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New Eigenvalue Inequalities for the Hadamard Product and Fan Product of Structured Tensors

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Abstract. The spectral radius of nonnegative tensors and the minimum H-eigenvalues of strong $\mathcal{M}$-tensors are two types of tensor eigenvalues with important research significance, which promotes the tensor eigenvalue inequality to become an important component in tensor analysis. In this paper, based on Brualdi-type and Brauer-type eigenvalue inclusion sets of tensors, some Brualdi-type inequalities on the spectral radius for Hadamard product of two nonnegative tensors and some Brauer-type inequalities on the minimum H-eigenvalue for the Fan product of two strong $\mathcal{M}$-tensors are provided, respectively. In addition, the theoretical comparisons between the newly obtained inequalities and some previous ones are investigated. Finally, some numerical examples are reported to show the feasibility and effectiveness of our theoretical results.

1. Introduction

Let $\mathbb{C}$ ($\mathbb{R}$) denote the set of all complex (real) numbers, $\mathbb{R}_+$ denote the set of all nonnegative numbers, and $\mathbb{R}^n_{++}$ denote be the set of n-dimensional positive vectors. An m-th order n-dimensional complex (real) tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ is a multidimensional array [19] with entries

$$a_{i_1i_2\cdots i_m} \in \mathbb{C} \ \ (a_{i_1i_2\cdots i_m} \in \mathbb{R}), \quad i_j \in N = \{1,2,\ldots,n\},\ j \in \{1,2,\ldots,m\}.$$

An m-th order n-dimensional tensor $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ is nonnegative if all its entries are nonnegative, i.e., $a_{i_1i_2\cdots i_m} \geq 0$. The m-th order unit tensor $\mathcal{I} = (\delta_{i_1i_2\cdots i_m})$ [19] is defined as a diagonal tensor with entries

$$\delta_{i_1i_2\cdots i_m} = \begin{cases} 1 & \text{if } i_1 = i_2 = \cdots = i_m, \\ 0 & \text{otherwise.} \end{cases}$$

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In 2005, the concept on eigenvalue-eigenvector of a tensor was proposed by Qi [19] and Lim [16], independently. For an m-order n-dimensional tensor $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$, if there is a $\lambda \in \mathbb{C}$ and a nonzero column vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{C}^n$ such that

$$\mathcal{A}x^{m-1} = \lambda x^{[m-1]},$$

where $\mathcal{A}x^{m-1}$ and $x^{[m-1]}$ are n-dimensional column vectors whose i-th component is defined as

$$(\mathcal{A}x^{m-1})_i = \sum_{i_2,\ldots,i_m \in N} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}, \quad x^{[m-1]} = (x_1^{m-1}, x_2^{m-1}, \ldots, x_n^{m-1})^T,$$

then the pair (λ, x) is called an eigenvalue-eigenvector pair of $\mathcal{A}$. In particular, the pair (λ, x) is called an H-eigenpair of $\mathcal{A}$ if $\lambda \in \mathbb{R}$ and $x \in \mathbb{R}^n$. The spectral radius of $\mathcal{A}$ is defined as $\rho(\mathcal{A}) = \max\{|\lambda| : \lambda \in \sigma(\mathcal{A})\}$, and the minimum H-eigenvalue of $\mathcal{A}$ is defined as $\tau(\mathcal{A}) = \min\{\mathrm{Re}(\lambda) : \lambda \in \sigma(\mathcal{A})\}$, where $\mathrm{Re}(\lambda)$ is the real part of the eigenvalue λ, and $\sigma(\mathcal{A})$ denotes the set of all eigenvalues of $\mathcal{A}$.

Tensor eigenvalue theory plays an important role in different fields, such as magnetic resonance imaging [19], positive definiteness of the multivariate polynomials [18], spectral hypergraph theory [9,23] and etc. However, the accurate calculation of tensor eigenvalues is a tedious and challenging task when the order and dimension of tensors are very large. As a result, some researchers consider investigating the eigenvalues of tensors by the form of inequality [2,3,10,11,13,32,34,35], which becomes one of the interesting topics in tensor analysis. The spectral radius of nonnegative tensors and the minimum H-eigenvalue of strong $\mathcal{M}$-tensors as two important tensor eigenvalues have attracted the attention of a large number of scholars, and the detailed inequalities can be found in [2,3,7,10,12–14,37].

The Hadamard product of two m-th order n-dimensional tensors $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m})$ [20] is defined by $\mathcal{A} \circ \mathcal{B} = \mathcal{C} = (c_{i_1 i_2 \cdots i_m})$, where $c_{i_1 i_2 \cdots i_m} = a_{i_1 i_2 \cdots i_m} b_{i_1 i_2 \cdots i_m}$. Under the Hadamard product of tensors, the closed properties of some structured tensors including complete Hankel tensors, strong Hankel tensors, completely positive tensors and strong $\mathcal{H}$-tensors, strictly diagonally dominant tensors, doubly strictly diagonally dominant tensors, $\mathcal{B}$-tensors and $\mathcal{C}$-tensors are obtained in [21, 22, 30, 39]. The Hadamard product of matrices, as we know, has been involved in estimations of spectral radius for nonnegative matrices, and readers can refer to the related literature [5, 8, 15, 17, 38]. As a high-order generalization of the matrix, some authors consider using the Hadamard product of tensors as an effective tool to investigate spectral radius inequalities of nonnegative tensors. By extending to some spectral radius inequalities of nonnegative matrices, some inequalities on spectral radius for Hadamard product of two nonnegative tensors are given in [25, 29–31, 33], but in some cases, the previous inequalities may not be well approximated the spectral radius for Hadamard product of two nonnegative tensors. Based the above facts, this paper will give some new inequalities on the spectral radius $\rho(\mathcal{A}\circ\mathcal{B})$ for two nonnegative tensors $\mathcal{A}$ and $\mathcal{B}$.

For a real number $s>0$ and a nonnegative tensor $\mathcal{B}$, a $\mathcal{Z}$-tensor $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ can be written as $\mathcal{A}=s\mathcal{I}-\mathcal{B}$, where $\mathcal{A}$ is called a $\mathcal{Z}$-tensor if its all off-diagonal entries are nonpositive, i.e., $a_{i_1i_2\cdots i_m}\le 0$. If $s\ge\rho(\mathcal{B})$, then $\mathcal{A}$ is called an $\mathcal{M}$-tensor. If $s>\rho(\mathcal{B})$, then $\mathcal{A}$ is called a strong $\mathcal{M}$-tensor. More results of $\mathcal{M}$-tensors can refer to the literature [4,36]. Given two m-th order n-dimensional tensors $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$. The Fan product of $\mathcal{A}$ and $\mathcal{B}$ [22] is defined as $\mathcal{A}\star\mathcal{B}=\mathcal{C}=(c_{i_1i_2\cdots i_m})$ with its entries

$$c_{i_1i_2\cdots i_m}=\begin{cases}a_{ii\cdots i}b_{ii\cdots i} & \text{if } i_1=i_2=\cdots=i_m=i,\\ -|a_{i_1i_2\cdots i_m}b_{i_1i_2\cdots i_m}| & \text{otherwise.}\end{cases}$$

As shown in [22], it is not difficult to know that the Fan product of two strong $\mathcal{M}$-tensors is a strong $\mathcal{M}$-tensor. The above facts are the higher-order generalization of the matrix cases. In the past few decades, the Fan product of matrices is a useful tool for investigating the minimum H-eigenvalue of matrices [5, 15, 17, 38]. Analogous with spectral radius inequalities for the Hadamard product of two nonnegative tensors, the minimum H-eigenvalue inequalities for the Fan product of two strong $\mathcal{M}$-tensors has also caused extensive attention from scholars, and some of the latest papers can be found in [24, 26–28, 33]. In this paper, we will continue to focus on the minimum H-eigenvalue inequalities for the Fan product of two strong $\mathcal{M}$-tensors.

This paper is organized as follows: In Section 2, we reviewed some existing concepts and results which are useful in the subsequent analysis. In Section 3, we provide some inequalities on spectral radius for the Hadamard product of two nonnegative tensors. In Section 4, some inequalities on minimum H-eigenvalue for the Fan product of two strong $\mathcal{M}$-tensors are obtained. To verify the effectiveness and sharpness of newly proposed results, some numerical examples are reported in Section 5.

2. Preliminaries

For the convenience, some used concepts on the directed graph are recalled as follows.

Definition 2.1. [1] Let $A=(a_{ij})$ with $n\ge 2$ be an $n\times n$ matrix. Then Γ_A is the directed graph on n vertices $V=\{v_i\}_{i=1}^n$, consisting of an arc $\overrightarrow{v_iv_j}$ from v_i to vertex v_j if and only if $a_{ij}\ne 0$ for any $i\ne j$. Moreover, a circuit of Γ_A is a sequence γ of vertices $i_1,i_2,\ldots,i_p$, $i_{p+1}=i_1$, where $p\ge 2$, $i_1,i_2,\ldots,i_p$ are distinct, and $\overrightarrow{v_{i_1}v_{i_2}},\overrightarrow{v_{i_2}v_{i_3}},\ldots,\overrightarrow{v_{i_p}v_{i_1}}$ are arcs of Γ_A.

Note that a digraph $\Gamma_{\mathcal{A}}$ [3,6] is associated with $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ as follows: the vertex set of $\Gamma_{\mathcal{A}}$ is $V(\mathcal{A})=\{1,2,\ldots,n\}$, and the arc set of $\Gamma_{\mathcal{A}}$ is $E(\mathcal{A})=\big\{(i,j): a_{ii_2\cdots i_m}\ne 0, j\in\{i_2,i_3,\ldots,i_m\}\ne\{i,i,\ldots,i\}\big\}$.

Definition 2.2. [1] Let $V = \{v_i\}_{i=1}^n$ be the set of n vertices. If there exist directed paths from v_i to v_j and v_j to v_i for any $i, j \in V$ $(i \neq j)$, then Γ is called strongly connected.

Definition 2.3. [1] Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an m-th order n-dimensional tensor. If $\Gamma_{\mathcal{A}}$ is strongly connected, then $\mathcal{A}$ is said to be weakly irreducible.

As a high-order generalization of matrix spectral results, we revisit Perron Frobenius theorem in [6] and spectral invariance under the diagonal similarity transformation of tensors in [35] as follows.

Lemma 2.4. [6] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an m-th order n-dimensional weakly irreducible nonnegative tensor. Then $\mathcal{A}$ has a positive eigenpair $(\rho(\mathcal{A}), x)$, and x is unique up to a multiplicative constant.*

Lemma 2.5. [35] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m})$ be two m-th order n-dimensional tensors. If there is a positive diagonal matrix $E = \mathrm{diag}(e_1, e_2, \ldots, e_n)$ such that $\mathcal{B} = \mathcal{A} \cdot E^{-(m-1)} \overbrace{E \cdots E}^{m-1}$, where $\mathcal{B}_{i_1 \cdots i_m} = a_{i_1 \cdots i_m} e_{i_1}^{1-m} e_{i_2} \cdots e_{i_m}$, then $\sigma(\mathcal{A}) = \sigma(\mathcal{B})$.*

Inspired by the spectral results of nonnegative tensors, there are the following minimum H-eigenvalue results of strong $\mathcal{M}$-tensors, and the readers can refer to the literature [7, 25, 28].

