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Existence of Ground State Solutions for the Schrödinger–Poisson System in $\mathbb{R}^2$

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Abstract. This paper concerns the following planar Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-2}u & \text{in } \mathbb{R}^2, \\ \Delta \phi = u^2, \end{cases}$$

where $p \geq 3$. By developing some new analytic techniques and variational methods, we establish a local compactness splitting lemma, and prove that this system possesses ground state solutions. We extend the case where $V(x)$ is a constant coefficient to the case where $V(x)$ is a variable coefficient. Some related results are improved.

1. Introduction

The purpose of this paper is to consider the following planar Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + \phi u = |u|^{p-2}u, & x \in \mathbb{R}^2, \\ \Delta \phi = u^2, & x \in \mathbb{R}^2, \end{cases} \tag{1.1}$$

where $V(x)$ is continuous on $\mathbb{R}^2$ and satisfies the following basic assumptions

(V1) $\langle \nabla V(x), x\rangle \in L^\infty(\mathbb{R}^2)$ and $4(p-2)V(x) + 2(8-p)\langle \nabla V(x), x\rangle \geq 4-p$ for a.e. $x \in \mathbb{R}^2$;

(V2) $V \in C(\mathbb{R}^2, [0,\infty))$, $V(x) := V(x_1, x_2) = V(|x_1|, |x_2|)$ for all $x = (x_1, x_2) \in \mathbb{R}^2$, $0 \leq V(x) \leq V(\infty) := \liminf_{|x|\to\infty} V(x) < +\infty$, $V(x) \not\equiv V(\infty)$ and the inequality is strict in a subset of positive measure.

Remark 1.1. It is easy to see that there exist lots of functions verifying (V1) and (V2), for example, set $V(x) = 2 - \frac{1}{1+x^2}$.

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The system (1.1) is a special form of the following nonlinear Schrödinger–Poisson system

$$\begin{cases} -\Delta u + V(x)u + m\phi u = f(u), & x \in \mathbb{R}^N, \\ \Delta\phi = u^2, & x \in \mathbb{R}^N, \end{cases} \tag{1.2}$$

where $m \in \mathbb{R} \setminus \{0\}$, V and f are continuous functions. As we know that solutions of (1.2) are related to the solitary wave solutions of the following form

$$\begin{cases} i\psi_t - \Delta\psi + W(x)\psi + m\phi\psi = f(\psi) & \text{in } \mathbb{R}^N \times \mathbb{R},\ t > 0, \\ \Delta\phi = |\psi|^2 & \text{in } \mathbb{R}^N,\ t > 0, \end{cases} \tag{1.3}$$

where $\psi\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{C}$ is the time dependent wave function, $W\colon \mathbb{R}^N \to \mathbb{R}$ is a real external potential, $W(x) = V(x) - a$, and $a \in \mathbb{R}$ denotes a real valued external potential. The function ϕ represents an internal potential for a nonlocal self-interaction of the wave function ψ. Evolution problem (1.3) mainly appears in physical phenomena, such as (1.3) arises in a quantum mechanical phenomenon (see [4, 6, 14]) and in semiconductor theory [17, 19]. More contents of physical aspects can be found in [2, 3]. The nonlinear term f describes the interaction effect among particles. The solution ϕ of system (1.2) can be solved by choosing $\phi = \Gamma_N * u^2$, where $*$ represents the convolution in $\mathbb{R}^N$, Γ_N is the fundamental solution of the Laplacian, which is defined by

$$\Gamma_N(x) = \begin{cases} \frac{1}{2\pi}\log|x|, & N = 2, \\ \frac{1}{N(2-N)\omega_N}|x|^{2-N}, & N \geq 3, \end{cases}$$

and ω_N denotes the volume of the unit N-ball. With this formal inversion, (1.2) is changed into the integro-differential equation

$$-\Delta u + V(x)u + m(\Gamma * u^2)u = f(u) \quad \text{in } \mathbb{R}^N. \tag{1.4}$$

The energy functional of (1.4) is defined by

$$\begin{aligned} I(u) = {} & \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x \\ & + \frac{m}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \Gamma_N(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y - \int_{\mathbb{R}^2} F(u)\,\mathrm{d}x, \end{aligned}$$

where $F(u) = \int_0^u f(t)\,\mathrm{d}t$. Set $\phi_{N,u} = (\Gamma_N * u^2)(x)$. If u is a critical point of I, then the pair $(u, \phi_{N,u})$ is a weak solution of (1.4). For convenience, we say that u, instead of $(u, \phi_{N,u})$, is a weak solutio of (1.4). Recently, the existence and multiplicity of solutions of (1.4) (or (1.2)) have been studied extensively. Most of the results focuses on investigating (1.4) with $N = 3$ and $m < 0$. In this situation, from the Hardy–Littlewood–Sobolev inequality, I is a well-defined C^1-functional on a weighted Sobolev space, and it is easy to verify the mountain pass geometry when $f(u)$ is super-cubic at $u = \infty$ and superlinear at $u = 0$. In this case, the existence, concentration, multiplicity, and ground state solutions of (1.4) were derived under various hypotheses on f and V. If $f(u)$ is super-quadratic at $t = \infty$, from the Nehari–Pohozaev manifold, the existence of ground state solutions of (1.4) were obtained, see [7, 12, 18, 21] and so on. However, in the case where $N = 2$ and $\Gamma_N = \frac{1}{2\pi} \log|x|$, the functional I is not well defined on the space $H^1(\mathbb{R}^2)$, which leads that the approaches used in the case where $N = 3$ cannot be applied easily to $N = 2$, even if $V \in L^\infty(\mathbb{R}^2)$ and $\inf_{\mathbb{R}^2} V(x) > 0$, since the logarithmic integral kernel $\frac{1}{2\pi} \log|x|$ is sign-changing, and is neither bounded from below nor from above. This is one of the reasons why there only exist few results studying the planar case. In order to overcome this difficulty, motivated by [20], Cingolani and Weth [10], using variational methods, considered the equation

$$-\Delta u + V(x)u + (\log(|\cdot| * |u|^2))u = |u|^{p-2}u \quad \text{in } \mathbb{R}^2. \tag{1.5}$$

The associated energy functional of problem (1.5) is

$$\begin{aligned} I_V(u) = {}& \frac{1}{2} \int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x \\ &+ \frac{1}{4} \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{p} \int_{\mathbb{R}^2} |u|^p\,\mathrm{d}x. \end{aligned}$$

With the smaller Hilbert space $E = \left\{u \in H^1(\mathbb{R}^2) : \int_{\mathbb{R}^2} \log(1+|x|)u^2\,\mathrm{d}x < \infty\right\}$, by the following decomposition

$$\begin{aligned} &\int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|)u(x)u(y)\,\mathrm{d}x\,\mathrm{d}y \\ &= \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \left[\log(1+|x-y|) - \log\left(1 + \frac{1}{|x-y|}\right)\right] u(x)u(y)\,\mathrm{d}x\,\mathrm{d}y, \end{aligned}$$

they proved that there exist high energy solutions in a periodic setting, where the corresponding functional $I(u)$ is invariant under $\mathbb{Z}^2$-translations. But this fails to verify a global PS-condition. They also derived that problem (1.5) has a non-radial solution with $V(x) = a$, where a is a positive constant. We need to point out that these results are unavailable for the case where $p \in (2,4)$. Later, Du and Weth [11] dealt with problem (1.5) with $V(x) = 1$, by a different variational approach and obtained a ground state solution. While the methods used in [11] are not suitable to study for nonautonomous problem (1.5) with $V \neq$ constant, because it needs to construct a Cerami sequence which satisfies asymptotically Pohozev identity. Recently, in [8], the authors proved problem (1.5) has a nontrivial solution and ground state solution possessing the least energy in the axially symmetric functions space, and they assumed that $V(x)$ is axially symmetric. More results can be found in [1, 5, 9, 23] and so on.

Based on the above works, we begin to study problem (1.5) with $V(x)=$ constant and $V(x)\neq$ constant, and establish the existence of ground state solution for problem (1.5) by employing Pohozaev–Nehari manifold, concentration compactness lemma (see [16,22]), and a global compactness lemma (see Lemma 4.5 below).

