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Multilinear Fractional Integral Operators with Generalized Kernels

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Abstract. In this article, we introduce a class of multilinear fractional integral operators with generalized kernels that are weaker than the Dini kernel condition. We establish the boundedness of multilinear fractional integral operators with generalized kernels on weighted Lebesgue spaces and variable exponent Lebesgue spaces, as well as the boundedness of multilinear commutators generated by multilinear fractional integral operators with generalized kernels and BMO functions. Even when the generalized kernels condition goes back to the Dini kernel condition, the conclusions on the commutators remain new.

1. Introduction

Serving as a generalization of the Hilbert transform to higher-dimensional Euclidean spaces, the Riesz transform, as a classical singular integral operator, is defined by setting, for any $f \in \mathscr{S}(\mathbb{R}^n)$ (the space of all Schwartz functions on $\mathbb{R}^n$) and $x \in \mathbb{R}^n$,

$$R_j(f)(x) := c_n \,\mathrm{p.\,v.} \int_{\mathbb{R}^n} \frac{x_j - y_j}{|x-y|^{n+1}} f(y)\, dy, \quad 1 \le j \le n,$$

where $c_n := \Gamma(\frac{n+1}{2})/\pi^{(n+1)/2}$. The Riesz transforms have wide applications in fields such as image processing, signal processing, and partial differential equations, and are commonly used for tasks such as edge detection and texture analysis [3, 8, 12, 25, 26].

The Riesz potential, which plays a crucial role in harmonic analysis, is also known as the fractional integral operator. Let $0 < \alpha < n$ and the Riesz potential is defined by setting, for any $f \in \mathscr{S}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$I_\alpha(f)(x) := \frac{1}{\gamma(\alpha)} \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}}\, dy,$$

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where $\gamma(\alpha) := \pi^{n/2} 2^\alpha \Gamma\left(\frac{\alpha}{2}\right) / \Gamma\left(\frac{n-\alpha}{2}\right)$. Riesz potentials have important applications in fields such as signal processing, partial differential equations, and probability theory, and have recently been used in image encryption [9].

Due to the different definitions of these two types of operators, they play very different and important roles in image processing. By comparing the above two operators, apart from the coefficients, we can easily observe two differences.

(i) The Riesz transform is a principal value integral, while the Riesz potential is not.

(ii) The Riesz potential involves an additional parameter α compared to the Riesz transform.

Yabuta [24] introduced non-convolution ω-type Calderón–Zygmund operators which are more general than the Riesz transform. In 2014, Lu and Zhang [14] introduced multilinear Calderón–Zygmund operators with Dini's type kernels and established the boundedness of these operators and multilinear commutators on weighted Lebesgue spaces. In 2016, Zhang and Sun [27] established the boundedness of multilinear iterated commutators generated by multilinear Calderón–Zygmund operators with Dini's type kernels and BMO functions on weighted Lebesgue spaces. For some other results, one can see [10, 18–20] and related references.

Recently, Wu and Zhang [22] proposed multilinear fractional integral operators with Dini's type kernels. They established the boundedness of these operators on weighted Lebesgue spaces and variable exponent Lebesgue spaces, whose weight functions belong to $A_{\vec{P},q}$. Suppose that $\omega(t)\colon [0,\infty) \to [0,\infty)$ and $\omega(t)$ is a non-negative, non-decreasing function with $0 < \omega(1) < \infty$. For any $a > 0$,

(i) ω is said to satisfy the Dini(a) condition, denoted as $\omega \in \mathrm{Dini}(a)$, if

$$|\omega|_{\mathrm{Dini}(a)} := \int_0^1 \omega^a(t) \, \frac{dt}{t} < \infty.$$

(ii) ω is said to satisfy the $\log^m$-Dini(a) condition if the following inequality

$$\int_0^1 \omega^a(t)(1 + \log t^{-1})^m \, \frac{dt}{t} < \infty$$

holds, where $m \in \mathbb{Z}^+ := \{1, 2, 3, 4, \ldots\}$.

It is straightforward to verify that the $\log^m$-Dini(a) condition is more stronger than the Dini(a) condition, and if $0 < a_1 < a_2 < \infty$, then $\mathrm{Dini}(a_1) \subset \mathrm{Dini}(a_2)$. In particular, if $\omega \in \mathrm{Dini}(1)$, then

$$\sum_{j=0}^{\infty} \omega(2^{-j}) \sim \int_0^1 \omega(t) \, \frac{dt}{t} < \infty.$$

And if $\omega \in$ log-Dini(1), meaning

$$\int_0^1 \omega(t)(1+\log t^{-1})\,\frac{dt}{t} < \infty,$$

then $\omega \in$ Dini(1) and

$$\sum_{k=1}^{\infty} k\omega(2^{-k}) \sim \int_0^1 \omega(t)(1+\log t^{-1})\,\frac{dt}{t} < \infty.$$

Definition 1.1. (see [22]) Let $0 < \alpha < mn$ and $K_\alpha(x, y_1, \ldots, y_m)$ be a locally integrable function defined away from the diagonal $x = y_1 = \cdots = y_m$ in $(\mathbb{R}^n)^{m+1}$, which is called the m-linear fractional integral kernel of type $\omega(t)$, if for some $A > 0$, it satisfies the following size condition

$$|K_\alpha(x,\vec{y})| \le \frac{A}{\big(\sum_{j=1}^m |x-y_j|\big)^{mn-\alpha}}, \tag{1.1}$$

and the smoothness condition

$$|K_\alpha(x,\vec{y}) - K_\alpha(x',\vec{y})| \le \frac{A}{\big(\sum_{j=1}^m |x-y_j|\big)^{mn-\alpha}}\,\omega\left(\frac{|x-x'|}{\sum_{j=1}^m |x-y_j|}\right), \tag{1.2}$$

where $|x-x'| \le \frac{1}{2}\max_{1\le j\le m}|x-y_j|$, and

$$\begin{aligned} &|K_\alpha(x, y_1, \ldots, y_j, \ldots, y_m) - K_\alpha(x, y_1, \ldots, y_j', \ldots, y_m)| \\ &\le \frac{A}{\big(\sum_{j=1}^m |x-y_j|\big)^{mn-\alpha}}\,\omega\left(\frac{|y_j-y_j'|}{\sum_{j=1}^m |x-y_j|}\right), \end{aligned}$$

where $|y_j - y_j'| \le \frac{1}{2}\max_{1\le j\le m}|x-y_j|$.

We say $T_\alpha\colon \mathscr{S}(\mathbb{R}^n)\times\cdots\times\mathscr{S}(\mathbb{R}^n) \to \mathscr{S}'(\mathbb{R}^n)$ an m-linear fractional integral operator with kernel of type $\omega(t)$, $K_\alpha(x,\vec{y})$, if

$$T_\alpha(\vec{f})(x) := \int_{(\mathbb{R}^n)^m} K_\alpha(x,\vec{y}) \prod_{j=1}^m f_j(y_j)\,d\vec{y}, \tag{1.3}$$

where $x \notin \bigcap_{j=1}^m \operatorname{supp} f_j$ and each $f_j \in \mathscr{S}(\mathbb{R}^n)$, $j = 1, \ldots, m$.

Lin and Xiao [13] introduced a kind of multilinear singular integral operators with generalized kernels and they established the weighted norm inequalities on the product of weighted Lebesgue spaces. Inspired by [13,22], we focus on a class of generalized fractional integral kernels by attenuating the condition (1.2). For $k \in \mathbb{Z}^+$,

$$\begin{aligned} &\left(\int_{Q(x,2^{k+2}\sqrt{mn}|x-x'|)^m \setminus Q(x,2^{k+1}\sqrt{mn}|x-x'|)^m} |K_\alpha(x,\vec{y}) - K_\alpha(x',\vec{y})|^{p_0}\,d\vec{y}\right)^{1/p_0} \\ &\le CC_k 2^{k\left(\alpha-\frac{mn}{p_0'}\right)}|x-x'|^{\alpha-\frac{mn}{p_0'}}, \end{aligned} \tag{1.4}$$

where $Q(x, 2^{k+2}\sqrt{mn}|x-x'|)$ denotes the cube centered at x with the sidelength $2^{k+2}\sqrt{mn}$ $|x-x'|$, (p_0, p_0') is a pair of positive numbers satisfying $1/p_0 + 1/p_0' = 1$, $1 < p_0 < \infty$ and C_k is a positive constant depending on k.

Definition 1.2. Let $0 < \alpha p_0' < mn$ and T_α be an m-linear fractional integral operator defined by (1.3). Then T_α is called an m-linear fractional integral operator with generalized kernel if the following conditions are satisfied.

(1) The kernel function satisfies the size condition (1.1) and the smoothness condition (1.4).

(2) For fixed $1 \le s_1, \ldots, s_m \le p_0'$ with $\frac{1}{s} = \frac{1}{s_1} + \cdots + \frac{1}{s_m}$, T_α maps $L^{s_1} \times \cdots \times L^{s_m}$ to $L^{\frac{sn}{n-s\alpha},\infty}$.

