TAIWANESE JOURNAL OF MATHEMATICS
Vol. 29, No. 1, pp. 129–169, February 2025
DOI: 10.11650/tjm/240706

Asymptotic Behavior of Solutions for the Three-dimensional Generalized Incompressible MHD Equations with Nonlinear Damping Terms

Le Tran Tinh

Abstract. This work is concerned with the three-dimensional generalized incompressible MHD equations with nonlinear damping terms (polynomial damping and exponential damping). We first establish the existence and uniqueness, and then we study the asymptotic behavior of weak solutions for these systems via attractors. The novelty is that the strength of nonlinearities and the degree of dissipations can work together to yield the global existence and uniqueness of the weak solutions of these systems. Since our systems might be not uniqueness, we could not use directly the classical scheme of the dynamical system to find attractors. Therefore, we use a new framework developed by Cheskidov and Lu which is called evolutionary system to obtain various attractors and its properties.

1. Introduction

We consider the following three-dimensional generalized incompressible MHD equations with polynomial damping

$$\begin{cases} \partial_t u + \nu(-\Delta)^{\alpha_1} u + (u\cdot\nabla)u - (v\cdot\nabla)v + \mu|u|^{2\alpha_2}u + \nabla p = f, \\ \partial_t v + \kappa(-\Delta)^{\beta_1} v + (u\cdot\nabla)v - (v\cdot\nabla)u + \eta|v|^{2\beta_2}v = g, \\ \nabla\cdot u = 0, \quad \nabla\cdot v = 0, \end{cases} \tag{1.1}$$

and the following three-dimensional generalized incompressible MHD equations with exponential damping

$$\begin{cases} \partial_t u + \nu(-\Delta)^{\alpha_1} u + (u\cdot\nabla)u - (v\cdot\nabla)v + \mu(e^{a|u|^r}-1)u + \nabla p = f, \\ \partial_t v + \kappa(-\Delta)^{\beta_1} v + (u\cdot\nabla)v - (v\cdot\nabla)u + \eta(e^{b|v|^s}-1)v = g, \\ \nabla\cdot u = 0, \quad \nabla\cdot v = 0, \end{cases} \tag{1.2}$$

where ν, κ, μ, η, α_1, β_1, a, b, r and s are positive real parameters, α_2 and β_2 nonnegative real parameters, $u(t,x)\in\mathbb{R}^3$, $p(t,x)\in\mathbb{R}$ and $v(t,x)\in\mathbb{R}^3$ denote the fluid velocity vector,

Received October 14, 2023; Accepted July 15, 2024.
Communicated by Cheng-Hsiung Hsu.
2020 *Mathematics Subject Classification.* 35Q35, 37L30, 76D03, 76F20, 76F65.
Key words and phrases. three-dimensional generalized incompressible MHD equations, fractional Laplacians, weak solutions, polynomial damping and exponential damping.

the scalar pressure and the magnetic field of the fluid, respectively. The operators $(-\Delta)^{\alpha_1}$ and $(-\Delta)^{\beta_1}$ are fractional Laplacians which will be defined below.

The MHD equations have always been of great interest, due to their mathematical importance and physical applications. Until now, the role of damping and fractional diffusion operators has been increasingly consolidated, developed and diversified (see, e.g., [1, 5, 6, 8–11, 22, 23, 25, 37, 39, 45, 54] and references therein). That is why the MHD equations with fractional dissipation and damping have recently attracted considerable interests. The existence and uniqueness of weak solutions for (1.1) and (1.2) is an open issue. The asymptotic behavior of weak solutions via attractors for (1.1) and (1.2) is also an open issue. Since our Cauchy problems could have non-unique solution, we cannot use directly the classical scheme of the dynamical system to find attractors. As we know that there are several abstract frameworks for studying dynamical systems without uniqueness. Therefore, the evolutionary systems which is a new framework work developed in recent years by Cheskidov and Lu in [13–16, 35] is used here.

The systems (1.1) and (1.2) have direct or indirect relations with many models. Let us review some very related works to the content of our articles. When the damping terms vanish, the systems (1.1) and (1.2) reduce to the 3D generalized MHD equations determining by

$$\begin{cases} \partial_t u + \nu(-\Delta)^{\alpha_1} u + (u\cdot\nabla)u - (v\cdot\nabla)v + \nabla p = f, \\ \partial_t v + \kappa(-\Delta)^{\beta_1} v + (u\cdot\nabla)v - (v\cdot\nabla)u = g, \\ \nabla\cdot u = 0, \quad \nabla\cdot v = 0. \end{cases} \tag{1.3}$$

The system (1.3) has been studied extensively which has many more exciting results on existence, uniqueness, non-uniqueness, regularity, decay rate and asymptotic behavior of solutions (see, e.g., [18, 19, 21, 30, 31, 38, 46–48, 50, 51, 53, 56, 59, 60] and references therein) and progress has been made. Especially, when $\alpha_1 = \beta_1 = 1$, the system (1.3) reduces to the usual 3D MHD equations that have a lot of rich results (see, e.g., [42, 49]). It is flawed because we cannot refer to the many exciting results of (1.3) in the 2D case and we can find a summary on some of recent results in a review paper [49].

In case of $\alpha_1 = \beta_1 = 1$, the system (1.1) reduces to the 3D MHD with damping terms determining by

$$\begin{cases} \partial_t u - \nu\Delta u + (u\cdot\nabla)u - (v\cdot\nabla)v + \mu|u|^{2\alpha_2}u + \nabla p = f, \\ \partial_t v - \kappa\Delta v + (u\cdot\nabla)v - (v\cdot\nabla)u + \eta|v|^{2\beta_2}v = g, \\ \nabla\cdot u = 0, \quad \nabla\cdot v = 0. \end{cases} \tag{1.4}$$

The system (1.4) was studied in [52, 55]. The existence, uniqueness, regularity and decay of strong solutions were proved with the relationship of parameters α_2 and β_2 (see also [45]).

