TAIWANESE JOURNAL OF MATHEMATICS
Vol. xx, No. x, pp. 1–11, xx 20xx
DOI: 10.11650/tjm/241104

Bounds of Eigenfunction for Some Diffusion Operators

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Abstract. In this paper, we consider the L^∞ bounds of the eigenfunctions for some diffusion operators. In case of classical Laplace eigenvalue problem $\Delta u = -\lambda u$ with Dirichlet boundary condition, we obtain the polynomial bound which is different from the result derived by heat kernel estimates. We then study the nonlocal dispersal eigenvalue problem $\int_\Omega J(x-y)u(y)\,dy - u(x) = -\lambda u(x)$ with bounded domain Ω and obtain an exponential bound by means of Fourier transform and nonlocal estimates.

1. Introduction

In the present paper, we are interested in the L^∞ bounds of positive eigenfunctions for some diffusion operators, aiming to obtain the bounds associated eigenvalue and L^2-norm of eigenfunctions.

Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a smooth bounded domain, we consider the classical eigenvalue problem

$$\begin{cases} \Delta u = -\lambda u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{1.1}$$

We say that $\lambda > 0$ is an eigenvalue of (1.1) if there exists a nontrivial solution $u \in W^{1,2}(\Omega)$. It follows from [7, 8, 17] that there is a sequence of eigenvalues

$$0 < \lambda_1 \leq \lambda_2 \leq \cdots \to \infty \quad \text{and} \quad \lambda_k = \min_{S \in S_{k+1}} \max_{u \in S \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2\,dx}{\int_\Omega u^2(x)\,dx},$$

where S_{k+1} is the family of all subspace of dimension $k+1$ in $W^{1,2}(\Omega)$. Moreover, we know that λ_1 is the only eigenvalue associated with a positive eigenfunction, which has attracted much more attention [11, 18]. Let u_k be the corresponding eigenfunction associated with λ_k for $k \geq 1$. In this case, a direct argument from heat kernel estimate (see for instance [6]) gives

$$\|u_k\|_{L^\infty(\Omega)} \leq \exp(\lambda_k t)(8\pi t)^{-n/4}\|u_k\|_{L^2(\Omega)}.$$

Received December 3, 2023; Accepted November 17, 2024.
Communicated by Cheng-Hsiung Hsu.
2020 *Mathematics Subject Classification.* 35P10, 45C05, 35K57.
Key words and phrases. eigenfunction, diffusion operator, nonlocal.
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Putting $t = 1/(8\pi\lambda_k)$, we obtain

$$\|u_k\|_{L^\infty(\Omega)} \leq C_n \lambda_k^{n/4} \|u_k\|_{L^2(\Omega)} \tag{1.2}$$

for some constant C_n. On the other hand, assume that $\|u_k\|_{L^2(\Omega)} = 1$. Then the pioneering works of Hörmander [14] and Avakumović [2] guarantee that

$$\|u_k\|_{L^\infty(\Omega)} \leq c_\Omega \lambda_k^{(n-1)/4}, \tag{1.3}$$

where c_Ω is a constant depending only on the domain Ω.

Since the diffusion process may take place by non-adjacent positions, various diffusion operators have been used to model diffusion with nonlocal effects [1,9,15]. Let $J\colon \mathbb{R}^n \to \mathbb{R}$ be a nonnegative and bounded function with unit integral, the integral operator of form

$$Du(x,t) = J * u(x,t) - u(x,t) = \int_{\mathbb{R}^n} J(x-y)u(y,t)\,dy - u(x,t)$$

has been widely used to model the diffusion in materials science, ecology and statistical mechanics [16], see also the seminal work of van der Waals [22]. As stated in [9], if $u(y,t)$ is thought of as the density at location y at time t, and $J(x-y)$ is thought of as the probability distribution of jumping from y to x, then $\int_{\mathbb{R}^n} J(x-y)u(y,t)\,dy$ denotes the rate at which individuals are arriving to location x from all other places and $u(x,t) = \int_{\mathbb{R}^n} J(y-x)u(x,t)\,dy$ is the rate at which they are leaving location x to all other places. This consideration, in the absence of external sources, leads immediately to that $u(x,t)$ satisfies

