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On a $p(x)$-Kirchhoff Problem with Variable Singular and Sublinear Exponents

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Abstract. In the present paper, we study a singular $p(x)$-Kirchhoff equation with combined effects of variable singular, $\beta(x)$, and sublinear, $q(x)$, nonlinearities. Using the Ekeland's variational principle and a constrained minimization, we show the existence of a positive solution for the case variable singularity $\beta(x)$ assumes its values in the interval $(1,\infty)$. We provide an example to illustrate our results.

1. Introduction

In this article, we study the following singular $p(x)$-Kirchhoff equation involving variable singular and sublinear nonlinearities

$$\begin{cases} -\mathcal{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx\right)\operatorname{div}(|\nabla u|^{p(x)-2}\nabla u) = f(x)u^{-\beta(x)} + g(x)u^{q(x)} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 3)$ with smooth boundary $\partial\Omega$; $f \in L^1(\Omega)$ is a positive function; $g \in L^\infty(\Omega)$ is a nonnegative function; $\beta \in C(\overline{\Omega})$ such that $1 < \beta(x) < \infty$ for any $x \in \overline{\Omega}$; $q \in C(\overline{\Omega})$ such that $0 < q(x) < 1$ for any $x \in \overline{\Omega}$.

To our best knowledge, there are few papers dealt with the equation of the type (1.1). For $p(x) = p$ a constant growth case, we refer to the paper [40] where a 2-Laplace equation with constant singular and sublinear nonlinearities was studied. As for singular $p(x)$-Kirchhoff-type equations; the author could find only the papers [9, 29–31, 35] except his own paper [8], while writing the present paper. Papers [8, 9, 29, 30, 35] are devoted to the case where singularity $\gamma(x)$ assumes its values in the interval $(0,1)$, that is, $\gamma(x)$ is from $C(\overline{\Omega},(0,1))$. In [31], on the other hand, the authors consider the case where the singularity, $q(x)$, is from $C(\overline{\Omega},(1,\infty))$, and they consider a singular $p(x)$-Kirchhoff type equation with Hardy-type term, that is, the singular term, w, is a weight function in the

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form $w = \operatorname{dist}(x, \partial\Omega)$ and doesn't depend on the unknown function. However, none of these papers studied combined effects of variable singular and sublinear nonlinearities.

We would also like to mention that the singular problems of the type

$$\begin{cases} -\Delta u = f(x)u^{-\beta} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1.2}$$

have been intensively studied because of their wide applications to physical models in the study of non-Newtonian fluids, boundary layer phenomena for viscous fluids, chemical heterogenous catalysts, glacial advance, etc. (see, e.g., [1,3,12,16,24,28,34,36]).

The studies, however, mainly have been focused on the case $0 < \beta < 1$, i.e., the weak singularity (see, e.g., [2,18,23,25,39,41]), and in this case the corresponding energy functional is continuous.

When $\beta > 1$, on the other hand, the situation changes dramatically, and numerous challenges emerge in the analysis of differential equations of the type (1.2), where the primary challenge encountered is due to the lack of integrability of $u^{-\beta}$ for $u \in H_0^1(\Omega)$ [10,32,40,42,44].

To overcome these challenges, a new approach (which is called compatibility relation, and based on the optimal relation between $f(x)$ and β) has been introduced in the recent studies [40,42,44]. As an alternative approach to the critical point theory, this method, used along with a constrained minimization and the Ekeland's variational principle [20], suggests a practical approach to obtain solutions to the problems of the type (1.2) when the corresponding energy functional is not of class C^1.

In [40], for example, the author study the singular nonlinear elliptic problem of the type

$$\begin{cases} -\Delta u = h(x)u^{-p} + k(x)u^{q} & \text{in } \Omega, \\ u > 0 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.3}$$

where Ω is a bounded domain in $\mathbb{R}^N$ $(N \geq 1)$ with smooth boundary $\partial\Omega$; $h \in L^1(\Omega)$ is a positive function; $k \in L^\infty(\Omega)$ is a nonnegative function; $0 < q < 1$ and $p > 1$. The author obtained $H_0^1(\Omega)$-solutions for the strong singularity case $p > 1$.

In a subsequent work [33], using the same approach, the author generalizes problem (1.3) to the p-Kirchhoff equation and obtain existence and uniqueness of solutions for the case $0 \leq q \leq p - 1$, $p > 1$.

In the present paper, we generalize these results to nonstandard $p(\cdot)$-growth, i.e., the variable exponent case. We apply the same reasoning and approach as in our recent works [6, 7], where we addressed different types of problems. The equations involving variable exponent have been intensively studied by many authors for the past two decades due to its significant role in many fields of mathematics, such as in the study of calculus of variations, partial differential equations [17, 19], but also for their use in a variety of physical and engineering contexts: the modeling of electrorheological fluids [38], the analysis of Non-Newtonian fluids [45], fluid flow in porous media [4], magnetostatics [13], image restoration [14], and capillarity phenomena [5] (see also, e.g., [11, 15, 26, 27, 43] and references therein).

The paper is organised as follows. In Section 2, we provide some fundamental information for the theory of variable Sobolev spaces since it's our work space. In Section 3, first we obtain the auxiliary results. Then, we present our main result and obtain the existence of a positive solution to problem (1.1). In Section 4, we provide an example to illustrate our results in a concrete way.

2. Preliminaries

We start with some basic concepts of variable Lebesgue–Sobolev spaces. For more details, and the proof of the following propositions, we refer the reader to [17, 19, 22, 37].

$$C_+(\overline{\Omega}) = \{p \in C(\overline{\Omega}) \mid \inf p(x) > 1 \text{ for all } x \in \overline{\Omega}\}.$$

For $p \in C_+(\overline{\Omega})$ denote

$$p^- := \inf_{x\in\overline{\Omega}} p(x) \le p(x) \le p^+ := \sup_{x\in\overline{\Omega}} p(x) < \infty.$$

For any $p \in C_+(\overline{\Omega})$, we define *the variable exponent Lebesgue space* by

$$L^{p(x)}(\Omega) = \left\{ u \;\middle|\; u\colon \Omega \to \mathbb{R} \text{ is measurable}, \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\},$$

then $L^{p(x)}(\Omega)$ endowed with the norm

$$|u|_{p(x)} = \inf\left\{ \lambda > 0 : \int_\Omega \left|\frac{u(x)}{\lambda}\right|^{p(x)} dx \le 1 \right\}$$

becomes a Banach space.

Proposition 2.1 (Hölder inequality)**.** *For any $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, we have*

$$\int_\Omega |uv|\, dx \le C(p^-, (p^-)')|u|_{p(x)}|v|_{p'(x)}$$

where $L^{p'(x)}(\Omega)$ is conjugate space of $L^{p(x)}(\Omega)$ such that $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$.

The convex functional $\Lambda\colon L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\Lambda(u) = \int_\Omega |u(x)|^{p(x)}\, dx$$

is called modular on $L^{p(x)}(\Omega)$.

Proposition 2.2. *If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, \ldots)$, we have*

(i) $|u|_{p(x)} < 1$ $(= 1; > 1)$ $\iff$ $\Lambda(u) < 1$ $(= 1; > 1)$.

(ii) $|u|_{p(x)} > 1 \implies |u|^{p^-}_{p(x)} \le \Lambda(u) \le |u|^{p^+}_{p(x)}$; $|u|_{p(x)} \le 1 \implies |u|^{p^+}_{p(x)} \le \Lambda(u) \le |u|^{p^-}_{p(x)}$.

(iii) $\lim_{n\to\infty} |u_n|_{p(x)} = 0 \iff \lim_{n\to\infty} \Lambda(u_n) = 0$; $\lim_{n\to\infty} |u_n|_{p(x)} = \infty \iff \lim_{n\to\infty} \Lambda(u_n) = \infty$.

Proposition 2.3. *If $u, u_n \in L^{p(x)}(\Omega)$ $(n = 1, 2, \ldots)$, then the following statements are equivalent:*

(i) $\lim_{n\to\infty} |u_n - u|_{p(x)} = 0$,

(ii) $\lim_{n\to\infty} \Lambda(u_n - u) = 0$,

(iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \Lambda(u_n) = \Lambda(u)$.

The variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ is defined by

$$W^{1,p(x)}(\Omega) = \{u \in L^{p(x)}(\Omega) \mid |\nabla u| \in L^{p(x)}(\Omega)\}$$

with the norm

$$\|u\|_{1,p(x)} = |u|_{p(x)} + |\nabla u|_{p(x)},$$

or equivalently

$$\|u\|_{1,p(x)} = \inf\left\{\lambda > 0 \;\middle|\; \int_\Omega \left(\left|\frac{\nabla u(x)}{\lambda}\right|^{p(x)} + \left|\frac{u(x)}{\lambda}\right|^{p(x)} \right) dx \le 1 \right\}$$

for all $u \in W^{1,p(x)}(\Omega)$. The space $W_0^{1,p(x)}(\Omega)$ is defined as $\overline{C_0^\infty(\Omega)}^{\|\cdot\|_{1,p(x)}} = W_0^{1,p(x)}(\Omega)$, and hence, $u \in W_0^{1,p(x)}(\Omega)$ if and only if there exists a sequence (u_n) of $C_0^\infty(\Omega)$ such that $\|u_n - u\|_{1,p(x)} \to 0$.

As a consequence of Poincaré inequality, $\|u\|_{1,p(x)}$ and $|\nabla u|_{p(x)}$ are equivalent norms on $W_0^{1,p(x)}(\Omega)$. Therefore, for any $u \in W_0^{1,p(x)}(\Omega)$ we can define an equivalent norm $\|u\|$ such that

$$\|u\| = |\nabla u|_{p(x)}.$$

Proposition 2.4. *If $1 < p^- \leq p^+ < \infty$, then the spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Proposition 2.5. *Let $q \in C(\overline{\Omega})$. If $1 \leq q(x) < p^*(x)$ for all $x \in \overline{\Omega}$, then the embedding $W^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$ is compact and continuous, where $p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{if } p(x) < N, \\ +\infty & \text{if } p(x) \geq N. \end{cases}$*

Remark 2.6. The convex functional $\rho(u) := \int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx$ is from $C^1(W_0^{1,p(x)}(\Omega), \mathbb{R})$ with the derivative

$$\langle \rho'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v\,dx$$

where $\langle\,\cdot\,,\cdot\,\rangle$ is the dual pairing between $W_0^{1,p(x)}(\Omega)$ and its dual $(W_0^{1,p(x)}(\Omega))^*$ [21].

3. The main results

We define the singular energy functional $\mathcal{J} \colon W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ corresponding to equation (1.1) by

$$\mathcal{J}(u) = \widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx\right) - \int_\Omega \frac{g(x)|u|^{q(x)+1}}{q(x)+1}\,dx + \int_\Omega \frac{f(x)|u|^{1-\beta(x)}}{\beta(x)-1}\,dx.$$

Definition 3.1. A function u is called a weak solution to problem (1.1) if $u \in W_0^{1,p(x)}(\Omega)$ such that $u > 0$ in Ω and

$$\mathcal{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla\varphi\,dx - \int_\Omega g(x)u^{q(x)}\varphi\,dx - \int_\Omega f(x)u^{-\beta(x)}\varphi\,dx = 0$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$.

Definition 3.2. Due to the singularity of $\mathcal{J}$ on $W_0^{1,p(x)}(\Omega)$, we apply a constrained minimization for problem (1.1). As such, we introduce the following constrains:

$$\mathcal{N}_1 = \left\{u \in W_0^{1,p(x)}(\Omega) \;\middle|\; \int_\Omega \left[\mathcal{M}(\rho(u))|\nabla u|^{p(x)} - g(x)|u|^{q(x)+1} - f(x)|u|^{1-\beta(x)}\right]dx \geq 0\right\}$$

and

$$\mathcal{N}_2 = \left\{u \in W_0^{1,p(x)}(\Omega) \;\middle|\; \int_\Omega \left[\mathcal{M}(\rho(u))|\nabla u|^{p(x)} - g(x)|u|^{q(x)+1} - f(x)|u|^{1-\beta(x)}\right]dx = 0\right\}.$$

Remark 3.3. $\mathcal{N}_2$ can be considered as a Nehari manifold, even though in general it may not be a manifold. Therefore, if we set

$$c_0 := \inf_{u\in\mathcal{N}_2} \mathcal{J}(u),$$

then one might expect that c_0 is attained at some $u \in \mathcal{N}_2$ (i.e., $\mathcal{N}_2 \neq \varnothing$) and that u is a critical point of $\mathcal{J}$.

Throughout the paper we assume the following conditions hold.

(A_1) $\beta\colon \overline{\Omega} \to (1,\infty)$ is a continuous function such that $1 < \beta^- \le \beta(x) \le \beta^+ < \infty$.

(A_2) $q\colon \overline{\Omega} \to (0,1)$ is a continuous function such that $0 < q^- \le q(x) \le q^+ < 1$ and $q^+ + 1 \le \beta^-$.

(A_3) $2 \le p^- \le p(x) \le p^+ < p^*(x)$.

(A_4) $f \in L^1(\Omega)$ is a positive function, that is, $f(x) > 0$ a.e. in Ω.

(A_5) $g \in L^\infty(\Omega)$ is a nonnegative function.

(A_6) $\mathcal{M}\colon (0,\infty) \to (0,\infty)$ is a C^1-continuous nondecreasing function and satisfies the polynomial growth condition

$$\kappa_1 s^{\alpha_1 - 1} \le \mathcal{M}(s) \le \kappa_2 s^{\alpha_2 - 1}, \tag{3.1}$$

where κ_i, $i = 1,2$ are real numbers such that $\kappa_2 \ge \kappa_1 > 0$; and α_i, $i = 1,2$ are positive integers such that $\alpha_2 \ge \alpha_1 > 1$.

Lemma 3.4. *For any $u \in W_0^{1,p(x)}(\Omega)$ satisfying $\int_\Omega f(x)|u|^{1-\beta(x)}\,dx < \infty$, the functional $\mathcal{J}$ is well-defined and coercive on $W_0^{1,p(x)}(\Omega)$.*

Proof. By Proposition 2.2 and the embedding $W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)+1}(\Omega)$ is compact and continuous, it follows

$$\begin{aligned} |\mathcal{J}(u)| &\le \frac{\kappa_2}{\alpha_2}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx\right)^{\alpha_2} + \frac{|g|_\infty}{q^- + 1}\int_\Omega |u|^{q(x)+1}\,dx + \frac{1}{\beta^- - 1}\int_\Omega f(x)|u|^{1-\beta(x)}\,dx \\ &\le \frac{\kappa_2}{\alpha_2 (p^-)^{\alpha_2}}\|u\|^{\alpha_2 p^+} + \frac{|g|_\infty}{q^- + 1}\|u\|^{q^+ + 1} + \frac{1}{\beta^- - 1}\int_\Omega f(x)|u|^{1-\beta(x)}\,dx \end{aligned}$$

which shows that $\mathcal{J}$ is well-defined on $W_0^{1,p(x)}(\Omega)$. Applying similar steps gives

$$\begin{aligned} \mathcal{J}(u) &\ge \frac{\kappa_1}{\alpha_1}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\,dx\right)^{\alpha_1} - \frac{|g|_\infty}{q^- + 1}\int_\Omega |u|^{q(x)+1}\,dx + \frac{1}{\beta^+ - 1}\int_\Omega f(x)|u|^{1-\beta(x)}\,dx \\ &\ge \frac{\kappa_1}{\alpha_1 (p^+)^{\alpha_1}}\|u\|^{\alpha_1 p^-} - \frac{|g|_\infty}{q^- + 1}\|u\|^{q^+ + 1} + \frac{1}{\beta^+ - 1}\int_\Omega f(x)|u|^{1-\beta(x)}\,dx. \end{aligned}$$

That is, $\mathcal{J}$ is coercive ($\mathcal{J}(u) \to \infty$ as $\|u\| \to \infty$), and bounded below on $W_0^{1,p(x)}(\Omega)$. □

Next, we provide a-priori estimate.