Lemma 2.6. [7,28] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an m-th order n-dimensional weakly irreducible strong $\mathcal{M}$-tensor. Then there is a positive vector u such that*

$$\mathcal{A} u^{m-1} = \tau(\mathcal{A}) u^{[m-1]}.$$

Lemma 2.7. [25] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an m-th order n-dimensional strong $\mathcal{M}$-tensor. Then*

$$\tau(\mathcal{A}) \leq \min_{i \in N} a_{ii \cdots i}.$$

The eigenvalue inclusion sets of tensors, which can be found in [2,3], play an important role in investigation of tensor eigenvalue inequalities. Before concluding this section, we restate the related results as the following lemmas.

Lemma 2.8. [3] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an m-th order n-dimensional tensor such that $\Gamma_{\mathcal{A}}$ is weakly connected. Then*

$$\sigma(\mathcal{A}) \subseteq \bigcup_{\gamma \in C(\mathcal{A})} \left\{ \lambda \in \mathbb{C} : \prod_{i \in \gamma} (\lambda - a_{ii \cdots i}) \leq \prod_{i \in \gamma} R_i(\mathcal{A}) \right\},$$

where $C(\mathcal{A})$ is the set of circuits of $\Gamma_{\mathcal{A}}$, and $R_i(\mathcal{A}) = \sum_{\substack{i_2, \ldots, i_m \in N, \\ \delta_{ii_2 \cdots i_m} = 0}} |a_{ii_2 \cdots i_m}|$.

Lemma 2.9. [2] *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ be an m-th order n-dimensional tensor. Then*

$$\sigma(\mathcal{A}) \subseteq \bigcup_{\substack{i, j \in N, \\ i \neq j}} \left\{ z \in \mathbb{C} : |z - a_{ii \cdots i}|^{m-1} |z - a_{jj \cdots j}| \leq (R_i(\mathcal{A}))^{m-1} R_j(\mathcal{A}) \right\}.$$

3. Brualdi-type inequalities for spectral radius of the Hadamard product of two nonnegative tensors

In this section, some Brualdi-type inequalities of the Hadamard product of two nonnegative tensors are given, and some theoretical comparisons between the newly proposed inequalities are established.

3.1. Brualdi-type inequalities for $\rho(\mathcal{A}\circ\mathcal{B})$

Theorem 3.1. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional nonnegative tensors such that $\Gamma_{\mathcal{A}\circ\mathcal{B}}$ is weakly connected. Then*

$$\prod_{i\in\gamma}(\rho(\mathcal{A}\circ\mathcal{B})-a_{ii\cdots i}b_{ii\cdots i})\le\max_{\gamma\in C(\mathcal{A}\circ\mathcal{B})}\left\{\prod_{i\in\gamma}(\rho(\mathcal{A})-a_{ii\cdots i})(\rho(\mathcal{B})-b_{ii\cdots i})\right\}. \tag{3.1}$$

Proof. We prove the inequality (3.1) by the following cases.

Case 1. If both $\mathcal{A}$ and $\mathcal{B}$ are weakly irreducible. According to Lemma 2.4, there exist two vectors $x=(x_1,x_2,\dots,x_n)^T\in\mathbb{R}^n_{++}$ and $y=(y_1,y_2,\dots,y_n)^T\in\mathbb{R}^n_{++}$ such that $\mathcal{A}x^{m-1}=\rho(\mathcal{A})x^{[m-1]}$ and $\mathcal{B}y^{m-1}=\rho(\mathcal{B})y^{[m-1]}$, which implies that

$$a_{ii\cdots i}x_i^{m-1}+\sum_{\substack{i_2,\dots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}=\rho(\mathcal{A})x_i^{m-1} \tag{3.2}$$

and

$$b_{ii\cdots i}y_i^{m-1}+\sum_{\substack{i_2,\dots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}b_{ii_2\cdots i_m}y_{i_2}\cdots y_{i_m}=\rho(\mathcal{B})y_i^{m-1} \tag{3.3}$$

for all $i\in N$. Define a positive diagonal matrix $D=\mathrm{diag}(x_1y_1,\dots,x_ny_n)$. It follows from Lemma 2.5 that $\sigma(\mathcal{A}\circ\mathcal{B})=\sigma((\mathcal{A}\circ\mathcal{B})D^{-(m-1)}\overbrace{D\cdots D}^{m-1})$. Combining Lemma 2.8 and the equalities (3.2) and (3.3), there is a circuit $\gamma\in C(\mathcal{A}\circ\mathcal{B})$ such that

$$\begin{aligned}
&\prod_{i\in\gamma}(\rho(\mathcal{A}\circ\mathcal{B})-a_{ii\cdots i}b_{ii\cdots i})\\
&\le\prod_{i\in\gamma}\sum_{\substack{i_2,\dots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}\frac{a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}b_{ii_2\cdots i_m}y_{i_2}\cdots y_{i_m}}{x_i^{m-1}y_i^{m-1}}\\
&\le\prod_{i\in\gamma}\left(\sum_{\substack{i_2,\dots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}\frac{a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}}{x_i^{m-1}}\right)\left(\sum_{\substack{i_2,\dots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}\frac{b_{ii_2\cdots i_m}y_{i_2}\cdots y_{i_m}}{y_i^{m-1}}\right)\\
&=\prod_{i\in\gamma}(\rho(\mathcal{A})-a_{ii\cdots i})(\rho(\mathcal{B})-b_{ii\cdots i})\le\max_{\gamma\in C(\mathcal{A}\circ\mathcal{B})}\left\{\prod_{i\in\gamma}(\rho(\mathcal{A})-a_{ii\cdots i})(\rho(\mathcal{B})-b_{ii\cdots i})\right\}.
\end{aligned} \tag{3.4}$$

Case 2. Either $\mathcal{A}$ or $\mathcal{B}$ is weakly reducible. Without loss of generality, suppose that both $\mathcal{A}$ and $\mathcal{B}$ are weakly reducible. Let $\mathcal{A}_\epsilon = \mathcal{A} + \frac{1}{\epsilon}\mathcal{K}$ and $\mathcal{B}_\epsilon = \mathcal{B} + \frac{1}{\epsilon}\mathcal{K}$, where $\mathcal{K} = (k_{i_1 i_2 \cdots i_m})$ is a tensor with its entries $k_{i_1 i_2 \cdots i_m} = 1$. Obviously, both $\mathcal{A}_\epsilon$ and $\mathcal{B}_\epsilon$ are weakly irreducible tensors for the sufficiently large number $\epsilon > 0$. Substituting $\mathcal{A}$ and $\mathcal{B}$ by $\mathcal{A}_\epsilon$ and $\mathcal{B}_\epsilon$ in the inequality (3.4) of Case 1, respectively. Then $\lim_{\epsilon\to+\infty}\mathcal{A}_\epsilon = \mathcal{A}$ and $\lim_{\epsilon\to+\infty}\mathcal{B}_\epsilon = \mathcal{B}$, and hence the inequality (3.1) is derived by the continuity [35] of $\rho(\mathcal{A}_\epsilon)$ and $\rho(\mathcal{B}_\epsilon)$ with letting $\epsilon \to +\infty$. The proof is completed. □

Suppose that $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m})$ are two m-th order n-dimensional nonnegative tensors, two useful notations are defined as $\alpha_i = \max_{\delta_{ii_2\cdots i_m}=0} a_{ii_2\cdots i_m}$ and $\beta_i = \max_{\delta_{ii_2\cdots i_m}=0} b_{ii_2\cdots i_m}$. With the help of the quantities α_i and β_i, we provide the following results.

Theorem 3.2. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m})$ be two m-th order n-dimensional nonnegative tensors such that $\Gamma_{\mathcal{A}\circ\mathcal{B}}$ is weakly connected. Then*

$$\prod_{i\in\gamma}(\rho(\mathcal{A}\circ\mathcal{B}) - a_{ii\cdots i}b_{ii\cdots i}) \leq \max_{\gamma\in C(\mathcal{A}\circ\mathcal{B})}\left\{\prod_{i\in\gamma}\left[\alpha_i\beta_i(\rho(\mathcal{A}) - a_{ii\cdots i})(\rho(\mathcal{B}) - b_{ii\cdots i})\right]^{1/2}\right\}. \tag{3.5}$$

Proof. For the inequality (3.5), we prove the result under two cases.

Case 1. If both $\mathcal{A}$ and $\mathcal{B}$ are weakly irreducible. Then it follows from Lemma 2.4 that there are two vectors $u = (x_1^2, x_2^2, \ldots, x_n^2)^T \in \mathbb{R}^n_{++}$ and $v = (y_1^2, y_2^2, \ldots, y_n^2)^T \in \mathbb{R}^n_{++}$ such that $\mathcal{A}u^{m-1} = \rho(\mathcal{A})u^{[m-1]}$ and $\mathcal{B}v^{m-1} = \rho(\mathcal{B})v^{[m-1]}$, and then for all $i \in N$, it yields

$$a_{ii\cdots i}x_i^{2(m-1)} + \sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} a_{ii_2\cdots i_m}x_{i_2}^2\cdots x_{i_m}^2 = \rho(\mathcal{A})x_i^{2(m-1)} \tag{3.6}$$

and

$$b_{ii\cdots i}y_i^{2(m-1)} + \sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} b_{ii_2\cdots i_m}y_{i_2}^2\cdots y_{i_m}^2 = \rho(\mathcal{B})y_i^{2(m-1)}. \tag{3.7}$$

Let $E = (e_{ij})$ be an $n\times n$ positive diagonal matrix with its entries $e_{ii} = x_i y_i$ for all $i \in N$. According to Lemma 2.5, $\sigma(\mathcal{A}\circ\mathcal{B}) = \sigma((\mathcal{A}\circ\mathcal{B})E^{-(m-1)}\overbrace{E\cdots E}^{m-1})$. By the Cauchy–Schwarz inequality and combining Lemma 2.8 and the equalities (3.6) and (3.7), there is a circuit $\gamma \in C(\mathcal{A}\circ\mathcal{B})$ such that

$$\begin{aligned}&\prod_{i\in\gamma}(\rho(\mathcal{A}\circ\mathcal{B}) - a_{ii\cdots i}b_{ii\cdots i})\\ \leq&\prod_{i\in\gamma}\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}\frac{a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}b_{ii_2\cdots i_m}y_{i_2}\cdots y_{i_m}}{x_i^{m-1}y_i^{m-1}}\end{aligned}$$

$$\le \prod_{i\in\gamma}\Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}\frac{a_{ii_2\cdots i_m}^2 x_{i_2}^2\cdots x_{i_m}^2}{x_i^{2(m-1)}}\Bigg)^{1/2}\Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}\frac{b_{ii_2\cdots i_m}^2 y_{i_2}^2\cdots y_{i_m}^2}{y_i^{2(m-1)}}\Bigg)^{1/2}$$
$$=\prod_{i\in\gamma}\big(\alpha_i\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})(\rho(\mathcal{B})-b_{ii\cdots i})\big)^{1/2}$$
$$\le \max_{\gamma\in C(\mathcal{A}\circ\mathcal{B})}\Big\{\prod_{i\in\gamma}\big(\alpha_i\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})(\rho(\mathcal{B})-b_{ii\cdots i})\big)^{1/2}\Big\}.$$