$$u_t(x,t) = \int_{\mathbb{R}^n} J(x-y)u(y,t)\,dy - u(x,t).$$

In this case, we can see that the change of $u(x,t)$ does not only depend on $u(x,t)$, but on all the values of $u(x,t)$ in a fixed spatial neighborhood of x through the integral term $J * u(x,t)$. For recent references on nonlocal dispersal equations, see [1, 3–5, 7, 15, 20] and references therein.

In the present paper, we consider the following nonlocal eigenvalue problem

$$\int_\Omega J(x-y)u(y)\,dy - u(x) = -\lambda u(x) \quad \text{in } \overline{\Omega}. \tag{1.4}$$

We say that (1.4) has a principal eigenvalue if there exists a positive solution to (1.4) for some $\lambda \in \mathbb{R}$ which is called principal eigenfunction. The nonlocal eigenvalue problem (1.4) has been well investigated, see [5, 10, 19]. We know that there exists a unique principal eigenvalue λ_p associated with eigenfunction $\phi \in C(\overline{\Omega})$ such that $\phi(x) > 0$ for $x \in \overline{\Omega}$.

Since the nonlocal dispersal operator shares many properties with the Laplace (local) operator (e.g. [1,4]), we want to obtain the L^∞ bounds of positive eigenfunctions associated with the eigenvalue and L^2-norm of eigenfunctions. However, unlike the local case (1.2), there is no general estimate for nonlocal operator as in the case of heat kernel method. We cannot prove a bound for nonlocal problem by a similar argument. So we need to use new method to establish the bound for eigenfunctions. In this paper, we use the Fourier transform and comparison method to get the bounds for nonlocal problem.

We are ready to state our main result. Throughout this paper, we assume that $J \in C(\mathbb{R}^n)$ verifies $J(x) > 0$ for $x \in B$ (the unit ball), $J(x) = 0$ for $x \in \mathbb{R}^n \setminus B$, J is radial and $\int_{\mathbb{R}^n} J(x)\,dx = 1$.

Theorem 1.1. *Let $\phi \in C(\overline{\Omega})$ be the positive eigenfunction of* (1.4) *associated eigenvalue λ_p, then*

$$\|\phi\|_{L^\infty(\Omega)} \le e^{C\lambda_p}\|\phi\|_{L^2(\Omega)},$$

where $C = C(n, \Omega)$ depends only on n and Ω. Meanwhile, we have

$$\|\phi\|_{L^\infty(\Omega)} \le \frac{C_{J,\Omega}}{(1-\lambda_p)}\|\phi\|_{L^2(\Omega)},$$

where $C = C(J, \Omega)$ depends only on J and Ω.

In the second part, we consider the local problem (1.1). More precisely, we establish a new bound of positive eigenfunctions for (1.1) using the initial value problem corresponding to (1.1).

Theorem 1.2. *Let $\phi \in W^{1,2}(\Omega)$ be the positive eigenfunction of* (1.1) *associated eigenvalue λ_1, then we have*

$$\|\phi\|_{L^\infty(\Omega)} \le C\lambda_1^{n/2}\|\phi\|_{L^2(\Omega)}, \tag{1.5}$$

where $C = C(\Omega)$ depends only on Ω. Moreover, then there is $\widetilde{C}$, independent of Ω, such that

$$\|\phi\|_{L^\infty(\Omega)} \le \widetilde{C}\lambda_1^{n/4}\|\phi\|_{L^2(\Omega)}. \tag{1.6}$$

The rest of this paper is organized as follows. In Section 2, we investigate the L^∞ bounds for nonlocal dispersal problem (1.4). Section 3 is devoted to the local problem (1.1).

