TAIWANESE JOURNAL OF MATHEMATICS
Vol. xx, No. x, pp. 1–25, xx 20xx
DOI: 10.11650/tjm/241001

Algebraic Gromov Ellipticity: A Brief Survey

Mikhail Zaidenberg

Abstract. We survey on algebraically elliptic varieties in the sense of Gromov.

1. Introduction

Gromov ellipticity is often considered to be a chapter of complex analysis. However, in his foundational paper [42] Gromov also presented and studied an algebraic counterpart of this notion. In this survey, we mainly focus on the algebraic part. All algebraic varieties considered below are defined over an algebraically closed field $\mathbb{K}$ with $\operatorname{char}\mathbb{K} = 0$. In what follows by *Gromov spray* and *Gromov ellipticity* we mean *algebraic Gromov spray* and *algebraic Gromov ellipticity*; otherwise we talk about *analytic Gromov spray* and *analytic Gromov ellipticity.* We address the comprehensive monograph [35] and the survey article [37] by F. Forstnerič for a thorough introduction to Gromov ellipticity, especially for its complex analytic counterpart.

1.1. Prehistory: the Oka–Grauert principle

The origin of Gromov ellipticity in complex analysis lies in the following Oka–Grauert principle; see [35, Sec. 5.1] for a historical account.

Theorem 1.1. (see Oka [66], Frenkel [38], Grauert [40,41], H. Cartan [18], Ramspott [75], Henkin–Leiterer [43]) *Given a complex Lie group G, the classifications of principal G-fiber bundles over a Stein complex manifold S in topological and holomorphic categories coincide. The same holds for the classes up to homotopy of sections of the associated fiber bundles with G-homogeneous fibers.*

Among important properties of a complex Lie group G linked to the Oka–Grauert principle, we distinguish the following.

Theorem 1.2. (Grauert [40], Gromov [42, Sec. 1.4D$'$]) *For a complex Lie group G and a complex Stein manifold S, the following hold.*

(A) *Every continuous map $S \to G$ is homotopic to a holomorphic map $S \to G$.*

(B) *Every holomorphic map $\overline{D} \to G$, where $D \subset \mathbb{C}^n$ is a bounded convex domain, can be approximated by holomorphic maps $\mathbb{C}^n \to G$ uniformly on $\overline{D}$.*

Received September 11, 2024; Accepted September 30, 2024.
Communicated by Ivan Cheltsov.
2020 *Mathematics Subject Classification.* Primary: 14J60, 14M25, 14M27; Secondary: 32Q56.
Key words and phrases. Gromov ellipticity, spray, unirationality, uniformly rational variety.

1.2. Analytic Gromov ellipticity

Loosely speaking, in his paper [42] Gromov answers the following question.

Question 1.3. What do you need to know about a complex manifold X to be sure that the analogues of (A) and (B) hold for holomorphic maps $S \to X$?

The following answer is a manifestation of Gromov's h-principle for complex manifolds (see Subsection 1.5 below):

Every analytically elliptic complex manifold X verifies analogues of (A) *and* (B).

In the following two subsections, we give the definitions of Gromov spray and Gromov ellipticity.

1.3. Gromov spray

Definitions 1.4. (Gromov [42]) Let X be a complex manifold. A *spray* on X is a triple (E, p, s), where

- $p\colon E \to X$ is a holomorphic vector bundle on X with zero section Z, and
- $s\colon E \to X$ is a holomorphic map such that $s|_Z = p|_Z$, i.e., $s(0_x) = x$ for all $x \in X$, where 0_x is the origin of the vector space $E_x = p^{-1}(x)$.

A spray (E, p, s) on X is called *dominating* if for every $x \in X$ the differential $ds\colon T_{0_x}E_x \to T_xX$ is onto.

1.4. Ellipticity and subellipticity

Definition 1.5. (cf. Gromov [42, Sec. 0.5 and 3.5B])

(1) X is called *analytically elliptic* if it admits a dominating spray (E, p, s).

(2) X is called *locally analytically elliptic* if it admits an open covering $(U_i)_i$ with dominating sprays (E_i, p_i, s_i) on U_i with values in X, i.e., s_i is a holomorphic map $U_i \to X$.

Definition 1.6. (Forstnerič [31, Definition 2]) X is called *analytically subelliptic* if it admits a dominating family of sprays (E_i, p_i, s_i) on X, i.e., for each $x \in X$ we have

$$T_xX = \operatorname{span}\left(\bigcup_i ds_i(T_{0_x}E_{i,x})\right). \tag{1.1}$$

1.5. Gromov's h-principle for complex manifolds

Definition 1.7. (Gromov [42, Sec. 0]) Let X and Y be complex manifolds. We say that holomorphic maps $Y \to X$ satisfy the h-principle (h for homotopy) if every continuous map $Y \to X$ is homotopic to a holomorphic map.

Theorem 1.8. (Gromov [42, Sec. 1.4D′], Forstnerič [33, Theorem 1.2]) *Let S be a Stein complex space and X be a complex manifold. If X is analytically elliptic, then the following hold.*

(A′) *Every continuous map $S \to X$ is homotopic to a holomorphic one; the same holds for sections of holomorphic fiber bundles over S with fiber X.*

(B′) *Every holomorphic map $f\colon \overline{D} \to X$, where $D \subset \mathbb{C}^n$ is a bounded convex domain, can be uniformly on $\overline{D}$ approximated by holomorphic maps $\mathbb{C}^n \to X$.*

A complex manifold X verifying (B′) is called an *Oka manifold*, see [32, 34] and [35, Definition 5.4.1]. An Oka manifold verifies a stronger *convex approximation property* (CAP, for short), see [35, Theorem 5.4.4]. Any analytically elliptic complex manifold is Oka, it verifies the CAP and a condition Ell_1 of Gromov, see [35, Proposition 8.8.11]. Actually, the Oka property is equivalent to the condition Ell_1 (Kusakabe [56, Theorem 1.3]) and does not imply analytic ellipticity, in general, see Kusakabe [55] and [60, Corollary 1.5.] for corresponding examples.

1.6. Algebraic Gromov ellipticity

As we have already mentioned, Gromov [42, Sec. 3.5A] also introduced the notions of algebraic spray, algebraic ellipticity, etc., where complex manifolds are replaced by smooth algebraic varieties (i.e., algebraic manifolds) defined over $\mathbb{K}$, holomorphic vector bundles by algebraic vector bundles and holomorphic maps by regular maps. In the algebraic category, we have the following equivalences.

Theorem 1.9. (Gromov [42, Sec. 3.5.B′], Kaliman–Zaidenberg [50, Theorem 1.1]) *For a smooth algebraic variety X, the following are equivalent:*

(1) *X is algebraically Gromov elliptic;*

(2) *X is locally algebraically Gromov elliptic;*

(3) *X is algebraically subelliptic.*

An analogue of (1) $\Leftrightarrow$ (3) in the analytic category is known to hold for complex Stein manifolds X, see [31, Lemma 2.2].

The proof of (2) $\Rightarrow$ (1) uses the Gromov's Localization Lemma [42, Sec. 3.5.B]; see also [35, Propositions 6.4.1–6.4.2] and [49, Proposition 8.1].

Lemma 1.10. *Let X be a smooth algebraic variety, D be a reduced effective divisor on X and (E,p,s) be a Gromov spray on $U = X \setminus \operatorname{supp}(D)$ with values in X such that $p\colon E \to U$ is a trivial vector bundle. Then there exists a spray $(\widetilde{E},\widetilde{p},\widetilde{s})$ on X whose restriction to U is isomorphic to (E,p,s). In particular, if (E,p,s) is dominating on U, then so is $(\widetilde{E},\widetilde{p},\widetilde{s})|_U$.*

Yet another important ingredient in the proofs of (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (1) is the composition of Gromov sprays, see Gromov [42, Sec. 1.3.B]. Given two sprays (E_1,p_2,s_1) and (E_2,p_2,s_2) on X, we consider their composition (E,p,s) where

- $E = \{(e_1,e_2) \in E_1 \times E_2 \mid s_1(e_1) = p_2(e_2)\}$, and so $\operatorname{pr}_1|_E\colon E = s_1^*E_2 \to E_1$ is the induced vector bundle;
- $p = p_1 \circ \operatorname{pr}_1\colon E \to X$;
- $s = s_2 \circ \operatorname{pr}_2\colon E \to X$.