It is worth to point out that some difficulties are faced. Firstly, the energy functional I_V is not coercive on X, and we have to partly adopt the method developed in [10]. Secondly, the norm of X is not translation invariant, while the energy functional I is translation invariant. Thirdly, the logarithmic convolution term of $(\log(|\cdot|*|u|^2))u$ does't have a definite sign on X in contrast to higher dimensional spaces, while this leads the difficulty of proving the boundedness of (PS)-sequence in the variational setting.

Our main results are presented below.

Theorem 1.2. *If $V(x)$ is a positive constant, then problem* (1.5) *has a positive ground state solution for $p\in[3,\infty)$.*

Theorem 1.3. *If $V(x)$ satisfies* (V1) *and* (V2)*, then problem* (1.5) *has a positive ground state solution for $p\in[3,\infty)$.*

Remark 1.4. (i) In [11], the authors only considered $V(x)=1$, here we extend it to the case that $V(x)$ is a variable function. By suitable assumptions and new variational method, a positive ground state solution of (1.5) is obtained. In a sense, Theorem 1.2 in [11] is our special case.

(ii) Although Theorem 1.2 is similar as that in [11], our methods are totally different from those in [11]. It seems that our analysis is more succinct.

This paper is organized as follows. In Section 2, we present some necessary preliminary knowledge. In Section 3, we give the proof of Theorem 1.2 with $V(x)=a$, where a is a positive constant. In Section 4, the existence of positive ground state solutions of problem (1.5) is proved with $V(x)$ being a function.

2. Preliminaries

We firstly give some notations. $(X,\|\cdot\|)$ denotes a (real) Banach space and $(X^*,\|\cdot\|_*)$ denotes its topological dual. C, and C_i $(i=1,2,\ldots)$ denote estimated constants (the concrete values may be different from line to line). "$\to$" means the stronger convergence in X and "$\rightharpoonup$" stands for the weak convergence in X. $|u|_p$ denotes the norm of $L^p(\mathbb{R}^2)$.

Let $H^1(\mathbb{R}^2)$ be the usual Sobolev space and

$$E=\left\{u\in H^1(\mathbb{R}^2):\int_{\mathbb{R}^2}(|\nabla u|^2+V(x)|u|^2)\,\mathrm{d}x<+\infty\right\}$$

equipped with the inner product and norm

$$\langle u, v\rangle := \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv)\, \mathrm{d}x, \quad \|u\| = \langle u, u\rangle^{1/2}.$$

We define the symmetric bilinear forms

$$\begin{aligned}
(u, v) &\mapsto T_1(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1 + |x-y|)u(x)u(y)\, \mathrm{d}x\, \mathrm{d}y,\\
(u, v) &\mapsto T_2(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1 + \frac{1}{|x-y|}\right) u(x)u(y)\, \mathrm{d}x\, \mathrm{d}y,\\
(u, v) &\mapsto T_0(u, v) = T_1(u, v) - T_2(u, v) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)u(x)u(y)\, \mathrm{d}x\, \mathrm{d}y,
\end{aligned}$$

where these definitions are restricted to measurable functions $u, v\colon \mathbb{R}^2 \to \mathbb{R}$ such that corresponding double integrals are well defined in Lebesgue sense. Noting that $0 < \log(1+r) < r$ for $r > 0$, by Hardy–Littlewood–Sobolev inequality [15] we infer that

$$|T_2(u, v)| \le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \frac{1}{|x-y|}|u(x)u(y)|\, \mathrm{d}x\, \mathrm{d}y \le C_0|u|_{4/3}|v|_{4/3} \quad \text{for all } v \in L^{4/3}(\mathbb{R}^2). \tag{2.1}$$

Then we define the functionals

$$\begin{aligned}
V_1\colon H^1(\mathbb{R}^2) \to [0, \infty], &\quad V_1(u) = T_1(u^2, u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1 + |x-y|)u^2(x)u^2(y)\, \mathrm{d}x\, \mathrm{d}y,\\
V_2\colon L^{8/3}(\mathbb{R}^2) \to [0, \infty), &\quad V_2(u) = T_2(u^2, u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log\left(1 + \frac{1}{|x-y|}\right) u^2(x)u^2(y)\, \mathrm{d}x\, \mathrm{d}y,\\
V_0\colon H^1(\mathbb{R}^2) \to \mathbb{R} \cap \{\infty\}, &\quad V_0(u) = T_0(u^2, u^2) = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)u^2(x)u^2(y)\, \mathrm{d}x\, \mathrm{d}y.
\end{aligned}$$

It follows from (2.1) that

$$|V_2(u)| \le C_0|u|_{8/3}^4 \quad \text{for all } u \in L^{8/3}(\mathbb{R}^2), \tag{2.2}$$

then T_2 only takes finite values on $L^{8/3}(\mathbb{R}^2) \subset H^1(\mathbb{R}^2)$.

Now, for any measurable function $u\colon \mathbb{R}^2 \to \mathbb{R}$, we define

$$|u|_* := \left(\int_{\mathbb{R}^2} \log(1 + |x|)u^2\, \mathrm{d}x\right)^{1/2} \in [0, \infty].$$

Noting that $\log(1 + |x-y|) \le \log(1 + |x| + |y|) \le \log(1 + |x|) + \log(1 + |y|)$ for $x, y \in \mathbb{R}^2$, we obtain the following estimate

$$\begin{aligned}
|T_1(uv, wz)| &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} [\log(1 + |x|) + \log(1 + |y|)]|u(x)v(x)||w(y)z(y)|\, \mathrm{d}x\, \mathrm{d}y\\
&\le |u|_*|v|_*|w|_2|z|_2 + |u|_2|v|_2|w|_*|z|_*
\end{aligned} \tag{2.3}$$

for $u, v, w, z \in L^2(\mathbb{R}^2)$.

Define the smaller Hilbert space

$$X := \{u \in E : |u|_* < \infty\}$$

equipped with the norm $\|u\|_X := \sqrt{\|u\|^2 + |u|_*^2}$. By (2.3) we know that the restriction of I_V to X only takes finite values in $\mathbb{R}$.

It is obvious to see that critical points of I_V are weak solutions to problem (1.5) and I_V is a C^1-functional whose derivative is defined by

$$\langle I'_V(u), v\rangle = \int_{\mathbb{R}^2} (\nabla u \nabla v + V(x)uv)\,\mathrm{d}x + T_0(u^2, uv) - \int_{\mathbb{R}^2} |u|^{p-2}uv\,\mathrm{d}x.$$

We now establish a inequality which is very important to estimate T_1.

Lemma 2.1. *If* (V2) *holds, then* $T_1(u^2, v^2) \geq \frac{1}{4}|u|_2^2|v|_*^2$ *for all* $u, v \in X$.

Proof. Set $\Omega_1 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 \geq 0\}$, $\Omega_2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \leq 0, x_2 \geq 0\}$, $\Omega_3 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0, x_2 \leq 0\}$ and $\Omega_4 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 < 0\}$. For all $(x, y) \in \Omega_1 \times \Omega_3$ or $(x, y) \in \Omega_2 \times \Omega_4$, we derive

$$|x - y|^2 = |x|^2 + |y|^2 - 2x \cdot y \geq |x|^2 + |y|^2.$$

Then

$$\begin{aligned}
&T_1(u^2, v^2)\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1 + |x - y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y\\
&\geq 2\left(\int_{\Omega_3} u^2(x)\,\mathrm{d}x \int_{\Omega_1} \log(1 + |x - y|)v^2(y)\,\mathrm{d}y + \int_{\Omega_4} u^2(x)\,\mathrm{d}x \int_{\Omega_2} \log(1 + |x - y|)v^2(y)\,\mathrm{d}y\right)\\
&\geq 2\left(\int_{\Omega_3} u^2(x)\,\mathrm{d}x \int_{\Omega_1} \log(1 + |y|)v^2(y)\,\mathrm{d}y + \int_{\Omega_4} u^2(x)\,\mathrm{d}x \int_{\Omega_2} \log(1 + |y|)v^2(y)\,\mathrm{d}y\right)\\
&= \frac{1}{4}\int_{\mathbb{R}^2} u^2(x)\,\mathrm{d}x \int_{\mathbb{R}^2} \log(1 + |y|)v^2(y)\,\mathrm{d}y\\
&= \frac{1}{4}|u|_2^2|u|_*^2.
\end{aligned}$$

□

In the following we give a Pohozaev type identity for equation (1.5). The proof is similar as that in [11]. Here we omit its proof.