2. L^∞ bounds of nonlocal operator

In this section, we study the L^∞ bounds of positive eigenfunctions for nonlocal problem (1.4). Throughout this section, we assume that $\phi(x) > 0$ is the positive eigenfunction of principal eigenvalue λ_p. We can see that

$$\int_\Omega J(x-y)\phi(y)\,dy - \phi(x) = -\lambda_p\phi(x) \quad \text{in } \overline{\Omega}. \tag{2.1}$$

We first obtain the L^∞ bound of $\phi(x)$ by a direct argument.

Lemma 2.1. *Let $\Omega \subset \mathbb{R}^n$ $(n \geq 1)$ be a bounded domain. Then for every positive eigenfunction $\phi \in C(\overline{\Omega})$ of* (2.1) *with associated eigenvalue λ_p, we have*

$$\|\phi\|_{L^\infty(\Omega)} \leq \frac{C_{J,\Omega}}{(1-\lambda_p)}\|\phi\|_{L^2(\Omega)},$$

where the constant $C_{J,\Omega}$ depends only on J and Ω.

Proof. It follows from (2.1) that

$$(1-\lambda_p)\phi(x) = \int_\Omega J(x-y)\phi(y)\,dy \leq \left[\int_\Omega J^2(x-y)\,dy\right]^{1/2} \|\phi\|_{L^2(\Omega)}.$$

Thus we have

$$\|\phi\|_{L^\infty(\Omega)} \leq \frac{C}{(1-\lambda_p)}\|\phi\|_{L^2(\Omega)},$$

where

$$C = C(J,\Omega) = \max_{\overline{\Omega}}\left[\int_\Omega J^2(x-y)\,dy\right]^{1/2}. \qquad \square$$

In what follows, we shall give new bounds for the positive eigenfunctions of (2.1) by Fourier transform and comparison argument. Let us denote $\omega(x,t) = e^{-\lambda_p t}\phi(x)$, then

$$\begin{cases} \omega_t(x,t) = \int_\Omega J(x-y)\omega(y,t)\,dy - \omega(x,t) & \text{in } \overline{\Omega}, \\ \omega(x,0) = \phi(x) & \text{in } \overline{\Omega}. \end{cases} \tag{2.2}$$

Since $\phi \in C(\overline{\Omega})$, we can find a smooth nonnegative function $\phi^\varepsilon \in C(\mathbb{R}^n)$ for $\varepsilon > 0$ such that

$$\phi^\varepsilon(x) = \begin{cases} \phi(x) & \text{if } x \in \overline{\Omega}, \\ 0 & \text{if } \operatorname{dist}(x,\Omega) > 1, \end{cases} \quad \text{and} \quad \lim_{\varepsilon\to 0+}\phi^\varepsilon(x) = \phi(x) \quad \text{uniformly in } \overline{\Omega}.$$

Consider the nonlocal initial problem

$$\begin{cases} u_t(x,t) = J * u(x,t) - u(x,t) & \text{in } \mathbb{R}^n, \\ u(x,0) = \phi^\varepsilon(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{2.3}$$

We know from [1] that there exists a unique bounded solution $v^\varepsilon \in C(\mathbb{R}^n \times [0,\infty))$ to (2.3).

Lemma 2.2. *Let $v^\varepsilon(x,t)$ be the unique bounded solution of* (2.3) *for $\varepsilon > 0$. Then we have*

$$\omega(x,t) \leq v^\varepsilon(x,t) \tag{2.4}$$

for $(x,t) \in \overline{\Omega} \times [0,\infty)$.