Lemma 3.5. *Assume that $(u_n) \subset \mathcal{N}_1$ is a nonnegative minimizing sequence for the minimization problem $\lim_{n\to\infty} \mathcal{J}(u_n) = \inf_{\mathcal{N}_1} \mathcal{J}$. Then, there exist positive real numbers δ_1, δ_2 such that*

$$\delta_1 \le \|u_n\| \le \delta_2.$$

Proof. Assume by contradiction that there exists a subsequence (u_n) (not relabelled) such that $u_n \to 0$ in $W_0^{1,p(x)}(\Omega)$. Then, using the reversed Hölder's inequality, we have
(3.2)
$$\left(\int_\Omega f(x)^{1/\beta^-}\,dx\right)^{\beta^-}\left(\int_\Omega |u_n|\,dx\right)^{1-\beta^-} \le \int_\Omega f(x)|u_n|^{1-\beta^-}\,dx \le \int_\Omega f(x)|u_n|^{1-\beta(x)}\,dx.$$

By assumption, $(u_n)\subset \mathcal{N}_1$. Thus, using (3.2) along with (A$_4$), Proposition 2.2 and the necessary embeddings lead to

$$\begin{aligned}&\left(\int_\Omega f(x)^{1/\beta^-}\,dx\right)^{\beta^-}\left(\int_\Omega |u_n|\,dx\right)^{1-\beta^-}\\ &\le \int_\Omega f(x)|u_n|^{1-\beta^-}\,dx \le \frac{\kappa_2}{(p^+)^{\alpha_1-1}}\|u_n\|^{\alpha_2 p^-} - \frac{|g|_\infty}{q^++1}\|u_n\|^{q^++1}\to 0.\end{aligned}$$

Considering the assumption (A$_2$), this can only happen if $\int_\Omega |u_n|\,dx\to\infty$, which is not possible. Therefore, there exists a positive real number δ_1 such that $\|u_n\|\ge\delta_1$. Notice that, this also means that unbounded set $\mathcal{N}_1$ lies in the exterior of $W_0^{1,p(x)}(\Omega)$, i.e., its intersection with a ball centred at $u=0$ is empty.

Let's now assume, on the contrary, that $\|u_n\|>1$ for any n. We know, by the coerciveness of $\mathcal{J}$, that the infimum of $\mathcal{J}$ is attained, that is, $\infty < m := \inf_{u\in W_0^{1,p(x)}(\Omega)}\mathcal{J}(u)$. Moreover, due to the assumption $\lim_{n\to\infty}\mathcal{J}(u_n)=\inf_{\mathcal{N}_1}\mathcal{J}$, $(\mathcal{J}(u_n))$ is bounded. Then,

$$C\|u_n\|+\mathcal{J}(u_n)\ge \frac{\kappa_1}{\alpha_1(p^+)^{\alpha_1}}\|u_n\|^{\alpha_1 p^-} - \frac{|g|_\infty}{q^-+1}\|u_n\|^{q^++1} + \frac{1}{\beta^+-1}\int_\Omega f(x)|u_n|^{1-\beta(x)}\,dx$$

for some constant $C>0$. If we drop the nonnegative terms, we obtain

$$C\|u_n\|+\mathcal{J}(u_n)\ge \frac{\kappa_1}{\alpha_1(p^+)^{\alpha_1}}\|u_n\|^{\alpha_1 p^-} - \frac{|g|_\infty}{q^-+1}\|u_n\|^{q^++1}.$$

Dividing the both sides of the above inequality by $\|u_n\|^{q^++1}$ and passing to the limit as $n\to\infty$ leads to a contradiction. Therefore, there exists a positive real number δ_2 such that $\|u_n\|\le\delta_2$. □

Lemma 3.6. *$\mathcal{N}_1$ is closed in $W_0^{1,p(x)}(\Omega)$.*

Proof. Assume that $(u_n)\subset\mathcal{N}_1$ such that $u_n\to\widehat{u}$ (strongly) in $W_0^{1,p(x)}(\Omega)$. Thus, $u_n(x)\to\widehat{u}(x)$ a.e. in Ω. Then, using Fatou's lemma, it reads

$$\begin{aligned}&\liminf_{n\to\infty}\left[\mathcal{M}\left(\int_\Omega\frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega|\nabla u_n|^{p(x)}\,dx - \int_\Omega g(x)|u_n|^{q(x)+1}\,dx\right]\\ &\ge \liminf_{n\to\infty}\left[\int_\Omega f(x)|u_n|^{1-\beta(x)}\,dx\right],\end{aligned}$$

and hence

$$\mathcal{M}\left(\int_\Omega \frac{|\nabla \widehat{u}|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla \widehat{u}|^{p(x)}\,dx - \int_\Omega g(x)|\widehat{u}|^{q(x)+1}\,dx \geq \int_\Omega f(x)|\widehat{u}|^{1-\beta(x)}\,dx$$

which means $\widehat{u} \in \mathcal{N}_1$. $\mathcal{N}_1$ is closed in $W_0^{1,p(x)}(\Omega)$. □

Lemma 3.7. *For any $u \in W_0^{1,p(x)}(\Omega)$ satisfying $\int_\Omega f(x)|u|^{1-\beta(x)}\,dx < \infty$, there exists a unique continuous scaling function $W_0^{1,p(x)}(\Omega) \to (0,\infty)\colon u \mapsto t(u)$ such that $t(u)u \in \mathcal{N}_2$, and $t(u)u$ is the minimizer of the functional $\mathcal{J}$ along the ray $\{tu : t > 0\}$, that is, $\inf_{t>0} \mathcal{J}(tu) = \mathcal{J}(t(u)u)$.*

Proof. Fix $u \in W_0^{1,p(x)}(\Omega)$ such that $\int_\Omega f(x)|u|^{1-\beta(x)}\,dx < \infty$. For any $t > 0$, the scaled functional, $\mathcal{J}(tu)$, determines a curve that can be characterized by

$$\Phi(t) := \mathcal{J}(tu), \quad t \in [0,\infty).$$

Then, for a $t \in [0,\infty)$, $tu \in \mathcal{N}_2$ if and only if

$$\Phi'(t) = \frac{d}{dt}\Phi(t)\Big|_{t=t(u)} = 0.$$

First, we show that $\Phi(t)$ attains its minimum on $[0,\infty)$ at some point $t = t(u)$. Considering the fact $0 < \int_\Omega f(x)|u|^{1-\beta(x)}\,dx < \infty$, we will examine two cases for t.

For $0 < t < 1$:

$$\begin{aligned}\Phi(t) = \mathcal{J}(tu) &\geq \frac{\kappa_1 \delta_1^{\alpha_1 p_m}}{\alpha_1 (p^+)^{\alpha_1}} t^{\alpha_1 p^+} - \frac{t^{q^-+1}}{q^-+1}\int_\Omega g(x)|u|^{q(x)+1}\,dx + \frac{t^{1-\beta^-}}{\beta^+-1}\int_\Omega f(x)|u|^{1-\beta(x)}\,dx \\ &:= \Psi_0(t),\end{aligned}$$

where $p_m := \begin{cases} p^- & \text{if } \|u\| \geq 1, \\ p^+ & \text{if } \|u\| \leq 1. \end{cases}$

Then, the comparison function $\Psi_0\colon (0,1) \to \mathbb{R}$ is continuous. Taking the derivative of Ψ_0 gives

$$\begin{aligned}\Psi_0'(t) = {} & \frac{\kappa_1 \delta_1^{\alpha_1 p_m} p^+}{(p^+)^{\alpha_1}} t^{\alpha_1 p^+ - 1} - t^{q^-}\int_\Omega g(x)|u|^{q(x)+1}\,dx \\ & + \left(\frac{1-\beta^-}{\beta^+-1}\right) t^{-\beta^-}\int_\Omega f(x)|u|^{1-\beta(x)}\,dx.\end{aligned} \tag{3.3}$$

It is easy to see from (3.3) that $\Psi_0'(t) < 0$ when $t > 0$ is small enough. Therefore, $\Psi_0(t)$ is decreasing when $t > 0$ is small enough. In the similar way,

$$\begin{aligned}\Phi(t) = \mathcal{J}(tu) &\leq \frac{\kappa_2 \delta_2^{\alpha_2 p_M}}{\alpha_2 (p^-)^{\alpha_2}} t^{\alpha_2 p^-} - \frac{t^{q^++1}}{q^++1}\int_\Omega g(x)|u|^{q(x)+1}\,dx + \frac{t^{1-\beta^+}}{\beta^--1}\int_\Omega f(x)|u|^{1-\beta(x)}\,dx \\ &:= \Psi_1(t),\end{aligned}$$

where $p_M := \begin{cases} p^- & \text{if } \|u\| \le 1, \\ p^+ & \text{if } \|u\| \ge 1. \end{cases}$

Then, the comparison function $\Psi_1\colon (0,1) \to \mathbb{R}$ is continuous. Taking the derivative of Ψ_1 gives

$$\begin{aligned} \Psi_1'(t) = {} & \frac{\kappa_2 \delta_2^{\alpha_2 p_M} \alpha_2 p^-}{\alpha_2 (p^-)^{\alpha_2}} t^{\alpha_2 p^- - 1} - t^{q^+} \int_\Omega g(x)|u|^{q(x)+1}\,dx \\ & + \left(\frac{1-\beta^+}{\beta^+ - 1}\right) t^{-\beta^+} \int_\Omega f(x)|u|^{1-\beta(x)}\,dx. \end{aligned} \tag{3.4}$$

But (3.4) also suggests that $\Psi_1'(t) < 0$ when $t > 0$ is small enough. Thus, $\Psi_1(t)$ is decreasing when $t > 0$ is small enough. Therefore, since $\Psi_0(t) \le \Phi(t) \le \Psi_1(t)$ for $0 < t < 1$, $\Phi(t)$ is decreasing when $t > 0$ is small enough.

