ of X and define $X_j := \mathbb{R}e_j$, then Z_k and Y_k can be defined as that in Theorem 3.4. Obviously, $\Phi \in C^1(X,\mathbb{R})$ is even. Next we will check that all conditions in Theorem 3.4 are satisfied.

Step 1. We verify condition (A$_2$) in Theorem 3.4. Let $\lambda_k = \sup_{u\in Z_k, \|u\|=1} \|u\|_2$ and $\beta_k = \sup_{u\in Z_k, \|u\|=1} \|u\|_\nu$, then $\lambda_k \to 0$ and $\beta_k \to 0$ as $k \to \infty$. The proof is similar to the proof of Lemma 2.9. In view of (3.4) and (B_4), we have

$$\begin{aligned}\Phi(u) &= \frac{1}{2}\|u\|^2 - \frac{a}{2}\|u\|_2^2 - \sum_{n\in\mathbb{Z}} W(n,u(n)) \\ &\ge \frac{1}{2}\|u\|^2 - \frac{a}{2}\|u\|_2^2 - M_0\left(\|u\|_2^2 + \|u\|_\nu^\nu\right) \\ &\ge \frac{1}{2}\|u\|^2 - \lambda_k^2\left(\frac{a}{2} + M_0\right)\|u\|^2 - D\beta_k^\nu \|u\|^\nu .\end{aligned} \tag{3.6}$$

Since $\lambda_k \to 0$ as $k \to \infty$, there exists a positive constant N_1 such that

$$\lambda_k^2\left(\frac{a}{2} + M_0\right) \le \frac{1}{4}, \quad \forall\, k \ge N_1. \tag{3.7}$$

By (3.6) and (3.7), we obtain

$$\Phi(u) \ge \frac{1}{4}\|u\|^2 - M_0\beta_k^\nu \|u\|^\nu, \quad \forall\, k \ge N_1. \tag{3.8}$$

We choose $r_k = (8M_0\beta_k^\nu)^{1/(2-\nu)}$, then

$$b_k = \inf_{\substack{u\in Z_k \\ \|u\|=r_k}} \Phi(u) \ge \frac{1}{8}r_k^2, \quad \forall\, k \ge N_1. \tag{3.9}$$

Since $\beta_k \to 0$ as $k \to \infty$ and $\gamma > 2$, we have

$$b_k \to +\infty \quad \text{as } k \to \infty.$$

Step 2. We verify condition (A_1) in Theorem 3.4. Since $\dim Y_k < \infty$ and all norms of a finite-dimensional normed space are equivalent, there exists $\overline{M}_1 > 0$ such that

$$\|u\| \le \overline{M}_1 \|u\|_\infty, \quad \forall\, u \in Y_k. \tag{3.10}$$

In view of (H_1) and (B_4), there exists $\widetilde{\tau} > 0$ such that

$$W(n,u) \ge -\widetilde{\tau}\,|u|^2, \quad \forall\, (n,u) \in \mathbb{Z}\times\mathbb{R}^N. \tag{3.11}$$

By (H_1), for $\overline{M}_2 = \left(1 + \frac{\widetilde{\tau}}{\varrho_2} + \frac{a}{2\varrho_2}\right)\overline{M}_1^2$, there exists $\overline{\delta} = \overline{\delta}(\overline{M}_2) > 0$ such that

$$W(n,u) \ge \overline{M}_2\,|u|^2, \quad \forall\, |u| \ge \overline{\delta},\ \forall\, n \in \mathbb{Z}, \tag{3.12}$$

where $\widetilde{\tau}$ is given in (3.11) and a is given in Remark 3.1. For any $u \in Y_k$, there exists $n_0 = n_0(u) \in \mathbb{Z}$ such that $|u(n_0)| = \|u\|_\infty$. In view of (3.2), (3.4) and (3.10)–(3.12), we get