In general, $p\colon E \to X$ is not a projection of a vector bundle. However, p happens to be such a projection provided that $p_2\colon E_2 \to X$ is a line bundle, see [50, Proposition 2.1].

If X is algebraically subelliptic, then we can find $m \ge \dim(X)$ rank 1 sprays (L_i,p_i,s_i) on X which satisfy an analogue of (1.1). The iterated composition of these sprays yields a dominating Gromov spray (E,p,s) on X of rank m and provides the ellipticity of X.

It is easily seen that the product of Gromov elliptic smooth algebraic varieties is Gromov elliptic. The converse is also true.

Lemma 1.11. (Lárusson's Lemma, see e.g. [5, Lemma 3.6]) *If the product $X_1 \times X_2$ of two smooth algebraic varieties is Gromov elliptic, then X_1 and X_2 are Gromov elliptic.*

Lemma 1.12. (Forstnerič [35, Proposition 6.4.10], Lárusson–Truong [62, Theorem 1 and Remark 2(a)]) *Let $\widetilde{X} \to X$ be a proper étale morphism of smooth complex algebraic varieties. If X is Gromov elliptic, then also $\widetilde{X}$ is.*

2. Flexible varieties

2.1. Flexibility versus Gromov ellipticity

Example 2.1. Let $\mathbb{G}_a = (\mathbb{K},+)$ be the additive group of the field $\mathbb{K}$, let X be a smooth algebraic $\mathbb{G}_a$-variety and $s\colon \mathbb{G}_a \times X \to X$ be the action morphism. Consider the trivial line bundle $p\colon L = \mathbb{G}_a \times X \to X$ of rank 1, where p is the second projection. Then (L,p,s) is a rank 1 spray on X dominating in directions of the one dimensional $\mathbb{G}_a$-orbits $s(E_x) = \mathbb{G}_a.x$. The latter means that $\operatorname{rank}(ds|_{T_{0_x}E_x}) = 1$ provided $\dim(\mathbb{G}_a.x) = 1$.

Definition 2.2. A smooth quasiaffine algebraic variety X is called *flexible* if there exists a collection $U_1, \ldots, U_k$ of $\mathbb{G}_a$-subgroups of $\mathrm{Aut}(X)$ such that for each $x \in X$ the velocity vectors of $U_1, \ldots, U_k$ at x span $T_x X$. It is said to be *locally flexible* is X admits a Zariski open covering by flexible quasiaffine varieties.

Given a flexible algebraic manifold X, consider rank 1 sprays (L_i, p_i, s_i) associated with the $\mathbb{G}_a$-subgroups $U_1, \ldots, U_k$ as above, see Example 2.1. The composition of these sprays provides the subellipticity of X. Together with Theorem 1.9 this leads to the following result, see Gromov [42, Sec. 0.5.B] and also [4, Appendix], [35, Proposition 5.6.22(C)] and [46, Theorem 3.1].

Proposition 2.3. *A locally flexible smooth algebraic variety is Gromov elliptic.*

Definition 2.4. One says that a smooth algebraic variety X is *stably flexible* if $X \times \mathbb{A}^k$ is flexible for some $k \geq 0$, and *locally stably flexible* if X admits an open covering by stably flexible affine charts.

If X admits a covering by copies of $\mathbb{A}^n$, then it is certainly locally flexible. Using Theorem 1.9, Proposition 2.3 and Lárusson's Lemma 1.11 one deduces the following strengthening of Proposition 2.3.

Theorem 2.5. (cf. [46, Corollary 3.2]) *Every locally stably flexible smooth algebraic variety is Gromov elliptic.*

Let us give examples of non-flexible, but stably flexible affine surfaces.

Example 2.6. [46, Example 0.4] Consider the smooth affine Danielewski surfaces S_k given in $\mathbb{C}^3$ by equations $\{x^k y - z^2 + 1 = 0\}$. The surface F_1 is flexible, while F_k with $k > 1$ is not. This follows e.g. from the description of the automorphism groups $\mathrm{Aut}(F_k)$, see Makar-Limanov [64]. For every $k \geq 2$ we have $F_k \times \mathbb{C} \cong F_1 \times \mathbb{C}$ (Danielewski [27, Theorem 1]). Since $F_1 \times \mathbb{C}$ is flexible, F_k with $k > 1$ is stably flexible, while being non-flexible. Due to Theorem 2.5, F_k is Gromov elliptic for every $k \geq 1$.

Another characteristic property of flexible varieties are their homogeneity properties, see [4, Theorem 1.1], [29, Theorem 1.1], and [1, Theorem 11].

Theorem 2.7. *Let X be a smooth quasiaffine variety of dimension ≥ 2 and $\mathrm{SAut}(X)$ be the subgroup of $\mathrm{Aut}(X)$ generated by all $\mathbb{G}_\mathrm{a}$-subgroups of $\mathrm{Aut}(X)$. Then X is flexible if and only if $\mathrm{SAut}(X)$ acts transitively on X, if and only if it acts m-transitively on X for any $m \geq 1$.*

Definition 2.8. (Bogomolov–Karzhemanov–Kuyumzhiyan [14, Definition 1.2]) An algebraic variety X is *birationally stably flexible* if the field extension $\mathbb{K}(X)(y_1, \ldots, y_n)$ admits a flexible model.

See [14, Theorems 2.1 and 2.2] for criteria of the birational stable flexibility.

Clearly, any stably rational variety is birationally stably flexible. On the other hand, a birationally stably flexible variety is unirational. There is the following conjecture.

Conjecture 2.9. [14, Conjecture 1.4] *Any unirational algebraic variety X is birationally stably flexible.*

2.2. Examples of flexible varieties and of Gromov elliptic varieties

(1) Let G be a connected complex Lie group with Lie algebra $\mathfrak{g}$, let $Y = G/H$ be a homogeneous manifold of G and $\exp\colon \mathfrak{g} \to G$ be the exponential map. Then the map $Y \times \mathfrak{g} \to Y$, $(y, v) \mapsto \exp(v)y$ is a dominating analytic spray on Y, see [37, Sec. 3.1]. Hence, the homogeneous space Y is analytically elliptic.

    In particular, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is elliptic in analytic sense, but is not elliptic in algebraic sense. The same holds for any smooth projective curve of genus 1.

(2) If G is a semisimple linear algebraic group, then G/H is flexible, see [4, Proposition 5.4], and so Gromov elliptic. In particular, every flag variety G/P, where $P \subset G$ is a parabolic subgroup, is Gromov elliptic.

(3) The affine space $\mathbb{A}^n$ is Gromov elliptic.

(4) Let X be an algebraic variety. If X admits a Zariski open covering (U_i) where $U_i \simeq \mathbb{A}^n$, then X is locally Gromov elliptic, hence Gromov elliptic by Theorem 1.9.

(5) Every smooth complete spherical variety X admits an open covering by affine spaces (Brion–Luna–Vust [15, Sec. 1.5, Corollaire]). By (4), X is Gromov elliptic. In particular, every smooth complete toric variety is Gromov elliptic. Moreover, a smooth toric variety with no torus factor is covered by affine spaces, hence is Gromov elliptic, cf. [5, Remark 4.7].

(6) A smooth hypersurface X in $\mathbb{A}^{n+2}$ given by equation

$$uv - p(x_1, \ldots, x_n) = 0,$$

    where $p \in \mathbb{K}[x_1, \ldots, x_n]$ is a nonconstant polynomial, is flexible, see [47, Theorem 5.1] and [4, Theorem 0.1], or alternatively [7, Theorem 0.2(3)]. So, X is Gromov elliptic (cf. Example 2.6).

(7) Every smooth complete rational surface S admits a covering by copies of $\mathbb{A}^2$. By (4), S is Gromov elliptic.

