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Complex m-Hessian Type Equations in Weighted Energy Classes of m-subharmonic Functions with Given Boundary Value

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Abstract. In this paper, we concern with the existence of solutions of the complex m-Hessian type equation $-\chi(u)H_m(u) = \mu$ in the class $\mathcal{E}_{m,\chi}(f,\Omega)$ if there exists subsolution in this class, where the given boundary value $f \in \mathcal{N}_m(\Omega) \cap MSH_m(\Omega)$.

1. Introduction

The complex Monge–Ampère operator plays a central role in pluripotential theory and has been extensively studied through the years. This operator was used to obtain many important results of the pluripotential theory in $\mathbb{C}^n$, $n > 1$. In [3], Bedford and Taylor have shown that this operator is well defined in the class of locally bounded plurisubharmonic functions with range in the class of non-negative measures. Solving the Monge–Ampère equation $\mu = (dd^c u)^n$ in the class of plurisubharmonic functions is an important problem in pluripotential theory. In [16], the subsolution theorem was for the first time proved by Kołodziej in the class of bounded plurisubharmonic functions, i.e., if $\mu \leq (dd^c w)^n$, where $w \in PSH^-(\Omega) \cap L^\infty(\Omega)$, then there exists $u \in PSH^-(\Omega) \cap L^\infty(\Omega)$ such that $\mu = (dd^c u)^n$. In [6,7], Cegrell introduced the classes $\mathcal{F}(\Omega), \mathcal{E}(\Omega)$ which are not necessarily locally bounded and he proved that the complex Monge–Ampère operator is well defined in these classes. The subsolution theorem was proved in [1, Theorem 4.14] in the class $\mathcal{E}(\Omega)$. In [13, Theorem 3.1], the subsolution theorem for the complex Monge–Ampère type equation $\mu = -\chi(u)(dd^c u)^n$ in the class $\mathcal{E}_\chi(f,\Omega)$ was proved in the case when $f \equiv 0$. Note that, in [16], the measure μ vanishes on polar sets, whereas in [1,13], the condition that measure μ vanishes on all pluripolar sets of Ω is not required; it only needs to have finite mass. In [9], Czyż shown that the complex Monge–Ampère type equation $\mu = -\chi(u)(dd^c u)^n$ have solution in the class $\mathcal{E}_\chi(\Omega)$ when μ is a positive and finite measure which puts no mass on all pluripolar sets. In the case when maximal function $f \in \mathcal{E}(\Omega)$ and measure μ vanishes on all pluripolar sets of Ω, Benelkourchi [4] proved that the

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complex Monge–Ampère type equation $F(u,z)\mu = (dd^c u)^n$ has solution in $\mathcal{N}^a(f,\Omega)$ if it has subsolution in $\mathcal{N}^a(\Omega)$.