In the absence of magnetic field the system (1.1) reduces to the three dimensional generalized Navier–Stokes equations with polynomial damping

$$\begin{cases} \partial_t u + \nu(-\Delta)^{\alpha_1} u + (u \cdot \nabla)u + \mu|u|^{2\alpha_2} u + \nabla p = f, \\ \nabla \cdot u = 0, \end{cases} \tag{1.5}$$

and the system (1.2) reduces to the three dimensional generalized Navier–Stokes equations with exponential damping

$$\begin{cases} \partial_t u + \nu(-\Delta)^{\alpha_1} u + (u \cdot \nabla)u + \mu(e^{a|u|^r} - 1)u + \nabla p = f, \\ \nabla \cdot u = 0. \end{cases} \tag{1.6}$$

The existence, uniqueness and the asymptotic behavior of weak solutions via attractors for (1.5) and (1.6) have been investigated in [29, 44] (see also in [5, 11, 20, 26, 57, 58] and references therein).

We know that it is a mistake not to mention the results in two dimensional case. However, we are considering our problems in three dimensional case, so we will omit this here. Our motivation for studying (1.1) and (1.2) is mainly mathematical and the main purpose of this paper is to understand how the degree of dissipation and damping affect the existence, uniqueness and regularity of the weak solutions of (1.1) and (1.2). Moreover, we want to study the asymptotic behavior of these systems via attractors.

The paper is organized as follows. In Section 2, we recall some preliminary results. In Section 3, we study the existence and uniqueness for our problems. In Section 4, we investigate the attractors and its properties. We also denote by $A \lesssim \sum_{i=1}^n B_i$ an estimate of the form $A \leq \sum_{i=1}^n c_i B_i$ with some positive constants c_i.

2. Preliminaries

We will study (1.1) and (1.2) with periodic boundary conditions. Therefore, for simplicity, the spatial domain will be the torus $\mathbb{T} = [-\pi, \pi]^3$. As usual in the periodic setting and the divergence-free condition, we can restrict ourselves to deal with initial datum, f and g with vanishing spatial averages; then the solutions will enjoy the same property.

Therefore we can represent any divergence free vector u which are periodic and have zero spatial averages as follows:

$$u := \sum_{k \in J} u_k \phi_k \quad \text{with } u_k \in \mathbb{C}^3,\ u_k^* = u_{-k},\ u_k \cdot k = 0,\ \forall\, k \in J,$$

where $\phi_k = e^{ik \cdot x}$, $J = \mathbb{Z}^3 \setminus \{0\}$. For $s \in \mathbb{R}$, we define the following spaces

$$V^s := \left\{ u := \sum_{k \in J} u_k \phi_k \;\middle|\; u_k \in \mathbb{C}^3, u_k^* = u_{-k}, u_k \cdot k = 0, \phi_k = e^{ik \cdot x} \text{ and } \sum_{k \in J} |u_k|^2 |k|^{2s} < \infty \right\}.$$

These spaces are also Hilbert spaces with scalar product

$$\langle u, v\rangle_{V^s} = \sum_{k\in J} u_k \cdot v_{-k}|k|^{2s}.$$

For simplicity, we use the notation $\langle\,\cdot\,,\cdot\,\rangle$ denoted the scalar product in V^0 and also the dual pairing of $V^s - V^{-s}$ by $\langle u, v\rangle := \sum_{k\in J} u_k \cdot v_{-k}$.

The following compact embedding $V^{s+\varepsilon} \hookrightarrow\hookrightarrow V^s$, for any $\varepsilon > 0$, holds. Therefore, if $s_1 \le s_2$ and $u \in V^{s_2}$, then we get

$$\|u\|_{V^{s_1}} \lesssim \|u\|_{V^{s_2}}. \tag{2.1}$$

If $s = \gamma s_1 + (1-\gamma)s_2$, $0 \le \gamma \le 1$, then the following interpolation holds

$$\|u\|_{V^s} \lesssim \|u\|^{\gamma}_{V^{s_1}}\|u\|^{1-\gamma}_{V^{s_2}}.$$

If $0 \le s < 3/2$ and $1/p \ge 1/2 - s/3$, then $V^s \hookrightarrow L^p(\mathbb{T})$, i.e.,

$$\|u\|_{L^p(\mathbb{T})} \lesssim \|u\|_{V^s} \quad \text{for all } u \in V^s.$$

Moreover, if $s = 3/2$, then

$$\|u\|_{L^p(\mathbb{T})} \lesssim \|u\|_{V^s} \quad \text{for any finite } p \text{ and all } u \in V^s,$$

and if $s > 3/2$, then

$$\|u\|_{L^\infty(\mathbb{T})} \lesssim \|u\|_{V^s} \quad \text{for all } u \in V^s.$$

We define the linear operator $\Lambda = (-\Delta)^{1/2}$ and its powers as follows:

$$\Lambda u = \sum_{k\in J} |k|u_k\phi_k \quad \text{and} \quad \Lambda^s u = \sum_{k\in J} |k|^s u_k\phi_k,$$

where $u = \sum_{k\in J} u_k\phi_k$, $\phi_k = e^{ik\cdot x}$. Hence $(-\Delta)^s = \Lambda^{2s}$. It follows from the construction of Λ^s that Λ^s preserves the divergence free condition $k \cdot u_k = 0$ and Λ^s maps V^α onto $V^{\alpha-s}$. Moreover, we have

$$\|u\|_{V^s} = \|\Lambda^s u\|_{V^0}.$$

In particular, Λ^s maps V^s onto V^0 for all $s > 0$ and so $D(\Lambda^s) = V^s$.