Case 2. Either $\mathcal{A}$ or $\mathcal{B}$ is weakly reducible. Without loss of generality, suppose that $\mathcal{A}$ and $\mathcal{B}$ are two weakly reducible tensors. Similar to the proof of Case 2 in Theorem 3.1, the inequality (3.5) follows. The proof is completed. □

Theorem 3.3. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional nonnegative tensors such that $\Gamma_{\mathcal{A}\circ\mathcal{B}}$ is weakly connected. Then*

$$\prod_{i\in\gamma}(\rho(\mathcal{A}\circ\mathcal{B})-a_{ii\cdots i}b_{ii\cdots i})\le \max_{\gamma\in C(\mathcal{A}\circ\mathcal{B})}\Big\{\prod_{i\in\gamma}\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})\Big\}. \tag{3.8}$$

Proof. The presented argument is divided into two cases as follows.

Case 1. If $\mathcal{A}$ is weakly irreducible. From Lemma 2.4, it follows that there exists a vector $x=(x_1,x_2,\ldots,x_n)^T\in\mathbb{R}^n_{++}$ such that $\mathcal{A}x^{m-1}=\rho(\mathcal{A})x^{[m-1]}$, and for all $i\in N$, then

$$a_{ii\cdots i}x_i^{m-1}+\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}=\rho(\mathcal{A})x_i^{m-1}. \tag{3.9}$$

Define a positive diagonal matrix $D=\mathrm{diag}(x_1,x_2,\ldots,x_n)$. According to Lemma 2.5, $\sigma(\mathcal{A}\circ\mathcal{B})=\sigma((\mathcal{A}\circ\mathcal{B})D^{-(m-1)}\overbrace{D\cdots D}^{m-1})$. Combining Lemma 2.8 and the equality (3.9), there is a circuit $\gamma\in C(\mathcal{A}\circ\mathcal{B})$ such that

$$\begin{aligned}
&\prod_{i\in\gamma}(\rho(\mathcal{A}\circ\mathcal{B})-a_{ii\cdots i}b_{ii\cdots i})\\
&\le \prod_{i\in\gamma}\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}\frac{a_{ii_2\cdots i_m}b_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}}{x_i^{m-1}}\le \prod_{i\in\gamma}\beta_i\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}\frac{a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}}{x_i^{m-1}}\\
&=\prod_{i\in\gamma}\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})\le \max_{\gamma\in C(\mathcal{A}\circ\mathcal{B})}\Big\{\prod_{i\in\gamma}\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})\Big\}.
\end{aligned} \tag{3.10}$$

Case 2. If $\mathcal{A}$ is weakly reducible. Let $\mathcal{A}_\epsilon=\mathcal{A}+\frac{1}{\epsilon}\mathcal{K}$, where $\mathcal{K}=(k_{i_1i_2\cdots i_m})$ is an m-th order n-dimensional tensor with its entries $k_{i_1i_2\cdots i_m}=1$. Obviously, $\mathcal{A}_\epsilon$ is a weakly irreducible tensor for the sufficiently large number $\epsilon>0$. Substituting $\mathcal{A}$ by $\mathcal{A}_\epsilon$ in the inequality (3.10) of Case 1, it is evident that $\lim_{\epsilon\to+\infty}\mathcal{A}_\epsilon=\mathcal{A}$, and hence the inequality (3.8) is obtained by the continuity [35] of $\rho(\mathcal{A}_\epsilon)$ with respect to $\epsilon\to+\infty$. The proof is completed. □

Due to the fact that the Hadamard product of two nonnegative tensors satisfies the commutative law, the following result is immediately derived similar to the proof of Theorem 3.3.

Corollary 3.4. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional nonnegative tensors such that $\Gamma_{\mathcal{A}\circ\mathcal{B}}$ is weakly connected. Then*

$$\prod_{i\in\gamma}(\rho(\mathcal{A}\circ\mathcal{B})-a_{ii\cdots i}b_{ii\cdots i})\le\max_{\gamma\in C(\mathcal{A}\circ\mathcal{B})}\left\{\prod_{i\in\gamma}\alpha_i(\rho(\mathcal{B})-b_{ii\cdots i})\right\}. \tag{3.11}$$

Combining Theorem 3.3 and Corollary 3.4, we obtain the following inequality for $\rho(\mathcal{A}\circ\mathcal{B})$, which improves the inequalities (3.8) and (3.11).

Corollary 3.5. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional nonnegative tensors such that $\Gamma_{\mathcal{A}\circ\mathcal{B}}$ is weakly connected. Then*

$$\begin{aligned}&\prod_{i\in\gamma}(\rho(\mathcal{A}\circ\mathcal{B})-a_{ii\cdots i}b_{ii\cdots i})\\&\le\min\left\{\max_{\gamma\in C(\mathcal{A}\circ\mathcal{B})}\prod_{i\in\gamma}\beta_i(\rho(\mathcal{A})-a_{ii\cdots i}),\ \max_{\gamma\in C(\mathcal{A}\circ\mathcal{B})}\prod_{i\in\gamma}\alpha_i(\rho(\mathcal{B})-b_{ii\cdots i})\right\}.\end{aligned} \tag{3.12}$$

3.2. The comparisons of the Brualdi-type inequalities for $\rho(\mathcal{A}\circ\mathcal{B})$

Define $h(u)=\prod_{i\in\gamma}(u-a_{ii\cdots i}b_{ii\cdots i})$, where u belongs to $H=\left(\max_{i\in N}a_{ii\cdots i}b_{ii\cdots i},+\infty\right)$. For two nonnegative real numbers k_1 and k_2, set

$$S_{k_1}=\{u\in H:h(u)\le k_1\},\quad S_{k_2}=\{u\in H:h(u)\le k_2\}.$$

Then $S_{k_1}\subseteq S_{k_2}$ if $k_1\le k_2$. Given two m-th order n-dimensional nonnegative tensors $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ such that $\Gamma_{\mathcal{A}\circ\mathcal{B}}$ is weakly connected. For $\gamma\in C(\mathcal{A}\circ\mathcal{B})$ and $i\in\gamma$, if $(\rho(\mathcal{A})-a_{ii\cdots i})(\rho(\mathcal{B})-b_{ii\cdots i})\le(\ge)\alpha_i\beta_i$, then

$$\prod_{i\in\gamma}(\rho(\mathcal{A})-a_{ii\cdots i})(\rho(\mathcal{B})-b_{ii\cdots i})\le(\ge)\prod_{i\in\gamma}\left(\alpha_i\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})(\rho(\mathcal{B})-b_{ii\cdots i})\right)^{1/2}.$$

Based on the above analysis, we obtain the following comparisons between the upper bounds for $\rho(\mathcal{A}\circ\mathcal{B})$ characterized by the inequality (3.1) in Theorem 3.1 and the inequality (3.5) in Theorem 3.2.

Theorem 3.6. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m})$ be two m-th order n-dimensional nonnegative tensors such that $\Gamma_{\mathcal{A} \circ \mathcal{B}}$ is weakly connected. Then the following conclusions hold:*

(1) *If $(\rho(\mathcal{A}) - a_{ii\cdots i})(\rho(\mathcal{B}) - b_{ii\cdots i}) \leq \alpha_i \beta_i$ for all $i \in \gamma$ and $\gamma \in C(\mathcal{A} \circ \mathcal{B})$, then the upper bound for $\rho(\mathcal{A} \circ \mathcal{B})$ in Theorem 3.2 is not less than the one in Theorem 3.1.*

(2) *If $(\rho(\mathcal{A}) - a_{ii\cdots i})(\rho(\mathcal{B}) - b_{ii\cdots i}) \geq \alpha_i \beta_i$ for all $i \in \gamma$ and $\gamma \in C(\mathcal{A} \circ \mathcal{B})$, then the upper bound for $\rho(\mathcal{A} \circ \mathcal{B})$ in Theorem 3.1 is not less than the one in Theorem 3.2.*

The following is the theoretical comparison between the upper bound for $\rho(\mathcal{A} \circ \mathcal{B})$ in Corollary 3.5 and the one in Theorem 3.1 under certain conditions.

Theorem 3.7. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m})$ be two m-th order n-dimensional nonnegative tensors such that $\Gamma_{\mathcal{A} \circ \mathcal{B}}$ is weakly connected, and*

$$\min\left\{\prod_{i\in\gamma} \beta_i(\rho(\mathcal{A}) - a_{ii\cdots i}), \prod_{i\in\gamma} \alpha_i(\rho(\mathcal{B}) - b_{ii\cdots i})\right\} = \prod_{i\in\gamma} \beta_i(\rho(\mathcal{A}) - a_{ii\cdots i}).$$

Then the following conclusions hold:

(1) *If $(\rho(\mathcal{B}) - b_{ii\cdots i}) \leq \beta_i$ for all $i \in \gamma$ and $\gamma \in C(\mathcal{A} \circ \mathcal{B})$, then the upper bound for $\rho(\mathcal{A} \circ \mathcal{B})$ in Corollary 3.5 is not less than the one in Theorem 3.1.*

(2) *If $(\rho(\mathcal{B}) - b_{ii\cdots i}) \geq \beta_i$ for all $i \in \gamma$ and $\gamma \in C(\mathcal{A} \circ \mathcal{B})$, then the upper bound for $\rho(\mathcal{A} \circ \mathcal{B})$ in Corollary 3.5 is not larger than the one in Theorem 3.1.*

Proof. (1) For all $i \in \gamma$ and $\gamma \in C(\mathcal{A} \circ \mathcal{B})$, it follows that $\prod_{i\in\gamma}(\rho(\mathcal{A}) - a_{ii\cdots i})(\rho(\mathcal{B}) - b_{ii\cdots i}) \leq \prod_{i\in\gamma} \beta_i(\rho(\mathcal{A}) - a_{ii\cdots i})$, which yield that the upper bound for $\rho(\mathcal{A} \circ \mathcal{B})$ characterized by the inequality (3.1) in Theorem 3.1 is not more than the one characterized by the inequality (3.12) in Corollary 3.5. The proof is completed.