Błocki [5] and Sadullaev, Abdullaev [23] introduced m-subharmonic functions which are extensions of the plurisubharmonic functions and the complex m-Hessian operator $H_m(\cdot) = (dd^c\cdot)^m \wedge \beta^{n-m}$ which is more general than the Monge–Ampère operator $(dd^c\cdot)^n$. In [8], Chinh introduced the Cegrell classes $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$ which are not necessarily locally bounded and the complex m-Hessian operator is well defined in these classes. In the classes of m-subharmonic functions, the complex m-Hessian equation $\mu = H_m(u)$ plays important role. Besides solving the m-Hessian equation in the case when the measure μ vanishes on all m-polar sets, mathematicians are interested in solving the m-Hessian equation when it has subsolution. In [18], Cuong proved the subsolution theorem for the m-Hessian equation in the class of bounded m-subharmonic functions. In [17, Lemma 5.1], Chinh proved that the complex m-Hessian equation has solution in $\mathcal{E}^0_m(\Omega)$ if it has subsolution in $SH_m(\Omega) \cap L^\infty(\Omega)$. After that, Hung and Phu [15] proved that the subsolution theorem is true in the class $\mathcal{E}_m(\Omega)$. In [12], Gasmi extended this result, he solved complex m-Hessian equation in the class $\mathcal{N}_m(f)$ if it has subsolution in the class $\mathcal{N}_m(\Omega)$. Amal, Asserda, Gasmi [2] solved m-Hessian type equation $H_m(u) = F(u,\cdot)d\mu$ in the class $\mathcal{N}_m(f)$ if there exists subsolution in the class $\mathcal{N}_m(\Omega)$. Recently, in [22] the authors proved that the complex m-Hessian type equation $-\chi(u)H_m(u) = \mu$ has solution in the class $\mathcal{E}_{m,\chi}(\Omega)$ if it has subsolution in this class. Note that, in [17,18], the measure μ puts no mass on all m-polar sets, whereas in [2,12,15,22], the condition that measure μ vanishes on all m-polar sets of Ω is not required. In [14, Theorem 1.1], Hai and Quan solved complex m-Hessian type equation $H_m(u,z) = F(u,z)d\mu$ in the class $\mathcal{D}_m(\Omega)$ if it has subsolution in this class in the case when the given measure μ puts no mass on all m-polar set and the function $t \mapsto F(t,z)$ is continuous and non-decreasing. For any function $f \in \mathcal{E}_m(\Omega) \cap MSH_m(\Omega)$, in [21, Theorem 4.1], the authors proved that the equation $\chi(u,z)H_m(u) = \mu$ has solution in the class $\mathcal{F}^a_m(f,\Omega)$ if there exists $\varphi \in \mathcal{F}_m(f,\Omega) \cap L^1(\Omega,\mu)$ such that $\mu \le H_m(\varphi)$ and μ puts no mass on all m-polar sets. Note that in [21], the authors do not require $\chi(t,z)$ to be decreasing in the first variable for all $z \in \Omega$. Continuing the study in this direction, in Theorem 3.3 of this paper, authors will solve complex m-Hessian type equation $-\chi(u)H_m(u) = \mu$ in the class $\mathcal{E}_{m,\chi}(f,\Omega)$ if it has subsolution in this class, where the given boundary value $f \in \mathcal{N}_m(\Omega) \cap MSH_m(\Omega)$. This can be seen as Kołodziej's subsolution theorem in the class $\mathcal{E}_{m,\chi}(f,\Omega)$. Note that, when $f \equiv 0$, we get the result in [22]. This results seems to be new even in the plurisubharmonic case.

The paper is organized as follows. Besides the introduction, the paper has two other sections. In Section 2 we recall the definitions and results concerning the m-subharmonic functions which were introduced and investigated intensively in recent years by many authors (see [5, 10, 23]). We also recall the Cegrell classes of m-subharmonic functions $\mathcal{F}_m(\Omega)$, $\mathcal{N}_m(\Omega)$, $\mathcal{E}_m(\Omega)$ and $\mathcal{E}_{m,\chi}(\Omega)$ which were introduced and studied in [11, 17, 20]. Finally, in Section 3, we solve complex m-Hessian type equations $-\chi(u)H_m(u) = \mu$ in the class $\mathcal{E}_{m,\chi}(f, \Omega)$ in the case when measure μ is arbitrary.

2. Preliminaries

Some elements of the theory of m-subharmonic functions and the complex m-Hessian operator can be found, e.g., in [5, 10, 17, 20, 23]. By $\beta = dd^c\|z\|^2$ we denote the canonical Kähler form of $\mathbb{C}^n$ with the volume element $dV_{2n} = \frac{1}{n!}\beta^n$ where $d = \partial + \overline{\partial}$ and $d^c = \frac{\partial - \overline{\partial}}{4i}$.

Let $1 \le m \le n$ be integers. Let Ω be a bounded m-hyperconvex domain in $\mathbb{C}^n$, which mean there exists a negative continuous m-subharmonic function $\rho\colon \Omega \to (-\infty, 0)$ such that $\{z \in \Omega : \rho(z) < c\} \Subset \Omega$ for every $c < 0$. Such a function ρ is called the exhaustion function on Ω. Throughout this paper Ω will denote a bounded m-hyperconvex domain in $\mathbb{C}^n$.

We recall the classes $\mathcal{E}_m^0(\Omega)$, $\mathcal{F}_m(\Omega)$ and $\mathcal{E}_m(\Omega)$ introduced in [8].

Definition 2.1.

$$\mathcal{E}_m^0 = \mathcal{E}_m^0(\Omega) = \left\{u \in SH_m^-(\Omega) \cap L^\infty(\Omega) : \lim_{z\to\partial\Omega} u(z) = 0,\ \int_\Omega (dd^c u)^m \wedge \beta^{n-m} < \infty\right\},$$

$$\mathcal{F}_m = \mathcal{F}_m(\Omega) = \left\{u \in SH_m^-(\Omega) : \exists\, \mathcal{E}_m^0 \ni u_j \searrow u,\ \sup_j \int_\Omega (dd^c u_j)^m \wedge \beta^{n-m} < \infty\right\},$$

and

$$\mathcal{E}_m = \mathcal{E}_m(\Omega) = \Big\{u \in SH_m^-(\Omega) : \forall\, z_0 \in \Omega, \exists \text{ a neighborhood } \omega \ni z_0, \text{ and}$$
$$\mathcal{E}_m^0 \ni u_j \searrow u \text{ on } \omega,\ \sup_j \int_\Omega (dd^c u_j)^m \wedge \beta^{n-m} < \infty\Big\}.$$