Proof. Since $v^\varepsilon(x,t)$ is nonnegative, we can see that

$$\begin{cases} v_t^\varepsilon(x,t) \geq \int_\Omega J(x-y)v^\varepsilon(y,t)\,dy - v^\varepsilon(x,t) & \text{in } \overline{\Omega}, \\ v^\varepsilon(x,0) = \phi(x) & \text{in } \overline{\Omega}. \end{cases}$$

Thus $v^\varepsilon(x,t)$ is an upper-solution to (2.2) and (2.4) is followed by the comparison principle (see e.g. [1,9]). □

Lemma 2.3. *Let $\phi \in C(\overline{\Omega})$ be a positive eigenfunction of λ_p, we have*

$$\|\phi\|_{L^\infty(\Omega)} \leq e^{C\lambda_p}\|\phi\|_{L^2(\Omega)},$$

where $C = C(n,\Omega)$ depends only on n and Ω.

Proof. First, we can take continuous $\psi_\epsilon^\varepsilon(x)$ ($\epsilon > 0$) such that

$$\psi_\epsilon^\varepsilon(x) \begin{cases} > \phi^\varepsilon(x) & \text{for } \operatorname{dist}(x,\Omega) \leq 1, \\ = 0 & \text{for } \operatorname{dist}(x,\Omega) > 2, \end{cases}$$

and

$$\lim_{\epsilon\to 0+} \psi_\epsilon^\varepsilon(x) = \phi^\varepsilon(x) \quad \text{uniformly in } \mathbb{R}^n. \tag{2.5}$$

For any $\epsilon > 0$, we can find a smooth function $j_\delta(x)$ ($\delta > 0$) such that

$$\lim_{\delta\to 0+} j_\delta * \psi_\epsilon^\varepsilon(x) = \lim_{\delta\to 0+} \int_{\mathbb{R}^n} j_\delta(x-y)\psi_\epsilon^\varepsilon(y)\,dy = \psi_\epsilon^\varepsilon(x) \quad \text{uniformly in } \mathbb{R}^n.$$

Then we can find $\delta_0 > 0$ such that

$$j_{\delta_0} * \psi_\epsilon^\varepsilon(x) \geq \phi^\varepsilon(x)$$

for $x \in \mathbb{R}^n$. Let $V_\epsilon^\varepsilon(x,t)$ be the unique solution to

$$\begin{cases} u_t(x,t) = J * u(x,t) - u(x,t) & \text{in } \mathbb{R}^n, \\ u(x,0) = j_{\delta_0} * \psi_\epsilon^\varepsilon(x) & \text{in } \mathbb{R}^n. \end{cases} \tag{2.6}$$

Again by the comparison principle we get

$$v^\varepsilon(x,t) \leq V_\epsilon^\varepsilon(x,t) \tag{2.7}$$

for $(x,t) \in \mathbb{R}^n \times [0,\infty)$.

On the other hand, let $\widehat{V}(\xi,t)$ be the Fourier transform of $V_\epsilon^\varepsilon(x,t)$. Then in Fourier variables, we know from (2.6) that

$$\begin{cases} \widehat{V}_t(\xi,t) = (\widehat{J}(\xi) - 1)\widehat{V}(\xi,t) & \text{in } \mathbb{R}^n, \\ \widehat{V}(\xi,0) = \widehat{j}_{\delta_0}(\xi)\widehat{\psi}_\epsilon^\varepsilon(\xi) & \text{in } \mathbb{R}^n. \end{cases}$$

Hence,

$$\widehat{V}(\xi,t)=e^{(\widehat{J}(\xi)-1)t}\widehat{j}_{\delta_0}(\xi)\widehat{\psi}^{\varepsilon}_{\epsilon}(\xi).$$

Using the properties of Fourier transform, we obtain

$$\|V^{\varepsilon}_{\epsilon}(\,\cdot\,,t)\|_{L^\infty(\mathbb{R}^n)}\le\|\widehat{V}(\,\cdot\,,t)\|_{L^1(\mathbb{R}^n)}=\int_{\mathbb{R}^n}e^{(\widehat{J}(\xi)-1)t}\big|\widehat{j}_{\delta_0}(\xi)\widehat{\psi}^{\varepsilon}_{\epsilon}(\xi)\big|\,d\xi.$$