Remark 1.3. As a matter of fact, when $C_k = \omega(2^{-k})$, it is straightforward to deduce that the condition (1.2) implies the condition (1.4) for any $1 < p_0 < \infty$. Denote $\Omega_k := Q(x, 2^k\sqrt{mn}|x-x'|)$. We have

$$\begin{aligned}
&\left(\int_{Q(x,2^{k+2}\sqrt{mn}|x-x'|)^m \setminus Q(x,2^{k+1}\sqrt{mn}|x-x'|)^m} |K_\alpha(x,\vec{y}) - K_\alpha(x',\vec{y})|^{p_0}\, d\vec{y}\right)^{1/p_0} \\
&\le \left(\int_{(\Omega_{k+2})^m \setminus (\Omega_{k+1})^m} \left[\frac{A}{\left(\sum_{j=1}^{\infty}|x-y_j|\right)^{mn-\alpha}}\, \omega\left(\frac{|x-x'|}{\sum_{j=1}^{\infty}|x-y_j|}\right)\right]^{p_0} d\vec{y}\right)^{1/p_0} \\
&\le \left(\int_{(\Omega_{k+2})^m \setminus (\Omega_{k+1})^m} \left[\frac{A}{\left(\max_{1\le j\le m}|x-y_j|\right)^{mn-\alpha}}\, \omega\left(\frac{|x-x'|}{\max_{1\le j\le m}|x-y_j|}\right)\right]^{p_0} d\vec{y}\right)^{1/p_0} \\
&\le \left(\int_{(\Omega_{k+2})^m \setminus (\Omega_{k+1})^m} \left[\frac{A}{(2^k|x-x'|)^{mn-\alpha}}\, \omega(2^{-k})\right]^{p_0} d\vec{y}\right)^{1/p_0} \\
&\le C\omega(2^{-k})2^{-k(mn-\alpha)}|x-x'|^{-(mn-\alpha)} \left(\int_{(\Omega_{k+2})^m} d\vec{y}\right)^{1/p_0} \\
&= C\omega(2^{-k})2^{-k(mn-\alpha)}|x-x'|^{-(mn-\alpha)}(2^{k+2}\sqrt{mn}|x-x'|)^{mn/p_0} \\
&\le C\omega(2^{-k})2^{k\left(\alpha-\frac{mn}{p_0'}\right)}|x-x'|^{\alpha-\frac{mn}{p_0'}} \\
&\le CC_k 2^{k\left(\alpha-\frac{mn}{p_0'}\right)}|x-x'|^{\alpha-\frac{mn}{p_0'}}.
\end{aligned}$$

Remark 1.4. When $\alpha = 0$, the m-linear fractional integral operator with generalized kernel corresponds precisely to the multilinear Calderón–Zygmund operator with generalized kernel studied by Yang, et al. [23], as well as Gao, et al. [11].

The article is structured as follows. In Section 2, we establish the boundedness of multilinear fractional integral operators with generalized kernels and multilinear commutators on weighted Lebesgue spaces. Section 3 focuses on demonstrating the boundedness of multilinear fractional integral operators and multilinear commutators on variable exponent Lebesgue spaces.

Throughout this article, the letter C always denotes a constant independent of the main parameters involved, whose value may vary from line to line. A cube $Q \subset \mathbb{R}^n$ always refers to a cube with sides parallel to the coordinate axes, and we denote its side length by $l(Q)$. For $t > 0$, the notation tQ represents the cube with the same center as Q and side length $l(tQ) = tl(Q)$. Denote by $|S|$ the Lebesgue measure and χ_S represents the characteristic function for a measurable set $S \subset \mathbb{R}^n$. $B(x, r)$ denotes the ball centered at x with radius r. The notation $X \sim Y$ indicates that there exists a constant $C > 0$ such that $C^{-1}Y \le X \le CY$. For any index $1 < q(x) < \infty$, we denote its conjugate index by $q'(x) := \frac{q(x)}{q(x)-1}$. Occasionally, we use the notation $\vec{f} = (f_1, \dots, f_m)$, $T(\vec{f}) := T(f_1, \dots, f_m)$, $d\vec{y} := dy_1 \cdots dy_m$ and $(x, \vec{y}) := (x, y_1, \dots, y_m)$ for convenience. For a set E and a positive integer m, we use the notation $(E)^m := \underbrace{E \times \cdots \times E}_{m}$ sometimes.

2. The boundedness on weighted Lebesgue spaces

The arrangement of this section is as follows. In Subsection 2.1, we present certain definitions and symbols that will be utilized later on. Subsection 2.2 is dedicated to establishing the pointwise estimate for the sharp maximal function. Subsection 2.3 addresses the boundedness of multilinear fractional integral operators with generalized kernels and multilinear commutators on weighted Lebesgue spaces.

2.1. Definitions and lemmas

Definition 2.1. (see [16]) Let $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and denote by M the usual Hardy–Littlewood maximal operator. For a cube $Q \subset \mathbb{R}^n$ and $\delta > 0$, the maximal function M_δ is defined by

$$M_\delta(f)(x) := \left[M(|f|^\delta)(x)\right]^{1/\delta} = \left(\sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y)|^\delta \, dy\right)^{1/\delta}.$$

Let $M^\#$ be the standard sharp maximal function, that is,

$$M^\# f(x) := \sup_{Q \ni x} \frac{1}{|Q|} \int_Q |f(y) - f_Q| \, dy \sim \sup_{Q \ni x} \inf_c \frac{1}{|Q|} \int_Q |f(y) - c| \, dy,$$

where, as usual, f_Q denotes the average of f over the cube Q containing the point x. The operator $M^\#_\delta$ is defined as $M^\#_\delta(f)(x) := \left[M^\#(|f|^\delta)(x)\right]^{1/\delta}$.

Definition 2.2. For $0 < \alpha < n$ and $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, the standard fractional maximal operator $\mathscr{M}_\alpha$ is defined by

$$\mathscr{M}_\alpha(f)(x) := \sup_{Q \ni x} \frac{1}{|Q|^{1-\frac{\alpha}{n}}} \int_Q |f(y)| \, dy.$$

For any $r > 1$, $\mathscr{M}_{\alpha,r}$ is defined by

$$\mathscr{M}_{\alpha,r}(f)(x) := \sup_{Q \ni x} |Q|^{\alpha/n} \left(\frac{1}{|Q|} \int_Q |f(y)|^r \, dy \right)^{1/r}.$$

Definition 2.3. (see [22]) Let $0 < \alpha < mn$ and the multilinear fractional maximal operator $\mathcal{M}_\alpha$ is defined by setting. For $\vec{f} = (f_1, \ldots, f_m)$ with $f_i \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$, $i = 1, \ldots, m$, and $x \in \mathbb{R}^n$,

$$\mathcal{M}_\alpha(\vec{f})(x) := \sup_{Q \ni x} |Q|^{\alpha/n} \prod_{j=1}^m \frac{1}{|Q|} \int_Q |f_j(y_j)| \, dy_j = \sup_{Q \ni x} \prod_{j=1}^m \frac{1}{|Q|^{1-\frac{\alpha}{mn}}} \int_Q |f_j(y_j)| \, dy_j.$$

For any $r > 1$, $\mathcal{M}_{\alpha,r}$ is defined by

$$\mathcal{M}_{\alpha,r}(\vec{f})(x) := \sup_{Q \ni x} |Q|^{\alpha/n} \prod_{j=1}^m \left(\frac{1}{|Q|} \int_Q |f_j(y_j)|^r \, dy_j \right)^{1/r}.$$

Definition 2.4. (see [17]) Let $\vec{P} := (p_1, \ldots, p_m)$ and $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 \le p_j < \infty$ $(j = 1, \ldots, m)$. Given $\vec{\omega} := (\omega_1, \ldots, \omega_m)$ with non-negative functions $\omega_1, \ldots, \omega_m$ on $\mathbb{R}^n$ and $p \le q < \infty$, we say that $\vec{\omega}$ satisfies the $A_{\vec{P},q}$ condition if

$$\sup_Q \left(\frac{1}{|Q|} \int_Q v_{\vec{\omega}}(x)^q \, dx \right)^{1/q} \prod_{j=1}^m \left(\frac{1}{|Q|} \int_Q \omega_j(x)^{-p'_j} \, dx \right)^{1/p'_j} < \infty,$$

where the supremum is taken over all cubes $Q \subset \mathbb{R}^n$, $v_{\vec{\omega}} = \prod_{j=1}^m \omega_j$, and $\big(\frac{1}{|Q|} \int_Q \omega_j(x)^{-p'_j} \, dx \big)^{1/p'_j}$ is understood as $\big(\inf_Q \omega_j \big)^{-1}$ when $p_j = 1$.

Definition 2.5. (see [21]) Suppose $\vec{b} := (b_1, \ldots, b_m)$ with $b_j \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ $(j = 1, \ldots, m)$. We define the m-linear commutator $T_{\vec{b},\alpha}$ to be

$$T_{\vec{b},\alpha}(\vec{f})(x) := \sum_{i=1}^m T^i_{\vec{b},\alpha}(\vec{f})(x).$$

Each term represents the commutator of T_α in the i-th entry with b_i, which is defined by

$$T^i_{\vec{b},\alpha}(\vec{f})(x) := b_i(x) T_\alpha(f_1, \ldots, f_i, \ldots, f_m)(x) - T_\alpha(f_1, \ldots, b_i f_i, \ldots, f_m)(x).$$

Definition 2.6. (see [15]) Suppose that $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$. Let

$$\|b\|_{\mathrm{BMO}} := \sup_Q \frac{1}{|Q|} \int_Q |b(x) - b_Q| \, dx,$$

where the supreme is taken over all cubes $Q \subset \mathbb{R}^n$, and

$$f_Q := \frac{1}{|Q|} \int_Q f(y) \, dy.$$

Define

$$\mathrm{BMO}(\mathbb{R}^n) := \{f : \|f\|_{\mathrm{BMO}} < \infty\}.$$

Let $\vec{b} = (b_1, \dots, b_m)$, if $b_j \in \mathrm{BMO}$ for $1 \le j \le m$, then $\vec{b} \in \mathrm{BMO}^m$.