From Theorem 3.14 in [8] it follows that if $u \in \mathcal{E}_m(\Omega)$, the complex m-Hessian $H_m(u) = (dd^c u)^m \wedge \beta^{n-m}$ is well defined and it is a Radon measure on Ω. On the other hand, by Remark 3.6 in [8] the following description of $\mathcal{E}_m(\Omega)$ may be given

$$\mathcal{E}_m = \mathcal{E}_m(\Omega) = \big\{u \in SH_m^-(\Omega) : \forall\, U \Subset \Omega, \exists\, v \in \mathcal{F}_m(\Omega), v = u \text{ on } U\big\}.$$

Definition 2.2. We say that an m-subharmonic function u is m-maximal if for every relatively compact open set K on Ω and for each upper semicontinuous function v on $\overline{K}$, $v \in SH_m(K)$ and $v \le u$ on ∂K, we have $v \le u$ on K. The family of m-maximal m-subharmonic function defined on Ω will be denoted by $MSH_m(\Omega)$.

Next, we recall the definition of the class $\mathcal{N}_m(\Omega)$ as in Subsection 2.6 in [19].

Definition 2.3. Let $u \in SH_m(\Omega)$, and let Ω_j be a fundamental sequence of Ω, which means that Ω_j is strictly m-hyperconvex domain, $\Omega_j \Subset \Omega_{j+1}$ and $\bigcup_{j=1}^{\infty} \Omega_j = \Omega$. Set

$$u^j(z) = \big(\sup\{\varphi(z) : \varphi \in SH_m(\Omega), \varphi \leq u \text{ on } \Omega_j^c\}\big)^*,$$

where Ω_j^c denotes the complement of Ω_j on Ω.

We can see that $u^j \in SH_m(\Omega)$ and $u^j = u$ on $(\overline{\Omega_j})^c$. From the definition of u^j we see that $\{u^j\}$ is an increasing sequence and therefore $\lim_{j\to\infty} u^j$ exists everywhere except on an m-polar subset on Ω. Hence, the function $\widetilde{u}$ defined by $\widetilde{u} = \big(\lim_{j\to\infty} u^j\big)^*$ is an m-subharmonic function on Ω. Obviously, we have $\widetilde{u} \geq u$. Moreover, if $u \in \mathcal{E}_m(\Omega)$ then $\widetilde{u} \in \mathcal{E}_m(\Omega)$ and $\widetilde{u} \in MSH_m(\Omega)$.

Definition 2.4. Set

$$\mathcal{N}_m = \mathcal{N}_m(\Omega) = \{u \in \mathcal{E}_m(\Omega) : \widetilde{u} = 0\}.$$

We have the following inclusion

$$\mathcal{F}_m(\Omega) \subset \mathcal{N}_m(\Omega) \subset \mathcal{E}_m(\Omega).$$

Let $\mathcal{K}$ be one of the classes $\mathcal{E}_m^0(\Omega)$, $\mathcal{F}_m(\Omega)$, $\mathcal{N}_m(\Omega)$, $\mathcal{E}_m(\Omega)$. Denote by $\mathcal{K}^a$ the set of all function in $\mathcal{K}$ whose Hessian measures vanish on all m-polar sets of Ω. We say that an m-subharmonic function defined on Ω belongs to the class $\mathcal{K}(f, \Omega)$, where $f \in \mathcal{E}_m$ if there exists a function $\varphi \in \mathcal{K}$ such that

$$f \geq u \geq f + \varphi.$$

Note that $\mathcal{K}(0, \Omega) = \mathcal{K}$.

Definition 2.5. Let $E \subset \Omega$ be a Borel subset. The m-capacity of E with respect to Ω is defined in [17] by

$$C_m(E) = C_m(E, \Omega) = \sup\left\{\int_E H_m(u) : u \in SH_m(\Omega), -1 \leq u \leq 0\right\}.$$

As in [8], we have the following definition.

Definition 2.6. A sequence $u_j \in SH_m^-(\Omega)$ converges to u in C_m-capacity if

$$C_m(K \cap \{|u_j - u| > \delta\}) \to 0 \quad \text{as } j \to +\infty, \forall\, K \subset\subset \Omega, \forall\, \delta > 0.$$

We recall some results on weighted m-energy classes in [11].