But $\int_{\mathbb{R}^n}J(x)\,dx=1$, we have $|\widehat{J}(\xi)|\le 1$. Thanks to the Hölder inequality, we derive

$$\|V^{\varepsilon}_{\epsilon}(\,\cdot\,,t)\|_{L^\infty(\mathbb{R}^n)}\le\int_{\mathbb{R}^n}\big|\widehat{j}_{\delta_0}(\xi)\widehat{\psi}^{\varepsilon}_{\epsilon}(\xi)\big|\,d\xi\le\|\widehat{j}_{\delta_0}\|_{L^2(\mathbb{R}^n)}\|\widehat{\psi}^{\varepsilon}_{\epsilon}\|_{L^2(\mathbb{R}^n)}=\|j_{\delta_0}\|_{L^2(\mathbb{R}^n)}\|\psi^{\varepsilon}_{\epsilon}\|_{L^2(\mathbb{R}^n)}.$$

In view of Lemma 2.2 and (2.7), we have

$$\|\omega(\,\cdot\,,t)\|_{L^\infty(\Omega)}\le\|V^{\varepsilon}_{\epsilon}(\,\cdot\,,t)\|_{L^\infty(\Omega)}\le\|j_{\delta_0}\|_{L^2(\mathbb{R}^n)}\|\psi^{\varepsilon}_{\epsilon}\|_{L^2(\mathbb{R}^n)}.$$

Letting $\epsilon\to 0+$, we get from (2.5) that

$$\|\omega(\,\cdot\,,t)\|_{L^\infty(\Omega)}\le\|j_{\delta_0}\|_{L^2(\mathbb{R}^n)}\|\phi^{\varepsilon}\|_{L^2(\mathbb{R}^n)}.$$

By the choice of ϕ^{ε}, we have

$$\|\omega(\,\cdot\,,t)\|_{L^\infty(\Omega)}\le\|j_{\delta_0}\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2(\Omega)}. \tag{2.8}$$

Notice that $\omega(x,t)=e^{-\lambda_p t}\phi(x)$, we know from (2.8) that

$$\|\phi\|_{L^\infty(\Omega)}\le e^{\lambda_p t}\|j_{\delta_0}\|_{L^2(\mathbb{R}^n)}\|\phi\|_{L^2(\Omega)}.$$

Since $t>0$ is arbitrary and $\|j_{\delta_0}\|_{L^2(\mathbb{R}^n)}>0$, we obtain that there exists $C=C(n,\Omega)>0$ such that

$$\|\phi\|_{L^\infty(\Omega)}\le e^{C\lambda_p}\|\phi\|_{L^2(\Omega)}.$$ □

The conclusion of Theorem 1.1 is followed by Lemmas 2.1 and 2.3.

3. L^∞ bounds of Laplace operator

In this section, we investigate the L^∞ bounds of positive eigenfunctions for local problem (1.1). We first obtain a bound that differs from the bound obtained by the heat kernel estimate.

Let λ_1 be the principal eigenvalue of (1.1) with a positive eigenfunction $\phi(x)$, we get

$$\begin{cases}\Delta\phi=-\lambda_1\phi & \text{in }\Omega,\\ \phi=0 & \text{on }\partial\Omega.\end{cases}$$

Setting $\omega(x,t) = e^{-\lambda_1 t}\phi(x)$, we then have

$$\begin{cases} \omega_t = \Delta\omega & \text{in } \Omega, \\ \omega = 0 & \text{on } \partial\Omega, \\ \omega(x,0) = \phi(x) & \text{in } \Omega. \end{cases} \tag{3.1}$$

Since Ω is smooth, we know that $\phi \in C_0^\infty(\overline{\Omega})$ and we can extend it by zero outside Ω, still denoted by $\phi(x)$.

We have the following lemma.

Lemma 3.1. *Let $v(x,t)$ be the unique bounded solution of Cauchy problem*

$$\begin{cases} u_t = \Delta u & \textit{in } \mathbb{R}^n, \\ u(x,0) = \phi(x) & \textit{in } \mathbb{R}^n. \end{cases} \tag{3.2}$$

Then we have

$$\omega(x,t) \le v(x,t) \tag{3.3}$$

for $(x,t) \in \overline{\Omega} \times [0,\infty)$.