Lemma 2.7. (see [7]) *Let $0 < p, \delta < \infty$ and ω be any A_∞-weight ($A_\infty := \bigcup_{p\ge 1} A_p$). Then there exists a constant $C > 0$ (depending on the A_∞ constant of ω), such that the inequalities*

$$\int \big(M_\delta f(x)\big)^p \omega(x)\, dx \le C \int \big(M_\delta^{\#} f(x)\big)^p \omega(x)\, dx$$

and

$$\|M_\delta(f)\|_{L^{p,\infty}(\omega)} \le C \|M_\delta^{\#}(f)\|_{L^{p,\infty}(\omega)}$$

hold for any function f for which the left hand side is finite.

Lemma 2.8. (see [14]) *Let $0 < p < u < \infty$. There exists a positive constant $C = C_{p,u}$ such that the following inequality holds:*

$$|Q|^{-1/p} \|f\|_{L^p(Q)} \le C |Q|^{-1/u} \|f\|_{L^{u,\infty}(Q)}.$$

Lemma 2.9. (see [17]) *Let $0 < \alpha < mn$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 \le p_j < \infty$ ($j = 1, \dots, m$), and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} > 0$.*

(i) *If $1 < p_j < \infty$ for all $j = 1, \dots, m$, then for $\vec{\omega} \in A_{\vec{P},q}$, there exists a constant $C > 0$ independent of $\vec{f}$ such that*

$$\big\|\mathcal{M}_\alpha(\vec{f})\big\|_{L^q(v_{\vec{\omega}}^q)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j^{p_j})}.$$

(ii) *If $1 \le p_j < \infty$ for all $j = 1, \dots, m$ and at least one $p_j = 1$ for some $j = 1, \dots, m$, then for $\vec{\omega} \in A_{\vec{P},q}$, there exists a constant $C > 0$ independent of $\vec{f}$ such that*

$$\big\|\mathcal{M}_\alpha(\vec{f})\big\|_{L^{q,\infty}(v_{\vec{\omega}}^q)} \le C \prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j^{p_j})}.$$

Lemma 2.10. (see [17]) *Let $0 < \alpha < mn$, $\vec{\omega} = (\omega_1, \dots, \omega_m)$, $v_{\vec{\omega}} = \prod_{j=1}^m \omega_j$, $\vec{P} = (p_1, \dots, p_m)$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $1 \le p_j < \infty$ ($j = 1, \dots, m$), and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} > 0$. Suppose $\vec{\omega} \in A_{\vec{P},q}$, then*

$$\omega_j^{-p_j'} \in A_{mp_j'} \ \text{ for } j = 1, \dots, m \quad \text{and} \quad v_{\vec{\omega}}^q \in A_{mq}.$$

Lemma 2.11. (see [1]) *Let $a \ge 1$. If $b \in \mathrm{BMO}$, then for all cubes Q, we have*

(i) $\big(\frac{1}{|Q|}\int_Q |b(y) - b_Q|\, dy\big)^{1/a} \le C\|b\|_{\mathrm{BMO}}$.

(ii) $\big(\frac{1}{|2^k Q|}\int_{2^k Q} |b(y) - b_Q|\, dy\big)^{1/a} \le Ck\|b\|_{\mathrm{BMO}}$, *for $k \in \mathbb{N}^+$.*

2.2. Sharp maximal pointwise estimates

In this subsection, we initially obtain the sharp maximal estimates for multilinear fractional integral operators with generalized kernels and multilinear commutators generated by BMO functions and these operators.

Theorem 2.12. *Let $m \geq 2$, $0 < \alpha p_0' < mn$, T_α be an m-linear fractional integral operator with generalized kernel as defined in Definition 1.2 and $\sum_{k=1}^{\infty} C_k < \infty$. Assuming $0 < \delta < \min\{1, \frac{sn}{n-s\alpha}\}$, there exists a constant $C > 0$ such that for all m-tuples of bounded measurable functions $\vec{f} = (f_1, \dots, f_m)$ with compact support, the following holds:*

$$M_\delta^{\#}(T_\alpha(\vec{f}))(x) \leq C\mathcal{M}_{\alpha,p_0'}(\vec{f})(x).$$

Proof. Since $\big||a|^e - |b|^e\big| \leq |a-b|^e$ for $0 < e < 1$, for any $Q \ni x$, we can estimate

$$\left(\frac{1}{|Q|}\int_Q \big||T_\alpha(\vec{f})(z)|^\delta - |c|^\delta\big|\, dz\right)^{1/\delta} \leq C\left(\frac{1}{|Q|}\int_Q |T_\alpha(\vec{f})(z) - c|^\delta\, dz\right)^{1/\delta}.$$

Let $Q^* := 14n\sqrt{mn}Q$, we decompose $f_j := f_j^0 + f_j^\infty$ with $f_j^0 := f_j\chi_{Q^*}$, then

$$\begin{aligned}\prod_{j=1}^m f_j(y_j) &= \prod_{j=1}^m \left(f_j^0(y_j) + f_j^\infty(y_j)\right)\\ &= \sum_{(\rho_1,\dots,\rho_m)\in\{0,\infty\}} f_1^{\rho_1}(y_1)\times\cdots\times f_m^{\rho_m}(y_m)\\ &= \prod_{j=1}^m f_j^0(y_j) + \sum_{(\rho_1,\dots,\rho_m)\in\rho} f_1^{\rho_1}(y_1)\times\cdots\times f_m^{\rho_m}(y_m),\end{aligned}$$

where $\rho = \{(\rho_1,\dots,\rho_m) :$ there is at least one $\rho_j = \infty\}$. It is easy to see that

$$T_\alpha(\vec{f})(z) = T_\alpha(f_1^0,\dots,f_m^0)(z) + \sum_{(\rho_1,\dots,\rho_m)\in\rho} T_\alpha(f_1^{\rho_1},\dots,f_m^{\rho_m})(z).$$

Let $c = \sum_{(\rho_1,\dots,\rho_m)\in\rho} c_{\rho_1,\dots,\rho_m}$, then we can get

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q |T_\alpha(\vec{f})(z) - c|^\delta\, dz\right)^{1/\delta}\\ &\leq C\left(\frac{1}{|Q|}\int_Q |T_\alpha(f_1^0,\dots,f_m^0)(z)|^\delta\, dz\right)^{1/\delta}\\ &\quad + C\sum_{(\rho_1,\dots,\rho_m)\in\rho}\left(\frac{1}{|Q|}\int_Q \left|T_\alpha(f_1^{\rho_1},\dots,f_m^{\rho_m})(z) - c_{\rho_1,\dots,\rho_m}\right|^\delta dz\right)^{1/\delta}\\ &:= I + \sum_{(\rho_1,\dots,\rho_m)\in\rho} II_{\rho_1,\dots,\rho_m}.\end{aligned}$$

From Definition 1.2(2), Hölder's inequality and Lemma 2.8, it follows that

$$\begin{aligned}
I &\le C|Q|^{-(n-s\alpha)/(sn)}\|T_\alpha(f_1^0,\dots,f_m^0)\|_{L^{\frac{sn}{n-s\alpha},\infty}(Q)}\\
&\le C|Q|^{\alpha/n}|Q|^{-1/s}\left(\int_{Q^*}|f_1(y_1)|^{s_1}\,dy_1\right)^{1/s_1}\times\cdots\times\left(\int_{Q^*}|f_m(y_m)|^{s_m}\,dy_m\right)^{1/s_m}\\
&\le C|Q^*|^{\alpha/n}\left(\frac{1}{|Q^*|}\int_{Q^*}|f_1(y_1)|^{s_1}\,dy_1\right)^{1/s_1}\times\cdots\times\left(\frac{1}{|Q^*|}\int_{Q^*}|f_m(y_m)|^{s_m}\,dy_m\right)^{1/s_m}\\
&\le C|Q^*|^{\alpha/n}\prod_{j=1}^m\left(\frac{1}{|Q^*|}\int_{Q^*}|f_j(y_j)|^{p_0'}\,dy_m\right)^{1/p_0'}\\
&\le C\mathcal{M}_{\alpha,p_0'}(\vec f)(x).
\end{aligned}$$

To estimate $II_{\rho_1,\dots,\rho_m}$, we choose $c_{\rho_1,\dots,\rho_m} := T_\alpha(f_1^{\rho_1},\dots,f_m^{\rho_m})(z_0)$ for a fixed $z_0\in 4Q\backslash 3Q$. Let $\Omega_k := Q(z_0, 2^k\sqrt{mn}|z-z_0|)$, $k\in\mathbb{N}^+$. Note that $\Omega_2\subset Q^*$, so $(\mathbb{R}^n)^m\backslash(Q^*)^m\subset(\mathbb{R}^n)^m\backslash(\Omega_2)^m$. When $z\in Q$, we obtain $l(Q)\le|z-z_0|\le\frac{5}{2}\sqrt{n}l(Q)$. By Hölder's inequality, we conclude that

$$\begin{aligned}
II_{\rho_1,\dots,\rho_m} &\le \frac{C}{|Q|}\int_Q\left|T_\alpha(f_1^{\rho_1},\dots,f_m^{\rho_m})(z)-T_\alpha(f_1^{\rho_1},\dots,f_m^{\rho_m})(z_0)\right|dz\\
&\le \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n)^m\backslash(Q^*)^m}|K_\alpha(z,\vec y)-K_\alpha(z_0,\vec y)|\prod_{j=1}^m|f_j(y_j)|\,d\vec y\,dz\\
&\le \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n)^m\backslash(\Omega_2)^m}|K_\alpha(z,\vec y)-K_\alpha(z_0,\vec y)|\prod_{j=1}^m|f_j(y_j)|\,d\vec y\,dz\\
&\le \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\int_{(\Omega_{k+2})^m\backslash(\Omega_{k+1})^m}|K_\alpha(z,\vec y)-K_\alpha(z_0,\vec y)|\prod_{j=1}^m|f_j(y_j)|\,d\vec y\,dz\\
&\le \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty\left(\int_{(\Omega_{k+2})^m\backslash(\Omega_{k+1})^m}|K_\alpha(z,\vec y)-K_\alpha(z_0,\vec y)|^{p_0}\,d\vec y\right)^{1/p_0}\\
&\qquad\times\left(\int_{(\Omega_{k+2})^m}\prod_{j=1}^m|f_j(y_j)|^{p_0'}\,d\vec y\right)^{1/p_0'}dz\\
&\le \frac{C}{|Q|}\int_Q\sum_{k=1}^\infty C_k|z-z_0|^{\alpha-\frac{mn}{p_0'}}2^{k\left(\alpha-\frac{mn}{p_0'}\right)}|\Omega_{k+2}|^{\frac{m}{p_0'}-\frac{\alpha}{n}}\\
&\qquad\times|\Omega_{k+2}|^{\alpha/n}\prod_{j=1}^m\left(\frac{1}{|\Omega_{k+2}|}\int_{\Omega_{k+2}}|f_j(y_j)|^{p_0'}\,dy_j\right)^{1/p_0'}dz\\
&\le C\mathcal{M}_{\alpha,p_0'}(\vec f)(x).
\end{aligned}$$