Definition 2.7. Let $\chi\colon [-\infty, 0] \to [-\infty, 0]$ be an increasing continuous function. We put

$$\mathcal{E}_{m,\chi}(\Omega) = \left\{u \in SH_m(\Omega) : \exists\, (u_j) \in \mathcal{E}_m^0(\Omega), u_j \searrow u, \sup_j \int_\Omega (-\chi) \circ u_j H_m(u_j) < +\infty\right\}.$$

Note that the weighted m-energy classes generalize Cegrell energy classes $\mathcal{F}_{m,p}, \mathcal{F}_m$.

- When $\chi \equiv -1$, then $\mathcal{E}_{m,\chi}(\Omega)$ is the class $\mathcal{F}_m(\Omega)$.
- When $\chi(t) = -(-t)^p$, then $\mathcal{E}_{m,\chi}(\Omega)$ is the class $\mathcal{E}_{m,p}(\Omega)$.

According to Theorem 3.3 in [11], if $\chi \not\equiv 0$ then $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{E}_m(\Omega)$ which means that the complex m-Hessian operator is well-defined on class $\mathcal{E}_{m,\chi}(\Omega)$ and if $\chi(t) < 0$ for all $t < 0$ then we have $\mathcal{E}_{m,\chi}(\Omega) \subset \mathcal{N}_m(\Omega)$.

We recall some results that will frequently be used in this paper.

Theorem 2.8. (Main theorem in [12]) *Let $\Omega \subset \mathbb{C}^n$ be a bounded m-hyperconvex domain and μ be a positive Borel measure on Ω. Assume that there exists a function $w \in \mathcal{E}_m(\Omega)$ such that $\mu \leq H_m(w)$ in the sense of currents on Ω. Then for every function $f \in \mathcal{E}_m(\Omega) \cap MSH_m(\Omega)$, there exists a function $u \in \mathcal{E}_m(\Omega)$ such that $H_m(u) = \mu$ and $f \geq u \geq f + w$. In particular, if we require w to be in $\mathcal{N}_m(\Omega)$ then $u \in \mathcal{N}_m(f)$.*

We recall a version of the comparison principle for a weighted m-Hessian operator (see Theorem 3.8 in [21]).

Theorem 2.9. *Let $u \in \mathcal{N}_m(f)$, $v \in \mathcal{E}_m(f)$ be such that*

$$-\chi(u)H_m(u) \leq -\chi(v)H_m(v).$$

Assume also that $H_m(u)$ puts no mass on m-polar sets. Then we have $u \geq v$ on Ω.

We also note the following convergence result for weighted m-Hessian operator. Recall that by Lemma 2.9 in [20], if a sequence of m-subharmonic functions $\{u_j\}$ converges monotonically to an m-subharmonic function u then $u_j \to u$ in C_m as $j \to \infty$.

Corollary 2.10. (see [22, Corollary 3.3]) *Let $\chi\colon \mathbb{R}^- \to \mathbb{R}^-$ be an increasing continuous function with $\chi(-\infty) > -\infty$. Let $\{u_j, u\} \subset \mathcal{E}_m(\Omega)$, be such that $u_j \geq v$, $\forall\, j \geq 1$ for some $v \in \mathcal{E}_m(\Omega)$ and that $u_j \to u \in \mathcal{E}_m(\Omega)$ in C_m. Then $-\chi(u_j)H_m(u_j) \to -\chi(u)H_m(u)$ weakly.*

Proposition 2.11. (see [19, Proposition 2.9]) *Assume that $u, v, u_k \in \mathcal{E}_m(\Omega)$, $k = 1, 2, \ldots, m-1$ with $u \geq v$ on Ω and $T = dd^c u_1 \wedge \cdots \wedge dd^c u_{m-1} \wedge \beta^{n-m}$. Then we have*

$$\mathbb{1}_{\{u=-\infty\}} dd^c u \wedge T \leq \mathbb{1}_{\{v=-\infty\}} dd^c v \wedge T.$$

In particular, if $u, v \in \mathcal{E}_m(\Omega)$ are such that $u \geq v$ then for every m-polar set $A \subset \Omega$ we have

$$\int_A H_m(u) \leq \int_A H_m(v).$$

We need the following useful approximation result in the class $\mathcal{F}_m(f,\Omega)$.