(2) It is similar to the proof of Theorem 3.7(1), and hence we will not elaborate further. The proof is completed. □

Remark 3.8. When $\min\left\{\prod_{i\in\gamma} \beta_i(\rho(\mathcal{A}) - a_{ii\cdots i}), \prod_{i\in\gamma} \alpha_i(\rho(\mathcal{B}) - b_{ii\cdots i})\right\} = \prod_{i\in\gamma} \alpha_i(\rho(\mathcal{B}) - b_{ii\cdots i})$, and then similar to Theorem 3.7, the following comparisons can be obtained.

(1) If $(\rho(\mathcal{A}) - a_{ii\cdots i}) \leq \alpha_i$ for all $i \in \gamma$ and $\gamma \in C(\mathcal{A} \circ \mathcal{B})$, then the upper bound for $\rho(\mathcal{A} \circ \mathcal{B})$ characterized by the inequality (3.1) in Theorem 3.1 is not more than the one characterized by the inequality (3.12) in Corollary 3.5.

(2) If $(\rho(\mathcal{A}) - a_{ii\cdots i}) \geq \alpha_i$ for all $i \in \gamma$ and $\gamma \in C(\mathcal{A} \circ \mathcal{B})$, the upper bound for $\rho(\mathcal{A} \circ \mathcal{B})$ characterized by the inequality (3.1) in Theorem 3.1 is not smaller than the one characterized by the inequality (3.12) in Corollary 3.5.

To illustrate the sharpness of the upper bound for $\rho(\mathcal{A}\circ\mathcal{B})$ in Corollary 3.5, the relationship between the upper bounds for $\rho(\mathcal{A}\circ\mathcal{B})$ characterized by the inequality (3.12) in Corollary 3.5 and the inequality (3.5) in Theorem 3.2 is obtained as follows.

Theorem 3.9. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional nonnegative tensors such that $\Gamma_{\mathcal{A}\circ\mathcal{B}}$ is weakly connected. Then the upper bound for $\rho(\mathcal{A}\circ\mathcal{B})$ characterized by the inequality* (3.12) *in Corollary* 3.5 *is sharper than the one characterized by the inequality* (3.5) *in Theorem* 3.2.

Proof. If $\min\left\{\prod_{i\in\gamma}\beta_i(\rho(\mathcal{A})-a_{ii\cdots i}),\prod_{i\in\gamma}\alpha_i(\rho(\mathcal{B})-b_{ii\cdots i})\right\}=\prod_{i\in\gamma}\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})$ for all $\gamma\in C(\mathcal{A}\circ\mathcal{B})$, and obviously, $\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})\le\alpha_i(\rho(\mathcal{B})-b_{ii\cdots i})$ for all $i\in\gamma$, then

$$\prod_{i\in\gamma}\left(\alpha_i\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})(\rho(\mathcal{B})-b_{ii\cdots i})\right)^{1/2}\ge\prod_{i\in\gamma}\beta_i(\rho(\mathcal{A})-a_{ii\cdots i}).$$

If $\min\left\{\prod_{i\in\gamma}\beta_i(\rho(\mathcal{A})-a_{ii\cdots i}),\prod_{i\in\gamma}\alpha_i(\rho(\mathcal{B})-b_{ii\cdots i})\right\}=\prod_{i\in\gamma}\alpha_i(\rho(\mathcal{B})-b_{ii\cdots i})$ for all $\gamma\in C(\mathcal{A}\circ\mathcal{B})$, then $\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})\ge\alpha_i(\rho(\mathcal{B})-b_{ii\cdots i})$ for all $i\in\gamma$, and hence

$$\prod_{i\in\gamma}\left(\alpha_i\beta_i(\rho(\mathcal{A})-a_{ii\cdots i})(\rho(\mathcal{B})-b_{ii\cdots i})\right)^{1/2}\ge\prod_{i\in\gamma}\alpha_i(\rho(\mathcal{B})-b_{ii\cdots i}).$$

Therefore, the upper bound for $\rho(\mathcal{A}\circ\mathcal{B})$ characterized by the inequality (3.12) in Corollary 3.5 is sharper than the one characterized by the inequality (3.5) in Theorem 3.2. □

4. Brauer-type inequalities on the minimum H-eigenvalue for the Fan product of two strong $\mathcal{M}$-tensors

In this section, we give some Brauer-type inequalities for the Fan product of two strong $\mathcal{M}$-tensors, and establish the theoretical comparisons between the lower bounds for $\tau(\mathcal{A}\star\mathcal{B})$ characterized by newly obtained inequalities and some existing ones in [24, 26, 33].

4.1. Brauer-type inequalities for $\tau(\mathcal{A}\star\mathcal{B})$

Theorem 4.1. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional strong $\mathcal{M}$-tensors. Then*

$$\begin{aligned}&(a_{ii\cdots i}b_{ii\cdots i}-\tau(\mathcal{A}\star\mathcal{B}))^{m-1}(a_{jj\cdots j}b_{jj\cdots j}-\tau(\mathcal{A}\star\mathcal{B}))\\ &\le\max_{(i,j)\in Q}\left\{(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))(b_{jj\cdots j}-\tau(\mathcal{B}))\right\},\end{aligned}\tag{4.1}$$

where $Q=\{(i,j)\mid i\in N,j\in N,i\ne j\}$ is a double indicator set.

Proof. In order to prove the inequality (4.1), we consider two cases as follows.

Case 1. If both $\mathcal{A}$ and $\mathcal{B}$ are weakly irreducible. Then both $\mathcal{A}$ and $\mathcal{B}$ are two weakly irreducible strong $\mathcal{M}$-tensors. According to Lemma 2.6, there are two vectors $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n_{++}$ and $y = (y_1, y_2, \ldots, y_n)^T \in \mathbb{R}^n_{++}$ such that $\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}$ and $\mathcal{B}y^{m-1} = \tau(\mathcal{B})y^{[m-1]}$, and for all $i \in N$, we obtain

$$a_{ii\cdots i}x_i^{m-1} + \sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m} = \tau(\mathcal{A})x_i^{m-1} \tag{4.2}$$

and

$$b_{ii\cdots i}y_i^{m-1} + \sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} b_{ii_2\cdots i_m}y_{i_2}\cdots y_{i_m} = \tau(\mathcal{B})y_i^{m-1}. \tag{4.3}$$

Define an $n\times n$ diagonal matrix $E = \operatorname{diag}(x_1y_1, \ldots, x_ny_n)$, and it follows from Lemma 2.5 that $\sigma(\mathcal{A}\star\mathcal{B}) = \sigma((\mathcal{A}\star\mathcal{B})E^{-(m-1)}\overbrace{E\cdots E}^{m-1})$. By Lemma 2.9 and the equalities (4.2) and (4.3), we have

$$\begin{aligned}
&(a_{ii\cdots i}b_{ii\cdots i} - \tau(\mathcal{A}\star\mathcal{B}))^{m-1}(a_{jj\cdots j}b_{jj\cdots j} - \tau(\mathcal{A}\star\mathcal{B}))\\
&\le \Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}b_{ii_2\cdots i_m}y_{i_2}\cdots y_{i_m}}{x_i^{m-1}y_i^{m-1}}\Bigg)^{m-1}\\
&\quad\times\Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m}b_{ji_2\cdots i_m}y_{i_2}\cdots y_{i_m}}{x_j^{m-1}y_j^{m-1}}\Bigg)\\
&\le \Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{-a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}}{x_i^{m-1}}\Bigg)^{m-1}\Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{-b_{ii_2\cdots i_m}y_{i_2}\cdots y_{i_m}}{y_i^{m-1}}\Bigg)^{m-1}\\
&\quad\times\Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{-a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m}}{x_j^{m-1}}\Bigg)\Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{-b_{ji_2\cdots i_m}y_{i_2}\cdots y_{i_m}}{y_j^{m-1}}\Bigg)\\
&= (a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1}(b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1}(a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B}))\\
&\le \max_{(i,j)\in Q}\big\{(a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1}(b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1}(a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B}))\big\}.
\end{aligned}$$

Case 2. Either $\mathcal{A}$ or $\mathcal{B}$ is weakly reducible. Without loss of generality, assume that $\mathcal{A}$ and $\mathcal{B}$ are two weakly reducible strong $\mathcal{M}$-tensors. Let $\mathcal{K}' = (k'_{i_1i_2\cdots i_m})$ be an m-order n-dimensional tensor, where

$$k'_{ii_2\cdots i_m} = \begin{cases} 1 & \text{if } i_2 = i_3 = \cdots = i_m \neq i,\\ 0 & \text{otherwise.}\end{cases}$$

For every positive value of ϵ, both $\mathcal{A}-\frac{1}{\epsilon}\mathcal{K}'$ and $\mathcal{B}-\frac{1}{\epsilon}\mathcal{K}'$ are two weakly irreducible tensors. Assert that for sufficiently small positive value of $\frac{1}{\epsilon}$. Given that $\mathcal{A}$ and $\mathcal{B}$ are strong $\mathcal{M}$-tensors, there are two positive diagonal matrices $E=\operatorname{diag}(e_1,e_2,\ldots,e_n)$ and $F=\operatorname{diag}(f_1,f_2,\ldots,f_n)$ such that

$$\mathcal{E}=(\mathcal{E}_{i_1i_2\cdots i_m})=\mathcal{A}\cdot E^{-(m-1)}\overbrace{E\cdots E}^{m-1}\quad\text{and}\quad \mathcal{F}=(\mathcal{F}_{i_1i_2\cdots i_m})=\mathcal{B}\cdot F^{-(m-1)}\overbrace{F\cdots F}^{m-1}$$

with

$$|\mathcal{E}_{ii\cdots i}|=|a_{ii\cdots i}|>\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|=\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}|\mathcal{E}_{ii_2\cdots i_m}|e_i^{1-m}e_{i_2}\cdots e_{i_m}$$

and

$$|\mathcal{F}_{ii\cdots i}|=|b_{ii\cdots i}|>\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}|b_{ii_2\cdots i_m}|=\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}|\mathcal{F}_{ii_2\cdots i_m}|f_i^{1-m}f_{i_2}\cdots f_{i_m}.$$

Suppose that

$$T=\max_{i,j\in N,\, i\neq j}\left\{\frac{e_j^{m-1}}{e_i^{m-1}},\frac{f_j^{m-1}}{f_i^{m-1}}\right\}$$

and

$$\epsilon_0=\min_{\substack{i,j\in N,\\ i\neq j}}\left\{\frac{|a_{ii\cdots i}|-\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}|a_{ii_2\cdots i_m}|e_i^{1-m}e_{i_2}\cdots e_{i_m}}{(n-1)T},\right.$$
$$\left.\frac{|b_{ii\cdots i}|-\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}}|b_{ii_2\cdots i_m}|f_i^{1-m}f_{i_2}\cdots f_{i_m}}{(n-1)T}\right\}.$$

Then for any $0<\frac{1}{\epsilon}<\epsilon_0$, $\mathcal{A}-\frac{1}{\epsilon}\mathcal{K}'$ and $\mathcal{B}-\frac{1}{\epsilon}\mathcal{K}'$ are also two strong $\mathcal{M}$-tensors. By replacing $\mathcal{A}$ with $\mathcal{A}-\frac{1}{\epsilon}\mathcal{K}'$ and $\mathcal{B}$ with $\mathcal{B}-\frac{1}{\epsilon}\mathcal{K}'$ in Case 1, and then taking the limit as $\epsilon\to+\infty$, we achieve the desired inequality utilizing the continuity of $\tau\big(\mathcal{A}-\frac{1}{\epsilon}\mathcal{K}'\big)$ and $\tau\big(\mathcal{B}-\frac{1}{\epsilon}\mathcal{K}'\big)$ deduced from Lemma 2.4 in [24]. The proof is completed. □

For two m-th order n-dimensional strong $\mathcal{M}$-tensors $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$, we define two useful quantities $\mu_i=\max_{\delta_{ii_2\cdots i_m}=0}-a_{ii_2\cdots i_m}$ and $\nu_i=\max_{\delta_{ii_2\cdots i_m}=0}-b_{ii_2\cdots i_m}$, which contributes to achieve the following inequalities (4.4) and (4.7) on $\tau(\mathcal{A}\star\mathcal{B})$ for two strong $\mathcal{M}$-tensors $\mathcal{A}$ and $\mathcal{B}$.