Combining the above estimates we get the desired result. □

Theorem 2.13. *Let $m \geq 2$, $0 < \alpha p_0' < mn$, T_α be an m-linear fractional integral operator with generalized kernel as defined in Definition 1.2 and $\sum_{k=1}^{\infty} kC_k < \infty$. Assuming $0 < \delta < \min\{1, \frac{sn}{n-s\alpha}\}$, $p_0' < t < \infty$, $\delta < \varepsilon < \infty$ and $\vec{b} \in \mathrm{BMO}^m$, there exists a constant $C > 0$ such that for all bounded measurable functions with compact support m-tuples $\vec{f} = (f_1, \dots, f_m)$, we have*

$$M_\delta^{\#}(T_{\vec{b},\alpha}(\vec{f}))(x) \leq C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(\mathcal{M}_{\alpha,t}(\vec{f})(x) + M_\varepsilon(T_\alpha(\vec{f}))(x)\big),$$

where $\|\vec{b}\|_{(\mathrm{BMO})^m} := \max_{1\leq i\leq m}\|b_i\|_{\mathrm{BMO}}$.

Proof. Let $Q^* := 14n\sqrt{mn}Q$, we decompose $f_j := f_j^0 + f_j^\infty$ with $f_j^0 := f_j\chi_{Q^*}$, then

$$\begin{aligned}\prod_{j=1}^m f_j(y_j) &= \prod_{j=1}^m \big(f_j^0(y_j) + f_j^\infty(y_j)\big)\\ &= \sum_{(\rho_1,\dots,\rho_m)\in\{0,\infty\}} f_1^{\rho_1}(y_1) \times \cdots \times f_m^{\rho_m}(y_m)\\ &= \prod_{j=1}^m f_j^0(y_j) + \sum_{(\rho_1,\dots,\rho_m)\in\rho} f_1^{\rho_1}(y_1) \times \cdots \times f_m^{\rho_m}(y_m),\end{aligned}$$

where $\rho = \{(\rho_1, \dots, \rho_m) :$ there is at least one $\rho_j = \infty\}$. It is easy to see that

$$\begin{aligned}&T_\alpha(f_1, \dots, (b_j - \lambda)f_j, \dots, f_m)(z)\\ &= T_\alpha(f_1^0, \dots, (b_j - \lambda)f_j^0, \dots, f_m^0)(z) + \sum_{(\rho_1,\dots,\rho_m)\in\rho} T_\alpha(f_1^{\rho_1}, \dots, (b_j - \lambda)f_j^{\rho_j}, \dots, f_m^{\rho_m})(z).\end{aligned}$$

Fix $x \in \mathbb{R}^n$ and let Q be a cube containing x. Denote $\lambda = b_{j_{Q^*}}$. Let $c_j = -\sum_{(\rho_1,\dots,\rho_m)\in\rho} c_{j,\rho_1,\dots,\rho_m}$, then

$$\begin{aligned}&\left(\frac{1}{|Q|}\int_Q \big|T_{\vec{b},\alpha}^j(\vec{f})(z) - c_j\big|^\delta dz\right)^{1/\delta}\\ &\leq C\left(\frac{1}{|Q|}\int_Q \big|(b_j(z) - \lambda)T_\alpha(\vec{f})(z)\big|^\delta dz\right)^{1/\delta}\\ &\quad + C\left(\frac{1}{|Q|}\int_Q \big|T_\alpha(f_1, \dots, (b_j - \lambda)f_j, \dots, f_m)(z) + c_j\big|^\delta dz\right)^{1/\delta}\\ &\leq C\left(\frac{1}{|Q|}\int_Q \big|(b_j(z) - \lambda)T_\alpha(\vec{f})(z)\big|^\delta dz\right)^{1/\delta}\\ &\quad + C\left(\frac{1}{|Q|}\int_Q \big|T_\alpha(f_1^0, \dots, (b_j - \lambda)f_j^0, \dots, f_m^0)(z)\big|^\delta dz\right)^{1/\delta}\\ &\quad + C\sum_{(\rho_1,\dots,\rho_m)\in\rho}\left(\frac{1}{|Q|}\int_Q \big|T_\alpha(f_1^{\rho_1}, \dots, (b_j - \lambda)f_j^{\rho_j}, \dots, f_m^{\rho_m})(z) - c_{j,\rho_1,\dots,\rho_m}\big|^\delta dz\right)^{1/\delta}\\ &:= III + IV + \sum_{(\rho_1,\dots,\rho_m)\in\rho} V_{\rho_1,\dots,\rho_m}.\end{aligned}$$

Choosing $1 < h_1' < \min\left\{\frac{1}{1-\delta}, \frac{\varepsilon}{\delta}\right\}$ with $\frac{1}{h_1'} + \frac{1}{h_1} = 1$, by Hölder's inequality, we conclude that

$$\begin{aligned} III &\le C\left(\frac{1}{|Q|}\int_Q |b_j(z)-\lambda|^{\delta h_1}\,dz\right)^{1/(\delta h_1)}\left(\frac{1}{|Q|}\int_Q |T_\alpha(\vec{f})|^{\delta h_1'}\,dz\right)^{1/(\delta h_1')} \\ &\le C\left(\frac{1}{|Q^*|}\int_{Q^*} |b_j(z)-\lambda|^{\delta h_1}\,dz\right)^{1/(\delta h_1)}\left(\frac{1}{|Q|}\int_Q |T_\alpha(\vec{f})|^{\delta h_1'}\,dz\right)^{1/(\delta h_1')} \\ &\le C\|b_j\|_{\mathrm{BMO}} M_\varepsilon(T_\alpha(\vec{f}))(x). \end{aligned}$$

Let $h_2 = \frac{t}{s_j}$ with $\frac{1}{h_2} + \frac{1}{h_2'} = 1$. From Definition 1.2(2), Hölder's inequality, Lemma 2.8 and $s_i \le p_0' < t$ $(i = 1, \ldots, m)$, it follows that

$$\begin{aligned} IV &\le C|Q|^{-(n-s\alpha)/(sn)}\left\|T_\alpha(f_1^0, \ldots, (b_j-\lambda)f_j^0, \ldots, f_m^0)\right\|_{L^{\frac{sn}{n-s\alpha},\infty}(Q)} \\ &\le C|Q^*|^{\alpha/n}|Q^*|^{-1/s}\prod_{i=1,\, i\ne j}^m \|f_i\|_{L^{s_i}(Q^*)}\|(b_j-\lambda)f_j\|_{L^{s_j}(Q^*)} \\ &= C|Q^*|^{\alpha/n}\prod_{i=1,\, i\ne j}^m\left(\frac{1}{|Q^*|}\int_{Q^*}|f_i(y_i)|^{s_i}\,dy_i\right)^{1/s_i} \\ &\quad\times\left(\frac{1}{|Q^*|}\int_{Q^*}|b_j(y_j)-\lambda|^{s_j}|f_j(y_j)|^{s_j}\,dy_j\right)^{1/s_j} \\ &\le C|Q^*|^{\alpha/n}\prod_{i=1,\, i\ne j}^m\left(\frac{1}{|Q^*|}\int_{Q^*}|f_i(y_i)|^{s_i}\,dy_i\right)^{1/s_i}\left(\frac{1}{|Q^*|}\int_{Q^*}|b_j(y_j)-\lambda|^{s_jh_2'}\,dy_j\right)^{1/(s_jh_2')} \\ &\quad\times\left(\frac{1}{|Q^*|}\int_{Q^*}|f_j(y_j)|^{s_jh_2}\,dy_j\right)^{1/(s_jh_2)} \\ &\le C\|b_j\|_{\mathrm{BMO}}|Q^*|^{\alpha/n}\prod_{i=1}^m\left(\frac{1}{|Q^*|}\int_{Q^*}|f_i(y_i)|^t\,dy_i\right)^{1/t} \\ &\le C\|b_j\|_{\mathrm{BMO}}\mathcal{M}_{\alpha,t}(\vec{f})(x). \end{aligned}$$