Lemma 2.12. *Let $u \in \mathcal{F}_m(f,\Omega)$ with $f \in \mathcal{N}_m(\Omega) \cap MSH_m(\Omega)$ and $\int_\Omega H_m(u) < +\infty$. Then there exists a sequence $\{u_j\} \in \mathcal{F}_m(f,\Omega)$ such that*

(a) *$u_j \downarrow u$ on Ω;*

(b) *$H_m(u_j)$ has compact support in Ω;*

(c) *$H_m(u_j) \uparrow H_m(u)$;*

(d) *$H_m(u_j)$ puts no mass on m-polar sets in Ω.*

Proof. We fix an element $\rho \in \mathcal{E}^0_m(\Omega) \cap C(\Omega)$ and let $\Omega_j \uparrow \Omega$ be an increasing sequence of relatively compact subsets of Ω. For $j \geq 1$ we set $\mu_j := 1\!\!1_{\{u>j\rho\}\cap\Omega_j} H_m(u)$. Then the measures μ_j have the following properties:

(i) μ_j has compact support in Ω;

(ii) $\mu_j \leq \mu_{j+1} \leq H_m(u)$;

(iii) μ_j puts no mass on m-polar sets in Ω (by Lemma 2.16 in [12]);

(iv) $\int_\Omega d\mu_j \leq \int_{\{u>j\rho\}} H_m(u) \leq j^m \int_\Omega H_m(\rho) < \infty$ (by Lemma 5.5 in [20]).

It follows from the hypothesis $u \in \mathcal{F}_m(f,\Omega)$ and $f \in \mathcal{N}_m(\Omega)$ that $u \in \mathcal{N}_m(\Omega)$. Thus using (ii), (iv) and the main theorem in [12] we can find $u_j \in \mathcal{N}_m(f,\Omega)$ such that $H_m(u_j) = \mu_j$. We have $\int_\Omega H_m(u_j) = \int_\Omega d\mu_j < \infty$. Thus, by Theorem 3.1 in [19] $u_j \in \mathcal{F}_m(f,\Omega)$. Obviously, u_j satisfies properties (b), (c) and (d). So it remains to prove that $u_j \downarrow u$ on Ω.

Indeed, by Theorem 3.8 in [21], (ii) and (iii) we get $u_j \geq u_{j+1} \geq u$. Set $v := \lim_{j\to\infty} u_j$ then we have $v \geq u$. Moreover, by Corollary 2.10 we deduce that $H_m(u_j) \to H_m(v)$ weakly as $j \to \infty$. Coupling with the construction of u_j we have $H_m(v) = H_m(u)$. By Theorem 2.10 in [12] (see Theorem 3.6 in [1] for the case of plurisubharmonic functions) we obtain $u = v$, and so we have $u_j \downarrow u$ on Ω. The proof is completed. □

In connection to Lemma 2.12 we present the following result which might be of independent interest. This result was also used implicitly in the proof of [22].

Lemma 2.13. *Let $u \in \mathcal{F}_m(f,\Omega)$, $f \in \mathcal{E}_m(\Omega) \cap MSH_m(\Omega)$. Assume that the support of $H_m(u)$ is a compact subset of Ω, then there exist an open subset $\Omega' \Subset \Omega$ a sequence $\{u_j\} \in \mathcal{E}^0_m(f,\Omega)$ having the following properties:*

(a) *$u_j \downarrow u$ on Ω;*

(b) *$H_m(u_j)$ has compact support in $\overline{\Omega'}$;*

(c) *$H_m(u_j)$ puts no mass on m-polar subsets of Ω.*

Proof. Choose a domain $\Omega' \Subset \Omega$ such that Ω' contains the support of $H_m(u)$, $\partial\Omega'$ is $\mathcal{C}^1$-smooth and $H_m(u)$ puts no mass on $\partial\Omega'$. According to Proposition 2.12 in [12], we may find a sequence $v_j \in \mathcal{E}_m^0(\Omega, f)$ such that $v_j \downarrow u$ on Ω. Set

$$u_j := \sup\{\varphi : \varphi \in SH_m^-(\Omega), \varphi|_{\Omega'} \le v_j, \varphi \le f\}.$$

By the maximality of f we see that $H_m(u_j) = 0$ on $\Omega \setminus \overline{\Omega'}$. Since $u_j \ge v_j$ on Ω there exists a function $\xi_j \in \mathcal{E}_m^0(\Omega)$ such that $u_j \ge f + \xi_j$. According to Proposition 2.11 with note that $f \in MSH_m(\Omega)$ for all m-polar set $A \subset \Omega$ we also have

$$\int_A H_m(u_j) \le \int_A H_m(f + \xi_j) = 0,$$

where the last inequality is due to Lemma 5.6 in [15]. This means that $H_m(u_j)$ vanishes on all m-polar sets. So we are done.