(8) A flexible smooth quasiaffine algebraic variety remains flexible after deleting a closed subvariety of codimension ≥ 2 (Flenner–Kaliman–Zaidenberg [29, Theorem 0.1]). Likewise, a locally flexible algebraic variety remains locally flexible after deleting a closed subvariety of codimension ≥ 2.

(9) Let X be a smooth algebraic variety covered by open charts isomorphic to $\mathbb{A}^n$ and $A \subset X$ be a closed algebraic subvariety of codimension ≥ 2. Then $X \setminus A$ is Gromov elliptic (see Forstnerič [35, Proposition 6.4.5]). Actually, this follows immediately from (8) due to Theorem 1.9 and Proposition 2.3. Cf. also Gromov [42, Sec. 3.5C] and Kusakabe [56] for stronger results.

(10) The configuration spaces of a flexible quasiaffine variety X of dimension $\dim(X) \geq 2$ are flexible (Kusakabe [60, Proposition 3.4]).

For more examples of flexible varieties, see [19, Sec. 4.1]. See also Theorem 5.4 below for examples of non-flexible Gromov elliptic quasiaffine varieties.

For the following classes of birationally stably flexible varieties, see [14, Sections 3.1–3.4].

Examples 2.10. (1) Let $G \subset \mathrm{PGL}(n+1, \mathbb{K})$ be a finite subgroup. Then the quotient $\mathbb{P}^n/G$ is birationally stably flexible.

(2) Let X be an algebraic variety. Assume that X carries a collection of distinct birational structures of $\mathbb{P}^{m_i}$-bundles, $\pi_i \colon X \to S_i$ such that the tangent spaces of generic fibers of π_i span the tangent space of X at the generic point. Then X is birationally stably flexible.

(3) Every smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$, $n \geq 2$, is birationally stably flexible. The same holds for quartic hypersurfaces $X \subset \mathbb{P}^{n+1}$, $n \geq 3$ that have a line of double singularities, and for smooth complete intersections of three quadrics in $\mathbb{P}^6$.

3. Properties of Gromov elliptic varieties

3.1. Approximation results

In the algebraic category we have the following analogue of the Oka property (B′).

Theorem 3.1. (Forstnerič [35, Corollary 6.15.2]) *Let X be a smooth complex algebraic variety and D be a bounded convex domain in $\mathbb{C}^n$. If X is Gromov elliptic, then any holomorphic map $\overline{D} \to X$ can be approximated by morphisms $\mathbb{C}^n \to X$ uniformly on $\overline{D}$.*

Remark 3.2. The approximation property in the conclusion of Theorem 3.1 is called the *algebraic convex approximation property*, abbreviated as *aCAP*. Thus, according to Theorem 3.1 Gromov ellipticity implies aCAP. We don't know if the converse is true.

According to Forstnerič [32, Theorem 1.1 and Corollary 1.2] and Lárusson–Truong [62, Theorem 1], the algebraic Gromov ellipticity is equivalent to two other important properties.

Definition 3.3. [57, Definition 1.1] A smooth algebraic variety X over $\mathbb{K}$ is called *algebraically Oka* (abbreviated as *aOka*, or *aEll*$_1$ after Gromov) if for each morphism $f\colon Y \to X$, where Y is an affine algebraic manifold, there exists a morphism $F\colon Y\times\mathbb{A}^N \to X$ such that $F|_{Y\times\{0\}} = f$ and $F|_{\{y\}\times\mathbb{A}^N}\colon \mathbb{A}^N \to X$ is a submersion at $0\in\mathbb{A}^N$ for each $y\in Y$, i.e.,

$$dF|_{T_{(y,0)}(\{y\}\times\mathbb{C}^m)}\colon T_{(y,0)}(\{y\}\times\mathbb{A}^m_{\mathbb{C}}) \to T_{f(y)}X$$

is onto for every $y\in Y$.

Theorem 3.4. (Lárusson–Truong [62, Theorem 1]) *For a smooth complex algebraic variety X, the following are equivalent.*

(1) *X is Gromov elliptic;*

(2) *X is aOka;*

(3) *given a morphism $f_0\colon Y\to X$ from a complex affine variety Y, a holomorphically convex compact subset $K\subset Y$, a subset $U\supset K$ of X open in the complex topology of X, and a homotopy of holomorphic maps $f_t\colon U\to X$, $t\in[0,1]$, there is a morphism $F\colon Y\times\mathbb{A}^1_{\mathbb{C}}\to X$ with $F(\,\cdot,0) = f_0$ and $F(\,\cdot,t)$ as close to f_t as desired, uniformly on K.*

In the analytic setup, the properties analogous to (1)–(3) of Theorem 3.4 are known to be equivalent provided that the complex manifold X is Stein, see e.g. [35, Sec. 5.5], [62] and the references therein.

Remark 3.5. In fact, the equivalence (1) $\Leftrightarrow$ (2) of Theorem 3.4 holds for algebraic varieties defined over a general algebraically closed field $\mathbb{K}$ of characteristic zero. This can be shown by inspecting the arguments in the proof. For the reader's convenience, we provide a proof of the equivalence (1) $\Leftrightarrow$ (2) in this generality.

Proof of Theorem 3.4. (1) $\Rightarrow$ (2). Assume that X is Gromov elliptic, and let (E,p,s) be a dominating spray over X. Let Y be an affine variety and $f\colon Y\to X$ be a morphism. Consider the pullback $(\widetilde{E},\widetilde{p},\widetilde{s}) = f^*(E,p,s)$, where $\widetilde{p}\colon \widetilde{E}\to Y$ is the induced vector bundle and $\widetilde{s} = s\circ f\colon \widetilde{E}\to X$. Since Y is affine, the vector bundle $\widetilde{p}\colon\widetilde{E}\to Y$ is globally generated, see Serre [77, Sec. 45, p. 238, Corollaire 1]. Let $\eta_1,\dots,\eta_m$ be global sections of $\widetilde{p}\colon\widetilde{E}\to Y$ that span every fiber $\widetilde{p}^{-1}(y)$, $y\in Y$. Then the morphism of vector bundles

$$\varphi\colon Y\times\mathbb{A}^m \to_Y \widetilde{E},\quad (y,(a_1,\dots,a_m))\mapsto \sum_{i=1}^m a_i\eta_i(y)$$

identical on the base Y yields a fiberwise linear surjection. It is easily seen that the conditions of Definition 3.3 are fulfilled for the morphism

$$F = \widetilde{s} \circ \varphi = s \circ f \circ \varphi \colon Y \times \mathbb{A}^m \to X.$$

Thus, X is aOka.

(2) ⇒ (1). Suppose X is aOka. Let $Y \subset X$ be an affine dense open subset, $f \colon Y \to X$ be the identical embedding, and $F \colon Y \times \mathbb{A}^m \to X$ be the extension of f that satisfies the conditions of Definition 3.3. Then (E, p, s) with $E = Y \times \mathbb{A}^m$, $p = \mathrm{pr}_1 \colon E \to Y$, and $s = F$ is a dominating spray over Y. It follows that X is locally Gromov elliptic, hence Gromov elliptic by Theorem 1.9. □

From Theorems 2.5 and 3.4 we deduce the following corollary, cf. [37, Theorem 6.2].

Corollary 3.6. *Every locally stably flexible smooth algebraic variety X over $\mathbb{K}$ is aOka.*

The following approximation result concerns a (not necessary complete) homogeneous spaces; cf. (3) in Theorem 3.4.

Theorem 3.7. (Bochnak–Kucharz [10, Theorem 1.1 and Corollaries 1.2–1.3]) *Let $X = G/H$ be a homogeneous space of a linear complex algebraic group G, Y be a complex affine algebraic manifold, and K be a holomorphically convex compact set in Y. Given a holomorphic map $f \colon K \to X$, the following conditions are equivalent:*

(1) *f can be uniformly approximated by regular maps $K \to X$.*

(2) *f is homotopic to a regular map $K \to X$.*

In particular, every null homotopic holomorphic map $K \to X$ can be approximated by regular maps $K \to X$.

By a *regular map* $K \to X$ one means the restriction to K of a morphism $U \to X$, where U is a Zariski open neighborhood of K in X.

It is worth mentioning also the following approximation theorem.