To estimate $V_{\rho_1,\ldots,\rho_m}$, we choose $c_{j,\rho_1,\ldots,\rho_m} = T_\alpha(f_1^{\rho_1}, \ldots, (b_j-\lambda)f_j^{\rho_j}, \ldots, f_m^{\rho_m})(z_0)$ for a fixed $z_0 \in 4Q \setminus 3Q$. Since $\Omega_2 \subset Q^*$, we can get $\Omega_{k+2} \subset 2^kQ^*$. Since $|z - z_0| \sim l(Q)$ when $z \in Q$, then $|\Omega_{k+2}| \sim |2^kQ^*|$. Let $h_3 = \frac{t}{p_0'}$ with $\frac{1}{h_3} + \frac{1}{h_3'} = 1$. By Hölder's inequality, Lemma 2.11 and $s_i \le p_0' < t$ $(i = 1, \ldots, m)$, we conclude that

$$\begin{aligned} &V_{\rho_1,\ldots,\rho_m} \\ &\le \frac{C}{|Q|}\int_Q\left|T_\alpha(f_1^{\rho_1}, \ldots, (b_j-\lambda)f_j^{\rho_j}, \ldots, f_m^{\rho_m})(z) - T_\alpha(f_1^{\rho_1}, \ldots, (b_j-\lambda)f_j^{\rho_j}, \ldots, f_m^{\rho_m})(z_0)\right|dz \\ &\le \frac{C}{|Q|}\int_Q\int_{(\mathbb{R}^n)^m\setminus(Q^*)^m}|K_\alpha(z,\vec{y}) - K_\alpha(z_0,\vec{y})|\prod_{i=1}^m|f_i(y_i)||b_j(y_j)-\lambda|\,d\vec{y}\,dz \end{aligned}$$

$$
\begin{aligned}
&\leq \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty}\int_{(\Omega_{k+2})^m\setminus(\Omega_{k+1})^m} |K_\alpha(z,\vec{y})-K_\alpha(z_0,\vec{y})|\prod_{i=1}^{m}|f_i(y_i)||b_j(y_j)-\lambda|\,d\vec{y}\,dz\\
&\leq \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty}\left(\int_{(\Omega_{k+2})^m\setminus(\Omega_{k+1})^m} |K_\alpha(z,\vec{y})-K_\alpha(z_0,\vec{y})|^{p_0}\,d\vec{y}\right)^{1/p_0}\\
&\qquad\times\left(\int_{(\Omega_{k+2})^m}\left(\prod_{i=1}^{m}|f_i(y_i)||b_j(y_j)-\lambda|\right)^{p_0'}d\vec{y}\right)^{1/p_0'}dz\\
&\leq \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty} C_k|z-z_0|^{\alpha-\frac{mn}{p_0'}}2^{k\left(\alpha-\frac{mn}{p_0'}\right)}|\Omega_{k+2}|^{\frac{m}{p_0'}-\frac{\alpha}{n}}|\Omega_{k+2}|^{\alpha/n}\\
&\qquad\times\prod_{i=1,\, i\neq j}^{m}\left(\frac{1}{|\Omega_{k+2}|}\int_{\Omega_{k+2}}|f_i(y_i)|^{p_0'}\,dy_i\right)^{1/p_0'}\\
&\qquad\qquad\times\left(\frac{1}{|\Omega_{k+2}|}\int_{\Omega_{k+2}}|b_j(y_j)-\lambda|^{p_0'}|f_j(y_j)|^{p_0'}\,dy_j\right)^{1/p_0'}dz\\
&\leq \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty} C_k|\Omega_{k+2}|^{\alpha/n}\prod_{i=1,\, i\neq j}^{m}\left(\frac{1}{|\Omega_{k+2}|}\int_{\Omega_{k+2}}|f_i(y_i)|^{t}\,dy_i\right)^{1/t}\\
&\qquad\times\left(\frac{1}{|\Omega_{k+2}|}\int_{\Omega_{k+2}}|f_j(y_j)|^{p_0'h_3}\,dy_j\right)^{1/(p_0'h_3)}\\
&\qquad\times\left(\frac{1}{|\Omega_{k+2}|}\int_{\Omega_{k+2}}|b_j(y_j)-\lambda|^{p_0'h_3'}\,dy_j\right)^{1/(p_0'h_3')}dz\\
&\leq \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty} C_k|\Omega_{k+2}|^{\alpha/n}\prod_{i=1,\, i\neq j}^{m}\left(\frac{1}{|\Omega_{k+2}|}\int_{\Omega_{k+2}}|f_i(y_i)|^{t}\,dy_i\right)^{1/t}\\
&\qquad\times\left(\frac{1}{|\Omega_{k+2}|}\int_{\Omega_{k+2}}|f_j(y_j)|^{t}\,dy_i\right)^{1/t}\left(\frac{1}{|2^kQ^*|}\int_{2^kQ^*}|b_j(y_j)-\lambda|^{p_0'h_3}\,dy_j\right)^{1/(p_0'h_3')}dz\\
&\leq \frac{C}{|Q|}\int_Q \sum_{k=1}^{\infty} C_k|\Omega_{k+2}|^{\alpha/n}\prod_{i=1}^{m}\left(\frac{1}{|\Omega_{k+2}|}\int_{\Omega_{k+2}}|f_i(y_i)|^{t}\,dy_i\right)^{1/t}k\|b_j\|_{\mathrm{BMO}}\,dz\\
&\leq C\|b_j\|_{\mathrm{BMO}}\mathcal{M}_{\alpha,t}(\vec{f})(x).
\end{aligned}
$$

Then, we have

$$
\begin{aligned}
\inf_c\left(\frac{1}{|Q|}\int_Q\big||T_{\vec{b},\alpha}(\vec{f})(z)|^\delta-c\big|\,dz\right)^{1/\delta} &\leq \left(\frac{1}{|Q|}\int_Q\left|\Big|\sum_{i=1}^{m}T^i_{\vec{b},\alpha}(\vec{f})(z)\Big|^\delta-\Big|\sum_{i=1}^{m}c_i\Big|^\delta\right|dz\right)^{1/\delta}\\
&\leq C\sum_{i=1}^{m}\left(\frac{1}{|Q|}\int_Q\big|T^i_{\vec{b},\alpha}(\vec{f})(z)-c_i\big|^\delta\,dz\right)^{1/\delta}\\
&\leq C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(M_\varepsilon(T_\alpha(\vec{f}))(x)+\mathcal{M}_{\alpha,t}(\vec{f})(x)\big)
\end{aligned}
$$

and

$$M_\delta^{\#}(T_{\vec{b},\alpha}(\vec{f}))(x) \sim \sup_{Q\ni x}\inf_{c}\left(\frac{1}{|Q|}\int_Q \big||T_{\vec{b},\alpha}(\vec{f})(z)|^\delta - c\big|\,dz\right)^{1/\delta}$$
$$\leq C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(M_\varepsilon(T_\alpha(\vec{f}))(x)+\mathcal{M}_{\alpha,t}(\vec{f})(x)\big).$$

Thus, Theorem 2.13 is proved. □

2.3. The boundedness on weighted Lebesgue spaces

In this subsection, we use Theorems 2.12 and 2.13 to derive boundedness results for multilinear fractional integral operators with generalized kernels and multilinear commutators on weighted Lebesgue spaces. The main theorem of this subsection is presented below.

Theorem 2.14. *Let $m \geq 2$, $0 < \alpha p_0' < mn$, T_α be an m-linear fractional integral operator with generalized kernel as in Definition* 1.2 *and $\sum_{k=1}^\infty C_k < \infty$. Suppose that $\vec{P} = (p_1, \dots, p_m)$, $\frac{1}{p} = \frac{1}{p_1} + \cdots + \frac{1}{p_m}$ with $p_0' \leq p_j < \infty$ $(j = 1, \dots, m)$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} > 0$ and $\vec{\omega} \in A_{\frac{\vec{P}}{p_0'},\frac{q}{p_0'}}$.*

(i) *If $p_0' < p_j < \infty$ for all $j = 1, \dots, m$, then*

$$\|T_\alpha(\vec{f})\|_{L^q(v_{\vec{\omega}}^{q/p_0'})} \leq C\prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j^{p_j/p_0'})}.$$

(ii) *If $p_0' \leq p_j < \infty$ for all $j = 1, \dots, m$ and at least one of $p_j = p_0'$, then*

$$\|T_\alpha(\vec{f})\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/p_0'})} \leq C\prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j^{p_j/p_0'})}.$$

Proof. Firstly, we choose δ such that $0 < \delta < \min\{1, \frac{sn}{n-s\alpha}\}$ and let $\beta = \alpha p_0'$.

(i) By Lemma 2.10, for $\vec{\omega} \in A_{\frac{\vec{P}}{p_0'},\frac{q}{p_0'}}$ there is $v_{\vec{\omega}}^{q/p_0'} \in A_\infty$. Then, by Lemmas 2.7, 2.9(i) and Theorem 2.12, we conclude that

$$\begin{aligned}\|T_\alpha(\vec{f})\|_{L^q(v_{\vec{\omega}}^{q/p_0'})} &\leq \big\|M_\delta(T_\alpha(\vec{f}))\big\|_{L^q(v_{\vec{\omega}}^{q/p_0'})} \leq C\big\|M_\delta^{\#}(T_\alpha(\vec{f}))\big\|_{L^q(v_{\vec{\omega}}^{q/p_0'})}\\
&\leq C\big\|\mathcal{M}_{\alpha,p_0'}(\vec{f})\big\|_{L^q(v_{\vec{\omega}}^{q/p_0'})} = C\big\|\big[\mathcal{M}_\beta(\vec{f}^{p_0'})\big]^{1/p_0'}\big\|_{L^q(v_{\vec{\omega}}^{q/p_0'})}\\
&= C\big\|\mathcal{M}_\beta(\vec{f}^{p_0'})\big\|^{1/p_0'}_{L^{q/p_0'}(v_{\vec{\omega}}^{q/p_0'})} \leq C\prod_{j=1}^m \big\||f_j|^{p_0'}\big\|^{1/p_0'}_{L^{p_j/p_0'}(\omega_j^{p_j/p_0'})}\\
&= C\prod_{j=1}^m \|f_j\|_{L^{p_j}(\omega_j^{p_j/p_0'})}.\end{aligned}$$