Proof. Since $\phi(x) \ge 0$ for $x \in \mathbb{R}^n$ and $\phi(x) > 0$ for $x \in \Omega$, we know that $v(x,t) > 0$ for $(x,t) \in \overline{\Omega} \times (0,\infty)$. Note that $v(x,t)$ is an upper-solution of (3.1), the comparison principle gives (3.3). □

Lemma 3.2. *Let $v(x,t)$ be the unique bounded solution of Cauchy problem* (3.2)*. Then we have*

$$|v(x,t)| \le \frac{|\Omega|^{1/2}}{(4\pi t)^{n/2}} \|\phi\|_{L^2(\Omega)} \tag{3.4}$$

for $(x,t) \in \overline{\Omega} \times [0,\infty)$, where $|\Omega|$ is the measure of domain Ω.

Proof. Using the fact that $v(x,t)$ is the unique bounded solution of (3.2), we obtain that

$$v(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbb{R}^n} e^{-\frac{|x-y|^2}{4t}} \phi(y)\, dy.$$

But $\phi(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$, we know from Hölder inequality that

$$v(x,t) = \frac{1}{(4\pi t)^{n/2}} \int_\Omega e^{-\frac{|x-y|^2}{4t}} \phi(y)\, dy \le \frac{1}{(4\pi t)^{n/2}} \int_\Omega \phi(y)\, dy \le \frac{|\Omega|^{1/2}}{(4\pi t)^{n/2}} \left[\int_\Omega \phi^2(y)\, dy \right]^{1/2},$$

which proves (3.4). □

Now, we show that (1.5) in Theorem 1.2 holds. It follows from Lemmas 3.1 and 3.2 that

$$\|\phi\|_{L^\infty(\Omega)} \leq \frac{|\Omega|^{1/2} e^{\lambda_1 t}}{(4\pi t)^{n/2}} \|\phi\|_{L^2(\Omega)}$$

for $t > 0$. Putting $t = (4\pi\lambda_1)^{-1}$, we obtain

$$\|\phi\|_{L^\infty(\Omega)} \leq |\Omega|^{1/2} e^{1/(4\pi)} \lambda_1^{n/2} \|\phi\|_{L^2(\Omega)}.$$

This implies (1.5).

In the proof of (1.5), we transform the eigenvalue problem into evolution problem (3.1). Then by the comparison argument, we obtain a direct bound. Note that the bound is different to the case of heat kernel estimates as obtained in [6].

At the end of this section, we consider the weighted eigenvalue problem

$$\begin{cases} \Delta u - a(x)u(x) = -\lambda m(x)u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{3.5}$$

where the coefficient $a(x)$ is nonnegative and $m(x)$ is a weighted function such that $a, m \in C^\theta(\overline{\Omega})$ for some $0 < \theta < 1$, and there exists $x_0 \in \Omega$ such that $m(x_0) > 0$. The eigenvalue problem (3.5) has been well investigated, see the seminal work of Hess and Kato [12, 13] and references therein. We know that there exists a positive principal eigenvalues $\widetilde{\lambda}_1 = \widetilde{\lambda}_1(m) > 0$ of (3.5). Moreover, $\widetilde{\lambda}_1$ is the only positive eigenvalue whose associated eigenspace contains a positive function.

Inspired by the argument as in the proof of Theorem 1.1, we can obtain a different bound as in (1.3) such that the coefficient is independent of Ω.

Theorem 3.3. *Let $\phi(x)$ be the eigenfunction of* (3.5) *with associated eigenvalue $\widetilde{\lambda}_1$ such that $\|\phi\|_{L^2(\Omega)} = 1$, then*

$$\|\phi\|_{L^\infty(\Omega)} \leq c_m \widetilde{\lambda}_1^{n/4},$$

where c_m is a constant depending only on $m(x)$.