For $t > 1$: Following the same arguments shows that $\Psi_0'(t) > 0$ and $\Psi_1'(t) > 0$ when $t > 1$ is large enough, and therefore, both $\Psi_0(t)$ and $\Psi_1(t)$ are increasing. Thus, $\Phi(t)$ is increasing when $t > 1$ is large enough. In conclusion, since $\Phi(0) = 0$, $\Phi(t)$ attains its minimum on $[0, \infty)$ at some point, say $t = t(u)$. That is, $\frac{d}{dt}\Phi(t)\big|_{t=t(u)} = 0$. Then, $t(u)u \in \mathcal{N}_2$ and $\inf_{t>0} \mathcal{J}(tu) = \mathcal{J}(t(u)u)$.

Next, we show that scaling function $t(u)$ is continuous on $W_0^{1,p(x)}(\Omega)$. Let $u_n \to u$ in $W_0^{1,p(x)}(\Omega) \setminus \{0\}$, and $t_n = t(u_n)$. Then, by the definition, $t_n u_n \in \mathcal{N}_2$. Defined in this way, the sequence t_n is bounded. Assume on the contrary that $t_n \to \infty$ (up to a subsequence). Then, using the fact $t_n u_n \in \mathcal{N}_2$ along with (A_4), it follows

$$\begin{aligned} & \mathcal{M}\left(\int_\Omega \frac{|\nabla t_n u_n|^{p(x)}}{p(x)}\,dx\right) \int_\Omega |\nabla t_n u_n|^{p(x)}\,dx - \int_\Omega g(x)|t_n u_n|^{q(x)+1}\,dx \\ & = \int_\Omega f(x)|t_n u_n|^{1-\beta(x)}\,dx, \\ & \frac{\kappa_1 t_n^{\alpha_1 p^-}}{(p^+)^{\alpha_1 - 1}} \left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^{\alpha_1} - t_n^{q^-+1} \int_\Omega g(x)|u_n|^{q(x)+1}\,dx \\ & \le t_n^{1-\beta^-} \int_\Omega f(x)|u_n|^{1-\beta(x)}\,dx, \end{aligned}$$

which suggests a contradiction as $t_n \to \infty$ since $p^- > q^- + 1 > 1 - \beta^-$. Hence, sequence t_n is bounded. Therefore, there exists a subsequence t_n (not relabelled) such that $t_n \to t_0$, $t_0 \ge 0$. On the other hand, from Lemma 3.5, $\|t_n u_n\| \ge \delta_1 > 0$. Thus, $t_0 > 0$ and $t_0 u \in \mathcal{N}_2$. By the uniqueness of the map $t(u)$, $t_0 = t(u)$, which concludes the continuity of $t(u)$. In conclusion, for any $u \in W_0^{1,p(x)}(\Omega)$ satisfying $\int_\Omega f(x)|u|^{1-\beta(x)}\,dx < \infty$, the function $t(u)$ scales $u \in W_0^{1,p(x)}(\Omega)$ continuously to a point such that $t(u)u \in \mathcal{N}_2$. □

Lemma 3.8. *Assume that $(u_n) \subset \mathcal{N}_1$ is the nonnegative minimizing sequence for the minimization problem $\lim_{n\to\infty} \mathcal{J}(u_n) = \inf_{\mathcal{N}_1} \mathcal{J}$. Then, there exists a subsequence (u_n) (not relabelled) such that $u_n \to u^*$ (strongly) in $W_0^{1,p(x)}(\Omega)$.*

Proof. Since (u_n) is bounded in $W_0^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ is reflexive, there exists a subsequence (u_n), not relabelled, and $u^* \in W_0^{1,p(x)}(\Omega)$ such that

- $u_n \rightharpoonup u^*$ (weakly) in $W_0^{1,p(x)}(\Omega)$,
- $u_n \to u^*$ in $L^{s(x)}(\Omega)$, $1 \le s(x) < p^*(x)$,
- $u_n(x) \to u^*(x)$ a.e. in Ω.

Since the norm $\|\cdot\|$ is a continuous convex functional, it is weakly lower semicontinuous. Using this fact along with the Fatou's lemma, Lemmas 3.5 and 3.7, it reads

$$\begin{aligned}
\inf_{\mathcal{N}_1} \mathcal{J} &= \lim_{n\to\infty} \mathcal{J}(u_n) \\
&\ge \liminf_{n\to\infty} \left[\widehat{\mathcal{M}} \left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\, dx \right) - \int_\Omega \frac{g(x)|u_n|^{q(x)+1}}{q(x)+1}\, dx \right] \\
&\quad + \liminf_{n\to\infty} \left[\int_\Omega \frac{f(x)|u_n|^{1-\beta(x)}}{\beta(x)-1}\, dx \right] \\
&\ge \widehat{\mathcal{M}} \left(\int_\Omega \frac{|\nabla u^*|^{p(x)}}{p(x)}\, dx \right) - \int_\Omega \frac{g(x)|u^*|^{q(x)+1}}{q(x)+1}\, dx + \int_\Omega \frac{f(x)|u^*|^{1-\beta(x)}}{\beta(x)-1}\, dx \\
&= \mathcal{J}(u^*) \ge \mathcal{J}(t(u^*)u^*) \ge \inf_{\mathcal{N}_2} \mathcal{J} \ge \inf_{\mathcal{N}_1} \mathcal{J}.
\end{aligned}$$

The above result implies, up to subsequences, that

$$\lim_{n\to\infty} \|u_n\| = \|u^*\|. \tag{3.5}$$

Thus, (3.5) along with $u_n \rightharpoonup u^*$ in $W_0^{1,p(x)}(\Omega)$ show that $u_n \to u^*$ in $W_0^{1,p(x)}(\Omega)$. □

The following is the main result of the present paper.

Theorem 3.9. *Assume that the conditions* (A_1)–(A_6) *hold. Then, problem* (1.1) *has at least one positive* $W_0^{1,p(x)}(\Omega)$*-solution if only if there exists* $\overline{u} \in W_0^{1,p(x)}(\Omega)$ *satisfying* $\int_\Omega f(x)|\overline{u}|^{1-\beta(x)}\, dx < \infty$.

Proof. ($\Rightarrow$) Assume that the function $u \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem (1.1). Then, letting $u = \varphi$ in Definition 3.1 gives

$$\begin{aligned}
\int_\Omega f(x)|u|^{1-\beta(x)}\, dx &= \mathcal{M}\left(\int_\Omega \frac{|\nabla u|^{p(x)}}{p(x)}\, dx \right) \int_\Omega |\nabla u|^{p(x}\, dx - \int_\Omega g(x)|u|^{q(x)+1}\, dx \\
&\le \frac{\kappa_2}{(p^-)^{\alpha_2-1}} \|u\|^{\alpha_2 p_M} - |g|_\infty |u|_{q(x)+1}^{q_M} < \frac{\kappa_2}{(p^-)^{\alpha_2-1}} \|u\|^{\alpha_2 p_M} \\
&< \infty.
\end{aligned}$$

($\Leftarrow$) Assume that there exists $\overline{u} \in W_0^{1,p(x)}(\Omega)$ such that $\int_\Omega f(x)|\overline{u}|^{1-\beta(x)}\,dx < \infty$. Then, by Lemma 3.7, there exists a unique number $t(\overline{u}) > 0$ such that $t(\overline{u})\overline{u} \in \mathcal{N}_2$.