For (a), we first observe that $u_j \downarrow := v \ge u$ on Ω. Moreover, since $u_j \ge v_j$ on Ω we infer that $u_j = v_j$ on Ω'. Thus $v = u$ on Ω', and so $v = u$ on $\partial\Omega'$. Now we define

$$\widetilde{u} := \big(\sup\{\varphi \in SH_m^-(\Omega') : \varphi^*|_{\partial\Omega'} \le u\}\big)^*.$$

Since $\partial\Omega'$ is $\mathcal{C}^1$-smooth we have $\widetilde{u} \in SH_m(\Omega')$ and $\widetilde{u} \ge u$ on Ω'. Hence the function

$$\widehat{u} := \begin{cases} u & \text{on } \Omega \setminus \Omega', \\ \widetilde{u} & \text{on } \Omega' \end{cases}$$

belongs to $SH_m(\Omega)$, and since $\widehat{u} \ge u$ on Ω, we infer that $\widehat{u} \in \mathcal{E}_m(\Omega)$. Observe also that $H_m(\widehat{u})$ is supported on $\partial\Omega'$. Observe that by Proposition 2.11 and the choice of Ω' we have

$$\int_{\{\widehat{u}=-\infty\}\cap\partial\Omega'} H_m(\widehat{u}) \le \int_{\{\widehat{u}=-\infty\}\cap\partial\Omega'} H_m(u) = 0. \tag{2.1}$$

Since $\{v > \widehat{u} > -\infty\} \cap \partial\Omega'$ is empty, in view of (2.1), we may apply Lemma 3.1 in [12] to conclude that $\widehat{u} \ge v$ on Ω. So in particular $u \ge v$ on $\Omega \setminus \Omega'$. Therefore $u = v$ on Ω. Thus we obtain $u_j \downarrow v = u$ on Ω. This completes the proof of the lemma. □

3. Complex m-Hessian equations in the class $\mathcal{E}_{m,\chi}(f, \Omega)$

In this section, we assume that $\chi\colon [-\infty, 0] \to [-\infty, 0]$ is a nondecreasing continuous function such that $\chi(t) < 0$ for all $t < 0$. We first concern with the complex m-Hessian equations $-\chi(u)H_m(u) = \mu$ in the class $\mathcal{E}_{m,\chi}(f, \Omega)$ when μ puts no mass on m-polar sets.

Theorem 3.1. *Let μ be a nonnegative, finite measure which puts no mass on m-polar sets. Then the complex m-Hessian type equation $-\chi(u)H_m(u) = \mu$ has solution in the class $\mathcal{E}_{m,\chi}(f,\Omega)$, where $f \in \mathcal{E}_m(\Omega) \cap MSH_m(\Omega)$.*

Proof. By Theorem 5.3 in [17] we can find $\varphi \in \mathcal{E}_m^0(\Omega)$ and $0 \le h \in L^1_{\text{loc}}(H_m(\varphi))$ such that $\mu = hH_m(\varphi)$. Set $\mu_j = 1_{\Omega_j}\min(h,j)H_m(\varphi)$, where $\{\Omega_j\}$ is a fundamental sequence of Ω.

Choose nondecreasing functions $\chi_j \in C^\infty(\mathbb{R}^-)$ such that $-\chi_j \searrow -\chi$. Put $\gamma(t) = \frac{1}{\chi(t)}$ and $\gamma_j(t) = \frac{1}{\chi_j(t)}$. We have that $\gamma(t)$ is a nonincreasing function and nonincreasing functions $\gamma_j \in C^\infty(\mathbb{R}^-)$ satisfying $-\gamma_j(t) \nearrow -\gamma(t)$.

Note that $-\gamma_j$ is bounded above on Ω_j so using Proposition 3.4 in [2], we can find $u_j \in \mathcal{N}_m(f)$ such that

$$H_m(u_j) = -\gamma_j(u_j)d\mu_j = \frac{d\mu_j}{-\chi_j(u_j)}.$$

It follows that

$$-\chi_j(u_j)H_m(u_j) = \mu_j.$$

Therefore, we have

$$-\chi_j(u_j)H_m(u_j) = \mu_j \le \mu_{j+1} = -\chi_{j+1}(u_{j+1})H_m(u_{j+1}) \le -\chi_j(u_{j+1})H_m(u_{j+1}).$$