Lemma 2.2. *Assume that* $u \in X$ *is a weak solution of* (1.5). *Then*

$$\begin{aligned}
P(u) := &\int_{\mathbb{R}^2} V(x)|u|^2\,\mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^2} \langle \nabla V(x), x\rangle |u|^2\,\mathrm{d}x + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x - y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y\\
&+ \frac{1}{4}\left(\int_{\mathbb{R}^2} u^2\,\mathrm{d}x\right)^2 - \frac{2}{p}\int_{\mathbb{R}^2} |u|^p\,\mathrm{d}x = 0.
\end{aligned}$$

If $V(x)$ is a positive constant V, the Pohozaev identity can be rewritten as follows:

$$
\begin{aligned}
P(u) := &\int_{\mathbb{R}^2} V|u|^2 \,\mathrm{d}x + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y \\
&+ \frac{1}{4}\left(\int_{\mathbb{R}^2} u^2\,\mathrm{d}x\right)^2 - \frac{2}{p}\int_{\mathbb{R}^2} |u|^p\,\mathrm{d}x = 0.
\end{aligned} \tag{2.4}
$$

From Lemma 2.1 in [8], we can derive the following results.

Lemma 2.3. *Assume that $V(x) \in C(\mathbb{R}^2, [0,\infty))$ with $\liminf_{|x|\to V(x)} > 0$. Then there exists*

$$C_1\|u\|_{H^1(\mathbb{R}^2)} \le C_2\|u\| \le \|u\|_X, \quad \forall\, u \in X.$$

Furthermore, the embedding $X \hookrightarrow L^q(\mathbb{R}^2)$ is compact for $q \in [2,\infty)$.

The following proposition is very important in proving our main results.

Proposition 2.4. [13] *Let X be a Banach space, interval $\Lambda \subset \mathbb{R}^+$. Consider a family $\{I_\lambda\}_{\lambda\in\Lambda}$ of C^1-functional on X of the form*

$$I_\lambda(u) = A(u) - \lambda B(u), \quad \forall\, \lambda \in \Lambda,$$

where $B(u) \ge 0$, $\forall\, u \in X$, and such that either $A(u) \to +\infty$ or $B(u) \to +\infty$ as $\|u\|_X \to \infty$. Assume that there exist two points v_1, v_2 in X such that

$$c_\lambda := \inf_{\gamma\in\Gamma}\max_{t\in[0,1]} I_\lambda(\gamma(t)) > \max\{I_\lambda(v_1), I_\lambda(v_2)\},$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = v_1, \gamma(1) = v_2\}.$$

Then, for a.e. $\lambda \in \Lambda$, there exists a bounded $(PS)_{c_\lambda}$ a sequence of I_λ, i.e., there is sequence such that

(i) *$\{u_n\}$ is bounded in X;*

(ii) *$I_\lambda(u_n(\lambda)) \to c_\lambda$;*

(iii) *$I'_\lambda(u_n(\lambda)) \to 0$ in X^*, where X^* is the dual of X.*

Furthermore, c_λ is non-increasing on $\lambda \in \Lambda$.

3. Case $V(x) = a$

In this section, we assume that $V(x)$ is a positive constant a and prove Theorem 1.2. So the purpose of this section is to show that the ground state solution can be derived on some manifold. Since $V(x) = a$, the functional I_V is reduced to be

$$I(u) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + au^2)\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{p}\int_{\mathbb{R}^2}|u|^p\,\mathrm{d}x.$$

In order to prove Theorem 1.2, we introduce the following auxiliary functional $J\colon X \to \mathbb{R}$ defined by

$$\begin{aligned} J(u) &= \int_{\mathbb{R}^2}(2|\nabla u|^2 + au^2)\,\mathrm{d}x + \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \frac{1}{4}\left(\int_{\mathbb{R}^2}|u|^2\,\mathrm{d}x\right)^2 - \frac{2(p-1)}{p}\int_{\mathbb{R}^2}|u|^p\,\mathrm{d}x, \\ &= 2\langle I'(u), u\rangle - P(u), \end{aligned}$$

where $P(u)$ is given by (2.4). Define the Nehari–Pohozaev manifold

$$\mathscr{M} := \{u \in X \setminus \{0\} : J(u) = 0\}.$$

We now construct the following properties of $\mathscr{M}$.

Lemma 3.1. (i) *Assume $p \in [3, +\infty)$, for any $u \in X \setminus \{0\}$, then there exists a unique $t_0 = t_0(u) > 0$ such that $u_{t_0} = t_0^2(u(t_0x)) \in \mathscr{M}$. Moreover, $I(u_{t_0}) = \max_{t\geq 0} I(u_t)$, where $u_t = t^2(u(tx))$;*

(ii) $0 \notin \partial\mathscr{M}$ *and* $\inf I|_{\mathscr{M}} > 0$;

(iii) *$\mathscr{M}$ is a nature constraint of I. That is every critical point of $I|_{\mathscr{M}}$ is a critical point of I.*

Proof. (i) For all $u \in X \setminus \{0\}$ and $t > 0$, let $u_t = t^2(u(tx))$. Consider

$$\begin{aligned} H(t) = I(t^2(u(tx))) &= \frac{t^4}{2}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x + \frac{t^2}{2}\int_{\mathbb{R}^2}au^2\,\mathrm{d}x \\ &\quad + \frac{t^4}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y - \frac{t^4\log t}{4}\left(\int_{\mathbb{R}^2}|u|^2\,\mathrm{d}x\right)^2 \\ &\quad - \frac{t^{2p-2}}{p}\int_{\mathbb{R}^2}|u|^p\,\mathrm{d}x = 0. \end{aligned}$$

It follows from Lemma 4.1 in [11] that $H(t)$ has a unique critical point $t_0 > 0$ corresponding to its maximum. Thus $H'(t) = 0$ and $H(t_0) = \max_{t\geq 0} I(u_t)$. Then we have

$$\begin{aligned} &2t_0^3\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x + t_0\int_{\mathbb{R}^2}au^2\,\mathrm{d}x + t^3\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \left(t_0^3\log t_0 + \frac{t_0^3}{4}\right)\left(\int_{\mathbb{R}^2}|u|^2\,\mathrm{d}x\right)^2 - \frac{(2p-2)t_0^{2p-3}}{p}\int_{\mathbb{R}^2}|u|^p\,\mathrm{d}x = 0. \end{aligned}$$

Since

$$J(u_{t_0}) = 2t_0^4 \int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x + t_0^2 \int_{\mathbb{R}^2} au^2) \, \mathrm{d}x + t_0^4 \int_{\mathbb{R}^2} \int_{\mathbb{R}^2} \log(|x-y|) u^2(x) u^2(y) \, \mathrm{d}x \, \mathrm{d}y$$
$$- t_0^4 \log t_0 \left(\int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}x \right)^2 - \frac{t_0^4}{4} \left(\int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}x \right)^2 - \frac{2(p-1)t_0^{2p-2}}{p} \int_{\mathbb{R}^2} |u|^p \, \mathrm{d}x,$$

from the above equalities, we derive $J(u_{t_0}) = 0$ and $u_{t_0} \in \mathscr{M}$.

(ii) By $u \in \mathscr{M}$, (2.2) and Lemma 2.3, we have

$$\begin{aligned} 0 = J(u) &\geq \min\{2, a\} \|u\|^2 - V_2(u) - \frac{1}{4} |u|_2^4 - \frac{2(p-1)}{p} |u|_p^p \\ &\geq \min\{2, a\} \|u\|^2 - C_1 \|u\|^4 - \frac{2(p-1)}{p} \|u\|^p \\ &= \|u\|^2 \left(\min\{2, a\} - C_1 \|u\|^2 - \frac{2(p-1)}{p} \|u\|^{p-2} \right), \end{aligned}$$

which implies $0 \notin \partial \mathscr{M}$ for $\|u\|$ small enough. This demonstrates that $\mathscr{M}$ is complete. Choosing any $u \in \mathscr{M}$, we infer that

$$\begin{aligned} I(u) &= I(u) - \frac{1}{4} J(u) \\ &= \frac{1}{4} \int_{\mathbb{R}^2} au^2 \, \mathrm{d}x + \frac{1}{16} \left(\int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}x \right)^2 + \left(\frac{p-1}{2p} - \frac{1}{p} \right) \int_{\mathbb{R}^2} |u|^p \, \mathrm{d}x > 0 \end{aligned}$$

as $p \geq 3$, from which we obtain that $\inf I|_{\mathscr{M}} > 0$.