Theorem 4.2. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional strong $\mathcal{M}$-tensors. Then*

$$\begin{aligned}&(a_{ii\cdots i}b_{ii\cdots i}-\tau(\mathcal{A}\star\mathcal{B}))^{m-1}(a_{jj\cdots j}b_{jj\cdots j}-\tau(\mathcal{A}\star\mathcal{B}))\\ &\le\max_{(i,j)\in Q}\big\{\big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}\\ &\qquad\times(a_{jj\cdots j}-\tau(\mathcal{A}))(b_{jj\cdots j}-\tau(\mathcal{B}))\big)^{1/2}\big\}.\end{aligned}\tag{4.4}$$

Proof. For the proof of the inequality (4.4), it needs to consider the following cases.

Case 1. If both $\mathcal{A}$ and $\mathcal{B}$ are weakly irreducible. Obviously, both $\mathcal{A}$ and $\mathcal{B}$ are weakly irreducible strong $\mathcal{M}$-tensors. From Lemma 2.6, it follows that there are two positive vectors $r = (x_1^2, x_2^2, \ldots, x_n^2)^T \in \mathbb{R}^n_{++}$ and $s = (y_1^2, y_2^2, \ldots, y_n^2)^T \in \mathbb{R}^n_{++}$ associated with $\tau(\mathcal{A})$ and $\tau(\mathcal{B})$ respectively, satisfying the following relations

$$a_{ii\cdots i}x_i^{2(m-1)} + \sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} a_{ii_2\cdots i_m}x_{i_2}^2\cdots x_{i_m}^2 = \tau(\mathcal{A})x_i^{2(m-1)} \tag{4.5}$$

and

$$b_{ii\cdots i}y_i^{2(m-1)} + \sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} b_{ii_2\cdots i_m}y_{i_2}^2\cdots y_{i_m}^2 = \tau(\mathcal{B})y_i^{2(m-1)}. \tag{4.6}$$

Define an $n\times n$ positive diagonal matrix $E = \mathrm{diag}(x_1y_1, x_2y_2, \ldots, x_ny_n)$. From Lemma 2.5, it yields that $\sigma(\mathcal{A}\star\mathcal{B}) = \sigma((\mathcal{A}\star\mathcal{B})E^{-(m-1)}\overbrace{E\cdots E}^{m-1})$, and then by the Cauchy–Schwarz inequality and combining Lemma 2.9 with the equalities (4.5) and (4.6), we derive

$$\begin{aligned}
&(a_{ii\cdots i}b_{ii\cdots i} - \tau(\mathcal{A}\star\mathcal{B}))^{m-1}(a_{jj\cdots j}b_{jj\cdots j} - \tau(\mathcal{A}\star\mathcal{B}))\\
&\le \Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{a_{ii_2\cdots i_m}x_{i_2}\cdots x_{i_m}b_{ii_2\cdots i_m}y_{i_2}\cdots y_{i_m}}{x_i^{m-1}y_i^{m-1}}\Bigg)^{m-1}\\
&\quad\times\Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{a_{ji_2\cdots i_m}x_{i_2}\cdots x_{i_m}b_{ji_2\cdots i_m}y_{i_2}\cdots y_{i_m}}{x_j^{m-1}y_j^{m-1}}\Bigg)\\
&\le \Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{a^2_{ii_2\cdots i_m}x^2_{i_2}\cdots x^2_{i_m}}{x_i^{2(m-1)}}\Bigg)^{(m-1)/2}\Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{b^2_{ii_2\cdots i_m}y^2_{i_2}\cdots y^2_{i_m}}{y_i^{2(m-1)}}\Bigg)^{(m-1)/2}\\
&\quad\times\Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{a^2_{ji_2\cdots i_m}x^2_{i_2}\cdots x^2_{i_m}}{x_j^{2(m-1)}}\Bigg)^{1/2}\Bigg(\sum_{\substack{i_2,\ldots,i_m\in N,\\ \delta_{ii_2\cdots i_m}=0}} \frac{b^2_{ji_2\cdots i_m}y^2_{i_2}\cdots y^2_{i_m}}{y_j^{2(m-1)}}\Bigg)^{1/2}\\
&\le \big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))(b_{jj\cdots j}-\tau(\mathcal{B}))\big)^{1/2}\\
&\le \max_{(i,j)\in Q}\big\{\big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}\\
&\qquad\qquad\times(a_{jj\cdots j}-\tau(\mathcal{A}))(b_{jj\cdots j}-\tau(\mathcal{B}))\big)^{1/2}\big\}.
\end{aligned}$$

Case 2. Either $\mathcal{A}$ or $\mathcal{B}$ is weakly reducible. Without loss of generality, assume that $\mathcal{A}$ and $\mathcal{B}$ are two weakly reducible strong $\mathcal{M}$-tensors. Similar to the proof of Case 2 in Theorem 4.1, we obtain the inequality (4.4). The proof is completed. □

Theorem 4.3. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m})$ be two m-th order n-dimensional strong $\mathcal{M}$-tensors. Then*

$$\begin{aligned} &(a_{ii\cdots i} b_{ii\cdots i} - \tau(\mathcal{A} \star \mathcal{B}))^{m-1} (a_{jj\cdots j} b_{jj\cdots j} - \tau(\mathcal{A} \star \mathcal{B})) \\ &\leq \max_{(i,j)\in Q} \left\{ \nu_i^{m-1} \nu_j (a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1} (a_{jj\cdots j} - \tau(\mathcal{A})) \right\}. \end{aligned} \tag{4.7}$$

Proof. To prove the inequality (4.7), we divide it into two distinct cases as follows.

Case 1. If $\mathcal{A}$ is weakly irreducible. Then $\mathcal{A}$ is a weakly irreducible strong $\mathcal{M}$-tensor. According to Lemma 2.6, there is a vector $x = (x_1, x_2, \ldots, x_n)^T \in \mathbb{R}^n_{++}$ corresponding to $\tau(\mathcal{A})$ such that $\mathcal{A}x^{m-1} = \tau(\mathcal{A})x^{[m-1]}$, and for all $i \in N$, this implies that

$$a_{ii\cdots i} x_i^{m-1} + \sum_{\substack{i_2,\ldots,i_m \in N, \\ \delta_{i i_2 \cdots i_m} = 0}} a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} = \tau(\mathcal{A}) x_i^{m-1}. \tag{4.8}$$

Define an $n \times n$ positive diagonal matrix $D = \mathrm{diag}(x_1, x_2, \ldots, x_n)$. By Lemma 2.5, $\sigma(\mathcal{A} \star \mathcal{B}) = \sigma((\mathcal{A} \star \mathcal{B}) D^{-(m-1)} \overbrace{D \cdots D}^{m-1})$. Combining Lemma 2.9 and the equality (4.8), we have

$$\begin{aligned} &(a_{ii\cdots i} b_{ii\cdots i} - \tau(\mathcal{A} \star \mathcal{B}))^{m-1} (a_{jj\cdots j} b_{jj\cdots j} - \tau(\mathcal{A} \star \mathcal{B})) \\ &\leq \Bigg(\sum_{\substack{i_2,\ldots,i_m \in N, \\ \delta_{i i_2 \cdots i_m} = 0}} \frac{a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} b_{i i_2 \cdots i_m}}{x_i^{m-1}} \Bigg)^{m-1} \Bigg(\sum_{\substack{i_2,\ldots,i_m \in N, \\ \delta_{i i_2 \cdots i_m} = 0}} \frac{a_{j i_2 \cdots i_m} x_{i_2} \cdots x_{i_m} b_{j i_2 \cdots i_m}}{x_j^{m-1}} \Bigg) \\ &\leq \nu_i^{m-1} \nu_j \Bigg(\sum_{\substack{i_2,\ldots,i_m \in N, \\ \delta_{i i_2 \cdots i_m} = 0}} \frac{-a_{i i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}}{x_i^{m-1}} \Bigg)^{m-1} \Bigg(\sum_{\substack{i_2,\ldots,i_m \in N, \\ \delta_{i i_2 \cdots i_m} = 0}} \frac{-a_{j i_2 \cdots i_m} x_{i_2} \cdots x_{i_m}}{x_j^{m-1}} \Bigg) \\ &= \nu_i^{m-1} \nu_j (a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1} (a_{jj\cdots j} - \tau(\mathcal{A})) \\ &\leq \max_{(i,j)\in Q} \left\{ \nu_i^{m-1} \nu_j (a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1} (a_{jj\cdots j} - \tau(\mathcal{A})) \right\}. \end{aligned}$$

Case 2. If $\mathcal{A}$ is weakly reducible. Then $\mathcal{A}$ is a weakly reducible strong $\mathcal{M}$-tensor. Similar to the proof of Case 2 in Theorem 4.1, the inequality (4.7) holds. The proof is completed. □

Due to the Fan product of two strong $\mathcal{M}$-tensors satisfying the commutative law, we derive the following result, which is similar to the proof of Theorem 4.3.

Corollary 4.4. *Let $\mathcal{A} = (a_{i_1 i_2 \cdots i_m})$ and $\mathcal{B} = (b_{i_1 i_2 \cdots i_m})$ be two m-th order n-dimensional strong $\mathcal{M}$-tensors. Then*

$$\begin{aligned} &(a_{ii\cdots i} b_{ii\cdots i} - \tau(\mathcal{A} \star \mathcal{B}))^{m-1} (a_{jj\cdots j} b_{jj\cdots j} - \tau(\mathcal{A} \star \mathcal{B})) \\ &\leq \max_{(i,j)\in Q} \left\{ \mu_i^{m-1} \mu_j (b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1} (b_{jj\cdots j} - \tau(\mathcal{B})) \right\}. \end{aligned}$$

Combining Theorem 4.3 and Corollary 4.4, we give the following inequality (4.9) for $\tau(\mathcal{A}\star\mathcal{B})$.