By Theorem 2.9 we have $u_j \searrow u$. We will prove that $u \in \mathcal{E}_{m,\chi}(f,\Omega)$ which satisfies

$$-\chi(u)H_m(u) = \mu.$$

Firstly, we prove that $u_j \in \mathcal{E}_m^0(f)$. Applying Proposition 3.4 in [2] once again (in the case $f \equiv 0$), we can find $\varphi_j \in \mathcal{F}_m^a(\Omega)$ such that

$$H_m(\varphi_j) = -\gamma_j(\varphi_j)d\mu_j = -\frac{d\mu_j}{\chi_j(\varphi_j)}.$$

This implies that

$$-\chi_j(\varphi_j)H_m(\varphi_j) = \mu_j.$$

On the other hand, since $\Omega_j \Subset \Omega$ and φ_j is an upper semicontinuous function, there exists a constant $A(j)$ which depends only on Ω_j and φ_j such that $\varphi_j \le A(j) < 0$ on Ω_j. Note that $-\gamma_j$ is a nondecreasing continuous function, we deduce that $-\gamma_j(\varphi_j) \le -\gamma_j(A(j)) \le -\gamma(A(j)) = B(j)$ on Ω_j, where $B(j)$ is a constant depending only on Ω_j, φ_j and γ. Hence

$$H_m(\varphi_j) = -\gamma_j(\varphi_j)\mu_j \le jB(j)H_m(\varphi) = H_m(\sqrt[m]{jB(j)}\varphi).$$

It follows from Theorem 2.9 that $\varphi_j \ge \sqrt[m]{jB(j)}\varphi$. Since $\varphi \in \mathcal{E}_m^0(\Omega)$ we obtain $\varphi_j \in \mathcal{E}_m^0(\Omega)$. Moreover, we have

$$-\chi_j(u_j)H_m(u_j) = \mu_j = -\chi_j(\varphi_j)H_m(\varphi_j) \le -\chi_j(f+\varphi_j)H_m(f+\varphi_j)$$

and $u_j, f+\varphi_j \in \mathcal{N}_m^a(f)$ then Theorem 2.9 implies that $u_j \geq f+\varphi_j$. So we have $u_j \in \mathcal{E}_m^0(f)$ as the desired.

Secondly, we prove that $u \in \mathcal{E}_{m,\chi}(f)$. Indeed, we have

$$-\chi_j(\varphi_j)H_m(\varphi_j) = d\mu_j \leq d\mu_{j+1} = -\chi_{j+1}(\varphi_{j+1})H_m(\varphi_{j+1}) \leq -\chi_j(\varphi_{j+1})H_m(\varphi_{j+1}).$$

According to Theorem 2.9 we see that $\{\varphi_j\}$ is decreasing and we assume that $\psi = \lim_{j\to\infty} \varphi_j$. Note that $\varphi_j \in \mathcal{E}_m^0(\Omega)$. Moreover, we have

$$\sup_{j\geq 1} \int_\Omega -\chi(\varphi_j)H_m(\varphi_j) \leq \sup_{j\geq 1} \int_\Omega -\chi_j(\varphi_j)H_m(\varphi_j) = \sup_{j\geq 1} \int_\Omega d\mu_j \leq \mu(\Omega) < \infty.$$

Therefore, we obtain $\psi \in \mathcal{E}_{m,\chi}(\Omega)$. It follows from $f \geq u_j \geq f + \varphi_j$ that $f \geq u \geq f + \psi$ and we get $u \in \mathcal{E}_{m,\chi}(f)$ as desired.

Finally, we prove that $-\chi(u)H_m(u) = \mu$. Indeed, we have $-\chi(u_j)H_m(u_j) = \mu_j$. Repeating the argument as in the last part in the proof of Theorem 4.1 in [22] we have $\lim_{j\to\infty} H_m(u_j) = -\gamma(u)\mu$. On the other hand, since $u_j \searrow u \in \mathcal{E}_{m,\chi}(f) \subset \mathcal{E}_m(\Omega)$, according to Theorem 3.8 in [15] we obtain $H_m(u_j)$ converges weakly to $H_m(u)$ as $j \to \infty$. So we have

$$H_m(u) = -\eta(u)\mu \quad \Longrightarrow \quad -\chi(u)H_m(u) = \mu.$$

The proof is complete. □

Remark 3.2. According to Theorem 3.1 in [2], if μ is a nonnegative measure which puts no mass on m-polar sets and the equation $-\chi(u)H_m(u) = \mu$ has a subsolution in $\mathcal{N}_m^a(\Omega)$ then it has a solution in $\mathcal{N}_m(f,\Omega)$. The main theorem in [14] also proved that if μ is a nonnegative measure which puts no mass on m-polar sets and if the equation $-\chi(u)H_m(u) = \mu$ has a subsolution in $\mathcal{E}_m(\Omega)$ then it has solution which belongs to $\mathcal{E}_m(\Omega)$. Theorem 3.1 does not require the existence of a subsolution, but instead we need finiteness of the measure μ. On the other hand, the solution we found is somewhat more precise since it is contained in $\mathcal{E}_{m,\chi}(f,\Omega) \subset \mathcal{N}_m(f,\Omega) \subset \mathcal{E}_m(\Omega)$.