Denote by P_σ the Leray–Helmholtz projection. It is the orthogonal projection from $L^2(\mathbb{T})$ onto V^0 and $P_\sigma\Lambda^s = \Lambda^s P_\sigma$. Set

$$b(u, v, w) = \int_{\mathbb{T}} \sum_{i,j=1}^{3} u_i \frac{\partial v_j}{\partial x_i} w_j \, dx.$$

We define $\mathcal{F}$ the space of formal Fourier series

$$\left\{ u := \sum_{k\in J} \widehat{u}_k \phi_k \;\middle|\; \widehat{u}_k \in \mathbb{C}^3, \phi_k = e^{ik\cdot x} \right\},$$

and

$$H^s := \left\{ u \in \mathcal{F} : \|u\|_{H^s}^2 := \sum_{k\in J} |u_k|^2 |k|^{2s} < \infty, \widehat{u^*}_k = \widehat{u}_{-k} \text{ and } \widehat{u}_0 = 0 \right\}.$$

Assume that $\mathcal{V}$ is the space of divergence free trigonometric polynomials consisting of all $u \in \mathcal{F}$ such that $k \cdot \widehat{u}_k = 0$ for all $k \in J$ and $\widehat{u}_k = 0$ for all but finitely many values of $k \in J$. It follows that V^s is the closure of $\mathcal{V}$ in H^s with respect to the $\|\cdot\|_{H^s}$ norm. We recall the following lemma in [7, 17, 27, 28] for the properties of the trilinear form b.

Lemma 2.1. *Let $u, v, w \in \mathcal{V}$, it holds that*

(i) $b(u, v, v) = 0$,

(ii) $b(u, v, w) = -b(u, w, v)$,

(iii) $b(u - v, u, u - v) = b(u, u, u - v) - b(v, v, u - v)$.

In fact, this result may be extended to larger spaces by the density of $\mathcal{V}$ in V^σ for the appropriate values of σ that the trilinear forms are continuous. The following proposition is taken from [24, Proposition 2.5] (see also [4]).

Proposition 2.2. *The trilinear form $b\colon V^{\sigma_1} \times V^{\sigma_2} \times V^{\sigma_3} \to \mathbb{R}$ is bounded provided that all following conditions hold:*

(i) $\sigma_1 + \sigma_2 + \sigma_3 > 5/2$, (ii) $\sigma_1 + \sigma_2 \geq s$, (iii) $\sigma_2 + \sigma_3 \geq 1$, (iv) $\sigma_1 + \sigma_3 \geq 1 - s$

for some $s \in \{0, 1\}$. If the last three conditions are satisfied and if σ_i is a nonpositive integer for some $i \in \{1, 2, 3\}$, then the condition (i) *can be replaced by the nonstrict version of the inequality. The nonstrict inequality is also allowed if for some $s \in \{0, 1\}$,*

$$\sigma_1 \geq 0, \quad \sigma_2 \geq s, \quad \sigma_3 \geq 1 - s.$$

Using the periodic setting and the projection operator P_σ, we can rewrite (1.1) and (1.2) in the following abstract form

$$\begin{cases} \partial_t u + \nu\Lambda^{2\alpha_1} u + B(u, u) - B(v, v) + \mu P_\sigma(|u|^{2\alpha_2} u) = P_\sigma f, \\ \partial_t v + \kappa\Lambda^{2\beta_1} v + B(u, v) - B(v, u) + \eta P_\sigma(|v|^{2\beta_2} v) = P_\sigma g, \end{cases} \tag{2.2}$$

and

$$\begin{cases} \partial_t u + \nu\Lambda^{2\alpha_1} u + B(u, u) - B(v, v) + \mu P_\sigma((e^{a|u|^r} - 1)u) = P_\sigma f, \\ \partial_t v + \kappa\Lambda^{2\beta_1} v + B(u, v) - B(v, u) + \eta P_\sigma((e^{b|v|^s} - 1)v) = P_\sigma g, \end{cases} \tag{2.3}$$

where $B(u,v) := P_\sigma\{(u\cdot\nabla)v\}$. We start our investigation with the definition of weak solutions for (2.2) and (2.3) with L^2 initial data:

$$u(\tau,x)=u_\tau(x), \quad v(\tau,x)=v_\tau(x).$$

Definition 2.3. Let ν, κ, μ, η, α_1, β_1 be positive real parameters and let α_2, β_2 be nonnegative real parameters. Given $f\in L^2_{\rm loc}(\mathbb{R};V^0)$, $g\in L^2_{\rm loc}(\mathbb{R};V^0)$, $u_\tau\in V^0$, $v_\tau\in V^0$ and a fixed $T>\tau$. A weak solution of (2.2) is a pair of functions (u,v) satisfying

$$\begin{aligned} &u\in L^\infty(\tau,T;V^0)\cap L^2(\tau,T;V^{\alpha_1})\cap L^{2\alpha_2+2}(\tau,T;L^{2\alpha_2+2}(\mathbb{T}))\cap C_w([\tau,T];V^0),\\ &v\in L^\infty(\tau,T;V^0)\cap L^2(\tau,T;V^{\beta_1})\cap L^{2\beta_2+2}(\tau,T;L^{2\beta_2+2}(\mathbb{T}))\cap C_w([\tau,T];V^0).\end{aligned}$$