Corollary 4.5. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional strong $\mathcal{M}$-tensors. Then*

$$\begin{aligned}&(a_{ii\cdots i}b_{ii\cdots i}-\tau(\mathcal{A}\star\mathcal{B}))^{m-1}(a_{jj\cdots j}b_{jj\cdots j}-\tau(\mathcal{A}\star\mathcal{B}))\\ \leq \min\Big\{&\max_{(i,j)\in Q}\nu_i^{m-1}\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A})),\\ &\max_{(i,j)\in Q}\mu_i^{m-1}\mu_j(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))\Big\}.\end{aligned}\tag{4.9}$$

4.2. The comparisons of the Brauer-type inequalities for $\tau(\mathcal{A}\star\mathcal{B})$

To compare the inequalities in Theorems 4.1 and 4.2 with the ones of Theorem 4.3 in [33] (i.e., Theorem 3.6 in [26] and Theorem 3.3 in [24]) and Theorem 3.8 in [26], we revisited the above results on $\tau(\mathcal{A}\star\mathcal{B})$ as follows:

$$\tau(\mathcal{A}\star\mathcal{B})\geq\min_{i\in N}\{(a_{ii\cdots i}\tau(\mathcal{B})+b_{ii\cdots i}\tau(\mathcal{A}))-\tau(\mathcal{A})\tau(\mathcal{B})\}$$

and

$$\tau(\mathcal{A}\star\mathcal{B})\geq\min_{i\in N}\left\{a_{ii\cdots i}b_{ii\cdots i}-\big(\mu_i\nu_i(a_{ii\cdots i}-\tau(\mathcal{A}))(b_{ii\cdots i}-\tau(\mathcal{B}))\big)^{1/2}\right\}.$$

The following presents Theorems 4.6 and 4.7, which illustrates the theoretical comparisons between the lower bounds for $\tau(\mathcal{A}\star\mathcal{B})$ characterized by newly obtained inequalities in Theorems 4.1 and 4.2 and some existing ones in [26, 33].

Theorem 4.6. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional strong $\mathcal{M}$-tensors. Then the lower bound for $\tau(\mathcal{A}\star\mathcal{B})$ in Theorem* 4.1 *provides the more accurate estimate compared to the one of Theorem* 4.3 *in* [33].

Proof. By Theorem 4.3 in [33], it is seen that

$$\tau(\mathcal{A}\star\mathcal{B})\geq\min_{i\in N}\{(a_{ii\cdots i}\tau(\mathcal{B})+b_{ii\cdots i}\tau(\mathcal{A}))-\tau(\mathcal{A})\tau(\mathcal{B})\}.$$

Let η be the lower bound of $\tau(\mathcal{A}\star\mathcal{B})$ in Theorem 4.1. Using the method of contradiction, assume that

$$\eta<\min_{i\in N}\{(a_{ii\cdots i}\tau(\mathcal{B})+b_{ii\cdots i}\tau(\mathcal{A}))-\tau(\mathcal{A})\tau(\mathcal{B})\},$$

and immediately, we obtain

$$\eta<a_{ii\cdots i}\tau(\mathcal{B})+b_{ii\cdots i}\tau(\mathcal{A})-\tau(\mathcal{A})\tau(\mathcal{B})+a_{ii\cdots i}b_{ii\cdots i}-a_{ii\cdots i}b_{ii\cdots i},$$

which is equivalent to

$$a_{ii\cdots i}b_{ii\cdots i} - \eta > (a_{ii\cdots i} - \tau(\mathcal{A}))(b_{ii\cdots i} - \tau(\mathcal{B})). \tag{4.10}$$

From the above condition and Lemma 2.7, it follows that

$$\eta \leq \tau(\mathcal{A} \star \mathcal{B}) \leq \min_{i\in N} a_{ii\cdots i}b_{ii\cdots i},$$

and further, we have

$$a_{ii\cdots i}b_{ii\cdots i} - \eta \geq 0, \quad \forall\, i \in N. \tag{4.11}$$

Combining the inequalities (4.10) and (4.11), for all $(i,j) \in Q$, we obtain

$$\begin{aligned}&(a_{ii\cdots i}b_{ii\cdots i} - \eta)^{m-1}(a_{jj\cdots j}b_{jj\cdots j} - \eta)\\ &> (a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1}(b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1}(a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B})),\end{aligned}$$

which implies that there is the pair $(i_0, j_0) \in Q$ such that

$$\begin{aligned}&(a_{i_0i_0\cdots i_0}b_{i_0i_0\cdots i_0} - \eta)^{m-1}(a_{j_0j_0\cdots j_0}b_{j_0j_0\cdots j_0} - \eta)\\ &> \max_{(i,j)\in Q}\left\{(a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1}(b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1}(a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B}))\right\}.\end{aligned}$$

Obviously, this is a contradiction with the inequality (4.1) in Theorem 4.1, and therefore the desired conclusion holds. The proof is completed. □

Theorem 4.7. *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ and $\mathcal{B} = (b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional strong $\mathcal{M}$-tensors. Then the lower bound for $\tau(\mathcal{A} \star \mathcal{B})$ in Theorem 4.2 provides a more accurate estimate compared to the one of Theorem 3.8 in [26].*

Proof. According to Theorem 3.8 in [26], we derive the following inequality

$$\tau(\mathcal{A} \star \mathcal{B}) \geq \min_{i\in N}\left\{a_{ii\cdots i}b_{ii\cdots i} - \left(\mu_i\nu_i(a_{ii\cdots i} - \tau(\mathcal{A}))(b_{ii\cdots i} - \tau(\mathcal{B}))\right)^{1/2}\right\}.$$

Let ζ be the lower bound of $\tau(\mathcal{A} \star \mathcal{B})$ in Theorem 4.2. The following proof will use the method of contradiction, and suppose that

$$\zeta < \min_{i\in N}\left\{a_{ii\cdots i}b_{ii\cdots i} - \left(\mu_i\nu_i(a_{ii\cdots i} - \tau(\mathcal{A}))(b_{ii\cdots i} - \tau(\mathcal{B}))\right)^{1/2}\right\},$$

and then it is obvious that

$$a_{ii\cdots i}b_{ii\cdots i} - \zeta > \left(\mu_i\nu_i(a_{ii\cdots i} - \tau(\mathcal{A}))(b_{ii\cdots i} - \tau(\mathcal{B}))\right)^{1/2}, \quad \forall\, i \in N. \tag{4.12}$$

By the above condition and Lemma 2.7, we obtain

$$\zeta \leq \tau(\mathcal{A} \star \mathcal{B}) \leq \min_{i\in N} a_{ii\cdots i}b_{ii\cdots i},$$

which means that

$$a_{ii\cdots i}b_{ii\cdots i} - \zeta \geq 0, \quad \forall\, i \in N. \tag{4.13}$$

From the inequalities (4.12) and (4.13), for all $(i, j) \in Q$, it follows that

$$\begin{aligned}&(a_{ii\cdots i}b_{ii\cdots i} - \zeta)^{m-1}(a_{jj\cdots j}b_{jj\cdots j} - \zeta)\\ &> \big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1}(b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1}(a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B}))\big)^{1/2}.\end{aligned}$$

Then there exists the pair $(i_0, j_0) \in Q$ such that

$$\begin{aligned}&(a_{i_0i_0\cdots i_0}b_{i_0i_0\cdots i_0} - \zeta)^{m-1}(a_{j_0j_0\cdots j_0}b_{j_0j_0\cdots j_0} - \zeta)\\ &> \max_{(i,j)\in Q}\big\{\big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1}(b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1}\\ &\qquad\qquad \times (a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B}))\big)^{1/2}\big\},\end{aligned}$$

which contradicts with the inequality (4.4) in Theorem 4.2. Therefore, the desired result is obtained. The proof is completed. □

Let $g(v) = (a_{ii\cdots i}b_{ii\cdots i} - v)^{m-1}(a_{jj\cdots j}b_{jj\cdots j} - v)$ for all $v \in G = \big(-\infty, \min_{i\in N} a_{ii\cdots i}b_{ii\cdots i}\big)$. If there exist two nonnegative real numbers satisfying $k_1 \leq k_2$, then

$$T_{k_1} = \{v \in G : g(v) \leq k_1\} \subseteq T_{k_2} = \{v \in G : g(v) \leq k_2\}.$$

For two m-th order n-dimensional strong $\mathcal{M}$-tensors $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ and $\mathcal{B} = (b_{i_1i_2\cdots i_m})$, if $\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j \leq (\geq)\big(a_{ii\cdots i} - \tau(\mathcal{A})\big)^{m-1}\big(b_{ii\cdots i} - \tau(\mathcal{B})\big)^{m-1}(a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B}))$ for all $(i, j) \in Q$, then

$$\begin{aligned}&\big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1}(b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1}(a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B}))\big)^{1/2}\\ &\leq (\geq)\ (a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1}(b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1}(a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B})).\end{aligned}$$

Based on the above facts, the following provides the comparison between the lower bound for $\tau(\mathcal{A} \star \mathcal{B})$ in Theorem 4.1 and the one in Theorem 4.2 as follows.

Theorem 4.8. *Let $\mathcal{A} = (a_{i_1i_2\cdots i_m})$ and $\mathcal{B} = (b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional strong $\mathcal{M}$-tensors. Then the following conclusions hold.*

(1) *If $\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j \leq (a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1}(b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1}(a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B}))$ for all $(i, j) \in Q$, then the lower bound for $\tau(\mathcal{A} \star \mathcal{B})$ in Theorem 4.2 is not less than the one in Theorem 4.1.*

(2) *If $\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j \geq (a_{ii\cdots i} - \tau(\mathcal{A}))^{m-1}(b_{ii\cdots i} - \tau(\mathcal{B}))^{m-1}(a_{jj\cdots j} - \tau(\mathcal{A}))(b_{jj\cdots j} - \tau(\mathcal{B}))$ for all $(i, j) \in Q$, then the lower bound for $\tau(\mathcal{A} \star \mathcal{B})$ in Theorem 4.2 is not greater than the one in Theorem 4.1.*

Under certain conditions, we establish the relationship between the lower bound for $\tau(\mathcal{A}\star\mathcal{B})$ in Theorem 4.1 and the one in Corollary 4.5, which is stated as the following theorem.

Theorem 4.9. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional strong $\mathcal{M}$-tensors, and*

$$\nu_i^{m-1}\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))\le \mu_i^{m-1}\mu_j(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))$$

for all $(i,j)\in Q$. Then the following conclusions hold.