Theorem 3.8. (Demailly–Lempert–Shiffman [28, Theorem 1.1]) *Let X and Y be smooth algebraic varieties, where Y is affine and X is quasiprojective. Let $D \subset X$ be a Runge domain, i.e., every holomorphic function on D can be approximated, uniformly on compacts in D, by holomorphic functions on Y. Then every holomorphic map $D \to X$ can be approximated, uniformly on compacts in D, by Nash algebraic maps.*

Recall that a holomorphic map $f \colon U \to X$ from an open domain $U \subset Y$ is *Nash algebraic* if its graph $\Gamma(f) \subset U \times X$ is contained in a closed algebraic subset $Z \subset Y \times$

X of dimension $\dim(Z) = \dim(Y)$. In fact, the approximation in Theorem 3.8 can be accompanied by an interpolation on a fixed submanifold of Y, see [28, Theorem 1.1].

Finally, we address Kusakabe [57] for a complex algebraic version of Thom's jet transversality theorem and its applications.

3.2. Domination by affine spaces

According to Theorems 3.9 and 3.10 below, Gromov elliptic algebraic manifolds are dominated by affine spaces. We use the following notation. Given a surjective morphism $f\colon Y \to X$ of smooth algebraic varieties, we let $D_{\text{smooth}}(f)$ stand for the subset of points $y \in Y$ such that $df|_{T_yY}\colon T_yY \to T_{f(y)}X$ is onto. If $\dim(X) = \dim(Y)$ and f is dominant, then $D_{\text{smooth}}(f)$ is the complement of the ramification divisor of f. In general, $D_{\text{smooth}}(f)$ is the maximal open subset U in Y such that the restriction of $f|_U$ is a smooth morphism onto its image.

Theorem 3.9. (Forstnerič [36, Theorem 1.1]) *Let X be a complete smooth complex algebraic variety of dimension n. If X is Gromov elliptic, then X admits a morphism $f\colon \mathbb{C}^n \to X$ such that the restriction $f|_{D_{\text{smooth}}(f)}\colon D_{\text{smooth}}(f) \to X$ is surjective.*

The proof exploits the approximation provided by Theorem 3.1, and the latter involves transcendental tools. The next result is valid over any algebraically closed field $\mathbb{K}$ of characteristic zero.

Theorem 3.10. (Kusakabe [58, Theorem 1.2]) *Every Gromov elliptic smooth algebraic variety X of dimension n admits a morphism $f\colon \mathbb{A}^{n+1} \to X$ such that the restriction $f|_{D_{\text{smooth}}(f)}$ is surjective.*

See also [5, Remark 1.9.4] for a modified and short proof of Theorem 3.10 in the case of a complete variety.

This theorem immediately leads to the following interpolation result.

Corollary 3.11. (Kusakabe [58, Corollary 1.5]) *Let X be a Gromov elliptic smooth algebraic variety, Y be a quasiaffine algebraic variety, and $Z \subset Y$ be a zero-dimensional subscheme. Then for every morphism $f\colon Z \to X$ there exists a morphism $\widetilde{f}\colon Y \to X$ such that $\widetilde{f}|_Z = f$.*

Using Corollary 3.11 we can deduce the following weak version of Theorem 2.7 for Gromov elliptic varieties. Let $\operatorname{End}(X)$ stand for the monoid of regular self-maps $X \to X$.

Corollary 3.12. (Kaliman–Zaidenberg [49, Proposition 6.1]) *Let X be a smooth quasi-affine algebraic variety. If X is Gromov elliptic, then* $\operatorname{End}(X)$ *acts m-transitively on X for every $m \geq 1$.*

See also Arzhantsev [2,3], Balch Barth [8] and Kusakabe [58, Corollary 1.4] for examples of affine varieties that admit surjective morphisms from affine spaces. However, in some of these examples the surjectivity of morphisms restricted to their smooth loci is not guaranteed.

4. Gromov ellipticity and birational geometry

4.1. Gromov ellipticity versus (uni)rationality

Recall that an algebraic variety is called *unirational* if it admits a dominant rational map from a projective space. An elliptic algebraic variety X is unirational. Indeed, let (E, p, s) be a dominating spray on X. Then each fiber $E_x = p^{-1}(x)$ is an affine space which dominates X. Gromov [42, Sec. 3.5E′′] asked whether the opposite is true:

Question 4.1. Is every smooth (uni)rational complete algebraic variety Gromov elliptic?

More generally, one can ask:

Question 4.2. Is every smooth rationally connected complete algebraic variety Gromov elliptic?

Since any Gromov elliptic manifold is unirational, an affirmative answer to Question 4.2 would imply that a rationally connected algebraic variety is unirational, thus resolving in the affirmative the old open problem on coincidence of the unirationality and the rational connectedness. However, the answer to the latter problem is expected to be negative.

We say that a projective variety is *special* if it does not admit a dominant rational map to a variety of general type, cf. Campana [16, Definition 2.1.2]. Gromov [42, Sec. 3.4.F] proposed “the most optimistic” conjecture:

Conjecture 4.3. *Every special smooth projective variety is analytically elliptic.*

See also Campana–Winkelmann [17] for some results and conjectures on the relationships between specialness properties of Campana and Gromov ellipticity.

4.2. Is Gromov ellipticity birationally invariant?

This is a question of Gromov, see [42, Remark 3.5.E′′′]. More specifically, we consider the following question.

Question 4.4. (cf. [62, Remark 2(f)]) Is Gromov ellipticity a birational property in the category of smooth complete algebraic varieties and compositions of blowups and blowdowns with smooth centers as morphisms?

Indeed, a birational map between smooth complete varieties can be factored in a sequence of blowups and blowdowns with smooth centers, see [80].

The behavior of Gromov ellipticity under blowdowns with smooth centers remains a mystery. However, Gromov ellipticity is preserved under blowups with smooth centers modulo certain additional assumptions.

Theorem 4.5. (Kaliman–Kutzschebauch–Truong [46, Theorem 0.6])[1] *Let X be an algebraic manifold and $Z \subset X$ be a smooth closed subvariety of codimension ≥ 2. Suppose X is locally stably flexible. Then X blown up along Z is Gromov elliptic.*

In general, a blowup of X along a smooth center Z does not need to preserve local stable flexibility, even if X admits a covering by copies of $\mathbb{A}^n$. However, this is the case if for any element $U_i \cong \mathbb{A}^n$ of the latter covering, the pair $(U_i, Z \cap U_i)$ with a nonempty intersection $Z \cap U_i$ is isomorphic to a pair $(\mathbb{A}^n, \mathbb{A}^k)$, where $\mathbb{A}^k \subset \mathbb{A}^n$ is a linear subspace, see [6].

The algebraic convex approximation property (the aCAP, see Remark 3.2) occurs to be stable under blowups with smooth centers.

Theorem 4.6. (Kusakabe [54, Corollary 4.3]) *Let X be a smooth complex algebraic variety and $A \subset X$ be a smooth closed algebraic subvariety of codimension ≥ 2. Then the blowup of X along A enjoys aCAP provided that X does so.*

It is not known whether the algebraic resp. analytic ellipticity of a smooth algebraic variety X are preserved under a blowup with smooth center. However, the blowup of X with smooth center is analytically elliptic provided that X is algebraically Gromov elliptic, see Kusakabe [54, Corollary 1.5].

4.3. The Poincaré group of a Gromov elliptic manifold

Let X be a complete smooth complex algebraic variety. If X is unirational, then $\pi_1(X) = 1$ by Serre's theorem, see [78]. Since every Gromov elliptic manifold is unirational, $\pi_1(X) = 1$ provided X is Gromov elliptic.

For a not necessary complete Gromov elliptic manifold X the following holds.

Theorem 4.7. (Kusakabe [59, Theorem 1.3], [60, Theorems 3.1 and 3.3]) *Let X be a smooth complex algebraic variety. If X is Gromov elliptic, then $\pi_1(X)$ is finite and the universal covering $\widetilde{X}$ of X is a Gromov elliptic algebraic variety. For any finite group Γ there exists a smooth complex quasiaffine variety X such that X is flexible, hence Gromov elliptic, and $\pi_1(X) = \Gamma$.*

[1]Cf. also Gromov [42, Sec. 3.5D″] and Lárusson–Truong [61, Main Theorem].