(iii) Since $J(u) = 2\langle I'(u), u \rangle - P(u)$, from Lemma 2.2 one has that every critical point of I is contained in $\mathscr{M}$. □

Lemma 3.2. *If $p \geq 3$, then there is a minimizer u of $\inf_{\mathscr{M}} I = c_{\mathscr{M}}$. Furthermore, $I'(u) = 0$ in X.*

Proof. We divided the proof into three steps.

Step 1. Let $\{u_n\} \subset \mathscr{M}$ be a sequence such that $I(u_n) \to \inf_{\mathscr{M}} I = c_{\mathscr{M}}$. We now prove the boundedness of $\{u_n\}$. Indeed, by using $u_n \in \mathscr{M}$ we have

$$c_{\mathscr{M}} + o_n(1) \geq I(u_n) - \frac{1}{4} J(u_n) = \frac{1}{4} \int_{\mathbb{R}^2} a|u|^2 \, \mathrm{d}x + \frac{1}{16} |u_n|_2^4 + \frac{p-3}{2p} |u_n|_p^p.$$

Similar as that in Propositions 3.3 and 3.4 in [11], we can obtain that $\{u_n\}$ is bounded in X.

Step 2. Passing to a subsequence, we assume that $u_n \rightharpoonup u$ in X. We will show that $u \in \mathscr{M}$ and $u_n \to u$ in X. Then $I|_{\mathscr{M}}$ derives its minimum at u. Set

$$\int_{\mathbb{R}^2} |\nabla u_n|^2 \, \mathrm{d}x = \alpha_n, \quad \int_{\mathbb{R}^2} |u_n|^2 \, \mathrm{d}x = \beta_n, \quad \int_{\mathbb{R}^2} |u_n|^p \, \mathrm{d}x = \delta_n,$$
$$\int_{\mathbb{R}^2} |\nabla u|^2 \, \mathrm{d}x = \alpha, \quad \int_{\mathbb{R}^2} |u|^2 \, \mathrm{d}x = \beta, \quad \int_{\mathbb{R}^2} |u|^p \, \mathrm{d}x = \delta,$$
$$\lim_{n\to\infty} \alpha_n = \overline{\alpha}, \quad \lim_{n\to\infty} \beta_n = \overline{\beta}, \quad \lim_{n\to\infty} \delta_n = \overline{\delta}.$$

Passing to a subsequence if necessary, we can assume that the above limits exists. From the compactness of the embedding $X \hookrightarrow L^p(\mathbb{R}^2)$ for $p \geq 2$, one has $\overline{\beta} = \beta$, $\overline{\delta} = \delta$. By the weak convergence, we have $\alpha \leq \overline{\alpha}$. We now prove that $\alpha = \overline{\alpha}$. Suppose $\alpha = \overline{\alpha}$ were false. Noting that $I(u_n) \to c_{\mathscr{M}}$ and $J(u_n) = 0$, then we could derive

$$\begin{cases} \frac{1}{2}\overline{\alpha} + \frac{1}{2}a\overline{\beta} + \frac{1}{4}V_0 - \frac{1}{p}\overline{\delta} = c_{\mathscr{M}}, \\ 2\overline{\alpha} + a\overline{\beta} + V_0 - \frac{1}{4}\overline{\beta}^2 - \frac{2(p-1)}{p}\overline{\delta} = 0, \end{cases}$$

from which we derive

$$c_{\mathscr{M}} = \frac{1}{4}a\overline{\beta} + \frac{1}{16}\overline{\beta}^2 + \frac{p-3}{2p}\overline{\delta} > 0.$$

From Lemma 3.1(ii), we have $\overline{\alpha} > 0$. As a result, $\alpha > 0$, $\beta > 0$ and $\delta > 0$. Define

$$\overline{g}(t) = \frac{t^4}{2}\overline{\alpha} + \frac{t^2}{2}a\overline{\beta} + \frac{t^4}{4}V_0 - \frac{t^4\log t}{4}\overline{\beta}^2 - \frac{t^{2p-2}}{p}\overline{\delta},$$
$$g(t) = \frac{t^4}{2}\alpha + \frac{t^2}{2}a\beta + \frac{t^4}{4}V_0 - \frac{t^4\log t}{4}\beta^2 - \frac{t^{2p-2}}{p}\delta.$$

According to Lemma 4.1 in [11], we deduce that $\overline{g}(t)$ and $g(t)$ have a unique critical point respectively, corresponding to their maximums. We infer that $\max \overline{g} = \inf I|_{\mathscr{M}} = c_{\mathscr{M}}$ as $t = 1$. Proceeding by contradiction we assume $\alpha+\beta+\delta < \overline{\alpha}+\overline{\beta}+\overline{\delta}$, $g(t) < \overline{g}(t)$ for all $t > 0$, and suppose that t_0 is the maximal point of g. Consequently, $g(t_0) \leq \max \overline{g} = \inf I_{\mathscr{M}} = c_{\mathscr{M}}$ and $g'(t_0) = 0$. Setting $u_0 = t_0^2 u(t_0 x)$, we have

$$I(u_0) = \frac{t_0^4}{2}\alpha + \frac{t_0^2}{2}a\beta + \frac{t_0^4}{4}V_0 - \frac{t_0^4\log t_0}{4}\beta^2 - \frac{t_0^p}{p}\delta,$$
$$J(u_0) = 2t_0^4\alpha + t_0^2 a\beta + t^4 V_0 - t_0^4 \log t_0 \beta^2 - \frac{t_0^4}{4}\beta^2 - \frac{2(p-1)t_0^{2p-2}}{p}\delta = t_0 g'(t_0) = 0,$$

which implies that $u_0 \in \mathscr{M}$ and $I(u_0) < \inf I|_{\mathscr{M}}$. This is a contradiction, thus $\alpha + \beta + \delta = \overline{\alpha} + \overline{\beta} + \overline{\delta}$, from which we derive $u_n \to u$ and $u \in \mathscr{M}$.

Step 3. From Lemma 4.2(iv) in [11], we obtain that $I'(u) = 0$. □

Proof of Theorem 1.2. It follows from Lemma 3.2 that there is a $u \in \mathscr{M}$ such that $I(u) = \inf I|_{\mathscr{M}}$ and $I'(u) = 0$. Thus u is a positive point of $I \mid \mathscr{M}$. Lemma 3.1 yields that u is a positive ground state solution of problem (1.5) with $V(x) = a$. □

4. Nonconstant potential $V(x)$

The main goal of this section is to show the proof of Theorem 1.3. From Proposition 2.4, we define the functional $I_{V,\lambda}\colon X \to \mathbb{R}$ by

$$I_{V,\lambda} = \frac{1}{2}\int_{\mathbb{R}^2} (|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y - \frac{\lambda}{p}\int_{\mathbb{R}^2} |u|^p\,\mathrm{d}x$$
$$= A(u) - \lambda B(u)$$

for $u \in X$, where

$$A(u) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + V(x)u^2)\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y,$$
$$B(u) = \frac{\lambda}{p}\int_{\mathbb{R}^2}|u|^p\,\mathrm{d}x$$

for $\lambda \in [1/2, 1]$. It is obvious to see that $I_{V,\lambda}$ is of C^1-class. For $u, v \in X$,

$$\langle I'_{V,\lambda}(u), v\rangle = \int_{\mathbb{R}^2}(\nabla u \nabla v + V(x)uv)\,\mathrm{d}x + B_0(u^2, uv) - \lambda\int_{\mathbb{R}^2}|u|^{p-2}uv\,\mathrm{d}x.$$

We also need to discuss the following limit problem

$$-\Delta u + V(\infty)u + (\log(|\cdot| * |u|^2))u = \lambda|u|^{p-2}u \quad \text{in } \mathbb{R}^2. \tag{4.1}$$

It is obvious to see that the Euler-Lagrange functional of (4.1) is defined by

$$I_{V,\lambda}(u) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + V(\infty)u^2)\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y - \frac{\lambda}{p}\int_{\mathbb{R}^2}|u|^p\,\mathrm{d}x.$$

Next we show that $I_{V,\lambda}$ has the mountain pass geometry denoted by $c_{V,\lambda}$.