(1) *If $\nu_i^{m-1}\nu_j\le(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))$ for all $(i,j)\in Q$, then the lower bound for $\tau(\mathcal{A}\star\mathcal{B})$ in Corollary 4.5 is not less than the one in Theorem 4.1.*

(2) *If $\nu_i^{m-1}\nu_j\ge(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))$ for all $(i,j)\in Q$, then the lower bound for $\tau(\mathcal{A}\star\mathcal{B})$ in Corollary 4.5 is not greater than the one in Theorem 4.1.*

Proof. (1) Since $\nu_i^{m-1}\nu_j\le(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))$ for all $(i,j)\in Q$, we have

$$\begin{aligned}&\nu_i^{m-1}\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))\\ \le\ &(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))(b_{jj\cdots j}-\tau(\mathcal{B})).\end{aligned}$$

This implies that the lower bound for $\tau(\mathcal{A}\star\mathcal{B})$ in Corollary 4.5 is not less than the one in Theorem 4.1. The proof is concluded.

(2) This follows similarly to the proof of Theorem 4.9(1), so we will not provide further details. The proof is concluded. □

Remark 4.10. When $\mu_i^{m-1}\mu_j(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))\le\nu_i^{m-1}\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}\times(a_{jj\cdots j}-\tau(\mathcal{A}))$ for all $(i,j)\in Q$. Similar to Theorem 4.9, we can conclude the following results.

(1) If $\mu_i^{m-1}\mu_j\le(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))$ for all $(i,j)\in Q$, then the lower bound for $\tau(\mathcal{A}\star\mathcal{B})$ in Corollary 4.5 is not less than the one in Theorem 4.1.

(2) If $\mu_i^{m-1}\mu_j\ge(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))$ for all $(i,j)\in Q$, then the lower bound for $\tau(\mathcal{A}\star\mathcal{B})$ in Corollary 4.5 is not greater than the one in Theorem 4.1.

In the following, we establish the comparison between the lower bound for $\tau(\mathcal{A}\star\mathcal{B})$ in Theorem 4.2 and the one in Corollary 4.5, which shows that the lower bound for $\tau(\mathcal{A}\star\mathcal{B})$ in Corollary 4.5 is sharper than the one in Theorem 4.2.

Theorem 4.11. *Let $\mathcal{A}=(a_{i_1i_2\cdots i_m})$ and $\mathcal{B}=(b_{i_1i_2\cdots i_m})$ be two m-th order n-dimensional strong $\mathcal{M}$-tensors. Then*

$$\max_{(i,j)\in Q}\big\{\big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}$$

$$
\begin{aligned}
&\times (a_{jj\cdots j}-\tau(\mathcal{A}))(b_{jj\cdots j}-\tau(\mathcal{B}))\big)^{1/2}\big\}\\
\geq \min\Bigg\{ &\max_{(i,j)\in Q}\big\{\nu_i^{m-1}\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))\big\},\\
&\max_{(i,j)\in Q}\big\{\mu_i^{m-1}\mu_j(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))\big\}\Bigg\}.
\end{aligned}
$$

Proof. We first divide the set Q into two disjoint subsets I and $Q\setminus I$, where $I=\big\{(i,j)\in Q \mid \mu_i^{m-1}\mu_j(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))\geq \nu_i^{m-1}\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))\big\}$. To prove the conclusion, two cases are considered as follows.

Case 1. For all $(i,j)\in I$, we have

$$
\begin{aligned}
&\big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))(b_{jj\cdots j}-\tau(\mathcal{B}))\big)^{1/2}\\
&\geq \nu_i^{m-1}\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A})),
\end{aligned}
$$

which implies that

$$
\begin{aligned}
&\max_{(i,j)\in I}\big\{\big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}\\
&\qquad\times (a_{jj\cdots j}-\tau(\mathcal{A}))(b_{jj\cdots j}-\tau(\mathcal{B}))\big)^{1/2}\big\}\\
&\geq \max_{(i,j)\in I}\big\{\nu_i^{m-1}\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))\big\}\\
&= \min\Bigg\{\max_{(i,j)\in I}\nu_i^{m-1}\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A})),\\
&\qquad \max_{(i,j)\in I}\mu_i^{m-1}\mu_j(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))\Bigg\}.
\end{aligned}
\tag{4.14}
$$

Case 2. For all $(i,j)\in Q\setminus I$, we obtain

$$
\begin{aligned}
&\big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A}))(b_{jj\cdots j}-\tau(\mathcal{B}))\big)^{1/2}\\
&> \mu_i^{m-1}\mu_j(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B})),
\end{aligned}
$$

and similarly, it yields that

$$
\begin{aligned}
&\max_{(i,j)\in Q\setminus I}\big\{\big(\mu_i^{m-1}\nu_i^{m-1}\mu_j\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}\\
&\qquad\times (a_{jj\cdots j}-\tau(\mathcal{A}))(b_{jj\cdots j}-\tau(\mathcal{B}))\big)^{1/2}\big\}\\
&> \max_{(i,j)\in Q\setminus I}\big\{\mu_i^{m-1}\mu_j(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))\big\}\\
&= \min\Bigg\{\max_{(i,j)\in Q\setminus I}\nu_i^{m-1}\nu_j(a_{ii\cdots i}-\tau(\mathcal{A}))^{m-1}(a_{jj\cdots j}-\tau(\mathcal{A})),\\
&\qquad \max_{(i,j)\in Q\setminus I}\mu_i^{m-1}\mu_j(b_{ii\cdots i}-\tau(\mathcal{B}))^{m-1}(b_{jj\cdots j}-\tau(\mathcal{B}))\Bigg\}.
\end{aligned}
\tag{4.15}
$$

Summarizing the deduction of Cases 1 and 2, the proved result follows according to the inequalities (4.14) and (4.15). The proof is completed. □

5. Numerical examples

To verify the rationality and validity of our main results, some numerical examples are given as follows.

Example 5.1. Consider two 4-th order 3-dimensional nonnegative tensors $\mathcal{A} = (a_{ijkl})$ and $\mathcal{B} = (b_{ijkl})$ defined as follows:

$$a_{1111} = 5, \quad a_{1222} = 0.5, \quad a_{1223} = a_{1211} = 0.2, \quad a_{2222} = 2, \quad a_{2333} = 0.2,$$
$$a_{2111} = 0.5, \quad a_{2122} = 0.3, \quad a_{3333} = 3, \quad a_{3111} = 0.5,$$
$$a_{3122} = 2, \quad a_{3113} = 0.2, \quad \text{other} \quad a_{ijkl} = 0,$$
$$b_{1111} = 2, \quad b_{1232} = 0.5, \quad b_{1222} = 2.5, \quad b_{2222} = 6, \quad b_{2111} = 2.5,$$
$$b_{2122} = 1, \quad b_{2333} = 0.1, \quad b_{3333} = b_{3111} = 5, \quad \text{other} \quad b_{ijkl} = 0.$$

Obviously, $\Gamma_{\mathcal{A}\circ\mathcal{B}}$ is weakly connected, and there exist the following circuits in $\Gamma_{\mathcal{A}\circ\mathcal{B}}$:

$$1 \to 2 \to 1, \quad 2 \to 2, \quad 3 \to 1 \to 2 \to 3.$$

By the calculations and analysis, the numerical comparisons between the newly proposed Brualdi-type inequalities for $\rho(\mathcal{A}\circ\mathcal{B})$ in Theorems 3.1, 3.2, Corollary 3.5 and some existing ones in [25, 29, 30, 33] is shown in Figure 5.1 below.

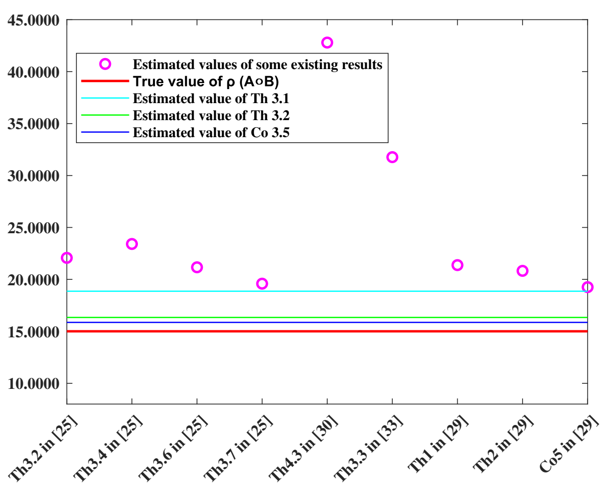

Figure 5.1: Numerical comparisons on the upper bounds for $\rho(\mathcal{A}\circ\mathcal{B})$.

By observing Figure 5.1, it is not difficult to conclude that the estimated values of the newly proposed inequalities for $\rho(\mathcal{A}\circ\mathcal{B})$ in Theorems 3.1, 3.2 and Corollary 3.5 are closed to the true values of $\rho(\mathcal{A}\circ\mathcal{B})$, and it also shows the feasibility of the newly proposed inequalities for $\rho(\mathcal{A}\circ\mathcal{B})$ in Theorems 3.1, 3.2 and Corollary 3.5. Comparing some existing results in [25, 29, 30, 33], we find that the upper bounds characterized by the inequality for $\rho(\mathcal{A}\circ\mathcal{B})$ in Theorems 3.1, 3.2 and Corollary 3.5 are sharper than the existing ones in Theorem 4.3 of [30], Theorems 3.2, 3.4, 3.6 and 3.7 of [25], Theorem 3.3 of [33] as well as Theorems 1–2 and Corollary 5 of [29].

Example 5.2. Consider two 4-th order 3-dimensional $\mathcal{Z}$-tensors $\mathcal{A} = (a_{ijkl})$ and $\mathcal{B} = (b_{ijkl})$ defined as follows:

$$a_{1111} = 5, \quad a_{1222} = -2.5, \quad a_{1223} = a_{1112} = -0.2, \quad a_{2222} = 1.5, \quad a_{2111} = -0.5,$$
$$a_{2122} = -0.3, \quad a_{2113} = -0.2, \quad a_{3333} = 3, \quad a_{3111} = -0.5,$$
$$a_{3122} = -0.3, \quad a_{3113} = -0.2, \quad \text{other} \quad a_{ijkl} = 0,$$
$$b_{1111} = 2, \quad b_{1113} = b_{1222} = -0.5, \quad b_{2222} = 4, \quad b_{2111} = -2.5, \quad b_{2122} = -0.8,$$
$$b_{2133} = -0.2, \quad b_{3333} = 7, \quad b_{3111} = -1, \quad \text{other} \quad b_{ijkl} = 0.$$

Obviously, both $\mathcal{A}$ and $\mathcal{B}$ are two strong $\mathcal{M}$-tensors. Moreover, $\Gamma_{\mathcal{A}\star\mathcal{B}}$ is weakly connected, and there exist the following circuits in $\Gamma_{\mathcal{A}\star\mathcal{B}}$:

$$1 \to 2 \to 1, \quad 2 \to 2, \quad 3 \to 1 \to 2 \to 3.$$

By the calculations and analysis, the numerical comparisons between the newly obtained Brauer-type inequalities for $\tau(\mathcal{A}\star\mathcal{B})$ in Theorems 4.1, 4.2, Corollary 4.5 and some existing ones in [24, 26, 28, 33] is shown in Figure 5.2 below.