The following question arises; cf. [35, Problem 6.4.11].

Question 4.8. Consider a finite morphism $X \to Y$ between smooth complete algebraic varieties. Suppose X is Gromov elliptic. Is it true that Y is Gromov elliptic?

4.4. Gromov ellipticity versus uniform rationality

Definition 4.9. An algebraic variety X of dimension n is called *uniformly rational*[2] if X can be covered by open sets isomorphic to open sets in $\mathbb{A}^n$. X is called *stably uniformly rational* if $X \times \mathbb{A}^k$ is uniformly rational for some $k \geq 0$.

Question 4.10. (Gromov [42, Sec. 3.5.E‴]) Is every rational smooth algebraic variety (stably) uniformly rational?

The affirmative answer is known for homogeneous spaces of connected affine algebraic groups, see V. L. Popov [73, Corollary 1]. As we have already mentioned, a complete smooth spherical variety admits an open covering by copies of affine spaces, see Brion–Luna–Vust [15, Sec. 1.5, Corollaire]. Furthermore, a smooth spherical variety is uniformly rational, see V. L. Popov [73, Theorem 4]. The total space of a locally trivial fiber bundle over a uniformly rational base with a uniformly rational general fiber is uniformly rational, cf. V. L. Popov [73, Theorem 2]. The next Theorems 4.11 and 4.12 provide more classes of uniformly rational varieties.

Theorem 4.11. (V. L. Popov [73, Theorem 1]) *Let X be a rational algebraic variety. If* $\mathrm{Aut}(X)$ *acts transitively on X, then X is uniformly rational.*

Theorem 4.12. (V. L. Popov [73, Theorem 3]) *Let G be a connected reductive algebraic group and X be a smooth affine G-variety. Assume that every G-invariant regular function on X is constant, and so there is a unique closed G-orbit in X. If this orbit is rational, then X is uniformly rational.*

The uniform rationality survives successive blowups with smooth centers.

Theorem 4.13. (Bogomolov, see [42, Proposition 3.5E], see also Bodnár–Hauser–Schicho–Villamayor U [11, Theorem 4.4] and Bogomolov–Böhning [13, Proposition 2.6]) *The blowup of a uniformly rational variety along a smooth center is uniformly rational.*

Theorem 4.14. (Arzhantsev–Kaliman–Zaidenberg [5, Theorem 1.3]) *A stably uniformly rational smooth complete algebraic variety X is Gromov elliptic.*

Due to Theorem 4.13, Gromov ellipticity of a uniformly rational smooth complete algebraic variety survives successive blowups with smooth irreducible centers.

From Theorems 4.11 and 4.14 we deduce such a corollary.

[2]There are many other names attributed to this same property; see [73, Sec. 3].

Corollary 4.15. *A (stably) locally flexible smooth rational algebraic variety X is (stably) uniformly rational. If such a variety X is complete, then it is Gromov elliptic.*

The latter conclusion also holds for complete smooth G-varieties verifying locally the assumptions of Theorem 4.12.

4.5. Unirationality versus uniform rationality

Proposition 4.16. (Arzhantsev–Kaliman–Zaidenberg [5, Theorem 1.7]) *Let X be a unirational complete variety of dimension n. Then there is a surjective morphism $\widetilde{X} \to X$ from a uniformly rational complete variety $\widetilde{X}$ of dimension n. If X is rational, then the morphism $\widetilde{X} \to X$ can be chosen to be birational.*

Proof. By Chow's Lemma we may assume that X is projective. Choose a generically finite dominant rational map $h\colon \mathbb{P}^n \dashrightarrow X$ which is birational if X is rational. By Hironaka's elimination of indeterminacy we have a commutative diagram

$$\begin{array}{ccc} & \widetilde{X} & \\ {}^{f}\swarrow & & \searrow^{g} \\ \mathbb{P}^n & \overset{h}{\dashrightarrow} & X \end{array}$$

where f is a composition of blowups with smooth irreducible centers. By Theorem 4.13, $\widetilde{X}$ is uniformly rational. □

Remark 4.17. In the case of a rational smooth projective variety X, Proposition 4.16 follows from [42, Proposition 3.5.E″] due to F. Bogomolov.

Corollary 4.18. [5, Corollary 1.8] *A complete algebraic variety X is unirational if and only if X admits a surjective morphism $\mathbb{A}^N \to X$ for some $N \geq \dim(X)$.*

Proof. By Proposition 4.16 a complete unirational X is dominated by a complete uniformly rational variety $\widetilde{X}$, where $\widetilde{X}$ is Gromov elliptic due to Theorem 4.14. By Forstnerič' and Kusakabe's Theorems 3.9–3.10, there is a surjective morphism $\mathbb{A}^{n+1} \to \widetilde{X}$ (resp., $\mathbb{A}^n \to \widetilde{X}$ if $\widetilde{X}$ is defined over $\mathbb{C}$). Anyway, there is a surjective morphism $\mathbb{A}^{n+1} \to X$. □

We address the article [14] for a closely related subject.

4.6. Examples of uniformly rational algebraic manifolds

Theorem 4.19. (Bogomolov–Böhning [13])

(a) *Every rational smooth cubic hypersurface in $\mathbb{P}^{n+1}$, $n \geq 2$ is uniformly rational. The same conclusion also applies to a small algebraic resolution of a nodal cubic threefold in $\mathbb{P}^4$.*

(b) *Every smooth complete intersection of two quadric hypersurfaces in* $\mathbb{P}^{n+2}$, $n \geq 3$, *is uniformly rational.*

(c) *The moduli space* $\overline{\mathcal{M}}_{0,n}$ *of stable* n*-pointed rational curves is a uniformly rational complete variety.*

Applying Theorem 4.14 we deduce the following corollary.

Corollary 4.20. *The varieties in Theorem* 4.19 *are Gromov elliptic.*

Cf. a discussion in Gromov [42, Sec. 3.5.E‴].

Example 4.21. A smooth cubic hypersurface of even dimension $n = 2k$ is rational provided that it contains a pair of skew linear k-spaces. By Theorem 4.19(a), such a hypersurface is uniformly rational, and so Gromov elliptic by Theorem 4.14. See Remarks 4.28 and the references therein for further examples of this type. Cf. also Theorem 4.26 below.

Example 4.22. (Prokhorov–Zaidenberg [74]) Every smooth Fano fourfold X with Picard number 1 and of genus 10, except at most one, up to isomorphism, such fourfold X_0, can be covered by copies of $\mathbb{A}^4$. The exceptional fourfold X_0 contains a projective line L such that $X_0 \setminus L$ is covered by copies of $\mathbb{A}^4$. Additionally, L is covered in X_0 by open $\mathbb{A}^2$-cylinders $S_i \times \mathbb{A}^2$, where the S_i are rational smooth affine surfaces. It follows that every such fourfold X, including X_0, is uniformly rational, and so Gromov elliptic.

Corollary 4.23. *Every smooth Fano fourfold* X *with Picard number* 1 *of genus* 10 *is Gromov elliptic.*

See also Liendo–Petitjean [63] and Petitjean [70] for examples of uniformly rational affine T-varieties.

4.7. Gromov ellipticity and irrationality

There are examples of irrational smooth affine and projective varieties that are Gromov elliptic. Let us start with an affine example.

Example 4.24. (cf. V. L. Popov [71, Example 1.22] and [72]) Recall that an algebraic variety X of dimension n is called *stably rational* if $X \times \mathbb{P}^k$ is birationally equivalent to $\mathbb{P}^{n+k}$ for a natural number k. There are irrational but stably rational varieties, see [9]. According to Saltman [76, Theorem 3.6] (see also e.g. [12] and [26]) for certain values of $n \geq 1$ and for some finite subgroups F of $\mathrm{SL}(n, \mathbb{C})$ the quotient $X = \mathrm{SL}(n, \mathbb{C})/F$ is stably irrational (i.e., is not stably rational). Since X is a homogeneous space of a semisimple algebraic group, it is flexible, see [4, Proposition 5.4]. So, by Proposition 2.3 X is Gromov elliptic. Thus, X is a Gromov elliptic smooth affine variety that is stably irrational.