The next result deals with the case μ is a arbitrary measure with finite total mass.

Theorem 3.3. *Let μ be a non-negative finite measure on Ω. Assume that there exists a function $w \in \mathcal{E}_{m,\chi}(f,\Omega)$ with $\mu \leq -\chi(w)H_m(w)$, where the given boundary $f \in \mathcal{N}_m(\Omega) \cap MSH_m(\Omega)$. Then there exists a function $u \in \mathcal{E}_{m,\chi}(f,\Omega)$ such that $u \geq w$ and $-\chi(u)H_m(u) = \mu$.*

Proof. We consider two cases.

Case 1. Assume that $\chi(-\infty) > -\infty$. Using Theorem 2.15 in [12] we may decompose $\mu = \alpha + \nu$, where α and ν are Radon measures defined on Ω such that α vanishes on all m-polar sets and ν is carried by an m-polar set. It follows from $w \in \mathcal{E}_{m,\chi}(f,\Omega)$ and

$f \in \mathcal{N}_m(\Omega)$ that $w \in \mathcal{N}_m(\Omega)$. Theorem 2.8 implies that there exists $v \in \mathcal{N}_m(f, \Omega)$ such that $v \geq f + w$ and $\nu = H_m(v)$. Note that,

$$\int_\Omega H_m(v) = \int_\Omega d\nu \leq \int_\Omega d\mu < +\infty.$$

Thus, by Theorem 3.1 in [19] we infer that $v \in \mathcal{F}_m(f, \Omega)$. According to Lemma 2.12, there exist $v_j \in \mathcal{F}_m(f, \Omega)$ such that $v_j \searrow v$, $\operatorname{supp} H_m(v_j) \Subset \Omega$, $H_m(v_j)$ puts no mass on m-polar sets and $\sup_{j\geq 1} \int_\Omega H_m(v_j) < \infty$.

Using Theorem 3.1 we can find $u_j \in \mathcal{E}_{m,\chi}(f, \Omega)$ that satisfies

$$-\chi(u_j)H_m(u_j) = \alpha + H_m(v_j). \tag{3.1}$$

Observe that for $j \geq 1$ we have

$$-\chi(u_j)H_m(u_j) \leq -\chi(u_{j+1})H_m(u_{j+1}) \leq -\chi(w)H_m(w).$$

It then follows from Theorem 2.9 that $u_j \searrow u \geq w \in \mathcal{E}_{m,\chi}(f, \Omega)$. This implies that $u \in \mathcal{E}_{m,\chi}(f, \Omega) \subset \mathcal{E}_m(\Omega)$. Note that we have $\chi(-\infty) > -\infty$, by letting $j \to \infty$ in (3.1) and using Corollary 2.10, we obtain

$$-\chi(u)H_m(u) = \alpha + H_m(v) = \mu.$$

Case 2. Assume that $\chi(-\infty) = -\infty$. It follows from the hypothesis $w \in \mathcal{E}_{m,\chi}(f, \Omega)$ that there exists a function $\psi \in \mathcal{E}_{m,\chi}(\Omega)$ such that $f \geq w \geq f + \psi$. By Theorem 3.7 in [11] we have $\psi \in \mathcal{E}^a_m(\Omega)$. Note that we have $f \in MSH_m(\Omega)$ so by Theorem 1.2 in [5] we obtain $H_m(f) = 0$. Therefore, for every m-polar set $A \subset \Omega$, by Proposition 2.11 and Lemma 5.6 in [15] we infer that

$$\int_A H_m(w) \leq \int_A H_m(f + \psi) = 0.$$

This means that $H_m(w)$ vanishes on pluripolar sets and so is μ. Thus, by Theorem 3.1 there exists a function $u \in \mathcal{E}_{m,\chi}(f, \Omega)$ such that $-\chi(u)H_m(u) = \mu$. The proof is complete. □

Remark 3.4. According to the main theorem in [2], we only achieve a solution $u \in \mathcal{N}_m(f)$ and $u \geq f + w$. In Theorem 3.3 we have finer information about this solution u.

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