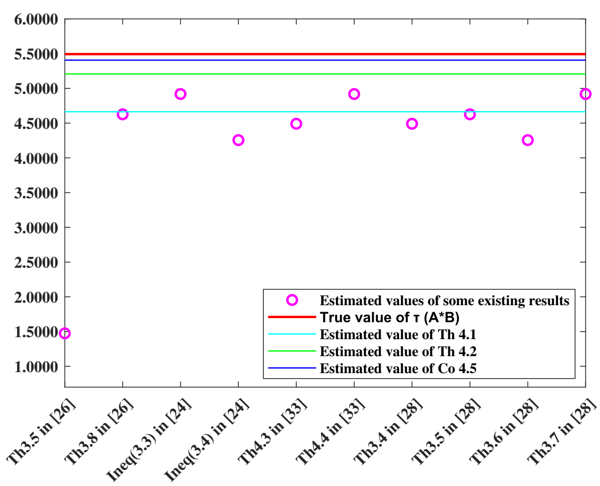

Figure 5.2: Numerical comparisons on the lower bounds for $\tau(\mathcal{A}\star\mathcal{B})$.

By observing Figure 5.2, we find that the estimated values of the newly obtained inequalities for $\tau(\mathcal{A}\star\mathcal{B})$ in Theorems 4.1, 4.2 and Corollary 4.5 are closed to the true value of $\tau(\mathcal{A}\star\mathcal{B})$, which verifies the correctness of the newly obtained inequalities for $\tau(\mathcal{A}\star\mathcal{B})$ in Theorems 4.1, 4.2 and Corollary 4.5. Comparing to some existing results in [24, 26, 28, 33], the detailed numerical comparison results are concluded as follows: (1) The lower bound described by the inequality for $\tau(\mathcal{A}\star\mathcal{B})$ in Theorem 4.1 is better than some existing ones in Theorem 4.3 of [33], Theorems 3.5 and 3.8 of [26], the inequality (3.4) from Theorem 3.6 of [24] and Theorems 3.4–3.6 of [28]; (2) The lower bounds characterized by the inequality for $\tau(\mathcal{A}\star\mathcal{B})$ in Theorem 4.2 and Corollary 4.5 are tighter than the previous ones in Theorems 4.3–4.4 of [33], Theorems 3.5 and 3.8 of [26], the inequalities (3.3) and (3.4) from Theorem 3.6 of [24] and Theorems 3.4–3.7 of [28].

6. Conclusions

In the presented paper, we have proposed some Brualdi-type inequalities on the spectral radius for Hadamard product of two nonnegative tensors and some Brauer-type inequalities on the minimum H-eigenvalue for the Fan product of two strong $\mathcal{M}$-tensors. Based on the newly proposed inequalities, we have established the theoretical and numerical comparisons between the newly obtained inequalities and some existing ones, which has shown the validity and effectiveness of the main results.

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References

[1] R. A. Brualdi, *Matrices, eigenvalues, and directed graphs*, Linear Multilinear Algebra **11** (1982), no. 2, 143–165.

[2] C. Bu, X. Jin, H. Li and C. Deng, *Brauer-type eigenvalue inclusion sets and the spectral radius of tensors*, Linear Algebra Appl. **512** (2017), 234–248.

[3] C. Bu, Y. Wei, L. Sun and J. Zhou, *Brualdi-type eigenvalue inclusion sets of tensors*, Linear Algebra Appl. **480** (2015), 168–175.

[4] W. Ding and L. Qi, *$\mathcal{M}$-tensors and nonsingular $\mathcal{M}$-tensors*, Linear Algebra Appl. **439** (2013), no. 10, 3264–3278.

[5] M. Fang, *Bounds on eigenvalues of the Hadamard product and the Fan product of matrices*, Linear Algebra Appl. **425** (2007), no. 1, 7–15.

[6] S. Friedland, S. Gaubert and L. Han, *Perron–Frobenius theorem for nonnegative multilinear forms and extensions*, Linear Algebra Appl. **438** (2013), no. 2, 738–749.

[7] J. He and T.-Z. Huang, *Inequalities for M-tensors*, J. Inequal. Appl. **2014** (2014), no. 114, 9 pp.

[8] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*, Cambridge University Press, Cambridge, 1991.

[9] S. Hu and L. Qi, *Algebraic connectivity of an even uniform hypergraph*, J. Comb. Optim. **24** (2012), no. 4, 564–579.

[10] C. Li, Z. Chen and Y. Li, *A new eigenvalue inclusion set for tensors and its applications*, Linear Algebra Appl. **481** (2015), 36–53.

[11] C. Li, Y. Li and X. Kong, *New eigenvalue inclusion sets for tensors*, Numer. Linear Algebra Appl. **21** (2014), no. 1, 39–50.

[12] C. Li, Y. Wang, J. Yi and Y. Li, *Bounds for the spectral radius of nonnegative tensors*, J. Ind. Manag. Optim. **12** (2016), no. 3, 975–990.

[13] C. Li, J. Zhou and Y. Li, *A new Brauer-type eigenvalue localization set for tensors*, Linear Multilinear Algebra. **64** (2016), no. 4, 727–736.

[14] S. Li, C. Li and Y. Li, *A new bound for the spectral radius of nonnegative tensors*, J. Inequal. Appl. **2017**, Paper No. 88, 12 pp.

[15] Y.-T. Li, Y.-Y. Li, R.-W. Wang and Y.-Q. Wang, *Some new bounds on eigenvalues of the Hadamard product and the Fan product of matrices*, Linear Algebra Appl. **432** (2010), no. 2-3, 536–545.

[16] L.-H. Lim, *Singular values and eigenvalues of tensors: a variational approach*, in: *1st IEEE International Workshop on Computational Advances in Multi-Sensor Adaptive Processing*, 129–132, New York, 2005.

[17] Q. Liu and G. Chen, *On two inequalities for the Hadamard product and the Fan product of matrices*, Linear Algebra Appl. **431** (2009), no. 5-7, 974–984.

[18] Q. Ni, L. Qi and F. Wang, *An eigenvalue method for testing positive definiteness of a multivariate form*, IEEE Trans. Automat. Control **53** (2008), no. 5, 1096–1107.

[19] L. Qi, *Eigenvalues of a real supersymmetric tensor*, J. Symbolic Comput. **40** (2005), no. 6, 1302–1324.

[20] ———, *Hankel tensors: Associated Hankel matrices and Vandermonde decomposition*, Commun. Math. Sci. **13** (2015), no. 1, 113–125.

[21] L. Qi, C. Xu and Y. Xu, *Nonnegative tensor factorization, completely positive tensors, and a hierarchical elimination algorithm*, SIAM J. Matrix Anal. Appl. **35** (2014), no. 4, 1227–1241.

[22] M. Rajesh Kannan, N. Shaked-Monderer and A. Berman, *Some properties of strong $\mathcal{H}$-tensors and general $\mathcal{H}$-tensors*, Linear Algebra Appl. **476** (2015), 42–55.

[23] S. Rota Bulò and M. Pelillo, *A generalization of the Motzkin–Straus theorem to hypergraphs*, Optim. Lett. **3** (2009), no. 2, 287–295.

[24] J. Shen, Y. Wang and C. Bu, *Some inequalities on the minimum H-eigenvalue of the Fan product of $\mathcal{Z}$-tensors*, Linear Algebra Appl. **579** (2019), 55–71.

[25] L. Sun, B. Zheng, J. Zhou and H. Yan, *Some inequalities for the Hadamard product of tensors*, Linear Multilinear Algebra **66** (2018), no. 6, 1199–1214.

[26] G. Wang, Y. Wang and L. Liu, *Bound estimations on the eigenvalues for Fan product of M-tensors*, Taiwanese J. Math. **23** (2019), no. 3, 751–766.

[27] G. Wang, Y. Wang and Y. Zhang, *Some inequalities for the Fan product of M-tensors*, J. Inequal. Appl. **2018**, Paper No. 257, 15 pp.

[28] G. Wang, Y. Wang and Y. Zhang, *Brualdi-type inequalities on the minimum eigenvalue for the Fan product of M-tensors*, J. Ind. Manag. Optim. **16** (2020), no. 5, 2551–2562.

[29] G. Wang, Y. Zhang and Y. Wang, *Brauer-type bounds for Hadamard product of nonnegative tensors*, Front. Math. China **15** (2020), no. 3, 555–570.

[30] Y. Xu, Y. Li and Z. Li, *Some results on the Hadamard product of tensors*, Bull. Iranian Math. Soc. **45** (2019), no. 4, 1193–1219.

[31] Y. Xu, L. Shao and G. He, *On some spectral radius inequalities for the Hadamard product of nonnegative tensors*, Bull. Malays. Math. Sci. Soc. **47** (2024), no. 1, Paper No. 10, 21 pp.

[32] Y. Xu, B. Zheng and R. Zhao, *Some results on Brauer-type and Brualdi-type eigenvalue inclusion sets for tensors*, Comput. Appl. Math. **38** (2019), no. 2, Paper No. 74, 13 pp.

[33] ______, *Some bounds of eigenvalues for Hadamard product and Fan product of tensors*, Bull. Iranian Math. Soc. **46** (2020), no. 4, 1003–1026.

[34] ______, *Some improved Ky Fan type eigenvalue inclusion sets for tensors*, Calcolo **57** (2020), no. 4, Paper No. 40, 28 pp.

[35] Y. Yang and Q. Yang, *Further results for Perron–Frobenius theorem for nonnegative tensors*, SIAM J. Matrix Anal. Appl. **31** (2010), no. 5, 2517–2530.

[36] L. Zhang, L. Qi and G. Zhou, *M-tensors and some applications*, SIAM J. Matrix Anal. Appl. **35** (2014), no. 2, 437–452.

[37] J. Zhao and C. Sang, *Two new lower bounds for the minimum eigenvalue of M-tensors*, J. Inequal. Appl. **2016**, Paper No. 268, 9 pp.

[38] D. Zhou, G. Chen, G. Wu and X. Zhang, *On some new bounds for eigenvalues of the Hadamard product and the Fan product of matrices*, Linear Algebra Appl. **438** (2013), no. 3, 1415–1426.

[39] J. Zhou, L. Sun, Y. Wei and C. Bu, *Some characterizations of M-tensors via digraphs*, Linear Algebra Appl. **495** (2016), 190–198.

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