To deduce a projective example of this kind, it is necessary to know the answer to the following question.

Question 4.25. Is it true that a Gromov elliptic smooth algebraic variety admits a Gromov elliptic smooth completion?

The following theorem provides examples of irrational Gromov elliptic projective varieties, see Corollary 4.27 below.

Theorem 4.26. (Kaliman–Zaidenberg [51]) *Every smooth cubic hypersurface $X \subset \mathbb{P}^{n+1}$, $n \geq 2$, is Gromov elliptic.*

Sketch of the proof. Let $X \subset \mathbb{P}^{n+1}$ be a smooth cubic hypersurface. The projection from a point $u \in X$ gives a generically 2-to-1 rational map $\pi_u \colon X \dashrightarrow \mathbb{P}^n$. By permuting the pair of points on a generic fiber of π_u we obtain a birational Galois involution τ_u on X.

Fix $y \in X$ and choose a general $x \in X$. The line (xy) intersects X at a third point u different from x and y. The involution τ_u biregularly sends a neighborhood U_x of x to a neighborhood U_y of y and sends a line L on X passing through x to a conic C on X passing through y.

Fix x and L and vary u' in a neighborhood V of u. Then the image $y' = \tau_{u'}(x)$ runs over the neighborhood $V' = \tau_x(V)$ of y. The image of L varies in a family of conics $C_{y'} = \tau_{u'}(L)$ passing through the points $y' = \tau_x(u') \in V'$.

Now letting $z = L \cap \mathbb{T}_u X$ and $L^* = L \setminus \{z\}$ and choosing x for zero of the vector line $L^* \cong \mathbb{A}^1$, we obtain a spray (E, p, s) of rank 1 on V' which is dominating along each orbit $s(E_{y'}) = \tau_{u'}(L^*) = C_{y'} \setminus \tau_{u'}(z)$.

We can choose n lines $L_1, \ldots, L_n$ on X passing through x in n independent directions. This gives a dominating family of n rank 1 sprays (E_i, p_i, s_i) on a neighborhood V_0 of y with values in X. By Gromov's Localization Lemma 1.10, each of these sprays admits an extension to a spray on X dominating its one-dimensional orbits inside V_0.

Choosing a finite covering of X by such neighborhoods we obtain a dominating family of rank 1 sprays on X. The composition of all these rank 1 sprays yields a dominating spray on X. Thus, X is Gromov elliptic. □

By Clemens–Griffiths' theorem [25] every smooth cubic threefold X in $\mathbb{P}^4$ is irrational. We can therefore deduce the following corollary. This confirms a conjecture of Gromov [42, Sec. 3.4.F].

Corollary 4.27. *The smooth cubic threefolds in $\mathbb{P}^4$ are irrational Gromov elliptic projective varieties.*

Remarks 4.28. For $n \geq 3$ each smooth cubic hypersurface in $\mathbb{P}^n$ is unirational, see e.g. Kollár [53]. However, we do not know if a general cubic threefold is stably irrational.

There is a 18-dimensional family of rational cubic fourfolds in $\mathbb{P}^5$ which contain a pair of skew planes. Another 19-dimensional family of rational cubic fourfolds in $\mathbb{P}^5$ consists of those cubic fourfolds which contain a quartic surface scroll. See e.g. [67, 79] and references therein for other examples. A general belief is that a very general cubic fourfold in $\mathbb{P}^5$ is irrational. However, at present, no example of an irrational smooth cubic fourfold is known.

4.8. Open questions

Likewise a smooth cubic threefold, a smooth quartic double solid admits a lot of birational involutions.

Question 4.29. Is there any Gromov elliptic smooth quartic double solid?

See [24] for examples of rational nodal quartic double solids that admit Gromov elliptic algebraic small resolutions.

Question 4.30. (cf. [62, Remark 2(g)]) Is there a birationally (super)rigid Fano manifold which is Gromov elliptic? Especially, is a unirational smooth Segre quartic threefold in $\mathbb{P}^4$ Gromov elliptic?

The superrigidity of smooth quartic threefolds in $\mathbb{P}^4$ was proven by Iskovskikh and Manin, see [44]; see also [ibid] for Segre examples of unirational smooth quartic threefolds.

Question 4.31. Is the Gromov ellipticity of a smooth complete algebraic variety stable under smooth deformations? Moreover, given a proper smooth deformation family $\mathcal{X} \to S$ over a smooth base S, is the locus of points $s \in S$ such that the fiber $\mathcal{X}_s$ is Gromov elliptic open, or closed, or constructible in the Zariski topology, or, over $\mathbb{C}$, in classical topology?

Question 4.32. Let $X \to S$ be a smooth morphism of smooth complete varieties. Suppose S and each fiber X_s, $s \in S$, are Gromov elliptic. Is it true that X is Gromov elliptic?

The last two questions can be addressed for (stable) uniform rationality and (stable) local flexibility replacing the Gromov ellipticity.

5. Ellipticity of affine cones versus flexibility

5.1. Gromov elliptic affine cones

Theorem 5.1. (Arzhantsev–Kaliman–Zaidenberg [5, Theorem 1.3]) *Let $X \subset \mathbb{P}^n$ be a uniformly rational projective variety and $\widehat{Y} \subset \mathbb{A}^{n+1}$ be the affine cone over X. Then the punctured cone $Y = \widehat{Y} \setminus \{0\}$ is Gromov elliptic.*

Note that $p\colon Y \to X$, where p is a natural projection, is a principal $\mathbb{G}_{\mathrm{m}}$-bundle. The conclusion of Theorem 5.1 stays true for every principal $\mathbb{G}_{\mathrm{m}}$-bundle $Y \to X$, provided that the associated line bundle $L \to X$ is ample or anti-ample. However, the assumption of ampleness is not necessary, due to the next stronger result.

Theorem 5.2. (Kaliman [45, Theorem 6.1]) *Let X be a complete uniformly rational algebraic variety and $L \to X$ be a nontrivial line bundle on X with zero section Z. Then $Y = L \setminus Z$ is Gromov elliptic.*

For the trivial line bundle $L = X \times \mathbb{C}$ over a complex smooth complete algebraic variety X we have $Y \cong X \times \mathbb{C}^*$. So, the group $\pi_1(Y)$ is infinite. Therefore, Y cannot be Gromov elliptic, because otherwise $\pi_1(Y)$ must be finite by Kusakabe's Theorem 4.7. Thus, the hypothesis of Theorem 5.2 that L is nontrivial is necessary.

There is the following analogue of Corollary 4.18 for affine cones (which are not complete!).

Theorem 5.3. (Arzhantsev [3, Theorem 1]) *The affine cone $\widehat{Y} \subset \mathbb{A}^{n+1}$ over a projective variety $X \subset \mathbb{P}^n$ admits a surjective morphism $\mathbb{A}^m \to \widehat{Y}$ for some positive integer m if and only if $\widehat{Y}$ is unirational or, equivalently, X is unirational. Furthermore, we can take $m = \dim(X) + 2$.*

5.2. Flexible affine cones

The following results certify that the flexibility of punctured affine cones is much stronger property than Gromov ellipticity.

Theorem 5.4. (Kishimoto–Prokhorov–Zaidenberg [52], Perepechko [69], Cheltsov–Park–Won [20]; cf. Park–Won [68]) *Let $X = X_d$ be a del Pezzo surface of degree d pluri-anticanonically embedded in $\mathbb{P}^n$, and let $Y \subset \mathbb{A}^{n+1}$ be the punctured affine cone over X. Then*

(1) *Y is flexible for $4 \le d \le 9$;*

(2) *Y admits no nontrivial $\mathbb{G}_{\mathrm{a}}$-action for $d \le 3$. In particular, Y is not flexible for $d \le 3$;*

(3) *Y is Gromov elliptic for every $d = 1, \ldots, 9$.*

The last statement follows from Theorem 5.1 due to uniform rationality of the smooth complete rational surface $X = X_d$, see Example (7) in Section 2.2.