$$\begin{aligned}
\Phi(u) &= \frac{1}{2}\|u\|^2 - \frac{a}{2}\|u\|_2^2 - \sum_{n\in\mathbb{Z}} W(n,u(n)) \\
&= \frac{1}{2}\|u\|^2 - \frac{a}{2}\|u\|_2^2 - \sum_{n\in\mathbb{Z}\setminus\{n_0\}} W(n,u(n)) - W(n_0,u(n_0)) \\
&\le \frac{1}{2}\|u\|^2 - \frac{a}{2}\|u\|_2^2 - \overline{M}_2|u(n_0)|^2 - \sum_{n\in\mathbb{Z}\setminus\{n_0\}} W(n,u(n)) \\
&\le \frac{1}{2}\|u\|^2 + \frac{a}{2}\|u\|_2^2 + \widetilde{\tau}\sum_{n\in\mathbb{Z}\setminus\{n_0\}} |u(n)|^2 - \overline{M}_2\|u\|_\infty^2 \\
&\le \left(\frac{1}{2} + \frac{\widetilde{\tau}}{\varrho_2} + \frac{a}{2\varrho_2}\right)\|u\|^2 - \overline{M}_2\|u\|_\infty^2 \\
&\le \left(\frac{1}{2} + \frac{\widetilde{\tau}}{\varrho_2} + \frac{a}{2\varrho_2} - \frac{\overline{M}_2}{\overline{M}_1^2}\right)\|u\|^2 \\
&\le -\frac{1}{2}\|u\|^2
\end{aligned}$$

for all $u \in Y_k$ with $\|u\| \ge \overline{M}_1\overline{\delta}$. Thus, we can choose $\|u\| = \rho_k$ large enough ($\rho_k > r_k$) such that

$$a_k = \max_{\substack{u\in Y_k \\ \|u\|=\rho_k}} \Phi(u) \le 0.$$

Step 3. We prove that Φ satisfies (C) condition. Let $\{u_i\}$ be a (C) condition sequence, that is, $\{\Phi(u_i)\}$ is bounded, and $(1+\|u_i\|)\|\Phi'(u_i)\| \to 0$ as $i \to \infty$. Hence there exists $D_2 > 0$ such that

$$D_2 \ge \frac{1}{2}\left\langle \Phi'(u_i), u_i \right\rangle - \Phi(u_i). \tag{3.13}$$

We now prove that $\{u_i\}$ is bounded in X. In fact, if not, we may assume by contradiction that $\|u_i\| \to \infty$ as $i \to \infty$. Let $v_i := u_i/\|u_i\|$. Clearly, $\|v_i\| = 1$ and there is $v_0 \in X$ such that, up to a subsequence,

$$v_i \rightharpoonup v_0 \quad \text{in } X, \tag{3.14}$$

$$v_i \to v_0 \quad \text{in } l^q(\mathbb{Z},\mathbb{R}^N),\ 2 \le q < +\infty \text{ as } i \to \infty. \tag{3.15}$$

Since $v_i \rightharpoonup v_0$ in X, it is easy to verify that $v_i(n)$ converges to $v_0(n)$ pointwise for all $n \in \mathbb{N}$, that is,

$$\lim_{i\to\infty} v_i(n) = v_0(n), \quad \forall\, n \in \mathbb{N}. \tag{3.16}$$

Case 1: $v_0 = 0$.

Case 1.1. Assume that for any $n \in \mathbb{Z}$, there exists a constant D_3 such that $|u_i(n)| \leq D_3$ as $i \to \infty$. By (B_4), there exists $D_4 > 0$ such that

$$W(n, u_i(n)) \leq \frac{M_0}{2} |u_i(n)|^2 + \frac{M_0 D_4^{\nu-2}}{\nu} |u_i(n)|^2, \quad \forall i \in \mathbb{N},\ n \in \mathbb{N}. \tag{3.17}$$

In view of (3.4) and (3.17), we get

$$\begin{aligned}\Phi(u_i) &= \frac{1}{2} \|u_i\|^2 - \frac{a}{2} \|u_i\|_2^2 - \sum_{n\in\mathbb{Z}} W(n, u_i(n)) \\ &\geq \frac{1}{2} \|u_i\|^2 - \frac{a}{2} \|u_i\|_2^2 - \sum_{n\in\mathbb{Z}} \left(\frac{M_0}{2} |u_i(n)|^2 + \frac{M_0 D_4^{\nu-2}}{\nu} |u_i(n)|^2 \right) \\ &\geq \frac{1}{2} \|u_i\|^2 - \left(\frac{a}{2} + \frac{M_0}{2} + \frac{M_0 D_4^{\nu-2}}{\nu} \right) \|u_i\|_2^2 .\end{aligned} \tag{3.18}$$

Divided by $\|u_i\|^2$ on both sides of (3.18), it follows from (3.15) and $\{\Phi(u_i)\}$ is bounded that there exit $\overline{\varepsilon} \in (0, 1/2)$ and $\overline{N_0} \in \mathbb{Z}$ such that

$$\frac{1}{2} > \overline{\varepsilon} \geq \frac{1}{2} \tag{3.19}$$

for $i \geq \overline{N_0}$, which is a contradiction.