Moreover, given any $t\in[\tau,T]$ and $\phi\in V^{\gamma_1}\cap L^{2\gamma_2+2}(\mathbb{T})$, $\gamma_1>\max\{5/2-\alpha_1,\alpha_1,5/2-\beta_1,\beta_1\}$, $\gamma_2=\max\{\alpha_2,\beta_2\}$, it satisfies $(u(\tau),v(\tau))=(u_\tau,v_\tau)$ and

$$\begin{aligned}&\langle u(t),\phi\rangle+\nu\int_\tau^t\langle\Lambda^{\alpha_1}u(s),\Lambda^{\alpha_1}\phi\rangle\,ds+\int_\tau^t\langle B(u(s),u(s)),\phi\rangle\,ds\\ &\quad-\int_\tau^t\langle B(v(s),v(s)),\phi\rangle\,ds+\mu\int_\tau^t\langle|u(s)|^{2\alpha_2}u(s),\phi\rangle\,ds\\ &=\langle u_\tau,\phi\rangle+\int_\tau^t\langle f(s),\phi\rangle\,ds\end{aligned}\tag{2.4}$$

for a.e. $t\in[\tau,T]$, and

$$\begin{aligned}&\langle v(t),\phi\rangle+\kappa\int_\tau^t\langle\Lambda^{\beta_1}v(s),\Lambda^{\beta_1}\phi\rangle\,ds+\int_\tau^t\langle B(u(s),v(s)),\phi\rangle\,ds\\ &\quad-\int_\tau^t\langle B(v(s),u(s)),\phi\rangle\,ds+\eta\int_\tau^t\langle|v(s)|^{2\beta_2}v(s),\phi\rangle\,ds\\ &=\langle v_\tau,\phi\rangle+\int_\tau^t\langle g(s),\phi\rangle\,ds\end{aligned}\tag{2.5}$$

for a.e. $t\in[\tau,T]$.

Definition 2.4. Let ν, κ, μ, η, α_1, β_1, a, b be positive real parameters and let $r\ge 1$ and $s\ge 1$. Given $f\in L^2_{\rm loc}(\mathbb{R};V^0)$, $g\in L^2_{\rm loc}(\mathbb{R};V^0)$, $u_\tau\in V^0$, $v_\tau\in V^0$ and a fixed $T>\tau$. A weak solution of (2.3) is a pair of functions (u,v) satisfying

$$\begin{aligned}&u\in L^\infty(\tau,T;V^0)\cap L^2(\tau,T;V^{\alpha_1})\cap\mathcal{G}_a^r(\tau,T;L^1(\mathbb{T}))\cap C_w([\tau,T];V^0),\\ &v\in L^\infty(\tau,T;V^0)\cap L^2(\tau,T;V^{\beta_1})\cap\mathcal{Q}_b^s(\tau,T;L^1(\mathbb{T}))\cap C_w([\tau,T];V^0),\end{aligned}$$

where

$$\begin{aligned}&\mathcal{G}_a^r(\tau,T;L^1(\mathbb{T})):=\big\{u:[\tau,T]\times\mathbb{T}\to\mathbb{R}^3\text{ measurable},(e^{a|u|^r}-1)|u|^2\in L^1(\tau,T;L^1(\mathbb{T}))\big\},\\ &\mathcal{Q}_b^s(\tau,T;L^1(\mathbb{T})):=\big\{v:[\tau,T]\times\mathbb{T}\to\mathbb{R}^3\text{ measurable},(e^{b|v|^s}-1)|v|^2\in L^1(\tau,T;L^1(\mathbb{T}))\big\}.\end{aligned}$$

Moreover, given any $t \in [\tau, T]$ and $\phi \in V^{\gamma_1}$, $\gamma_1 > \max\{5/2 - \alpha_1, \alpha_1, 5/2 - \beta_1, \beta_1\}$, it satisfies $(u(\tau), v(\tau)) = (u_\tau, v_\tau)$ and

$$\begin{aligned}
&\langle u(t), \phi\rangle + \nu \int_\tau^t \langle \Lambda^{\alpha_1} u(s), \Lambda^{\alpha_1}\phi\rangle\, ds + \int_\tau^t \langle B(u(s), u(s)), \phi\rangle\, ds \\
&\quad - \int_\tau^t \langle B(v(s), v(s)), \phi\rangle\, ds + \mu \int_\tau^t \langle (e^{a|u(s)|^r} - 1)u(s), \phi\rangle\, ds \\
&= \langle u_\tau, \phi\rangle + \int_\tau^t \langle f(s), \phi\rangle\, ds
\end{aligned}$$

for a.e. $t \in [\tau, T]$, and

$$\begin{aligned}
&\langle v(t), \phi\rangle + \kappa \int_\tau^t \langle \Lambda^{\beta_1} v(s), \Lambda^{\beta_1}\phi\rangle\, ds + \int_\tau^t \langle B(u(s), v(s)), \phi\rangle\, ds \\
&\quad - \int_\tau^t \langle B(v(s), u(s)), \phi\rangle\, ds + \eta \int_\tau^t \langle (e^{b|v(s)|^s} - 1)v(s), \phi\rangle\, ds \\
&= \langle v_\tau, \phi\rangle + \int_\tau^t \langle g(s), \phi\rangle\, ds
\end{aligned}$$

for a.e. $t \in [\tau, T]$.

Remark 2.5. (i) In the weak formulations above, we see that the trilinear terms are well defined. Indeed, it easily implies that $\gamma_1 > 1$ and using Proposition 2.2 we can estimate the trilinear terms as follows:

$$|\langle B(u,u), \phi\rangle| = |b(u,u,\phi)| = |b(u,\phi,u)| \le C\|u\|_{V^0}\|\phi\|_{V^{\gamma_1}}\|u\|_{V^{\alpha_1}}, \tag{2.6}$$

and

$$|\langle B(v,v), \phi\rangle| = |b(v,v,\phi)| = |b(v,\phi,v)| \le C\|v\|_{V^0}\|\phi\|_{V^{\gamma_1}}\|v\|_{V^{\beta_1}}, \tag{2.7}$$

and

$$|\langle B(u,v), \phi\rangle| = |b(u,v,\phi)| = |b(u,\phi,v)| \le C\|u\|_{V^0}\|\phi\|_{V^{\gamma_1}}\|v\|_{V^{\beta_1}}, \tag{2.8}$$

or

$$|\langle B(u,v), \phi\rangle| = |b(u,v,\phi)| = |b(u,\phi,v)| \le C\|u\|_{V^{\alpha_1}}\|\phi\|_{V^{\gamma_1}}\|v\|_{V^0}, \tag{2.9}$$

and

$$|\langle B(v,u), \phi\rangle| = |b(v,u,\phi)| = |b(v,\phi,u)| \le C\|v\|_{V^0}\|\phi\|_{V^{\gamma_1}}\|u\|_{V^{\alpha_1}}, \tag{2.10}$$

or

$$|\langle B(v,u), \phi\rangle| = |b(v,u,\phi)| = |b(v,\phi,u)| \le C\|v\|_{V^{\beta_1}}\|\phi\|_{V^{\gamma_1}}\|u\|_{V^0}. \tag{2.11}$$