The following example answers in negative the question in [30, Question 2.22].

Example 5.5. (Cheltsov–Park–Won [20, Corollary 1.8], Freudenburg–Moser-Jauslin [39, Theorem 8.1(c)]) Let $\overline{Y} \subset \mathbb{A}^4$ be the Fermat cubic cone

$$\overline{Y} = \{x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0\},$$

i.e., the affine cone over the Fermat cubic surface in $\mathbb{P}^3$. Then the coordinate ring $\mathcal{O}_{\overline{Y}}(\overline{Y})$ is rigid, i.e., it does not admit any locally nilpotent derivation. In particular, the punctured cone $Y = \overline{Y} \setminus \{0\}$ is not flexible. However, Y is Gromov elliptic by Theorem 5.1.

Example 5.6. More generally, consider a Pham–Brieskorn hypersurface $\overline{Y} \subset \mathbb{A}^{n+1}$, $n \geq 2$, defined by

$$x_0^{a_0} + x_1^{a_1} + \cdots + x_n^{a_n} = 0 \quad \text{where } 2 \leq a_0 \leq a_1 \leq \cdots \leq a_n.$$

There is a conjecture [19, Conjecture 1.22], based on [48] and [30, p. 551 and Example 2.21], claiming that the coordinate ring $\mathcal{O}_{\overline{Y}}(\overline{Y})$ is rigid if and only if $a_1 \geq 3$. This conjecture is known to hold for $n = 2$ [48, Lemma 4] and for $n = 3$ [23, Main theorem], see also [22] and [21, Theorem 4.8.3].

Example 5.7. Let X be a Fano fourfold with Picard number 1 of genus 10 embedded half-anticanonically in $\mathbb{P}^{12}$, and let Y be the punctured affine cone over X. Since X is uniformly rational, see Example 4.22, Y is Gromov elliptic by Theorem 5.1. Moreover, Y is flexible, see Prokhorov–Zaidenberg [74].

See Arzhantsev–Perepechko–Süß [6] and Michałek–Perepechko–Süß [65] for further examples of flexible affine cones and universal torsors, i.e., the spectra of Cox rings.

Acknowledgments

It is with pleasure that we thank F. Bogomolov, S. Kaliman and Yu. Prokhorov for sharing their expertise with the author.

References

[1] I. Arzhantsev, *Infinite transitivity and special automorphisms*, Ark. Mat. **56** (2018), no. 1, 1–14.

[2] ______, *On images of affine spaces*, Indag. Math. (N.S.) **34** (2023), no. 4, 812–819.

[3] ______, *Affine cones as images of affine spaces*, arXiv:2407.18580.

[4] I. Arzhantsev, H. Flenner, S. Kaliman, F. Kutzschebauch and M. Zaidenberg, *Flexible varieties and automorphism groups*, Duke Math. J. **162** (2013), no. 4, 767–823.

[5] I. Arzhantsev, S. Kaliman and M. Zaidenberg, *Varieties covered by affine spaces, uniformly rational varieties and their cones*, Adv. Math. **437** (2024), Paper No. 109449, 18 pp.

[6] I. Arzhantsev, A. Perepechko and H. Süß, *Infinite transitivity on universal torsors*, J. Lond. Math. Soc. (2) **89** (2014), no. 3, 762–778.

[7] I. V. Arzhantsev, M. G. Zaĭdenberg and K. G. Kuyumzhiyan, *Flag varieties, toric varieties, and suspensions: Three examples of infinite transitivity*, Sb. Math. **203** (2012), no. 7-8, 923–949.

[8] V. Balch Barth, *Surjective morphisms from affine space to its Zariski open subsets*, Internat. J. Math. **34** (2023), no. 12, Paper No. 2350075, 6 pp.

[9] A. Beauville, J.-L. Colliot-Thélène, J.-J. Sansuc and P. Swinnerton-Dyer, *Variétés stablement rationnelles non rationnelles*, Ann. of Math. (2) **121** (1985), no. 2, 283–318.

[10] J. Bochnak and W. Kucharz, *Rational approximation of holomorphic maps*, Ann. Inst. Fourier (Grenoble) **73** (2023), no. 3, 1115–1131.

[11] G. Bodnár, H. Hauser, J. Schicho and O. Villamayor U, *Plain varieties*, Bull. Lond. Math. Soc. **40** (2008), no. 6, 965–971.

[12] F. A. Bogomolov, *The Brauer group of quotient spaces of linear representations*, Izv. Akad. Nauk SSSR Ser. Mat. **51** (1987), no. 3, 485–516; translation in Math. USSR-Izv. **30** (1988), no. 3, 455–485.

[13] F. Bogomolov and C. Böhning, *On uniformly rational varieties*, in: *Topology, Geometry, Integrable Systems, and Mathematical Physics*, 33–48, Amer. Math. Soc. Transl. Ser. 2, **234**, Adv. Math. Sci. **67**, Amer. Math. Soc., Providence, RI, 2014.

[14] F. Bogomolov, I. Karzhemanov and K. Kuyumzhiyan, *Unirationality and existence of infinitely transitive models*, in: *Birational Geometry, Rational Curves, and Arithmetic*, 77–92, Simons Symp., Springer, Cham, 2013.

[15] M. Brion, D. Luna and Th. Vust, *Espaces homogènes sphériques*, Invent. Math. **84** (1986), no. 3, 617–632.

[16] F. Campana, *Orbifolds, special varieties and classification theory*, Ann. Inst. Fourier (Grenoble) **54** (2004), no. 3, 499–630.

[17] F. Campana and J. Winkelmann, *On the h-principle and specialness for complex projective manifolds*, Algebr. Geom. **2** (2015), no. 3, 298–314.

[18] H. Cartan, *Espaces fibrés analytiques, d'après H. Grauert*, in: *Séminaire Bourbaki*, Vol. 4, Exp. No. 137, 7–18, Soc. Math. France, Paris, 1995.

[19] I. Cheltsov, J. Park, Y. Prokhorov and M. Zaidenberg, *Cylinders in Fano varieties*, EMS Surv. Math. Sci. **8** (2021), no. 1-2, 39–105.

[20] I. Cheltsov, J. Park and J. Won, *Affine cones over smooth cubic surfaces*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 7, 1537–1564.

[21] M. Chitayat, *Rigidity of Pham–Brieskorn threefolds*, PhD thesis, Department of Mathematics and Statistics, Faculty of Science, University of Ottawa, Ottawa, Canada, 2023.

[22] M. Chitayat and D. Daigle, *On the rigidity of certain Pham–Brieskorn rings*, J. Algebra **550** (2020), 290–308.

[23] M. Chitayat and A. Dubouloz, *The rigid Pham–Brieskorn threefolds*, arXiv:2312.07587.

[24] C. Ciliberto and M. Zaidenberg, *Gromov elliptic resolutions of quartic double solid*, arXiv:2409.19367.

[25] C. H. Clemens and P. A. Griffiths, *The intermediate Jacobian of the cubic threefold*, Ann. of Math. (2) **95** (1972), 281–356.

[26] J.-L. Colliot-Thélène, *Non rationalité stable sur les corps quelconques*, in: *Birational Geometry of Hypersurfaces*, 73–110, Lect. Notes Unione Mat. Ital. **26**, Springer, Cham, 2019.

[27] W. Danielewski, *On the cancellation problem and automorphism group of affine algebraic varieties*, Unpublished preprint, 1989.

[28] J.-P. Demailly, L. Lempert and B. Shiffman, *Algebraic approximations of holomorphic maps from Stein domains to projective manifolds*, Duke Math. J. **76** (1994), no. 2, 333–363.

[29] H. Flenner, S. Kaliman and M. Zaidenberg, *A Gromov–Winkelmann type theorem for flexible varieties*, J. Eur. Math. Soc. (JEMS) **18** (2016), no. 11, 2483–2510.

[30] H. Flenner and M. Zaidenberg, *Rational curves and rational singularities*, Math. Z. **244** (2003), no. 3, 549–575.

[31] F. Forstnerič, *The Oka principle for sections of subelliptic submersions*, Math. Z. **241** (2002), no. 3, 527–551.