Case 1.2. Assume that there exists $n_0 \in \mathbb{Z}$ such that $|u_i(n_0)| \to \infty$ as $i \to \infty$. For $0 \leq \overline{a}_1 < \overline{a}_2$, let

$$\Omega_i(\overline{a}_1, \overline{a}_2) = \{ n \in \mathbb{Z} : \overline{a}_1 \leq |u_i(n)| < \overline{a}_2 \} . \tag{3.20}$$

Thus, $n_0 \in \Omega_i(T_0, +\infty)$ for large $i \in \mathbb{N}$, and it follows from (3.4), (3.5), (3.13), (H_2) and Fatou's Lemma that

$$\begin{aligned} D_2 &\geq \frac{1}{2} \langle \Phi'(u_i), u_i \rangle - \Phi(u_i) \\ &= \sum_{n\in\mathbb{Z}} \left(\frac{1}{2} (\nabla W(n, u_i(n)), u_i(n)) - W(n, u_i(n)) \right) \\ &\geq \widetilde{W}(n_0, u_i(n_0)) - \|h\|_1 \to +\infty \quad \text{as } i \to \infty, \end{aligned}$$

which is a contradiction.

Case 2: $v_0 \neq 0$. Since $\{\Phi(u_i)\}$ is bounded, there exists $M_3 > 0$ such that

$$\Phi(u_i) = \frac{1}{2} \|u_i\|^2 - \frac{a}{2} \|u_i\|_2^2 - \sum_{n\in\mathbb{Z}} W(n, u_i(n)) \geq -M_3. \tag{3.21}$$

Divided by $\|u_i\|^2$ on both sides of (3.21), it follows from Remark 3.1 that

$$\sum_{n\in\mathbb{Z}} \frac{W(n, u_i(n))}{\|u_i\|^2} \leq \frac{1}{2} + \frac{M_3}{\|u_i\|^2} < +\infty. \tag{3.22}$$

Let $\Lambda := \{n \in \mathbb{Z} : v_0(n) \neq 0\}$, then $\Lambda \neq \emptyset$. For any $n \in \Lambda$, we have $\lim_{i\to\infty} |u_i(n)| = +\infty$. Hence $\Lambda \subset \Omega_i(M, +\infty)$ for large $i \in \mathbb{N}$, it follows from (H_1), and Fatou's Lemma that

$$\begin{aligned}
\sum_{n\in\mathbb{Z}} \frac{W(n,u_i(n))}{\|u_i\|^2} &= \sum_{n\in\Omega_i(M,\infty)} \frac{W(n,u_i(n))}{\|u_i\|^2} + \sum_{n\in\Omega_i(0,M)} \frac{W(n,u_i(n))}{\|u_i\|^2} \\
&\geq \sum_{n\in\Omega_i(M,\infty)} \frac{W(n,u_i(n))}{\|u_i\|^2} - M_0\left(\frac{1}{2} + \frac{M^{\nu-2}}{\nu}\right) \sum_{n\in\Omega_i(0,M)} |v_i(n)|^2 \\
&\geq \sum_{n\in\Omega_i(M,\infty)} \frac{W(n,u_i(n))}{|u_i(n)|^2} |v_i(n)|^2 - \frac{M_0}{\varrho_2}\left(\frac{1}{2} + \frac{M^{\nu-2}}{\nu}\right) \to +\infty
\end{aligned}$$

as $i \to \infty$. This contradicts (3.22). Therefore, $\{u_i\}$ is bounded in X, that is, there exists $M_4 > 0$ such that

$$\|u_i\| \leq M_4. \tag{3.23}$$

In view of the boundedness of $\{u_i\}$, we may extract a weakly convergent subsequence that, for simplicity, we call $\{u_i\}$, $u_i \rightharpoonup u$ in X. Next we will verify that $u_i \to u$ in X. By virtue of (B_4), (3.2) and (3.23), we have