(ii) If $r \geq 1$, $s \geq 1$ and (u, v) is a weak solution of (2.3) determined by Definition 2.4, then

$$(e^{a|u|^r} - 1)u \in L^1(\tau, T; L^1(\mathbb{T})) \quad \text{and} \quad (e^{b|v|^s} - 1)v \in L^1(\tau, T; L^1(\mathbb{T})).$$

Indeed, we see that it is sufficient to examine for u and so v. We first define

$$\begin{aligned}
\Omega &:= [\tau, T] \times \mathbb{T}, \\
\Omega_1 &:= \{(t, x) \in [\tau, T] \times \mathbb{T} \mid 0 < |u(t, x)| < 1\}, \\
\Omega_2 &:= \{(t, x) \in [\tau, T] \times \mathbb{T} \mid |u(t, x)| \geq 1\}.
\end{aligned}$$

We then can estimate

$$\begin{aligned}
&\int_\Omega (e^{a|u(t)|^r} - 1)|u(t)|\, dxdt \\
&= \int_{\Omega_1 \cup \Omega_2} (e^{a|u(t)|^r} - 1)|u(t)|\, dxdt \\
&= \int_{\Omega_1} (e^{a|u(t)|^r} - 1)|u(t)|\, dxdt + \int_{\Omega_2} (e^{a|u(t)|^r} - 1)|u(t)|\, dxdt \\
&= \int_{\Omega_1} \frac{e^{a|u(t)|^r} - 1}{|u(t)|}|u(t)|^2\, dxdt + \int_{\Omega_2} \frac{1}{|u(t)|}(e^{a|u(t)|^r} - 1)|u(t)|^2\, dxdt \\
&\leq M_{ar} \int_{\Omega_1} |u(t)|^2\, dxdt + \int_{\Omega_2} (e^{a|u(t)|^r} - 1)|u(t)|^2\, dxdt \\
&\leq M_{ar}(T - \tau)\|u\|_{L^\infty(\tau,T;V^0)} + \int_\Omega (e^{a|u(t)|^r} - 1)|u(t)|^2\, dxdt \\
&\leq M_{ar}(T - \tau)\|u\|_{L^\infty(\tau,T;V^0)} + \left\|(e^{a|u|^r} - 1)|u|^2\right\|_{L^1(\tau,T;L^1(\mathbb{T}))},
\end{aligned}$$

where $M_{ar} := \sup_{0<\lambda\leq 1} \frac{e^{a\lambda^r} - 1}{\lambda} < \infty$ for $r \geq 1$, $a > 0$. This implies the desired results.

(iii) We have

$$(e^{a|u|^r} - 1)|u|^2 = \sum_{k=1}^{\infty} \frac{a^k}{k!}|u|^{rk+2} \quad \text{and} \quad (e^{b|v|^s} - 1)|v|^2 = \sum_{k=1}^{\infty} \frac{b^k}{k!}|v|^{sk+2}.$$

This implies that

$$\int_\tau^T \|(e^{a|u(t)|^r} - 1)|u(t)|^2\|_{L^1(\mathbb{T})}\, dt = \sum_{k=1}^{\infty} \frac{a^k}{k!} \int_\tau^T \|u(t)\|_{L^{rk+2}(\mathbb{T})}^{rk+2}\, dt, \tag{2.12}$$

$$\int_\tau^T \|(e^{b|v(t)|^s} - 1)|v(t)|^2\|_{L^1(\mathbb{T})}\, dt = \sum_{k=1}^{\infty} \frac{b^k}{k!} \int_\tau^T \|v(t)\|_{L^{sk+2}(\mathbb{T})}^{sk+2}\, dt. \tag{2.13}$$

We infer from (2.12) and (2.13) that

$$(e^{a|u|^r}-1)u \in L^1(\tau,T;L^1(\mathbb{T})) \text{ implies } u \in \bigcap_{k=1}^{\infty} L^{rk+2}(\tau,T;L^{rk+2}(\mathbb{T})), \tag{2.14}$$

$$(e^{b|v|^s}-1)v \in L^1(\tau,T;L^1(\mathbb{T})) \text{ implies } v \in \bigcap_{k=1}^{\infty} L^{sk+2}(\tau,T;L^{sk+2}(\mathbb{T})). \tag{2.15}$$

Let us recall the following weak continuity result in time (see [7, 32]) and strong continuity result in time (see [43]), respectively. Let $T > \tau$.

Lemma 2.6. *Let X and Y be Banach spaces such that $Y \hookrightarrow X$ with a continuous injection. Then*

$$L^\infty(\tau,T;Y) \cap C_w([\tau,T];X) = C_w([\tau,T];Y).$$

Lemma 2.7. *Assume that $u \in L^2(\tau,T;V^{\gamma+h})$ and $\frac{du}{dt} \in L^2(\tau,T;V^{\gamma-h})$ for $\gamma \in \mathbb{R}$ and $h > 0$, then $u \in C([\tau,T];V^\gamma)$ and*

$$\frac{d}{dt}\|u(t)\|_{V^\gamma}^2 = 2\left\langle \Lambda^{-h}\frac{du}{dt}(t), \Lambda^h u(t)\right\rangle_{V^\gamma}.$$

In particular, we also have the following important inequalities for the damping (see [3, Lemma 2.2] and [8, Lemma 2.3]).