Proof. Since $\widetilde{\lambda}_1$ is the positive eigenvalue of (3.5) associated with positive eigenfunction $\phi(x)$ such that $\|\phi\|_{L^2(\Omega)} = 1$, we get

$$\begin{cases} \Delta \phi - a(x)\phi = -\widetilde{\lambda}_1 m(x)\phi & \text{in } \Omega, \\ \phi = 0 & \text{on } \partial\Omega. \end{cases}$$

Setting

$$\mu = \widetilde{\lambda}_1 \max_{\overline{\Omega}} m(x) \quad \text{and} \quad \omega(x,t) = e^{-\mu t}\phi(x), \tag{3.6}$$

it becomes apparent that

$$
\begin{cases}
\omega_t = \Delta\omega + \big[\widetilde{\lambda}_1 m(x) - a(x) - \mu\big]\omega & \text{in } \Omega, \\
\omega = 0 & \text{on } \partial\Omega, \\
\omega(x,0) = \phi(x) & \text{in } \Omega.
\end{cases} \tag{3.7}
$$

Since Ω is smooth, we know that $\phi \in C_0^{2+\mu}(\overline{\Omega})$ and we can extend it by zero outside Ω, still denoted by $\phi(x)$. In this case, we consider the Cauchy problem

$$
\begin{cases}
u_t = \Delta u & \text{in } \mathbb{R}^n, \\
u(x,0) = \phi(x) & \text{in } \mathbb{R}^n.
\end{cases} \tag{3.8}
$$

It is standard that (3.8) admits a unique bounded solution $v(x,t)$. Since $\phi(x) \geq 0$ for $x \in \mathbb{R}^n$ and $\phi(x) > 0$ for $x \in \Omega$, we know that $v(x,t) > 0$ for $(x,t) \in \overline{\Omega} \times (0,\infty)$. Note that $v(x,t)$ is an upper-solution of (3.7), the comparison principle gives

$$
\omega(x,t) \leq v(x,t) \tag{3.9}
$$

for $(x,t) \in \overline{\Omega} \times [0,\infty)$. Using the fact that $v(x,t)$ is the unique bounded solution of (3.8), we obtain that

$$
v(x,t) = \int_{\mathbb{R}^n} G(x-y,t)\phi(y)\,dy,
$$

where

$$
G(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-\frac{|x|^2}{4t}}
$$

is the fundamental solution of the heat equation.

But $\phi(x) = 0$ for $x \in \mathbb{R}^n \setminus \Omega$, we know from the heat kernel estimate (see [6]) that

$$
v(x,t) \leq \|G(\,\cdot\,,t)\|_{L^2(\mathbb{R}^n)} \|\phi\|_{L^2(\Omega)} \leq c t^{-n/4},
$$

where the constant $c > 0$ is independent of Ω.

Note that $\omega(x,t) = e^{-\mu t}\phi(x)$, it follows from (3.9) that

$$
\|\phi\|_{L^\infty(\Omega)} \leq e^{\mu t} \|v\|_{L^\infty(\Omega)} \leq c e^{\mu t} t^{-n/4}
$$

for $t > 0$. Putting $t = \widetilde{\lambda}_1^{-1}$, we know from (3.6) that

$$
\|\phi\|_{L^\infty(\Omega)} \leq c e^{\max_{\overline{\Omega}} m(x)} \widetilde{\lambda}_1^{n/4}.
$$

Thus the proof is completed. □

In the special case $m(x) \equiv 1$, we know that (1.6) holds. In view of (1.3), we obtain the estimate for the positive eigenfunction, where the constant c is independent of Ω.

Remark 3.4. In our previous work [21], the L^∞ estimates of eigenfunction is proved by various order of eigenvalue, but the coefficient depends on the domain Ω.

Acknowledgments

This work was supported by NSF of China (12371170) and NSF of Gansu Province of China (21JR7RA535, 21JR7RA537).

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