Lemma 4.1. *If* (V1) *and* (V2) *hold,* $p \geq 3$*, then*

(i) *there exists a* $v \in X \setminus \{0\}$*, such that* $I_{V,\lambda}(v) < 0$ *for all* $\lambda \in [1/2, 1]$*;*

(ii) *for all* $\lambda \in [1/2, 1]$, $c_{V,\lambda} = \inf_{\gamma\in\Gamma}\max_{s\in[0,1]} I_{V,\lambda}(\gamma(s)) > \max\{I_{V,\lambda}(0), I_{V,\lambda}(v)\}$*, where* $\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0, \gamma(1) = v\}$.

Proof. From (V2) one has

$$I_{V,\lambda}(u) \leq I_{V,1/2}(u) = \frac{1}{2}\int_{\mathbb{R}^2}(|\nabla u|^2 + V(\infty)u^2)\,\mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y - \frac{1}{2p}\int_{\mathbb{R}^2}|u|^p\,\mathrm{d}x.$$

For $u \in X \setminus \{0\}$, set $u_t(x) = t^2u(tx)$, then

$$\begin{aligned} I_{\infty,1/2}(u_t) &= \frac{t^4}{2}\int_{\mathbb{R}^2}|\nabla u|^2\,\mathrm{d}x + \frac{t^2}{2}\int_{\mathbb{R}^2}V(\infty)u^2)\,\mathrm{d}x \\ &\quad + \frac{t^4}{4}\int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y \\ &\quad - \frac{t^4\log t}{4}\left(\int_{\mathbb{R}^2}|u|^2\,\mathrm{d}x\right)^2 - \frac{2p-2}{p}\int_{\mathbb{R}^2}|u|^p\,\mathrm{d}x \\ &\to -\infty \end{aligned}$$

as $t \to \infty$. Choosing $v = t^2 u(tx)$ for t large, we derive $I_{V,\lambda}(v) \le I_{\infty,1/2}(v) < 0$, which proves (i).

(ii) For all $u \in X$, by (2.2) and Sobolev embeddings, we obtain

$$\begin{aligned} I(u) &\ge \frac{\|u\|^2}{2} - \frac{V_2(u)}{4} - \frac{\lambda}{p}\int_{\mathbb{R}^2} |u|^p \,\mathrm{d}x \ge \frac{\|u\|^2}{2} - \frac{C_0}{4}|u|_{8/3}^4 - \frac{\lambda}{p}|u|_p^p \\ &\ge \|u\|^2(1 - C\|u\|^2 - C\|u\|^{p-2}), \end{aligned}$$

from which we infer that $I_{V,\lambda}(u)$ has a local minimum and $c_{V,\lambda} > 0$, which proves (ii). □

In the following we introduce the manifold

$$\mathscr{M}_{\infty,\lambda} = \{u \in X \setminus \{0\} : J_{\infty,\lambda}(u) = 0\},$$

where

$$\begin{aligned} J_{\infty,\lambda}(u) = &\int_{\mathbb{R}^2} (2|\nabla u|^2 + V(\infty)u^2)\,\mathrm{d}x + 2\int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|)u^2(x)u^2(y)\,\mathrm{d}x\,\mathrm{d}y \\ &- \frac{1}{4}\left(\int_{\mathbb{R}^2} |u|^2\,\mathrm{d}x\right)^2 - \frac{2(p-1)\lambda}{p}\int_{\mathbb{R}^2} |u|^p\,\mathrm{d}x. \end{aligned}$$

Set $m_{\infty,\lambda} := \inf_{u\in\mathscr{M}_{\infty,\lambda}} I_{\infty,\lambda}(u)$. Then $m_{\infty,\lambda}$ has some similar properties to those of the manifold $\mathscr{M}$, for example containing all the nontrivial critical points of $I_{\infty,\lambda}$ and the similar results to Lemma 3.1. Also, we have the following lemma.

Lemma 4.2. *If $p \ge 3$, $\lambda \in [1/2, 1]$, then $c_{\infty,\lambda}$ is attained at some $u_{\infty,\lambda} \in \mathscr{M}_{\infty,\lambda}$. Furthermore, $I'_{\infty,\lambda}(u_{\infty,\lambda}) = 0$ and $I_{\infty,\lambda}(u_{\infty,\lambda}) = c_{\infty,\lambda} = \inf\{I_{\infty,\lambda}(u) : u \ne 0, I'_{\infty,\lambda}(u_{\infty,\lambda}) = 0\}$.*

The proof is standard to that of Theorem 1.2. Here we omit its proof.

Lemma 4.3. *If $p \ge 3$ and (V1)–(V2) hold, then $c_{V,\lambda} < c_{\infty,\lambda}$ for any $\lambda \in [1/2, 1]$.*

Proof. Suppose that $V(x) \ne V(\infty)$. It follows from Lemma 3.1 that

$$I_{\infty,\lambda}(u_{\infty,\lambda}) = \max_{t>0} I_{\infty,\lambda}(t^2 u(tx)),$$

where $u_{\infty,\lambda}$ is the minimizer of $c_{\infty,\lambda}$. Hence by choosing $v = t^2 u_{\infty,\lambda}(tx)$ for t large enough in Lemma 4.1 we derive

$$c_{V,\lambda} \le \max_{t>0} I_{V,\lambda}(t^2 u_{\infty,\lambda}(tx)) < \max_{t>0} I_{\infty,\lambda}(t^2 u_{\infty,\lambda}(tx)) = c_{\infty,\lambda}.$$ □

To show that $I_{V,\lambda}$ satisfies $(PS)_{c_{V,\lambda}}$ for a.e. $\lambda \in [1/2, 1]$, we need to prove the following global compactness lemma.

Lemma 4.4. *Assume that* (V1)–(V2) *hold and* $p \geq 3$. *For any* $\lambda \in [1/2, 1]$, *let* $\{u_n\}$ *be a bounded* $(PS)_{c_{V,\lambda}}$ *sequence for* $I_{V,\lambda}$. *Then there exists a subsequence of* $\{u_n\}$, *still denoted by* $\{u_n\}$, u_0 *and integer* $\eta \in \mathbb{N} \cap \{0\}$, *sequence* $\{y_n^j\}$, $\{w^j\} \subset X$ *for* $1 \leq j \leq n$ *such that*

(i) $u_n \rightharpoonup u_0$ *with* $I'_{V,\lambda}(u_0) = 0$;

(ii) $|y_n^j| \to +\infty$, $|y_n^j - y_n^i| \to +\infty$ *if* $i \neq j$, $n \to +\infty$;

(iii) $w^j \neq 0$ *and* $I'_{\infty,\lambda}(w^j) = 0$ *for* $1 \leq j \leq \eta$;

(iv) $\left\| u_n - u_0 - \sum_{j=1}^{\eta} w^j(\cdot - y_n^j) \right\| \to 0$;

(v) $I_{V,\lambda}(u_n) \to I_{V,\lambda}(u_0) + \sum_{j=1}^{n} I_{\infty,\lambda}(w^j)$.

Also the case $\eta = 0$ *holds without* w^j *and* $\{y_n^j\}$.

Proof. The proof falls naturally into three steps.