Lemma 2.8. (1) *Assume that $p \in (1,\infty)$ and $\delta \geq 0$. There exist positive constants c_1 and c_2 such that for all $x, y \in \mathbb{R}^3$,*

$$\big||x|^{p-2}x - |y|^{p-2}y\big| \leq c_1|x-y|^{1-\delta}(|x|+|y|)^{p-2+\delta}$$

and

$$(|x|^{p-2}x - |y|^{p-2}y)\cdot(x-y) \geq c_2|x-y|^{2+\delta}(|x|+|y|)^{p-2-\delta}.$$

(2) *Assume that $a > 0$ and $r > 0$. There exists positive constant c_3 such that for all $x, y \in \mathbb{R}^3$,*

$$\big((e^{a|x|^r}-1)x - (e^{a|y|^r}-1)y\big)\cdot(x-y) \geq c_3|x-y|^2\big((e^{a|x|^r}-1)+(e^{a|y|^r}-1)\big).$$

3. Existence and uniqueness of weak solutions

This section is devoted to study the existence and uniqueness of weak solutions of (2.2) and (2.3). Our proof can be done by using the standard Faedo Galerkin approximation. Thus, we only focus on the main points of the method here.

We now denote

$$\mathfrak{M}_1 := \Big\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \,\Big|\, 1 \le \alpha_1 < \frac{3}{2}, 1 \le \beta_1 < \frac{5}{2} - \alpha_1, \alpha_2 \ge \max\Big\{\frac{5-2\alpha_1}{4\alpha_1 - 2}, \frac{5-2\beta_1}{4\beta_1-2}\Big\},$$
$$\beta_2 \ge \max\Big\{\frac{5-2\alpha_1}{2(\alpha_1+\beta_1-1)}, \frac{5-2\beta_1}{2(\alpha_1+\beta_1-1)}\Big\}\Big\},$$
$$\mathfrak{M}_2 := \Big\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \,\Big|\, 1 \le \alpha_1 < \frac{5}{4}, \beta_1 \ge \frac{5}{2} - \alpha_1, \alpha_2 \ge \frac{5-2\alpha_1}{4\alpha_1-2}, \beta_2 \ge 0\Big\},$$
$$\mathfrak{M}_3 := \Big\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \,\Big|\, \alpha_1 \ge \frac{5}{4}, \beta_1 \ge \max\Big\{\frac{5}{2} - \alpha_1, 1\Big\}, \alpha_2 \ge 0, \beta_2 \ge 0\Big\}.$$

The global well-posedness of (2.2) reads as follows.

Theorem 3.1. *Let ν, κ, μ, η, α_1, β_1 be positive real parameters and let α_2, β_2 be non-negative real parameters. Assume that $f \in L^2_{\mathrm{loc}}(\mathbb{R}; V^0)$, $g \in L^2_{\mathrm{loc}}(\mathbb{R}; V^0)$, $u_\tau \in V^0$ and $v_\tau \in V^0$. Then, the system (2.2) possesses a global weak solution obeying Definition 2.3 with initial condition (u_τ, v_τ). Furthermore, the global weak solution is unique and depends continuity on the initial data if $(\alpha_1,\beta_1,\alpha_2,\beta_2) \in \bigcup_{i=1}^3 \mathfrak{M}_i$.*

Proof. (i) Existence: Denote by Π_n the finite dimensional projectors onto V^0 which is determined by

$$\Pi_n u = \sum_{0<|k|\le n} u_k \phi_k \quad \text{for } u = \sum_{k\in J} u_k\phi_k \text{ and } \phi_k = e^{ik\cdot x}.$$

We define $B_n(u,v) := \Pi_n B(u,v)$ and we consider a sequence of the finite dimensional approximation of system (2.2) in the unknowns $u_n = \Pi_n u$ solving the equations

$$\begin{cases} \partial_t u_n + \nu\Lambda^{2\alpha_1} u_n + B_n(u_n,u_n) - B_n(v_n,v_n) + \mu\Pi_n P_\sigma(|u_n|^{2\alpha_2}u_n) = \Pi_n P_\sigma f, \\ \partial_t v_n + \kappa\Lambda^{2\beta_1} v_n + B_n(u_n,v_n) - B_n(v_n,u_n) + \eta\Pi_n P_\sigma(|v_n|^{2\beta_2}v_n) = \Pi_n P_\sigma g \end{cases} \tag{3.1}$$

with the initial condition

$$u_n(\tau) = \Pi_n u_\tau, \quad v_n(\tau) = \Pi_n v_\tau.$$

Obviously, $u_n(\tau)$ and $v_n(\tau)$ strongly converge to u_τ and v_τ in V^0, respectively. We take L^2-scalar product of the first equation in (3.1) with itself u_n; bearing in mind Lemma 2.1, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|u_n(t)\|^2_{V^0} + \nu\|u_n(t)\|^2_{V^{\alpha_1}} + \mu\|u_n(t)\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})} \\ &= b(v_n(t), v_n(t), u_n(t)) + \langle f(t), u_n(t)\rangle. \end{aligned} \tag{3.2}$$

Similar manner can be done for the remaining equation in (3.1), we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|v_n(t)\|^2_{V^0} + \kappa\|v_n(t)\|^2_{V^{\beta_1}} + \eta\|v_n(t)\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})} \\ &= b(v_n(t), u_n(t), v_n(t)) + \langle g(t), v_n(t)\rangle. \end{aligned} \tag{3.3}$$