The information we have had about $\mathcal{J}$ so far and the closeness of $\mathcal{N}_1$ allow us to apply Ekeland's variational principle to the problem $\inf_{\mathcal{N}_1} \mathcal{J}$. That is, it suggests the existence of a corresponding minimizing sequence $(u_n) \subset \mathcal{N}_1$ satisfying the following:

(E$_1$) $\mathcal{J}(u_n) - \inf_{\mathcal{N}_1} \mathcal{J} \leq \frac{1}{n}$,

(E$_2$) $\mathcal{J}(u_n) - \mathcal{J}(\nu) \leq \frac{1}{n}\|u_n - \nu\|$, $\forall\, \nu \in \mathcal{N}_1$.

Due to the fact $\mathcal{J}(|u_n|) = \mathcal{J}(u_n)$, it is not wrong to assume that $u_n \geq 0$ a.e. in Ω. Additionally, considering that $(u_n) \subset \mathcal{N}_1$ and following the same approach as it is done in the ($\Rightarrow$) part, we can obtain that $\int_\Omega f(x)|u_n|^{1-\beta(x)}\,dx < \infty$. If all these information and our assumptions are taken into consideration, it follows that $u_n(x) > 0$ a.e. in Ω. The rest of the proof is split into two cases.

Case 1*: $(u_n) \subset \mathcal{N}_1 \setminus \mathcal{N}_2$ for n large.* For a function $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \geq 0$, and $t \geq 0$, we have

$$0 < (u_n(x) + t\varphi(x))^{1-\beta(x)} \leq u_n(x)^{1-\beta(x)} \quad \text{a.e. in } \Omega.$$

Therefore, using (A$_1$) and (A$_2$) gives

$$\begin{aligned}\int_\Omega f(x)(u_n + t\varphi)^{1-\beta(x)}\,dx &\leq \int_\Omega f(x)u_n^{1-\beta(x)}\,dx \\ &< \mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega g(x)u_n^{q(x)+1}\,dx.\end{aligned}$$

Therefore, when $t > 0$ is sufficiently small, we obtain

$$\begin{aligned}&\int_\Omega f(x)(u_n + t\varphi)^{1-\beta(x)}\,dx \\ &< \mathcal{M}\left(\int_\Omega \frac{|\nabla(u_n + t\varphi)|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla(u_n + t\varphi)|^{p(x)}\,dx - \int_\Omega g(x)(u_n + t\varphi)^{q(x)+1}\,dx,\end{aligned}$$

which means that $\nu := u_n + t\varphi \in \mathcal{N}_1 \setminus \mathcal{N}_2$. Now, using (E$_2$), it reads

$$\begin{aligned}\frac{1}{n}\|t\varphi\| &\geq \mathcal{J}(u_n) - \mathcal{J}(\nu) \\ &= \widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right) - \widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla(u_n + t\varphi)|^{p(x)}}{p(x)}\,dx\right) \\ &\quad - \int_\Omega \frac{g(x)u_n^{q(x)+1}}{q(x)+1}\,dx + \int_\Omega \frac{g(x)(u_n + t\varphi)^{q(x)+1}}{q(x)+1}\,dx \\ &\quad + \int_\Omega \frac{f(x)u_n^{1-\beta(x)}}{\beta(x)-1}\,dx - \int_\Omega \frac{f(x)(u_n + t\varphi)^{1-\beta(x)}}{\beta(x)-1}\,dx.\end{aligned}$$

Dividing the above inequality by t and passing to the infimum limit as $t \to 0$ gives

$$\liminf_{t\to 0} \frac{\|\varphi\|}{n} + \underbrace{\liminf_{t\to 0} \left[\frac{1}{t} \left(\widehat{\mathcal{M}} \left(\int_\Omega \frac{|\nabla (u_n + t\varphi)|^{p(x)}}{p(x)}\, dx \right) - \widehat{\mathcal{M}} \left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\, dx \right) \right) \right]}_{:=I_1}$$
$$\underbrace{- \liminf_{t\to 0} \left[\int_\Omega g(x) \frac{\left[(u_n + t\varphi)^{q(x)+1} - u_n^{q(x)+1} \right]}{t(q(x)+1)}\, dx \right]}_{:=I_2}$$
$$\geq \underbrace{\liminf_{t\to 0} \left[\int_\Omega f(x) \frac{\left[(u_n + t\varphi)^{1-\beta(x)} - u_n^{1-\beta(x)} \right]}{t(1-\beta(x))}\, dx \right]}_{:=I_3}.$$

Calculations of I_1 and I_2 gives

$$\begin{aligned} I_1 &= \frac{d}{dt} \widehat{\mathcal{M}} \left(\int_\Omega \frac{|\nabla (u_n + t\varphi)|^{p(x)}}{p(x)}\, dx \right) \Big|_{t=0} \\ &= \mathcal{M} \left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\, dx \right) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi\, dx, \end{aligned}$$

and

$$I_2 = \frac{d}{dt} \left(\int_\Omega g(x) \frac{(u_n + t\varphi)^{q(x)+1}}{q(x)+1}\, dx \right) \Big|_{t=0} = \int_\Omega g(x) u_n^{q(x)} \varphi\, dx.$$

For I_3, since $t > 0$ it holds

$$u_n^{1-\beta(x)}(x) - (u_n(x) + t\varphi(x))^{1-\beta(x)} \geq 0 \quad \text{a.e. in } \Omega.$$

We can apply Fatou's lemma, that is,

$$\begin{aligned} I_3 &= \liminf_{t\to 0} \int_\Omega f(x) \frac{\left[(u_n + t\varphi)^{1-\beta(x)} - u_n^{1-\beta(x)} \right]}{t(1-\beta(x))}\, dx \\ &\geq \int_\Omega \liminf_{t\to 0} f(x) \frac{\left[(u_n + t\varphi)^{1-\beta(x)} - u_n^{1-\beta(x)} \right]}{t(1-\beta(x))}\, dx \\ &\geq \int_\Omega f(x) u_n^{-\beta(x)} \varphi\, dx. \end{aligned}$$

Now, substituting I_1, I_2, I_3 gives

$$\frac{\|\varphi\|}{n} + \mathcal{M} \left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\, dx \right) \int_\Omega |\nabla u_n|^{p(x)-2} \nabla u_n \cdot \nabla \varphi\, dx - \int_\Omega g(x) u_n^{q(x)} \varphi\, dx$$
$$\geq \int_\Omega f(x) u_n^{-\beta(x)} \varphi\, dx.$$

From Lemma 3.8, we know that $u_n \to u^*$ in $W_0^{1,p(x)}(\Omega)$. Thus, also considering Fatou's lemma, we obtain

$$\begin{aligned} &\mathcal{M}\left(\int_\Omega \frac{|\nabla u^*|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u^*|^{p(x)-2}\nabla u^* \cdot \nabla\varphi\,dx \\ &- \int_\Omega g(x)(u^*)^{q(x)}\varphi\,dx - \int_\Omega f(x)(u^*)^{-\beta(x)}\varphi\,dx \geq 0 \end{aligned} \tag{3.6}$$

for any $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \geq 0$. Letting $\varphi = u^*$ in (3.6) shows clearly that $u^* \in \mathcal{N}_1$. Lastly, from Lemma 3.8, we can conclude that

$$\lim_{n\to\infty} \mathcal{J}(u_n) = \mathcal{J}(u^*) = \inf_{\mathcal{N}_2} \mathcal{J},$$

which means

$$u^* \in \mathcal{N}_2 \quad \text{with } t(u^*) = 1. \tag{3.7}$$

Case 2: There exists a subsequence of (u_n) (not relabelled) contained in $\mathcal{N}_2$. For a function $\varphi \in W_0^{1,p(x)}(\Omega)$ with $\varphi \geq 0$, $t > 0$, and $u_n \in \mathcal{N}_2$, we have

$$\begin{aligned} \int_\Omega f(x)(u_n + t\varphi)^{1-\beta(x)}\,dx &\leq \int_\Omega f(x)u_n^{1-\beta(x)}\,dx \\ &= \mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega g(x)u_n^{q(x)+1}\,dx \\ &< \infty, \end{aligned}$$

and hence there exists a unique continuous scaling function, denoted by $\theta_n(t) := t(u_n + t\varphi) > 0$, corresponding to $(u_n + t\varphi)$ so that $\theta_n(t)(u_n + t\varphi) \in \mathcal{N}_2$ for $n = 1, 2, \ldots$. Obviously, $\theta_n(0) = 1$. Since $\theta_n(t)(u_n + t\varphi) \in \mathcal{N}_2$, we have

$$\begin{aligned} 0 &= \mathcal{M}\left(\int_\Omega \frac{|\nabla\theta_n(t)(u_n + t\varphi)|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla\theta_n(t)(u_n + t\varphi)|^{p(x)}\,dx \\ &\quad - \int_\Omega g(x)(\theta_n(t)(u_n + t\varphi))^{q(x)+1}\,dx - \int_\Omega f(x)(\theta_n(t)(u_n + t\varphi))^{1-\beta(x)}\,dx \\ &\geq \mathcal{M}\left(\int_\Omega \frac{|\nabla\theta_n(t)(u_n + t\varphi)|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla\theta_n(t)(u_n + t\varphi)|^{p(x)}\,dx \\ &\quad - \theta_n^{q_M+1}(t)\int_\Omega g(x)(u_n + t\varphi)^{q(x)+1}\,dx - \theta_n^{1-\beta_m}(t)\int_\Omega f(x)(u_n + t\varphi)^{1-\beta(x)}\,dx, \end{aligned} \tag{3.8}$$

and

$$0 = \mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx - \int_\Omega g(x)u_n^{q(x)+1}\,dx - \int_\Omega f(x)u_n^{1-\beta(x)}\,dx. \tag{3.9}$$