(ii) By Lemmas 2.7, 2.9(ii) and Theorem 2.12, we obtain

$$\begin{aligned}\|T_\alpha(\vec{f})\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/p_0'})} &\le \big\|M_\delta(T_\alpha(\vec{f}))\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/p_0'})} \le C\big\|M_\delta^{\#}(T_\alpha(\vec{f}))\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/p_0'})}\\ &\le C\big\|M_{\alpha,p_0'}(\vec{f})\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/p_0'})} = C\big\|\big[\mathcal{M}_\beta(\vec{f}^{p_0'})\big]^{1/p_0'}\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/p_0'})}\\ &= C\sup_{\lambda>0}\lambda\big|v_{\vec{\omega}}^{q/p_0'}\big(\big\{x\in\mathbb{R}^n:\big[\mathcal{M}_\beta(\vec{f}^{p_0'})\big]^{1/p_0'}>\lambda\big\}\big)\big|^{1/q}\\ &= C\sup_{\lambda>0}\lambda\big|v_{\vec{\omega}}^{q/p_0'}\big(\big\{x\in\mathbb{R}^n:\mathcal{M}_\beta(\vec{f}^{p_0'})>\lambda^{p_0'}\big\}\big)\big|^{1/q}\\ &= C\sup_{\lambda>0}\lambda^{1/p_0'}\big|v_{\vec{\omega}}^{q/p_0'}\big(\big\{x\in\mathbb{R}^n:\mathcal{M}_\beta(\vec{f}^{p_0'})>\lambda\big\}\big)\big|^{1/q}\\ &= C\left[\sup_{\lambda>0}\lambda\big|v_{\vec{\omega}}^{q/p_0'}\big(\big\{x\in\mathbb{R}^n:\mathcal{M}_\beta(\vec{f}^{p_0'})>\lambda\big\}\big)\big|^{p_0'/q}\right]^{1/p_0'}\\ &= C\big\|\mathcal{M}_\beta(\vec{f}^{p_0'})\big\|^{1/p_0'}_{L^{\frac{q}{p_0'},\infty}(v_{\vec{\omega}}^{q/p_0'})} \le C\prod_{j=1}^m\big\||f_j|^{p_0'}\big\|^{1/p_0'}_{L^{p_j/p_0'}(\omega_j^{p_j/p_0'})}\\ &= C\prod_{j=1}^m\|f_j\|_{L^{p_j}(\omega_j^{p_j/p_0'})}.\end{aligned}$$

Thus, Theorem 2.14 is proved. □

Theorem 2.15. *Let $m\ge 2$, $p_0'<t<\infty$, $0<\alpha t<mn$, T_α be an m-linear fractional integral operator with generalized kernel as defined in Definition 1.2 and $\sum_{k=1}^\infty kC_k<\infty$. Suppose that $\vec{b}\in \mathrm{BMO}^m$, $\vec{P}=(p_1,\dots,p_m)$, $\frac{1}{p}=\frac{1}{p_1}+\cdots+\frac{1}{p_m}$ with $t\le p_j<\infty$ ($j=1,\dots,m$), $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}>0$ and $\vec{\omega}\in A_{\frac{\vec{P}}{t},\frac{q}{t}}$.*

(i) *If $t<p_j<\infty$ for all $j=1,\dots,m$, then*

$$\|T_{\vec{b},\alpha}(\vec{f})\|_{L^q(v_{\vec{\omega}}^{q/t})}\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\omega_j^{p_j/t})}.$$

(ii) *If $t\le p_j<\infty$ for all $j=1,\dots,m$ and at least one of $p_j=t$, then*

$$\|T_{\vec{b},\alpha}(\vec{f})\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\omega_j^{p_j/t})}.$$

Proof. Firstly, we choose δ, ε such that $0<\delta<\varepsilon<\min\big\{1,\frac{sn}{n-s\alpha}\big\}$ and let $\beta=\alpha t$.

(i) By $\vec{\omega}\in A_{\frac{\vec{P}}{t},\frac{q}{t}}$, Lemmas 2.7, 2.9(i) and Theorems 2.12, 2.13, we conclude that

$$\begin{aligned}\|T_{\vec{b},\alpha}(\vec{f})\|_{L^q(v_{\vec{\omega}}^{q/t})} &\le \big\|M_\delta(T_{\vec{b},\alpha}(\vec{f}))\big\|_{L^q(v_{\vec{\omega}}^{q/t})}\le\big\|M_\delta^{\#}(T_{\vec{b},\alpha}(\vec{f}))\big\|_{L^q(v_{\vec{\omega}}^{q/t})}\\ &\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\big\|M_\varepsilon(T_\alpha(\vec{f}))+\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{L^q(v_{\vec{\omega}}^{q/t})}\end{aligned}$$

$$
\begin{aligned}
&\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(\big\|M_\varepsilon(T_\alpha(\vec{f}))\big\|_{L^q(v_{\vec{\omega}}^{q/t})}+\big\|\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{L^q(v_{\vec{\omega}}^{q/t})}\big)\\
&\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(\big\|M^{\#}_\varepsilon(T_\alpha(\vec{f}))\big\|_{L^q(v_{\vec{\omega}}^{q/t})}+\big\|\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{L^q(v_{\vec{\omega}}^{q/t})}\big)\\
&\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(\big\|\mathcal{M}_{\alpha,p_0'}(\vec{f})\big\|_{L^q(v_{\vec{\omega}}^{q/t})}+\big\|\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{L^q(v_{\vec{\omega}}^{q/t})}\big)\\
&\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\big\|\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{L^q(v_{\vec{\omega}}^{q/t})}=C\|\vec{b}\|_{(\mathrm{BMO})^m}\big\|\big[\mathcal{M}_\beta(\vec{f^t})\big]^{1/t}\big\|_{L^q(v_{\vec{\omega}}^{q/t})}\\
&=C\|\vec{b}\|_{(\mathrm{BMO})^m}\big\|\mathcal{M}_\beta(\vec{f^t})\big\|^{1/t}_{L^{q/t}(v_{\vec{\omega}}^{q/t})}\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\prod_{j=1}^m\big\||f_j|^t\big\|^{1/t}_{L^{p_j/t}(\omega_j^{p_j/t})}\\
&=C\|\vec{b}\|_{(\mathrm{BMO})^m}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\omega_j^{p_j/t})}.
\end{aligned}
$$

(ii) By Lemmas 2.7, 2.9(ii) and Theorems 2.12, 2.13, we conclude that

$$
\begin{aligned}
\|T_{\vec{b},\alpha}(\vec{f})\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}&\le\big\|M_\delta(T_{\vec{b},\alpha}(\vec{f}))\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}\le C\big\|M^{\#}_\delta(T_{\vec{b},\alpha}(\vec{f}))\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}\\
&\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\big\|M_\varepsilon(T_\alpha(\vec{f}))+M_{\alpha,t}(\vec{f})\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}\\
&\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(\big\|M_\varepsilon(T_\alpha(\vec{f}))\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}+\big\|M_{\alpha,t}(\vec{f})\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}\big)\\
&\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(\big\|M^{\#}_\varepsilon(T_\alpha(\vec{f}))\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}+\big\|M_{\alpha,t}(\vec{f})\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}\big)\\
&\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(\big\|M_{\alpha,p_0'}(\vec{f})\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}+\big\|M_{\alpha,t}(\vec{f})\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}\big)\\
&\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\big\|M_{\alpha,t}(\vec{f})\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}\\
&=C\|\vec{b}\|_{(\mathrm{BMO})^m}\big\|\big[\mathcal{M}_\beta(\vec{f^t})\big]^{1/t}\big\|_{L^{q,\infty}(v_{\vec{\omega}}^{q/t})}\\
&=C\|\vec{b}\|_{(\mathrm{BMO})^m}\sup_{\lambda>0}\lambda\big|v_{\vec{\omega}}^{q/t}\big(\big\{x\in\mathbb{R}^n:\big[\mathcal{M}_\beta(\vec{f^t})\big]^{1/t}(x)>\lambda\big\}\big)\big|^{1/q}\\
&=C\|\vec{b}\|_{(\mathrm{BMO})^m}\sup_{\lambda>0}\lambda\big|v_{\vec{\omega}}^{q/t}\big(\big\{x\in\mathbb{R}^n:\mathcal{M}_\beta(\vec{f^t})>\lambda^t\big\}\big)\big|^{1/q}\\
&=C\|\vec{b}\|_{(\mathrm{BMO})^m}\sup_{\lambda>0}\lambda^{1/t}\big|v_{\vec{\omega}}^{q/t}\big(\big\{x\in\mathbb{R}^n:\mathcal{M}_\beta(\vec{f^t})>\lambda\big\}\big)\big|^{1/q}\\
&=C\|\vec{b}\|_{(\mathrm{BMO})^m}\left[\sup_{\lambda>0}\lambda\big|v_{\vec{\omega}}^{q/t}\big(\big\{x\in\mathbb{R}^n:\mathcal{M}_\beta(\vec{f^t})>\lambda\big\}\big)\big|^{t/q}\right]^{1/t}\\
&=C\|\vec{b}\|_{(\mathrm{BMO})^m}\big\|\mathcal{M}_\beta(\vec{f^t})\big\|^{1/t}_{L^{\frac{q}{t},\infty}(v_{\vec{\omega}}^{q/t})}\\
&\le C\|\vec{b}\|_{(\mathrm{BMO})^m}\prod_{j=1}^m\big\||f_j|^t\big\|^{1/t}_{L^{p_j/t}(\omega_j^{p_j/t})}\\
&=C\|\vec{b}\|_{(\mathrm{BMO})^m}\prod_{j=1}^m\|f_j\|_{L^{p_j}(\omega_j^{p_j/t})}.
\end{aligned}
$$

Thus, Theorem 2.15 is proved. □

Remark 2.16. When the generalized kernel condition goes back to the Dini kernel condition, the boundedness of m-linear fractional integral operators with Dini's type kernels on Lebesgue spaces has been demonstrated by Wu and Zhang in [22].