Step 1. It follows from the boundedness of $\{u_n\}$ that, up to a subsequence, there exists u_0 such that $u_n \rightharpoonup u_0$ in X, $u_n \to u_0$ in $L^r(\mathbb{R}^2)$ for $r \geq 2$ and $u_n \to u_0$ a.e. in $\mathbb{R}^2$. We now prove that $I'_{V\lambda}(u_0) = 0$. Indeed, it suffices to prove that $\langle I'_{V\lambda}(u_0), v\rangle = 0$ for any fixed $v \in C_0^\infty(\mathbb{R}^2)$. Then, from Hölder inequality, we obtain that

$$\begin{aligned}
&\left| \int_{\mathbb{R}^2} (|u_n|^{q-2}u_n - |u_0|^{q-2}u_0)v \,\mathrm{d}x \right| \\
&\leq \int_{\mathbb{R}^2} (|u_n|^{q-2}|u_n - u_0|)v \,\mathrm{d}x + \int_{\mathbb{R}^2} (|u_n|^{q-2} - |u_0|^{q-2})|u_0 v| \,\mathrm{d}x \\
&\leq |v|_\infty |u_n|_{q-1}^{q-2} \left(\int_{\operatorname{supp} v} |u_n - u_0|^{q-1} \,\mathrm{d}x \right)^{\frac{1}{q-1}} + C|v|_\infty \left(\int_{\operatorname{supp} v} |u_n - u_0|^{q-1} \,\mathrm{d}x \right)^{\frac{q-2}{q-1}} \\
&\to 0 \quad \text{as } n \to \infty.
\end{aligned}$$

Since $u_n \rightharpoonup u_0$ in X, we derive

$$\langle u_n - u_0, v\rangle \to 0 \quad \text{as } n \to \infty.$$

Noting that $\{|u_n|_*\}$ and $\{|u_n|_2\}$ are bounded, we have

$$\begin{aligned}
&\int_{\mathbb{R}^2} (1 + |y|)|u_0||u_n - u_0| \,\mathrm{d}x \\
&\leq \log(1 + |R|)|u_0|_2 |u_n - u_0|_2 + |u_n - u_0|_* \left[\int_{\mathbb{R}^2 \setminus B_R(0)} \log(1 + |y|) u_0^2 \,\mathrm{d}x \right]^{1/2} \\
&= o_n(1) + o_R(1) \quad \text{as } n \to \infty,\ R \to \infty,
\end{aligned}$$

which means that

$$\int_{\mathbb{R}^2} (1 + |y|)|u_0||u_n - u_0| \,\mathrm{d}x = o_n(1).$$

It follows from $|u_n-u_0|_2 \to 0$ that

$$\begin{aligned}T_1(u_n^2, u_0(u_n-u_0)) &\le \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(1+|x|)\log(1+|y|)u_n^2(x)|u_0(y)||u_n(y)-u_0(y)|\,\mathrm{d}x\,\mathrm{d}y \\ &\le |u_n|_*^2|u_0|_2|u_n-u_0|_2 + |u_n|_2^2\int_{\mathbb{R}^2}\log(1+|y|)|u_0||u_n-u_0|\,\mathrm{d}y \\ &= o_n(1).\end{aligned}$$

Similarly, we have

$$T_1(u_0^2, u_0(u_n-u_0)) = o_n(1).$$

By virtue of (2.3), one has

$$T_2(u_n^2, u_0(u_n-u_0)) = o_n(1), \quad T_2(u_0^2, u_0(u_n-u_0)) = o_n(1).$$

It follows from the above inequalities that

$$\begin{aligned}&\langle I'_{V,\lambda}(u_n), v\rangle - \langle I'_{V,\lambda}(u_0), v\rangle \\ &= \langle u_n-u_0, v\rangle + T_1(u_n^2, (u_n-u_0)^2) + T_1(u_n^2, u_0(u_n-u_0)) \\ &\quad - T_1(u_0, u_0(u_n-u_0)) - T_2(u_n^2, u_n(u_n-u_0)) + T_2(u_0^2, u_0(u_n-u_0)) \\ &\quad - \lambda\int_{\mathbb{R}^2}(|u_n|^{p-2}u_n - |u_0|^{p-2}u_0)v\,\mathrm{d}x \\ &\to 0,\end{aligned}$$

which means that $I'_{V,\lambda}(u_0)=0$.

Step 2. We now prove that $I_{V,\lambda}(u_0) \ge 0$. Define

$$\alpha_1 = \int_{\mathbb{R}^2}|\nabla u_0|^2\,\mathrm{d}x, \quad \beta_1 = \int_{\mathbb{R}^2}V(x)|u_0|^2\,\mathrm{d}x, \quad \overline{\beta}_1 = \int_{\mathbb{R}^2}\langle\nabla V(x), x\rangle|u_0|^2\,\mathrm{d}x,$$
$$V_{0,1} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2}\log(|x-y|)u_0^2(x)u_0^2(y)\,\mathrm{d}x\,\mathrm{d}y, \quad \theta_1 = \int_{\mathbb{R}^2}|u_0|^2\,\mathrm{d}x, \quad \delta_1 = \int_{\mathbb{R}^2}|u_0|^p\,\mathrm{d}x.$$

Thus, from the definition of $I_{V,\lambda}(u_0)$, Lemma 2.2 and $\langle I_{V,\lambda}(u_0), u_0\rangle = 0$, we have

$$\begin{cases}\frac{1}{2}\alpha_1 + \frac{1}{2}\beta_1 + \frac{1}{4}V_{0,1} - \frac{\lambda}{p}\delta_1 = I_{V,\lambda}(u_0), \\ \beta_1 + \frac{1}{2}\overline{\beta}_1 + V_{0,1} + \frac{1}{4}\theta_1^2 - \frac{2\lambda}{p}\delta = 0, \\ \alpha_1 + \beta_1 + V_{0,1} - \lambda\delta_1 = 0.\end{cases}$$

From these relations, we obtain that

$$I_{V,\lambda}(u_0) = \frac{1}{2}\left(1-\frac{1}{p-2}\right)\alpha_1 + \frac{1}{4}\beta_1 + \frac{1}{p-2}\left(1-\frac{p}{8}\right)\overline{\beta}_1 + \frac{1}{4(p-2)}\left(1-\frac{p}{4}\right)\theta_1^2 \ge 0. \tag{4.2}$$

By (V1) we infer that $I_{V,\lambda}(u_0) \ge 0$.

Step 3. Set $v_n^1 = u_n - u_0$, $\mu = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{\mathbb{R}^2} |v_n^1|^2 \, \mathrm{d}x$. Then, we obtain $v_n^1 \rightharpoonup 0$ in X.

(i) *Vanishing.* If $\mu = 0$, then from Lemma 2.3 that $v_n^1 \to 0$ in $L^q(\mathbb{R}^2)$ for $q \geq 2$. Since $T_0(u_n^2, u_n(u_n - u_0)) \to 0$ and $T_0(u_0^2, u_0(u_n - u_0)) \to 0$ as $n \to \infty$, we have

$$\|v_n^1\|^2 = \langle I'_{V,\lambda}(u_n), v_n^1\rangle - \langle I'_{V,\lambda}(u_0), v_n^1\rangle \to 0,$$

which implies that $\|v_n^1\| \to 0$ as $n \to \infty$.

(ii) *Non-vanishing.* If $\mu > 0$, we can derive a sequence $\{y_n^1\} \subset \mathbb{R}^2$ such that

$$\int_{B_1(0)} |\widehat{v}_n^1|^2 \, \mathrm{d}x = \int_{B_1(y_n)} |\widehat{v}_n^1|^2 \, \mathrm{d}x > \frac{\mu}{2} > 0,$$

where $\widehat{v}_n^1 = v_n^1(\,\cdot + y_n^1)$. Since $\|\widehat{v}_n^1\| = \|v_n^1(\,\cdot + y_n^1)\|$, we have that $\{\widehat{v}_n^1\}$ is bounded. Passing to a subsequence if necessary, we obtain a $w^1 \in X$ such that $\widehat{v}_n^1 \rightharpoonup w^1$ in X, $\widehat{v}_n^1 \to w^1$ in $L^q(\mathbb{R}^2)$ and $\widehat{v}_n^1 \to w^1$ a.e. in $\mathbb{R}^2$. It follows from

$$\int_{B_1(0)} |\widehat{v}_n^1|^2 \, \mathrm{d}x > \frac{\mu}{2} > 0$$

that $w^1 \neq 0$. Furthermore, $v_n^1 \rightharpoonup 0$ in X means that $\{y_n^1\}$ must be unbounded. Consequently, we may suppose that $|y_n^1| \to +\infty$. Now we show that $I'_{\infty,\lambda}(w^1) = 0$. Similar to that in the proof of step 1, for any fixed $\eta \in C_0^\infty(\mathbb{R}^2)$, it suffices to prove that $\langle I'_{\infty,\lambda}(\widehat{v}_n), \eta\rangle \to 0$. It follows from (V2) and $|y_n^1| \to +\infty$, we have

$$\int_{\mathbb{R}^2} (V(x + y_n^1) - V(\infty))\widehat{v}_n^1 \eta \, \mathrm{d}x \to 0.$$