$$\begin{aligned}
&\sum_{n\in\mathbb{Z}} (\nabla W(n,u_i(n)) - \nabla W(n,u(n)), u_i(n) - u(n)) \\
&\leq \sum_{n\in\mathbb{Z}} (|\nabla W(n,u_i(n))| + |\nabla W(n,u(n))|)\, |u_i(n) - u(n)| \\
&\leq M_0 \sum_{n\in\mathbb{Z}} \left(|u_i(n)| + |u_i(n)|^{\nu-1}\right) |u_i(n) - u(n)| \\
&\quad + M_0 \sum_{n\in\mathbb{Z}} \left(|u(n)| + |u(n)|^{\nu-1}\right) |u_i(n) - u(n)| \\
&\leq M_0 \left(\|u_i\|_2 + \|u_i\|_{2\nu-2}^{\nu-1}\right) \|u_i - u\|_2 + M_0 \left(\|u\|_2 + \|u\|_{2\nu-2}^{\nu-1}\right) \|u_i - u\|_2 \\
&\leq M_0 \left(\frac{1}{\sqrt{\varrho_2}} \|u_i\| + C\|u_i\|^{\nu-1}\right) \|u_i - u\|_2 + M_0 \left(\|u\|_2 + \|u\|_{2\nu-2}^{\nu-1}\right) \|u_i - u\|_2 \\
&\leq M_5 \|u_i - u\|_2 \to 0 \quad \text{as } i \to \infty,
\end{aligned} \tag{3.24}$$

where $C = \left[\varrho_2^{-1}[(\varrho_2 + 4\varrho_1)\varrho_2]^{-2\nu/4}\right]^{1/2}$ and $M_5 = M_0(\frac{1}{\sqrt{\varrho_2}} M_4 + C M_4^{\nu-1} + \|u\|_2 + \|u\|_{2\nu-2}^{\nu-1})$. It follows from $u_i \rightharpoonup u$ and (3.24) that

$$\begin{aligned}
\|u_i - u\|^2 &= \left\langle \Phi'(u_i) - \Phi'(u), u_i - u \right\rangle \\
&\quad + \sum_{n\in\mathbb{Z}} (\nabla W(n,u_i(n)) - \nabla W(n,u(n)), u_i(n) - u(n)) \to 0,
\end{aligned}$$

as $i \to \infty$. Thus, Φ satisfies (C) condition.

It follows from Theorem 3.4 that Φ possesses an unbounded sequence $\{c_j\}$ of critical values with $c_j = \Phi(u_j)$, where u_j is such that $\Phi'(u_j) = 0$ for $j = 1, 2, \ldots$. If $\{\|u_j\|\}$ is bounded, then there exists $R > 0$ such that

$$\|u_j\| \le R \quad \text{for } j \in \mathbb{N}. \tag{3.25}$$

Thus, from (3.4) and (B_4), we have

$$\begin{aligned}
\frac{1}{2}\|u_j\|^2 &= c_j + \frac{a}{2}\|u_j\|_2^2 + \sum_{n\in\mathbb{Z}} W(n, u_j(n)) \\
&\ge c_j - M_0 \sum_{n\in\mathbb{Z}} \left(|u_j(n)|^2 + |u_j(n)|^\nu\right) \\
&\ge c_j - M_0 \left(\frac{1}{\varrho_2}\|u_j\|^2 + C_1 \|u_j\|^\nu\right),
\end{aligned}$$

where $C_1 = \varrho_2^{-1}\left[(\varrho_2 + 4\varrho_1)\varrho_2\right]^{(2-\nu)/4}$. It follows that

$$c_j \le \frac{1}{2}\|u_j\|^2 + M_0\left(\frac{1}{\varrho_2}\|u_j\|^2 + C_1\|u_j\|^\nu\right) < +\infty.$$

This contradicts the fact that $\{c_j\}$ is unbounded, and so $\{\|u_j\|\}$ is unbounded. The proof is completed. $\square$

Now we give the proof of Theorem 1.18.