Then, using (3.8) and (3.9) together and applying the Mean Value Theorem gives

(3.10)
$$\begin{aligned}
0 \geq \Big[&-(q_M+1)[\theta_n(0)+\tau_1(\theta_n(t)-\theta_n(0))]^{q_M}\int_\Omega g(x)(u_n+t\varphi)^{q(x)+1}\,dx \\
&-(1-\beta_m)[\theta_n(0)+\tau_2(\theta_n(t)-\theta_n(0))]^{-\beta_m}\int_\Omega f(x)(u_n+t\varphi)^{1-\beta(x)}\,dx\Big](\theta_n(t)-\theta_n(0)) \\
&+\mathcal{M}\left(\int_\Omega \frac{|\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}\,dx \\
&-\mathcal{M}\left(\int_\Omega \frac{|\nabla(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla(u_n+t\varphi)|^{p(x)}\,dx \\
&+\mathcal{M}\left(\int_\Omega \frac{|\nabla(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla(u_n+t\varphi)|^{p(x)}\,dx \\
&-\mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx \\
&-\left[\int_\Omega g(x)(u_n+t\varphi)^{q(x)+1}\,dx-\int_\Omega g(x)u_n^{q(x)+1}\,dx\right] \\
&-\left[\int_\Omega f(x)(u_n+t\varphi)^{1-\beta(x)}\,dx-\int_\Omega f(x)u_n^{1-\beta(x)}\,dx\right]
\end{aligned}$$

for some constants $\tau_1,\tau_2\in(0,1)$, where $\beta_m:=\min\{\beta^-,\beta^+\}$. To proceed, we assume that $\theta_n'(0)=\lim_{t\to 0}\frac{\theta_n(t)-\theta_n(0)}{t}\in[-\infty,\infty]$. In case this limit doesn't exist, we can consider a subsequence $t_k>0$ of t such that $t_k\to 0$ as $k\to\infty$.

Next, we show that $\theta_n'(0)\neq\infty$. Dividing the both sides of (3.10) by t and passing to the limit as $t\to 0$ leads to

$$\begin{aligned}
0 \geq \Big[&p^-\mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx+\mathcal{M}'\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^2 \\
&+(\beta_m-1)\int_\Omega f(x)u_n^{1-\beta(x)}\,dx-(q^++1)\int_\Omega g(x)u_n^{q(x)+1}\,dx\Big]\theta_n'(0) \\
&+\left[p^-\mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)+\mathcal{M}'\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx\right] \\
&\times\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\varphi\,dx \\
&-(q^++1)\int_\Omega g(x)u_n^{q(x)}\varphi\,dx+(\beta_m-1)\int_\Omega f(x)u_n^{-\beta(x)}\varphi\,dx,
\end{aligned}$$

or

$$\begin{aligned}
0 \geq \Big[&(p^--q^+-1)\mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx \\
&+\mathcal{M}'\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\left(\int_\Omega |\nabla u_n|^{p(x)}\,dx\right)^2+(\beta_m+q^+)\int_\Omega f(x)u_n^{1-\beta(x)}\,dx\Big]\theta_n'(0)
\end{aligned}$$

$$+\left[p^-\mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)+\mathcal{M}'\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx\right]$$
$$\times\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\varphi\,dx$$
$$-(q^++1)\int_\Omega g(x)u_n^{q(x)}\varphi\,dx+(\beta_m-1)\int_\Omega f(x)u_n^{-\beta(x)}\varphi\,dx$$

which along with Lemma 3.5 concludes that $-\infty\le\theta_n'(0)<\infty$, and hence, $\theta_n'(0)\le\bar{c}$, uniformly in all large n.

Next, we show that $\theta_n'(0)\neq-\infty$. First, we apply Ekeland's variational principle to the minimizing sequence $(u_n)\subset\mathcal{N}_2(\subset\mathcal{N}_1)$. Thus, letting $\nu:=\theta_n(t)(u_n+t\varphi)$ in (E_2) gives

$$\begin{aligned}
&\frac{1}{n}\big[|\theta_n(t)-1|\|u_n\|+t\theta_n(t)\|\varphi\|\big]\\
&\ge\mathcal{J}(u_n)-\mathcal{J}(\theta_n(t)(u_n+t\varphi))\\
&=\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)-\int_\Omega \frac{g(x)u_n^{q(x)+1}}{q(x)+1}\,dx+\int_\Omega \frac{f(x)u_n^{1-\beta(x)}}{\beta(x)-1}\,dx\\
&\quad-\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)+\int_\Omega \frac{g(x)[\theta_n(t)(u_n+t\varphi)]^{q(x)+1}}{q(x)+1}\,dx\\
&\quad-\int_\Omega \frac{f(x)[\theta_n(t)(u_n+t\varphi)]^{1-\beta(x)}}{\beta(x)-1}\,dx\\
&\ge\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)-\int_\Omega \frac{g(x)u_n^{q(x)+1}}{q(x)+1}\,dx+\int_\Omega \frac{f(x)u_n^{1-\beta(x)}}{\beta(x)-1}\,dx\\
&\quad-\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)+\int_\Omega \frac{g(x)[\theta_n(t)(u_n+t\varphi)]^{q(x)+1}}{q(x)+1}\,dx\\
&\quad-\frac{1}{\beta^--1}\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)+\frac{1}{\beta^--1}\int_\Omega g(x)[\theta_n(t)(u_n+t\varphi)]^{q(x)+1}\,dx
\end{aligned}$$

since $\beta^-\ge q^++1$, it reduces to

$$\begin{aligned}
&\frac{1}{n}\big[|\theta_n(t)-1|\|u_n\|+t\theta_n(t)\|\varphi\|\big]\\
&\ge\left[\left(\frac{1}{q^-+1}\right)[\theta_n(0)+\tau_1(\theta_n(t)-\theta_n(0))]^{q_m}\int_\Omega g(x)(u_n+t\varphi)^{q(x)+1}\,dx\right]\\
&\quad\times(\theta_n(t)-\theta_n(0))\\
&\quad+\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)-\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\\
&\quad+\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)-\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\\
&\quad+\frac{1}{q^-+1}\int_\Omega g(x)(u_n+t\varphi)^{q(x)+1}\,dx-\frac{1}{q^-+1}\int_\Omega g(x)u_n^{q(x)+1}\,dx
\end{aligned}\tag{3.11}$$

$$- \frac{1}{q^-}\mathcal{M}\left(\int_\Omega \frac{|\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}\,dx.$$

If we use Lemma 3.5 to manipulate the norm $\|u+t\varphi\|$, the integral in the last line of (3.11) can be written as follows:

$$\begin{aligned}
&\frac{1}{q^-}\mathcal{M}\left(\int_\Omega \frac{|\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}\,dx\\
&\le \frac{\theta_n^{p_M}(t)}{q^-}\mathcal{M}\left(\theta_n^{p_M}(t)\int_\Omega \frac{|\nabla(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla(u_n+t\varphi)|^{p(x)}\,dx\\
&\le \frac{\kappa_2\theta_n^{\alpha_2 p_M}(t)}{q^-(p^-)^{(\alpha_2-1)}}\|u_n+t\varphi\|^{\alpha_2 p_M}\\
&\le \frac{\kappa_2\theta_n^{\alpha_2 p_M}(t)C(\delta_2)^{\alpha_2 p_M}}{q^-(p^-)^{(\alpha_2-1)}}\|\varphi\|^{\alpha_2 p_M}t.
\end{aligned}$$