Corollary 2.17. *When the generalized kernel condition goes back to the Dini kernel condition, the boundedness of multilinear commutators of m-linear fractional integral operators with Dini's type kernels on Lebesgue spaces remains new.*

3. The boundedness on variable exponent Lebesgue spaces

Firstly, we introduce several definitions and notations that will be utilized subsequently. Then we establish the boundedness of multilinear fractional integral operators with generalized kernels and multilinear commutators on variable exponent Lebesgue spaces, respectively.

Definition 3.1. (see [5]) Let $q(\cdot)\colon \mathbb{R}^n \to [1,\infty)$ be a measurable function. The variable exponent Lebesgue space $L^{q(\cdot)}(\mathbb{R}^n)$ is defined by

$$L^{q(\cdot)}(\mathbb{R}^n) := \{f \text{ is a measurable function} : F_q(f/\eta) < \infty \text{ for some constant } \eta > 0\},$$

where $F_q(f) := \int_{\mathbb{R}^n} |f(x)|^{q(x)}\,dx$ is a convex functional modular.

The space $L^{q(\cdot)}(\mathbb{R}^n)$ is a Banach function space with respect to the Luxemburg type norm

$$\|f\|_{L^{q(\cdot)}(\mathbb{R}^n)} := \inf\left\{\eta > 0 : \int_{\mathbb{R}^n} \left(\frac{|f(x)|}{\eta}\right)^{q(x)} dx \le 1\right\}.$$

Definition 3.2. (see [5]) Given a measurable function $q(\cdot)$ defined on $\mathbb{R}^n$. For $E \subset \mathbb{R}^n$, we write

$$q_-(E) := \operatorname*{ess\,inf}_{x\in E} q(x), \quad q_+(E) := \operatorname*{ess\,sup}_{x\in E} q(x),$$

and write $q_-(\mathbb{R}^n) = q_-$ and $q_+(\mathbb{R}^n) = q_+$ simply.

(i) $q'_- = \operatorname{ess\,inf}_{x\in\mathbb{R}^n} q'(x) = \frac{q_+}{q_+-1}$, $q'_+ = \operatorname{ess\,sup}_{x\in\mathbb{R}^n} q'(x) = \frac{q_-}{q_--1}$.

(ii) Denote by $\mathscr{P}(\mathbb{R}^n)$ the set of all measurable functions $q(\cdot)\colon \mathbb{R}^n \to (1,\infty)$ such that

$$1 < q_- \le q(x) \le q_+ < \infty, \quad x \in \mathbb{R}^n.$$

(iii) The set $\mathscr{B}(\mathbb{R}^n)$ consists of all measurable functions $q(\cdot) \in \mathscr{P}(\mathbb{R}^n)$ such that the Hardy–Littlewood maximal operator M is bounded on $L^{q(\cdot)}(\mathbb{R}^n)$, i.e.,

$$\|Mf\|_{L^{q(\cdot)}} \le \|f\|_{L^{q(\cdot)}}.$$

Definition 3.3. (see [4]) Let $q(\cdot)$ be a real-valued function on $\mathbb{R}^n$.

(i) Denote by $\mathscr{L}^{\log}_{\rm loc}(\mathbb{R}^n)$ the set of all local log-Hölder continuous functions $q(\cdot)$ which satisfy
$$|q(x)-q(y)| \le \frac{-C}{\ln(|x-y|)}, \quad |x-y|\le 1/2,\ x,y\in\mathbb{R}^n.$$
Here, the positive constant C does not depend on x or y.

(ii) The set $\mathscr{L}^{\log}_{\infty}(\mathbb{R}^n)$ consists of all log-Hölder continuous functions $q(\cdot)$ at infinity, satisfying
$$|q(x)-q_\infty| \le \frac{C_\infty}{\ln(e+|x|)}, \quad x\in\mathbb{R}^n,$$
where $q_\infty := \lim_{|x|\to\infty} q(x)$.

(iii) Denote by $\mathscr{L}^{\log}(\mathbb{R}^n) := \mathscr{L}^{\log}_{\rm loc}(\mathbb{R}^n)\cap \mathscr{L}^{\log}_{\infty}(\mathbb{R}^n)$ the set of all global log-Hölder continuous functions $q(\cdot)$.

Remark 3.4. (see [22]) The condition $\mathscr{L}^{\log}_{\infty}(\mathbb{R}^n)$ is equivalent to the uniform continuity condition
$$|q(x)-q(y)| \le \frac{C}{\ln(e+|x|)}, \quad |y|\ge |x|,\ x,y\in\mathbb{R}^n.$$
In what follows, we denote
$$\mathscr{P}^{\log}(\mathbb{R}^n) := \mathscr{L}^{\log}(\mathbb{R}^n)\cap\mathscr{P}(\mathbb{R}^n).$$

Lemma 3.5. (see [5]) *Let $p(\cdot)\in\mathscr{P}(\mathbb{R}^n)$.*

(1) *If $p(\cdot)\in\mathscr{L}^{\log}(\mathbb{R}^n)$, then we have $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$.*

(2) *The following condition are equivalent:*

  (i) $p(\cdot)\in\mathscr{B}(\mathbb{R}^n)$.

  (ii) $(p(\cdot)/p_0)'\in\mathscr{B}(\mathbb{R}^n)$ *for some* $1<p_0<p_-$.

Lemma 3.6. (see [14]) *Let $q(\cdot), q_1(\cdot),\dots,q_m(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ satisfy the condition*
$$\frac{1}{q(x)} = \frac{1}{q_1(x)}+\cdots+\frac{1}{q_m(x)} \quad \textit{for a.e. } x\in\mathbb{R}^n.$$
Then, for any $f_j\in L^{q_j(\cdot)}(\mathbb{R}^n)$, $j=1,\dots,m$, one has
$$\|f_1\cdots f_m\|_{q(\cdot)} \le C\|f_1\|_{q_1(\cdot)}\cdots\|f_m\|_{q_m(\cdot)}.$$

Lemma 3.7. (see [6]) *Let $\mathcal{F}$ denote a family of ordered pairs of measurable functions (f,g). suppose that for some fixed q_0 with $0<q_0<\infty$ and every weight $\omega \in A_1$ such that*

$$\int_{\mathbb{R}^n} |f(x)|^{q_0}\omega(x)\,dx \le C_0 \int_{\mathbb{R}^n} |g(x)|^{q_0}\omega(x)\,dx.$$

Let $q(\cdot)\in\mathscr{P}(\mathbb{R}^n)$ with $q_0\le q_-$. If $(q(\cdot)/q_0)'\in\mathscr{B}(\mathbb{R}^n)$, then there exists a positive constant C, such that for all $(f,g)\in\mathcal{F}$,

$$\|f\|_{q(\cdot)} \le C\|g\|_{q(\cdot)}.$$

Lemma 3.8. (see [2]) *Let $0<\alpha<n$ and $p(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$ with $p_+<\frac{n}{\alpha}$ and $\frac{1}{q(\cdot)}=\frac{1}{p(\cdot)}-\frac{\alpha}{n}$, then*

$$\|\mathscr{M}_\alpha f\|_{q(\cdot)} \le C\|f\|_{p(\cdot)}.$$

Theorem 3.9. *Let $m\ge 2$, T_α be an m-linear fractional integral operator with generalized kernel as Definition 1.2 and $\sum_{k=1}^{\infty} C_k<\infty$. Given $\frac{p_i}{p_0'}(\cdot)\in\mathscr{P}^{\log}(\mathbb{R}^n)$, $0<\alpha_i<\frac{n}{p_{i,+}}$, $i=1,2,\dots,m$, $\alpha=\alpha_1+\cdots+\alpha_m$, $\frac{1}{p(\cdot)}=\frac{1}{p_1(\cdot)}+\cdots+\frac{1}{p_m(\cdot)}$, $\frac{1}{q(\cdot)}=\frac{1}{p(\cdot)}-\frac{\alpha}{n}$ and $q(\cdot)\in\mathscr{B}(\mathbb{R}^n)$, then there exists a positive constant C such that*

$$\|T_\alpha(\vec{f})\|_{L^{q(\cdot)}(\mathbb{R}^n)} \le C\prod_{j=1}^{m}\|f_j\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}.$$

Proof. Since $q(\cdot)\in\mathscr{B}(\mathbb{R}^n)$, by Lemma 3.5, there exists a q_0 with $1<q_0<q_-$ such that $(q(\cdot)/q_0)'\in\mathscr{B}(\mathbb{R}^n)$. For this q_0 and any $\omega\in A_1$, choosing $0<\delta<\min\{1,\frac{sn}{n-s\alpha}\}$, by Theorem 2.12, we obtain

$$\|T_\alpha(\vec{f})\|_{L^{q_0}(\omega)} \le \big\|M_\delta(T_\alpha(\vec{f}))\big\|_{L^{q_0}(\omega)} \le \big\|M_\delta^{\#}(T_\alpha(\vec{f}))\big\|_{L^{q_0}(\omega)} \le C\big\|\mathcal{M}_{\alpha,p_0'}(\vec{f})\big\|_{L^{q_0}(\omega)},$$

then we can get

$$\int_{\mathbb{R}^n} \big|T_\alpha(\vec{f})(x)\big|^{q_0}\omega(x)\,dx \le C\int_{\mathbb{R}^n} \big|\mathcal{M}_{\alpha,p_0'}(\vec{f})(x)\big|^{q_0}\omega(x)\,dx$$

for all m-tuples $\vec{f}=(f_1,\dots,f_m)$ of bounded functions with compact support.