Noting that $v_n^1 \rightharpoonup 0$ in X, it follows that

$$\langle I'_{V,\lambda}(\widehat{v}_n^1), \eta(\,\cdot - y_n^1)\rangle \to 0, \tag{4.3}$$

from which we have

$$\int_{\mathbb{R}^2} \big(\nabla \widehat{v}_n^1 \nabla \eta + V(x + y_n^1)\widehat{v}_n^1 \eta\big) \, \mathrm{d}x + T_0((\widehat{v}_n^1)^2, \widehat{v}_n^1 \eta) - \lambda \int_{\mathbb{R}^2} |\widehat{v}_n^1|^{p-2} \widehat{v}_n^1 \eta \, \mathrm{d}x \to 0 \tag{4.4}$$

as $n \to \infty$. Thus, from (4.3) and (4.4), we derive $\langle I'_{\infty,\lambda}(\widehat{v}_n^1), \eta\rangle \to 0$. Hence $I'_{\infty,\lambda}(w^1) > 0$. We now prove that

$$I_{V,\lambda}(u_n) - I_{V,\lambda}(u_0) - I_{\infty,\lambda}(u_n - u_0) \to 0 \quad \text{as } n \to \infty. \tag{4.5}$$

Indeed, it follows from Brezis–Lieb lemma that $V_0(u_n) - V_0(u_0) = V_0(u_n - u_0)$, which means that

$$T_0(u_n^2, u_n^2) - T_0(u_0^2, u_0^2) = T_0((u_n - u_0)^2, (u_n - u_0)^2) + o_n(1). \tag{4.6}$$

Also, we have

$$\|v_n^1\|^2 = \|u_n\|^2 - \|u_0\|^2 + o_n(1), \quad |v_n^1|_p^p = |u_n|_p^p - |u_0|_p^p + o_n(1). \tag{4.7}$$

By virtue of (V2) and the Sobolev inequality, we derive

$$\int_{\mathbb{R}^2} (V(x) - V(\infty))|u_n - u_0|^2 \,\mathrm{d}x \to 0 \quad \text{as } n \to \infty. \tag{4.8}$$

Then

$$\begin{aligned}
&I_{V,\lambda}(u_n) - I_{V,\lambda}(u_0) - I_{\infty,\lambda}(u_n - u_0) \\
&= \frac{1}{2}\int_{\mathbb{R}^2} \left(|\nabla u_n|^2 - |\nabla u_0|^2 - |\nabla(u_n - u_0)|^2\right) \mathrm{d}x \\
&\quad + \frac{1}{2}\left[\int_{\mathbb{R}^2} V(x)(|u_n|^2 - |u_0|^2)\,\mathrm{d}x - \int_{\mathbb{R}^2} V(\infty)(|u_n - u_0|^2)\,\mathrm{d}x\right] \\
&\quad + T_0(u_n^2, u_n^2) - T_0(u_0^2, u_0^2) - T_0((u_n - u_0)^2, (u_n - u_0)^2) \\
&\quad - \frac{\lambda}{p}\int_{\mathbb{R}^2} (|u_n|^p - |u_0|^p - |u_n - u_0|^p)\,\mathrm{d}x.
\end{aligned}$$

It follows from (4.6)–(4.8) that (4.5) holds.

Step 4. Set $v_{n,2} = v_n^1 - w^1(\cdot - y_n)$, then $v_{n,2} \rightharpoonup 0$ in X. Using Brezis–Lieb Lemma again, we infer that

$$\begin{aligned}
|\nabla v_{n,2}|_2^2 &= |\nabla u_n|_2^2 - |\nabla u_0|_2^2 - |\nabla w^1(\cdot - y_n)|_2^2 + o_n(1), \\
|v_{n,2}|_p^p &= |u_n|_p^p - |u_0|_p^p - |w^1(\cdot - y_n)|_p^p + o_n(1), \\
T_0(v_{n,2}^2, v_{n,2}^2) &= T_0(u_n^2, u_n^2) - T_0(u_0^2, u_0^2) \\
&\quad - T_0((w^1(\cdot - y_n))^2, (w^1(\cdot - y_n))^2) + o_n(1), \\
\int_{\mathbb{R}^2} V(x)|v_{n,2}|^2\,\mathrm{d}x &= \int_{\mathbb{R}^2} V(x)|u_n|^2\,\mathrm{d}x - \int_{\mathbb{R}^2} V(x)|u_0|^2\,\mathrm{d}x \\
&\quad - \int_{\mathbb{R}^2} V(x)|w^1(\cdot - y_n)|^2\,\mathrm{d}x + o_n(1).
\end{aligned} \tag{4.9}$$

It follows from (4.9), we can similarly obtain that

$$\begin{aligned}
I_{V,\lambda}(v_{n,2}) &= I_{V,\lambda}(u_n) - I_{V,\lambda}(u_0) - I_{\infty,\lambda}(w^1) + o_n(1), \\
I_{\infty,\lambda}(v_{n,2}) &= I_{V,\lambda}(v_n^1) - I_{\infty,\lambda}(w^1) + o_n(1), \\
\langle I'_{V,\lambda}(v_{n,2}), v_{n,2}\rangle &= \langle I'_{\infty,\lambda}(u_n), u_n\rangle - \langle I'_{V,\lambda}(u_0), u_0\rangle - \langle I'_{\infty,\lambda}(w^1), w^1\rangle = o_n(1).
\end{aligned} \tag{4.10}$$

By (4.5) and (4.10) we have

$$I_{V,\lambda}(u_n) = I_{V,\lambda}(u_0) + I_{\infty,\lambda}(v_n^1) + o_n(1) = I_{V,\lambda}(u_0) + I_{\infty,\lambda}(v_{n,2}) + I_{\infty,\lambda}(w^1) + o_n(1).$$

It follows from Lemma 3.1(ii) that $I_{\infty,\lambda}(w^1) \geq 0$.

Combining (4.2) one has

$$I_{V,\lambda}(v_{n,2}) = c_{V,\lambda} - I_{V,\lambda}(u_0) - I_{\infty,\lambda}(w^1) + o_n(1) \le c_{V,\lambda}.$$

Set $\mu_1 = \lim_{n\to\infty} \sup_{y\in\mathbb{R}^2} \int_{\mathbb{R}^2} |v_{n,2}|^2\,\mathrm{d}x$. In the same manner as Step 3, if the vanishing case occurs, then $\|v_{n,2}\| \to 0$ in X. Hence, Lemma 4.4 holds with $j = 1$. If $\{v_{n,2}\}$ is non-vanishing, then there is a sequence $\{y_{n,2}\}$ and $w^2 \in X$ such that $\widehat{v}_{n,2} = v_{n,2}(\,\cdot + y_{n,2}) \rightharpoonup w^2$ in X and $I_{\infty,\lambda}(w^2) = 0$. Moreover, $v_{n,2} \rightharpoonup 0$ in X implies that $|y_{n,2}| \to +\infty$ and $|y_{n,1} - y_{n,2}| \to +\infty$. Iterating this techniques, we derive $v_{n,j} = v_{n,j-1} - w^{j-1}$ with $j \ge 1$ such that

$$v_{n,j} \to w^j, \quad I'_{\infty,\lambda}(w^j) = 0,$$

and there exist sequences $\{y_{n,j}\} \subset \mathbb{R}^2$ such that $|y_{n,2}| \to +\infty$ and $|y_{n,i} - y_{n,j}|$ if $i \ne j$ as $n \to \infty$. Employing the properties of the weak convergence, we derive

$$\begin{aligned} &\|u_n\|^2 - \|u_0\|^2 - \sum_{k=1}^{j-1} \|w_k(\,\cdot - y_{n,k})\|^2 = \Big\| u_n - u_0 - \sum_{k=1}^{j-1} w_k(\,\cdot - y_{n,k}) \Big\| + o_n(1), \\ &I_{V,\lambda}(u_n) \to I_{V,\lambda}(u_0) + \sum_{k=1}^{j-1} I_{\infty,\lambda}(w^{k-1}) + I_{\infty,\lambda}(v_{n,j}). \end{aligned} \tag{4.11}$$

Recalling that $\{u_n\}$ is bounded in X, (4.11) means that the iteration stops at some finite index $\theta + 1$. This deduces that $v_{n,\theta+1} \to 0$ in X. Then we can verify that (iv) and (v) hold by (4.11). □

Lemma 4.5. *Assume that* (V1)–(V2) *hold. Let* $\{u_n\}$ *be a bounded* $(PS)_{c_{V,\lambda}}$ *of* $I_{V,\lambda}$. *Then there exists a nontrivial* $u_{V,\lambda} \in X$ *such that* $I'_{V,\lambda}(u_{V,\lambda}) = 0$ *and* $I_{V,\lambda}(u_{V,\lambda}) = c_{V,\lambda}$ *for a.e.* $\lambda \in [1/2, 1]$.