Summing-up (3.2) and (3.3) and using Lemma 2.1 again, we get

$$\begin{aligned}&\frac{1}{2}\frac{d}{dt}\{\|u_n(t)\|^2_{V^0}+\|v_n(t)\|^2_{V^0}\}+\nu\|u_n(t)\|^2_{V^{\alpha_1}}+\kappa\|v_n(t)\|^2_{V^{\beta_1}}\\ &+\mu\|u_n(t)\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})}+\eta\|v_n(t)\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\\ =\;&\langle f(t),u_n(t)\rangle+\langle g(t),v_n(t)\rangle.\end{aligned}\tag{3.4}$$

By using (2.1) and the Cauchy–Schwarz inequality, we deduce from (3.4) that

$$\begin{aligned}&\frac{d}{dt}\{\|u_n(t)\|^2_{V^0}+\|v_n(t)\|^2_{V^0}\}+\nu\|u_n(t)\|^2_{V^{\alpha_1}}+\kappa\|v_n(t)\|^2_{V^{\beta_1}}\\ &+2\mu\|u_n(t)\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})}+2\eta\|v_n(t)\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\\ \lesssim\;&\frac{1}{\nu}\|f(t)\|^2_{V^0}+\frac{1}{\kappa}\|g(t)\|^2_{V^0}.\end{aligned}$$

Integrating in time from τ to t, we obtain

$$\begin{aligned}&\|u_n(t)\|^2_{V^0}+\|v_n(t)\|^2_{V^0}+\nu\int_\tau^t\|u_n(s)\|^2_{V^{\alpha_1}}\,ds+\kappa\int_\tau^t\|v_n(s)\|^2_{V^{\beta_1}}\,ds\\ &+2\mu\int_\tau^t\|u_n(s)\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})}\,ds+2\eta\int_\tau^t\|v_n(s)\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\,ds\\ \lesssim\;&\|u_\tau\|^2_{V^0}+\|v_\tau\|^2_{V^0}+\frac{1}{\nu}\int_\tau^t\|f(s)\|^2_{V^0}\,ds+\frac{1}{\kappa}\int_\tau^t\|g(s)\|^2_{V^0}\,ds.\end{aligned}\tag{3.5}$$

Since $\|u_\tau\|^2_{V^0}$, $\|v_\tau\|^2_{V^0}$, $\int_\tau^t\|f(s)\|^2_{V^0}\,ds$ and $\int_\tau^t\|g(s)\|^2_{V^0}\,ds$ are bounded, it follows from (3.5) that the sequence $\{u_n\}$ is uniformly bounded in

$$L^\infty(\tau,T;V^0)\cap L^2(\tau,T;V^{\alpha_1})\cap L^{2\alpha_2+2}(\tau,T;L^{2\alpha_2+2}(\mathbb{T})),$$

and the sequence $\{v_n\}$ is uniformly bounded in

$$L^\infty(\tau,T;V^0)\cap L^2(\tau,T;V^{\beta_1})\cap L^{2\beta_2+2}(\tau,T;L^{2\beta_2+2}(\mathbb{T})).$$

It follows from (3.1) that

$$\begin{cases}\partial_t u_n=-\nu\Lambda^{2\alpha_1}u_n-B_n(u_n,u_n)+B_n(v_n,v_n)-\mu\Pi_nP_\sigma(|u_n|^{2\alpha_2}u_n)+\Pi_nP_\sigma f,\\ \partial_t v_n=-\kappa\Lambda^{2\beta_1}v_n-B_n(u_n,v_n)+B_n(v_n,u_n)-\eta\Pi_nP_\sigma(|v_n|^{2\beta_2}v_n)+\Pi_nP_\sigma g.\end{cases}$$

We deduce from the construction of $\Lambda^{2\alpha_1}$, $\Lambda^{2\beta_1}$, $\gamma_1\ge\alpha_1$ and $\gamma_1\ge\beta_1$ that $\Lambda^{2\alpha_1}u_n\in L^2(\tau,T;V^{-\gamma_1})$ and $\Lambda^{2\beta_1}v_n\in L^2(\tau,T;V^{-\gamma_1})$. It follows from (2.6)–(2.11) that $B_n(u_n,u_n)$, $B_n(v_n,v_n)$, $B_n(u_n,v_n)$ and $B_n(v_n,u_n)$ belong to $L^2(\tau,T;V^{-\gamma_1})$. Moreover, since $\gamma_2\ge\alpha_2$ and $\gamma_2\ge\beta_2$, we deduce that $\Pi_nP_\sigma(|u_n|^{2\alpha_2}u_n)$ and $\Pi_nP_\sigma(|v_n|^{2\beta_2}v_n)$ belong to $L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\tau,T;L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T}))$. Therefore, since $\frac{2\gamma_2+2}{2\gamma_2+1}\le 2$, $f\in L^2_{\text{loc}}(\mathbb{R};V^0)$ and $g\in L^2_{\text{loc}}(\mathbb{R};V^0)$, we infer that $\partial_t u_n$ and $\partial_t v_n$ are bounded uniformly in

$$L^2(\tau,T;V^{-\gamma_1})+L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\tau,T;L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T}))\subset L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\tau,T;V^{-\gamma_1}+L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T})).$$

Since

$$V^{\alpha_1} \cap L^{2\alpha_2+2}(\mathbb{T}) \hookrightarrow\hookrightarrow V^0 \hookrightarrow V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T})$$

and

$$V^{\beta_1} \cap L^{2\beta_2+2}(\mathbb{T}) \hookrightarrow\hookrightarrow V^0 \hookrightarrow V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T}),$$

we deduce from the Aubin–Lions lemma (see [43]) that the sequences $\{u_n\}$ and $\{v_n\}$ are compact in $L^2(\tau, T; V^0)$ and so we can extract subsequences, still denoted by u_n and v_n, respectively, such that

$$u_n \rightharpoonup u \ \text{ weakly in } L^2(\tau, T; V^{\alpha_1}), \tag{3.6}$$

$$v_n \rightharpoonup v \ \text{ weakly in } L^2(\tau, T; V^{\beta_1}), \tag{3.7}$$