Apply Lemma 3.7 to the pair $(T_\alpha(\vec{f}),M_{\alpha,p_0'}(\vec{f}))$ and obtain

$$\|T_\alpha(\vec{f})\|_{q(\cdot)} \le C\big\|\mathcal{M}_{\alpha,p_0'}(\vec{f})\big\|_{q(\cdot)}. \tag{3.1}$$

We can get $0<\alpha_i<\frac{n}{p_{i,+}}<\frac{n}{p_0'}$. Denote by $\frac{1}{q_i(\cdot)}=\frac{1}{p_i(\cdot)}-\frac{\alpha_i}{n}$ $(i=1,\dots,m)$, then $\frac{1}{q(\cdot)}=\frac{1}{q_1(\cdot)}+\cdots+\frac{1}{q_m(\cdot)}$. By Definition 2.3, it is easy to see that

$$\mathcal{M}_{\alpha,p_0'}(\vec{f})(x) \le \prod_{i=1}^{m}\mathscr{M}_{\alpha_i,p_0'}(f_i)(x) \quad \text{for } x\in\mathbb{R}^n. \tag{3.2}$$

Let $\beta_i = \alpha_i p_0'$. From (3.1), (3.2), Lemmas 3.6 and 3.8, it follows that

$$\begin{aligned}\|T_\alpha(\vec{f})(x)\|_{q(\cdot)} &\leq C\big\|\mathcal{M}_{\alpha,p_0'}(\vec{f})\big\|_{q(\cdot)} \leq C\prod_{i=1}^m \big\|\mathscr{M}_{\alpha_i,p_0'}(f_i)\big\|_{q_i(\cdot)}\\ &= C\prod_{i=1}^m \big\|\big[\mathscr{M}_{\beta_i}(f_i^{p_0'})\big]^{1/p_0'}\big\|_{q_i(\cdot)} = C\prod_{i=1}^m \|\mathscr{M}_{\beta_i}(f_i^{p_0'})\|^{1/p_0'}_{\frac{q_i}{p_0'}(\cdot)}\\ &\leq C\prod_{i=1}^m \||f_i|^{p_0'}\|^{1/p_0'}_{\frac{p_i}{p_0'}(\cdot)} = C\prod_{i=1}^m \|f_i\|_{p_i(\cdot)}.\end{aligned}$$

Thus, Theorem 3.9 is proved. □

Remark 3.10. When the generalized kernel condition goes back to the Dini kernel condition, the boundedness of m-linear fractional integral operators with Dini's type kernels on variable exponent Lebesgue spaces has been demonstrated by Wu and Zhang in [22].

Theorem 3.11. *Let $m \geq 2$, T_α be an m-linear fractional integral operator with generalized kernel as Definition* 1.2 *and $\sum_{k=1}^\infty kC_k < \infty$. Given $\frac{p_i}{p_0'}(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, $0 < \alpha_i < \frac{n}{p_{i,+}}$, $i = 1, 2, \ldots, m$, $\alpha = \alpha_1 + \cdots + \alpha_m$, $\frac{1}{p(\cdot)} = \frac{1}{p_1(\cdot)} + \cdots + \frac{1}{p_m(\cdot)}$, $\frac{1}{q(\cdot)} = \frac{1}{p(\cdot)} - \frac{\alpha}{n}$, $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$ and $\vec{b} \in \mathrm{BMO}^m$, then there exists a positive constant C such that*

$$\|T_{\vec{b},\alpha}(\vec{f})\|_{L^{q(\cdot)}(\mathbb{R}^n)} \leq C\|\vec{b}\|_{(\mathrm{BMO})^m}\prod_{j=1}^m \|f_j\|_{L^{p_j(\cdot)}(\mathbb{R}^n)}.$$

Proof. Since $q(\cdot) \in \mathscr{B}(\mathbb{R}^n)$, by Lemma 3.5 there exists a q_0 with $1 < q_0 < q_-$ such that $(q(\cdot)/q_0)' \in \mathscr{B}(\mathbb{R}^n)$. For this q_0 and any $\omega \in A_1$, choosing $0 < \delta < \varepsilon < \min\{1, \frac{sn}{n-s\alpha}\}$ and $p_0' < t < \min_{1\leq i\leq m} p_{i,-}$, by Theorem 2.13, we obtain

$$\begin{aligned}\|T_{\vec{b},\alpha}(\vec{f})\|_{L^{q_0}(\omega)} &\leq \big\|M_\delta(T_{\vec{b},\alpha}(\vec{f}))\big\|_{L^{q_0}(\omega)} \leq \big\|M_\delta^{\#}(T_{\vec{b},\alpha}(\vec{f}))\big\|_{L^{q_0}(\omega)}\\ &\leq C\|\vec{b}\|_{(\mathrm{BMO})^m}\big\|M_\varepsilon(T_\alpha(\vec{f})) + \mathcal{M}_{\alpha,t}(\vec{f})\big\|_{L^{q_0}(\omega)}\\ &\leq C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(\big\|M_\varepsilon(T_\alpha(\vec{f}))\big\|_{L^{q_0}(\omega)} + \big\|\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{L^{q_0}(\omega)}\big)\\ &\leq C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(\big\|M_\varepsilon^{\#}(T_\alpha(\vec{f}))\big\|_{L^{q_0}(\omega)} + \big\|\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{L^{q_0}(\omega)}\big)\\ &\leq C\|\vec{b}\|_{(\mathrm{BMO})^m}\big(\big\|\mathcal{M}_{\alpha,p_0'}(\vec{f})\big\|_{L^{q_0}(\omega)} + \big\|\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{L^{q_0}(\omega)}\big)\\ &\leq C\|\vec{b}\|_{(\mathrm{BMO})^m}\big\|\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{L^{q_0}(\omega)},\end{aligned}$$

then we can get

$$\int_{\mathbb{R}^n} \big|T_{\vec{b},\alpha}(\vec{f})(x)\big|^{q_0}\omega(x)\,dx \leq C\|\vec{b}\|^{q_0}_{(\mathrm{BMO})^m}\int_{\mathbb{R}^n} \big|\mathcal{M}_{\alpha,t}(\vec{f})(x)\big|^{q_0}\omega(x)\,dx$$

for all m-tuples $\vec{f} = (f_1, \ldots, f_m)$ of bounded functions with compact support.

Apply Lemma 3.7 to the pair $(T_{\vec{b},\alpha}(\vec{f}), M_{\alpha,t}(\vec{f}))$ and obtain

$$\|T_{\vec{b},\alpha}(\vec{f})\|_{q(\cdot)} \leq C\|\vec{b}\|_{(\mathrm{BMO})^m} \big\|\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{q(\cdot)}. \tag{3.3}$$

We can get $0 < \alpha_i < \frac{n}{p_{i,+}} < \frac{n}{t}$. Denote by $\frac{1}{q_i(\cdot)} = \frac{1}{p_i(\cdot)} - \frac{\alpha_i}{n}$ $(i = 1, \ldots, m)$, then $\frac{1}{q(\cdot)} = \frac{1}{q_1(\cdot)} + \cdots + \frac{1}{q_m(\cdot)}$. By Definition 2.3, it is easy to see that

$$\mathcal{M}_{\alpha,t}(\vec{f})(x) \leq \prod_{i=1}^{m} \mathscr{M}_{\alpha_i,t}(f_i)(x) \quad \text{for } x \in \mathbb{R}^n. \tag{3.4}$$

Let $\beta_i = \alpha_i t$. The fact $\frac{p_i}{p_0}(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$ implies that $\frac{p_i}{t}(\cdot) \in \mathscr{P}^{\log}(\mathbb{R}^n)$, $i = 1, \ldots, m$. From (3.3), (3.4), Lemmas 3.6 and 3.8, it follows that

$$\begin{aligned}
\|T_{\vec{b},\alpha}(\vec{f})(x)\|_{q(\cdot)} &\leq C\|\vec{b}\|_{(\mathrm{BMO})^m} \big\|\mathcal{M}_{\alpha,t}(\vec{f})\big\|_{q(\cdot)} \leq C\|\vec{b}\|_{(\mathrm{BMO})^m} \prod_{i=1}^{m} \big\|\mathscr{M}_{\alpha_i,t}(f_i)\big\|_{q_i(\cdot)} \\
&= C\|\vec{b}\|_{(\mathrm{BMO})^m} \prod_{i=1}^{m} \big\|\big[\mathscr{M}_{\beta_i}(f_i^t)\big]^{1/t}\big\|_{q_i(\cdot)} = C\|\vec{b}\|_{(\mathrm{BMO})^m} \prod_{i=1}^{m} \big\|\mathscr{M}_{\beta_i}(f_i^t)\big\|_{\frac{q_i}{t}(\cdot)}^{1/t} \\
&\leq C\|\vec{b}\|_{(\mathrm{BMO})^m} \prod_{i=1}^{m} \big\||f_i|^t\big\|_{\frac{p_i}{t}(\cdot)}^{1/t} = C\|\vec{b}\|_{(\mathrm{BMO})^m} \prod_{i=1}^{m} \|f_i\|_{p_i(\cdot)}.
\end{aligned}$$

Thus, Theorem 3.11 is proved. □

Corollary 3.12. *When the generalized kernel condition goes back to the Dini kernel condition, the boundedness of multilinear commutators generalized by m-linear fractional integral operators with Dini's type kernels and* BMO *functions on variable exponent Lebesgue spaces remains new.*

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