Proof. For $\lambda \in [1/2, 1]$, let $u_{\infty,\lambda}$ be the minimizer of $c_{\infty,\lambda}$. From Lemma 4.3 we derive

$$c_{V,\lambda} < c_{\infty,\lambda}. \tag{4.12}$$

Lemma 4.4 deduces that there exist $u_{V,\lambda}$ and integer $\theta \in \mathbb{N} \cap \{0\}$, sequence $\{y_{n,j}\}, w^j \subset X$ for $1 \le j \le \theta$ such that

$$I'_{V,\lambda}(u_{V,\lambda}) = 0, \quad u_n \rightharpoonup u_{V,\lambda}, \quad I_{V,\lambda}(u_n) \to I_{V,\lambda}(u_{V,\lambda}) + \sum_{j=1}^{\theta} I_{\infty,\lambda}(w^j), \tag{4.13}$$

where w^j is the critical point of $I_{\infty,\lambda}$. Define

$$\begin{aligned} &\alpha_2 = \int_{\mathbb{R}^2} |\nabla u_{V,\lambda}|^2\,\mathrm{d}x, \quad \beta_2 = \int_{\mathbb{R}^2} V(x)|u_{V,\lambda}|^2\,\mathrm{d}x, \quad \overline{\beta}_2 = \int_{\mathbb{R}^2} \langle \nabla V(x), x\rangle |u_{V,\lambda}|^2\,\mathrm{d}x, \\ &V_{0,2} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|) u_{V,\lambda}^2(x) u_{V,\lambda}^2(y)\,\mathrm{d}x\,\mathrm{d}y, \quad \theta_2 = \int_{\mathbb{R}^2} |u_{V,\lambda}|^2\,\mathrm{d}x, \quad \delta_2 = \int_{\mathbb{R}^2} |u_{V,\lambda}|^p\,\mathrm{d}x. \end{aligned} \tag{4.14}$$

Then, it follows from the definition of $I_{V,\lambda}(u_{V,\lambda})$, Lemma 2.3 and $\langle I_{V,\lambda}(u_{V,\lambda}), u_{V,\lambda}\rangle = 0$ that

$$\begin{cases} \frac{1}{2}\alpha_2 + \frac{1}{2}\beta_2 + \frac{1}{4}V_{0,2} - \frac{\lambda}{p}\delta_2 = I_{V,\lambda}(u_{V,\lambda}), \\ \beta_2 + \frac{1}{2}\overline{\beta}_2 + V_{0,2} + \frac{1}{4}\theta_2^2 - \frac{2\lambda}{p}\delta_2 = 0, \\ \alpha_2 + \beta_2 + V_{0,2} - \lambda\delta_2 = 0. \end{cases}$$

Similar arguments apply to (4.2), we derive $I_{V,\lambda}(u_{V,\lambda}) \geq 0$. If $\theta \neq 0$, then from (4.13) we have

$$c_{V,\lambda} = I_{V,\lambda}(u_{V,\lambda}) + \sum_{j=1}^{\theta} I_{\infty,\lambda}(w^j) \geq c_{\infty,\lambda},$$

which is a contradiction to (4.12). Thus $\theta = 0$, which leads that $u_n \to u_{V,\lambda}$ in X and $I_{V,\lambda}(u_{V,\lambda}) = c_{V,\lambda}$. □

Proof of Theorem 1.3. We divided the proof into two steps.

Step 1. By Proposition 2.4 and Lemma 4.1 that for a.e. $\lambda \in [1/2, 1]$ there is a bounded $(PS)_{c_{V,\lambda}}$ sequence of $I_{V,\lambda}$. Lemma 4.5 means that there exists a nontrivial critical point $u_{V,\lambda} \in X$ for $I_{V,\lambda}$ and $I_{V,\lambda}(u_{V,\lambda}) = c_{V,\lambda}$. Let $\lambda_n \to 1$ such that I_{V,λ_n} has a critical point u_{V,λ_n}, still denoted by $\{u_n\}$. We now prove that $\{u_n\}$ is bounded in X. Similar to (4.14), we set

$$\alpha_n = \int_{\mathbb{R}^2} |\nabla u_n|^2 \,\mathrm{d}x, \quad \beta_n = \int_{\mathbb{R}^2}, V(x)|u_n|^2 \,\mathrm{d}x, \quad \overline{\beta}_n = \int_{\mathbb{R}^2} \langle \nabla V(x), x\rangle |u_n|^2 \,\mathrm{d}x,$$
$$V_{0,n} = \int_{\mathbb{R}^2}\int_{\mathbb{R}^2} \log(|x-y|) u_n^2(x) u_n^2(y) \,\mathrm{d}x\,\mathrm{d}y, \quad \theta_n = \int_{\mathbb{R}^2} |u_n|^2 \,\mathrm{d}x, \quad \delta_n = \int_{\mathbb{R}^2} |u_n|^p \,\mathrm{d}x.$$

Then

$$\begin{cases} \frac{1}{2}\alpha_n + \frac{1}{2}\beta_n + \frac{1}{4}V_{0,n} - \frac{\lambda_n}{p}\delta_n = c_{V,\lambda_n}, \\ \beta_n + \frac{1}{2}\overline{\beta}_n + V_{0,n} + \frac{1}{4}\theta_n^2 - \frac{2\lambda_n}{p}\delta_n = 0, \\ \alpha_n + \beta_n + V_{0,n} - \lambda_n\delta_n = 0. \end{cases}$$

The same reasoning applies to (4.2), we get that $\alpha_n + \beta_n + \theta_n$ is bounded, which deduces that $\{u_n\}$ is bounded in E. It follows from (2.2) and Lemma 2.3 that $C|u_n|_2^2|u_n|_*^2 \leq V_1(u_n) \leq C\|u_n\|^4$. Then $C|u_n|_2^2|u|_*^2 \leq -(\alpha_n + \beta_n) + \lambda_n\delta_n + C\|u_n\|^4$, from which we infer that $\{u_n\}$ is bounded in X. Thus, by the fact that $\lambda \to c_{V,\lambda}$ is left-continuous (see Proposition 2.4), we derive

$$\lim_{n\to\infty} I_V(u_n) = \lim_{n\to\infty} \left(I_{V,\lambda_n}(u_n) + \frac{\lambda_n - 1}{p} \int_{\mathbb{R}^2} |u_n|^p \,\mathrm{d}x \right) = \lim_{n\to\infty} c_{V,\lambda_n} = c_{V,1} \tag{4.15}$$

and

$$\lim_{n\to\infty} \langle I'_{V,\lambda_n}(u_n), v\rangle = \lim_{n\to\infty} \left(\langle I'_V(u_n), v\rangle + (\lambda_n - 1) \int_{\mathbb{R}^2} |u_n|^p \,\mathrm{d}x \right) = 0. \tag{4.16}$$

It follows from (4.15) and (4.16) that $\{u_n\}$ is a bounded $(PS)_{c_{V,1}}$ sequence for $I_V := I_{V,1}$. Hence, from Lemma 4.5, one has a nontrivial critical point $u_0 \in X$ for $I_V(u_0) = c_{V,1}$.

Step 2. we show that problem (1.5) has a ground state solution. Set

$$c_V := \inf\{I_V(u) : u \neq 0, I'_V(u) = 0\}.$$

As in the proof of Step 2 in Lemma 4.4, we can infer that every critical point of I_V has a nonnegative energy. Consequently, $0 \leq c_V \leq I_V(u_0) < c_{V,1} < +\infty$. Let $\{u_n\}$ be a sequence of nontrivial critical points of I_V satisfying $I_V(u_n) \to c_V$. Recalling that $I_V(u_n)$ is bounded, we can conclude that $\{u_n\}$ is a bounded $(PS)_{c_V}$ sequence of I_V. Using the same methods as in Lemma 4.5, there exits a nontrivial $u_* \in X$ such that $I_V(u_*) = c_V$. The proof is completed. □

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