Proof of Theorem 1.18. The proof of Theorem 1.18 is similar to that of Theorem 1.14. In fact, we only need to prove that Φ satisfies (C) condition. Let $\{u_i\}$ be a (C) condition sequence, that is, $\{\Phi(u_i)\}$ is bounded, and $(1 + \|u_i\|)\|\Phi'(u_i)\| \to 0$ as $i \to \infty$. We now prove that $\{u_i\}$ is bounded in X. In fact, if not, we may assume by contradiction that $\|u_i\| \to \infty$ as $i \to \infty$. We take v_i as in the proof of Theorem 1.14.

Case 1: $v_0 = 0$. In view of Remark 3.1, (3.2), (3.4), (3.5), (B_4), (H_3) and the Hölder's inequality, we get

$$\begin{aligned}
D_5 &\ge \eta\Phi(u_i) - \langle \Phi'(u_i), u_i\rangle \\
&= \left(\frac{\eta}{2} - 1\right)\|u_i\|^2 - a\left(\frac{\eta}{2} - 1\right)\|u_i\|_2^2 + \sum_{n\in\mathbb{Z}} \left[(\nabla W(n, u_i(n)), u_i(n)) - \eta W(n, u_i(n))\right] \\
&\ge \left(\frac{\eta}{2} - 1\right)\|u_i\|^2 - a\left(\frac{\eta}{2} - 1\right)\|u_i\|_2^2 - M_0(1+\eta) \sum_{n\in\Omega_i(0,T)} \left(|u_i(n)|^2 + |u_i(n)|^\nu\right) \\
&\quad - \sum_{n\in\Omega_i(T,+\infty)} \left[k_0|u_i(n)|^2 + k_1(L(n)u_i(n), u_i(n)) + k_2(n)|u_i(n)|^{\overline{\kappa}} + k_3(n)\right] \\
&= \left(\frac{\eta}{2} - 1\right)\|u_i\|^2 - a\left(\frac{\eta}{2} - 1\right)\|u_i\|_2^2 - M_0(1+\eta) \sum_{n\in\Omega_i(0,T)} \left(|u_i(n)|^2 + |u_i(n)|^\nu\right) \qquad (3.26) \\
&\quad - \sum_{n\in\Omega_i(T,+\infty)} \Big[k_0|u_i(n)|^2 + k_1(\overline{L}(n)u_i(n), u_i(n))
\end{aligned}$$

$$
\begin{aligned}
&\quad + k_2(n)\,|u_i(n)|^{\overline{\kappa}} + k_3(n) - ak_1\,|u_i(n)|^2\Big] \\
&\geq \left(\frac{\eta-2}{2} - k_1\right)\|u_i\|^2 - \left[a\left(\frac{\eta}{2}-1\right) + M_0(1+\eta)\left(1+T^{\nu-2}\right) + k_0\right]\|u_i\|_2^2 \\
&\quad - \|k_2\|_{2/(2-\overline{\kappa})}\,\|u_i\|_2^{\overline{\kappa}} - \|k_3\|_1 \\
&\geq \left(\frac{\eta-2}{2} - k_1\right)\|u_i\|^2 - \left[a\left(\frac{\eta}{2}-1\right) + M_0(1+\eta)\left(1+T^{\nu-2}\right) + k_0\right]\|u_i\|_2^2 \\
&\quad - \left(\frac{1}{\varrho_2}\right)^{\overline{\kappa}/2}\|k_2\|_{2/(2-\overline{\kappa})}\,\|u_i\|^{\overline{\kappa}} - \|k_3\|_1
\end{aligned}
$$

for some $D_5 > 0$. Divided by $\|u_i\|^2$ on both sides of (3.26), noting that $0 \leq k_1 < (\eta-2)/2$ and $0 < \overline{\kappa} < 2$, we obtain

$$
\|v_i\|_2^2 \geq \frac{(\eta-2)/2 - k_1}{D_6} > 0 \quad \text{as } i \to \infty, \tag{3.27}
$$

where $D_6 = \left[(\frac{\eta}{2}-1)a + M_0(1+\eta)(1+T^{\nu-2}) + k_0\right]$. It follows from (3.15) and (3.27) that $v_0 \neq 0$. That is a contradiction.

Case 2: $v_0 \neq 0$. The proof is the same as the one in Theorem 1.14, and we omit it here. Therefore, $\{u_i\}$ is bounded in X. Similar to the proof of Theorem 1.14, we can prove that $\{u_i\}$ has a convergent subsequence in X. Hence, Φ satisfies (C) condition. The proof is completed. □

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