Then, putting all these together leads to

$$\begin{aligned}
&\frac{1}{n}\left[|\theta_n(t)-1|\|u_n\|+t\theta_n(t)\|\varphi\|\right]+\frac{\kappa_2\theta_n^{\alpha_2 p_M}(t)C(\delta_2)^{\alpha_2 p_M}}{q^-(p^-)^{(\alpha_2-1)}}\|\varphi\|^{\alpha_2 p_M}t\\
\ge&\left[\left(\frac{1}{q^-+1}\right)[\theta_n(0)+\tau_1(\theta_n(t)-\theta_n(0))]^{q_M}\int_\Omega g(x)(u_n+t\varphi)^{q(x)+1}\,dx\right](\theta_n(t)-\theta_n(0))\\
&-\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)+\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\\
&-\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)+\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla\theta_n(t)(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\\
&+\frac{1}{q^-+1}\int_\Omega g(x)\left[(u_n+t\varphi)^{q(x)+1}-u_n^{q(x)+1}\right]dx.
\end{aligned}$$

Dividing by t and passing to the limit as $t\to 0$ gives

$$\begin{aligned}
&\frac{1}{n}\|\varphi\|+\frac{\kappa_2\theta_n^{\alpha_2 p_M}(t)C(\delta_2)^{\alpha_2 p_M}}{q^-(p^-)^{(\alpha_2-1)}}\|\varphi\|^{\alpha_2 p_M}\\
&\ge\left[\left(-1+\frac{1}{q^-+1}\right)\mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx\right.\\
&\quad\left.-\frac{1}{q^-+1}\int_\Omega f(x)u_n^{1-\beta(x)}\,dx-\frac{\|u_n\|}{n}\operatorname{sgn}[\theta_n(t)-1]\right]\theta_n'(0)\\
&\quad-\mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\varphi\,dx+\int_\Omega g(x)u_n^{q(x)}\,dx
\end{aligned}$$

which concludes that $\theta_n'(0)\neq-\infty$. Thus, $\theta_n'(0)\ge \underline{c}$ uniformly in large n. In conclusion, there exists a constant, $C_0>0$ such that $|\theta_n'(0)|\le C_0$ when $n\ge N$, $N\in\mathbb{N}$.

Next, we show that $u^* \in \mathcal{N}_2$. Using (E_2) again, we have

$$
\begin{aligned}
&\frac{1}{n}\big[|\theta_n(t)-1|\|u_n\|+t\theta_n(t)\|\varphi\|\big]\\
&\geq \mathcal{J}(u_n)-\mathcal{J}(\theta_n(t)(u_n+t\varphi))\\
&=\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)-\int_\Omega \frac{g(x)u_n^{q(x)+1}}{q(x)+1}\,dx+\int_\Omega \frac{f(x)u_n^{1-\beta(x)}}{\beta(x)-1}\,dx\\
&\quad-\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla \theta_n(t)(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)+\int_\Omega \frac{g(x)[\theta_n(t)(u_n+t\varphi)]^{q(x)+1}}{q(x)+1}\,dx\\
&\quad-\int_\Omega \frac{f(x)[\theta_n(t)(u_n+t\varphi)]^{1-\beta(x)}}{\beta(x)-1}\,dx\\
&=-\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla (u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)+\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)+\int_\Omega \frac{g(x)(u_n+t\varphi)^{q(x)+1}}{q(x)+1}\,dx\\
&\quad-\int_\Omega \frac{g(x)u_n^{q(x)+1}}{q(x)+1}\,dx-\int_\Omega \frac{f(x)(u_n+t\varphi)^{1-\beta(x)}}{\beta(x)-1}\,dx+\int_\Omega \frac{f(x)u_n^{1-\beta(x)}}{\beta(x)-1}\,dx\\
&\quad-\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla \theta_n(t)(u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)+\widehat{\mathcal{M}}\left(\int_\Omega \frac{|\nabla (u_n+t\varphi)|^{p(x)}}{p(x)}\,dx\right)\\
&=\int_\Omega \frac{g(x)[\theta_n(t)(u_n+t\varphi)]^{q(x)+1}}{q(x)+1}\,dx-\int_\Omega \frac{g(x)(u_n+t\varphi)^{q(x)+1}}{q(x)+1}\,dx\\
&\quad-\int_\Omega \frac{f(x)[\theta_n(t)(u_n+t\varphi)]^{1-\beta(x)}}{\beta(x)-1}\,dx+\int_\Omega \frac{f(x)(u_n+t\varphi)^{1-\beta(x)}}{\beta(x)-1}\,dx.
\end{aligned}
$$

Dividing by t and passing to the limit as $t\to 0$ gives

$$
\begin{aligned}
&\frac{1}{n}\big[|\theta_n'(0)|\|u_n\|+\|\varphi\|\big]\\
&\geq \left[-\mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)}\,dx+\int_\Omega g(x)u_n^{q(x)+1}\,dx+\int_\Omega f(x)u_n^{1-\beta(x)}\,dx\right]\theta_n'(0)\\
&\quad-\mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\varphi\,dx+\int_\Omega g(x)u_n^{q(x)}\varphi\,dx\\
&\quad+\int_\Omega f(x)u_n^{-\beta(x)}\varphi\,dx\\
&=-\mathcal{M}\left(\int_\Omega \frac{|\nabla u_n|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u_n|^{p(x)-2}\nabla u_n\cdot\nabla\varphi\,dx+\int_\Omega g(x)u_n^{q(x)}\varphi\,dx\\
&\quad+\int_\Omega f(x)u_n^{-\beta(x)}\varphi\,dx.
\end{aligned}
$$

If we consider that $|\theta_n'(0)|\leq C_0$ uniformly in n, we obtain that $\int_\Omega f(x)u_n^{-\beta(x)}\,dx<\infty$.

Therefore, as $n \to \infty$, it holds

$$\begin{aligned} &\mathcal{M}\left(\int_\Omega \frac{|\nabla u^*|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u^*|^{p(x)-2}\nabla u^*\cdot\nabla\varphi\,dx \\ &-\int_\Omega g(x)(u^*)^{q(x)}\varphi\,dx-\int_\Omega f(x)(u^*)^{-\beta(x)}\varphi\,dx\ge 0 \end{aligned} \tag{3.12}$$

for all $\varphi \in W_0^{1,p(x)}(\Omega)$, $\varphi \geq 0$. Letting $\varphi = u^*$ in (3.12) shows clearly that $u^* \in \mathcal{N}_1$. This means, as done with Case 1, we have

$$u^* \in \mathcal{N}_2. \tag{3.13}$$

By taking into consideration the results (3.6), (3.7), (3.12) and (3.13), we infer that $u^* \in \mathcal{N}_2$ and (3.12) holds, in the weak sense, for both cases. Additionally, since $u^* \geq 0$ and $u^* \neq 0$, by the strong maximum principle for weak solutions, we must have $u^*(x) > 0$ almost everywhere in Ω.

Finally, we show that $u^* \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem (1.1). For a random function $\phi \in W_0^{1,p(x)}(\Omega)$, and $\varepsilon > 0$, let $\varphi = (u^* + \varepsilon\phi)^+ = \max\{0, u^* + \varepsilon\phi\}$. We split Ω into two sets as follows:

$$\Omega_{\geq} = \{x \in \Omega : u^*(x) + \varepsilon\phi(x) \geq 0\} \quad \text{and} \quad \Omega_{<} = \{x \in \Omega : u^*(x) + \varepsilon\phi(x) < 0\}.$$