[32] ______, *Holomorphic flexibility properties of complex manifolds*, Amer. J. Math. **128** (2006), no. 1, 239–270.

[33] ______, *Runge approximation on convex sets implies the Oka property*, Ann. of Math. (2) **163** (2006), no. 2, 689–707.

[34] ______, *Oka manifolds*, C. R. Math. Acad. Sci. Paris **347** (2009), no. 17-18, 1017–1020.

[35] ______, *Stein Manifolds and Holomorphic Mappings: The homotopy principle in complex analysis*, Second edition, Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics **56**, Springer, Cham, 2017.

[36] ______, *Surjective holomorphic maps onto Oka manifolds*, in: *Complex and Symplectic Geometry*, 73–84, Springer INdAM Ser. **21**, Springer, Cham, 2017.

[37] ______, *Recent developments on Oka manifolds*, Indag. Math. (N.S.) **34** (2023), no. 2, 367–417.

[38] J. Frenkel, *Cohomologie non abélienne et espaces fibrés*, Bull. Soc. Math. France **85** (1957), 135–220.

[39] G. Freudenburg and L. Moser-Jauslin, *Locally nilpotent derivations of rings with roots adjoined*, Michigan Math. J. **62** (2013), no. 2, 227–258.

[40] H. Grauert, *Holomorphe Funktionen mit Werten in komplexen Lieschen Gruppen*, Math. Ann. **133** (1957), 450–472.

[41] ______, *Analytische Faserungen über holomorph-vollständigen Räumen*, Math. Ann. **135** (1958), 263–273.

[42] M. Gromov, *Oka's principle for holomorphic sections of elliptic bundles*, J. Amer. Math. Soc. **2** (1989), no. 4, 851–897.

[43] G. Henkin and J. Leiterer, *The Oka–Grauert principle without induction over the base dimension*, Math. Ann. **311** (1998), no. 1, 71–93.

[44] V. A. Iskovskih and Ju. I. Manin, *Three-dimensional quartics and counterexamples to the Lüroth problem*, Mat. Sb. (N.S.) **86(128)** (1971), 140–166.

[45] S. Kaliman, *Gromov's ellipticity of principal* $\mathbb{G}_m$*-bundles*, arXiv:2406.10379.

[46] S. Kaliman, F. Kutzschebauch and T. T. Truong, *On subelliptic manifolds*, Israel J. Math. **228** (2018), no. 1, 229–247.

[47] S. Kaliman and M. Zaidenberg, *Affine modifications and affine hypersurfaces with a very transitive automorphism group*, Transform. Groups **4** (1999), no. 1, 53–95.

[48] ______, *Miyanishi's characterization of the affine 3-space does not hold in higher dimensions*, Ann. Inst. Fourier (Grenoble) **50** (2000), no. 6, 1649–1669.

[49] ______, *Gromov ellipticity of cones over projective manifolds*, Math. Res. Lett. (to appear), arXiv:2303.02036.

[50] ______, *Gromov ellipticity and subellipticity*, Forum Math. **36** (2024), no. 2, 373–376.

[51] ______, *Algebraic Gromov's ellipticity of cubic hypersurfaces*, arXiv:2402.04462.

[52] T. Kishimoto, Y. Prokhorov and M. Zaidenberg, *Group actions on affine cones*, in: *Affine Algebraic Geometry*, 123–163, CRM Proc. Lecture Notes **54**, Amer. Math. Soc., Providence, RI, 2011.

[53] J. Kollár, *Unirationality of cubic hypersurfaces*, J. Inst. Math. Jussieu **1** (2002), no. 3, 467–476.

[54] Y. Kusakabe, *An implicit function theorem for sprays and applications to Oka theory*, Internat. J. Math. **31** (2020), no. 9, 2050071, 9 pp.

[55] ______, *Oka complements of countable sets and nonelliptic Oka manifolds*, Proc. Amer. Math. Soc. **148** (2020), no. 3, 1233–1238.

[56] ______, *Elliptic characterization and localization of Oka manifolds*, Indiana Univ. Math. J. **70** (2021), no. 3, 1039–1054.

[57] ______, *Thom's jet transversality theorem for regular maps*, J. Geom. Anal. **31** (2021), no. 6, 6031–6041.

[58] ______, *Surjective morphisms onto subelliptic varieties*, arXiv:2212.06412.

[59] ______, *On the fundamental groups of subelliptic varieties*, arXiv:2212.07085.

[60] ______, *Oka properties of complements of holomorphically convex sets*, Ann. of Math. (2) **199** (2024), no. 2, 899–917.

[61] F. Lárusson and T. T. Truong, *Algebraic subellipticity and dominability of blow-ups of affine spaces*, Doc. Math. **22** (2017), 151–163.

[62] ______, *Approximation and interpolation of regular maps from affine varieties to algebraic manifolds*, Math. Scand. **125** (2019), no. 2, 199–209.

[63] A. Liendo and C. Petitjean, *Uniformly rational varieties with torus action*, Transform. Groups **24** (2019), no. 1, 149–153.

[64] L. Makar-Limanov, *On the group of automorphisms of a surface $x^n y = P(z)$*, Israel J. Math. **121** (2001), 113–123.

[65] M. Michałek, A. Perepechko and H. Süß, *Flexible affine cones and flexible coverings*, Math. Z. **290** (2018), 1457–1478.

[66] K. Oka, *Sur les fonctions analytiques de plusieurs variables III: Deuxième problème de Cousin*, J. Sci. Hiroshima Univ. Ser. A **9** (1939), 7–19.

[67] G. Ouchi, *Hilbert schemes of two points on K3 surfaces and certain rational cubic fourfolds*, Comm. Algebra **49** (2020), no. 3, 1173–1179.

[68] J. Park and J. Won, *Flexible affine cones over del Pezzo surfaces of degree 4*, Eur. J. Math. **2** (2016), no. 1, 304–318.

[69] A. Y. Perepechko, *Flexibility of affine cones over del Pezzo surfaces of degree 4 and 5*, Funct. Anal. Appl. **47** (2013), no. 4, 284–289.

[70] C. Petitjean, *Equivariantly uniformly rational varieties*, Michigan Math. J. **66** (2017), no. 2, 245–268.

[71] V. L. Popov, *On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties*, in: *Affine Algebraic Geometry*, 289–311, CRM Proc. Lecture Notes **54**, Amer. Math. Soc., Providence, RI, 2011.

[72] ———, *Rationality and the FML invariant*, J. Ramanujan Math. Soc. **28A** (2013), 409–415.

[73] ———, *Variations on the theme of Zariski's cancellation problem*, in: *Polynomial Rings and Affine Algebraic Geometry*, 233–250, Springer Proc. Math. Stat. **319**, Springer, Cham, 2020.

[74] Y. Prokhorov and M. Zaidenberg, *Affine cones over Fano–Mukai fourfolds of genus 10 are flexible*, in: *The Art of Doing Algebraic Geometry*, 363–383, Trends Math., Birkhäuser/Springer, Cham, 2023.

[75] K.-J. Ramspott, *Stetige und holomorphe Schnitte in Bündeln mit homogener Faser*, Math. Z. **89** (1965), 234–246.

[76] D. J. Saltman, *Noether's problem over an algebraically closed field*, Invent. Math. **77** (1984), no. 1, 71–84.

[77] J.-P. Serre, *Faisceaux algébriques cohérents*, Ann. of Math. (2) **61** (1955), no. 2, 197–278.

[78] ———, *On the fundamental group of a unirational variety*, J. London Math. Soc. **34** (1959), 481–484.

[79] S. L. Tregub, *Three constructions of rationality of a cubic fourfold*, Vestnik Moskov. Univ. Ser. I Mat. Mekh. **1984**, no. 3, 8–14.

[80] J. Włodarczyk, *Toroidal varieties and the weak factorization theorem*, Invent. Math. **154** (2003), no. 2, 223–331.

Mikhail Zaidenberg
Institut Fourier, UMR 5582, Laboratoire de Mathématiques, Université Grenoble Alpes, CS 40700, 38058 Grenoble cedex 9, France
E-mail address: `mikhail.zaidenberg@univ-grenoble-alpes.fr`