If we replace φ with $(u^* + \varepsilon\phi)$ in (3.12), it follows

$$\begin{aligned} 0 &\le \mathcal{M}\left(\int_\Omega \frac{|\nabla u^*|^{p(x)}}{p(x)}\,dx\right)\int_\Omega |\nabla u^*|^{p(x)-2}\nabla u^*\cdot\nabla\varphi\,dx \\ &\quad -\int_\Omega g(x)(u^*)^{q(x)}\varphi\,dx-\int_\Omega f(x)(u^*)^{-\beta(x)}\varphi\,dx \\ &= \int_{\Omega_\geq} \mathcal{M}(\rho(u^*))|\nabla u^*|^{p(x)-2}\nabla u^*\cdot\nabla(u^*+\varepsilon\phi)\,dx \\ &\quad -\int_{\Omega_\geq} g(x)(u^*)^{q(x)}(u^*+\varepsilon\phi)\,dx-\int_{\Omega_\geq} f(x)(u^*)^{-\beta(x)}(u^*+\varepsilon\phi)\,dx \\ &= \int_\Omega - \int_{\Omega_<} \big[\mathcal{M}(\rho(u^*))|\nabla u^*|^{p(x)-2}\nabla u^*\cdot\nabla(u^*+\varepsilon\phi) - g(x)(u^*)^{q(x)}(u^*+\varepsilon\phi) \\ &\qquad\qquad - f(x)(u^*)^{-\beta(x)}(u^*+\varepsilon\phi)\big]\,dx \\ &= \mathcal{M}(\rho(u^*))\int_\Omega |\nabla u^*|^{p(x)}\,dx + \varepsilon\int_\Omega |\nabla u^*|^{p(x)-2}\nabla u^*\cdot\nabla\phi\,dx \\ &\quad -\int_\Omega g(x)(u^*)^{q(x)+1}\,dx - \varepsilon\int_\Omega g(x)(u^*)^{q(x)}\phi\,dx \\ &\quad -\int_\Omega f(x)(u^*)^{1-\beta(x)}\,dx - \varepsilon\int_\Omega f(x)(u^*)^{-\beta(x)}\phi\,dx \end{aligned}$$

$$- \int_{\Omega_<} \big[\mathcal{M}(\rho(u^*))|\nabla u^*|^{p(x)-2}\nabla u^* \cdot \nabla(u^* + \varepsilon\phi) - g(x)(u^*)^{q(x)}(u^* + \varepsilon\phi) \\ - f(x)(u^*)^{-\beta(x)}(u^* + \varepsilon\phi)\big]\, dx.$$

If we consider that $u^* \in \mathcal{N}_2$ and drop the relevant terms, then we have

$$0 \leq \varepsilon \left[\int_\Omega \mathcal{M}(\rho(u^*))|\nabla u^*|^{p(x)-2}\nabla u^* \cdot \nabla\phi - g(x)(u^*)^{q(x)}\phi - f(x)(u^*)^{-\beta(x)}u^*\phi\right] dx \\ - \varepsilon \int_{\Omega_<} \mathcal{M}(\rho(u^*))|\nabla u^*|^{p(x)-2}\nabla u^* \cdot \nabla\phi\, dx + \varepsilon \int_{\Omega_<} g(x)(u^*)^{q(x)}\phi\, dx \\ + \varepsilon \int_{\Omega_<} f(x)(u^*)^{-\beta(x)}\phi\, dx.$$

Dividing by ε and passing to the limit as $\varepsilon \to 0$, and considering that $|\Omega_<| \to 0$ as $\varepsilon \to 0$ gives

$$\mathcal{M}\left(\int_\Omega \frac{|\nabla u^*|^{p(x)}}{p(x)}\, dx\right) \int_\Omega |\nabla u^*|^{p(x)-2}\nabla u^* \cdot \nabla\phi\, dx - \int_\Omega g(x)(u^*)^{q(x)}\phi\, dx \\ \geq \int_\Omega f(x)(u^*)^{-\beta(x)}\phi\, dx, \quad \forall\, \phi \in W_0^{1,p(x)}(\Omega).$$

However, since the function $\phi \in W_0^{1,p(x)}(\Omega)$ is chosen randomly, it follows that

$$\mathcal{M}\left(\int_\Omega \frac{|\nabla u^*|^{p(x)}}{p(x)}\, dx\right) \int_\Omega |\nabla u^*|^{p(x)-2}\nabla u^* \cdot \nabla\phi\, dx - \int_\Omega g(x)(u^*)^{q(x)}\phi\, dx \\ = \int_\Omega f(x)(u^*)^{-\beta(x)}\phi\, dx, \quad \forall\, \phi \in W_0^{1,p(x)}(\Omega)$$

which concludes that $u^* \in W_0^{1,p(x)}(\Omega)$ is a weak solution to problem (1.1). □

4. Example

In this section, we provide an abstract problem discussed in $\mathbb{R}^N$, $N \geq 3$, to illustrate the result of Theorem 3.9. Suppose that

$$p(x) = 3 + \frac{1}{2}\sin(\pi|x|^2), \quad g(x) = e^{\cos(\pi|x|)}, \quad f(x) = \frac{(1-|x|)^k}{\beta(x)}, \quad k > 0, \\ \mathcal{M}(t) = at, \quad \text{where } t, a > 0 \text{ are two real numbers.}$$

Then equation (1.1) turns into

$$\begin{cases} -M\left(\int_\Omega \frac{|\nabla u|^{3+\frac{1}{2}\sin(\pi|x|^2)}}{3+\frac{1}{2}\sin(\pi|x|^2)}\, dx\right) \operatorname{div}\left(|\nabla u|^{1+\frac{1}{2}\sin(\pi|x|^2)}\nabla u\right) = h(x,u) & \text{in } B_1, \\ u > 0 & \text{in } B_1, \\ u = 0 & \text{on } \partial B_1, \end{cases} \tag{4.1}$$

where $h(x,u) = \frac{(1-|x|)^k}{\beta(x)} u^{-\beta(x)} + e^{\cos(\pi|x|)} u^{q(x)}$, and B_1 is the unit ball $B(0,1) \subset \mathbb{R}^N$.

Theorem 4.1. *Assume that the conditions* (A_1) *and* (A_2) *hold. If* $1 < \beta^+ < 1 + \frac{k+1}{\alpha}$ *and* $\alpha > 1/2$, *then problem* (4.1) *has at least one positive* $W_0^{1,p(x)}(B_1)$*-solution.*

Proof. Since $p(x) = 3 + \frac{1}{2}\sin(\pi|x|^2)$, $p \in C(\overline{\Omega})$ and $p^- = \frac{5}{2} \geq 2$. If we let $\alpha_1 = \alpha_2 = 2$, then $\mathcal{M}\colon (0,\infty) \to (0,\infty)$, and $\mathcal{M}$ is a nondecreasing C^1-continuous function and satisfies the growth condition (3.1). For every $x \in B_1$, $g(x) > 0$, and $|g|_\infty \leq C$ for some $C > 0$, and thus, $g \in L^\infty(\Omega)$. Function $f(x) = \frac{(1-|x|)^k}{\beta(x)} \leq \frac{(1-|x|)^k}{\beta^-}$ is clearly non-negative and bounded above within B_1, and hence, $f(x) \in L^1(B_1)$. Therefore, conditions (A_3)–(A_6) hold.

Now, let's choose $\overline{u} = (1-|x|)^\alpha$, $x \neq 0$. Since $\overline{u}$ is also non-negative and bounded within B_1, it is in $\overline{u} \in L^{p(x)}(B_1)$. Indeed,

$$\int_{B_1} ((1-|x|)^\alpha)^{p(x)}\,dx \leq \int_{B_1} ((1-|x|)^\alpha)^{p^-}\,dx + \int_{B_1} ((1-|x|)^\alpha)^{p^+}\,dx < \infty.$$

Next, we show that $\nabla\overline{u} \in L^{p(x)}(B_1)$. Then, the derivative of $\overline{u}$ in B_1 is

$$\nabla(1-|x|)^\alpha = \alpha(1-|x|)^{\alpha-1}\frac{-x}{|x|}.$$

Considering that $x \in B_1$, we obtain

$$\int_{B_1} |\nabla(1-|x|)^\alpha|^{p(x)}\,dx \leq \alpha^{p_M} \int_{B_1} (1-|x|)^{(\alpha-1)p^-}\,dx < \infty$$

if $\alpha > \frac{p^- - 1}{p^-}$. Thus, $\nabla\overline{u} \in L^{p(x)}(B_1)$, and as a result, $\overline{u} \in W_0^{1,p(x)}(B_1)$. Lastly, we show that $\int_{B_1} \frac{(1-|x|)^k(1-|x|)^{\alpha(1-\beta(x))}}{\beta(x)}\,dx < \infty$. Then,

$$\int_{B_1} \frac{(1-|x|)^k(1-|x|)^{\alpha(1-\beta(x))}}{\beta(x)}\,dx \leq \frac{1}{\beta^-}\int_{B_1} (1-|x|)^{k+\alpha(1-\beta^+)}\,dx < \infty.$$

Thus, by Theorem 3.9, problem (4.1) has at least one positive $W_0^{1,p(x)}(B_1)$-solution. □

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