$$u_n \rightharpoonup u \ \text{ weakly in } L^{2\alpha_2+2}(\tau, T; L^{2\alpha_2+2}(\mathbb{T})), \tag{3.8}$$

$$v_n \rightharpoonup v \ \text{ weakly in } L^{2\beta_2+2}(\tau, T; L^{2\beta_2+2}(\mathbb{T})), \tag{3.9}$$

$$u_n \rightharpoonup^* u \ \text{ weakly star in } L^\infty(\tau, T; V^0), \tag{3.10}$$

$$v_n \rightharpoonup^* v \ \text{ weakly star in } L^\infty(\tau, T; V^0), \tag{3.11}$$

$$u_n \to u \ \text{ strongly in } L^2(\tau, T; V^0), \tag{3.12}$$

$$v_n \to v \ \text{ strongly in } L^2(\tau, T; V^0). \tag{3.13}$$

Using all convergences (3.6)–(3.13), it is classical result to pass to the limit in the variational formulations (2.4) and (2.5), and prove that the pair of functions (u, v) is the solution of (2.2) and inherits all the regularity from (u_n, v_n), i.e.,

$$u \in L^\infty(\tau, T; V^0) \cap L^2(\tau, T; V^{\alpha_1}) \cap L^{2\alpha_2+2}(\tau, T; L^{2\alpha_2+2}(\mathbb{T}))$$

and

$$v \in L^\infty(\tau, T; V^0) \cap L^2(\tau, T; V^{\beta_1}) \cap L^{2\beta_2+2}(\tau, T; L^{2\beta_2+2}(\mathbb{T})).$$

In addition, we integrate in time and we obtain

$$u(t) = u_\tau + \int_\tau^t [-\nu\Lambda^{2\alpha_1} u(s) - B(u(s), u(s)) + B(v(s), v(s)) - P_\sigma(|u(s)|^{2\alpha_2} u(s)) + P_\sigma f(s)]\, ds$$

and

$$v(t) = v_\tau + \int_\tau^t [-\nu\Lambda^{2\beta_1} u(s) - B(u(s), v(s)) + B(v(s), u(s)) - P_\sigma(|v(s)|^{2\beta_2} v(s)) + P_\sigma g(s)]\, ds.$$

This implies that $u \in C([\tau, T]; V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T}))$ and $v \in C([\tau, T]; V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T}))$. In addition, since $u \in L^\infty(\tau, T; V^0)$ and $v \in L^\infty(\tau, T; V^0)$, we deduce from Lemma 2.6 that

$$u \in L^\infty(\tau, T; V^0) \cap L^2(\tau, T; V^{\alpha_1}) \cap L^{2\alpha_2+2}(\tau, T; L^{2\alpha_2+2}(\mathbb{T})) \cap C_w([\tau, T]; V^0)$$

and

$$v \in L^\infty(\tau, T; V^0) \cap L^2(\tau, T; V^{\beta_1}) \cap L^{2\beta_2+2}(\tau, T; L^{2\beta_2+2}(\mathbb{T})) \cap C_w([\tau, T]; V^0).$$

(ii) Continuous dependence on the initial data: In this section, we will study the continuous dependence of the weak solutions on the initial data, in particular their uniqueness. Let (u_1, v_1) and (u_2, v_2) associated to the initial data $(u_{1\tau}, v_{1\tau})$ and $(u_{2\tau}, v_{2\tau})$, respectively, satisfy (2.2). We define $U = u_1 - u_2$ and $V = v_1 - v_2$, and thus (U, V) satisfies

(3.14)

$$\begin{cases} \partial_t U + \nu\Lambda^{2\alpha_1}U + B(u_1, U) + B(U, u_2) - B(v_1, V) - B(V, v_2) + \mu P_\sigma(|u_1|^{2\alpha_2}u_1 - |u_2|^{2\alpha_2}u_2) = 0, \\ \partial_t V + \kappa\Lambda^{2\beta_1}V + B(U, v_1) + B(u_2, V) - B(V, u_2) - B(v_1, U) + \eta P_\sigma(|v_1|^{2\beta_2}v_1 - |v_2|^{2\beta_2}v_2) = 0. \end{cases}$$

We take the L^2-scalar product of the first equation of (3.14) with U and using Lemma 2.1 leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|U\|^2_{V^0} + \nu\|U\|^2_{V^{\alpha_1}} + b(U, u_2, U) - b(v_1, V, U) - b(V, v_2, U) \\ &+ \mu\int_{\mathbb{T}}(|u_1|^{2\alpha_2}u_1 - |u_2|^{2\alpha_2}u_2)\cdot(u_1 - u_2)\,dx = 0. \end{aligned} \tag{3.15}$$

Similar manner can be done for the remain equation (3.14) and we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|V\|^2_{V^0} + \kappa\|V\|^2_{V^{\beta_1}} + b(U, v_1, V) - b(V, u_2, V) - b(v_1, U, V) \\ &+ \eta\int_{\mathbb{T}}(|v_1|^{2\beta_2}v_1 - |v_2|^{2\beta_2}v_2)\cdot(v_1 - v_2)\,dx = 0. \end{aligned} \tag{3.16}$$

It follows from Lemma 2.8 that

$$\int_{\mathbb{T}}(|u_1|^{2\alpha_2}u_1 - |u_2|^{2\alpha_2}u_2)\cdot(u_1 - u_2)\,dx \geq 0, \tag{3.17}$$

and

$$\int_{\mathbb{T}}(|v_1|^{2\beta_2}v_1 - |v_2|^{2\beta_2}v_2)\cdot(v_1 - v_2)\,dx \geq 0. \tag{3.18}$$

We deduce from (3.15)–(3.18) that

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big\{\|U\|^2_{V^0} + \|V\|^2_{V^0}\big\} + \nu\|U\|^2_{V^{\alpha_1}} + \kappa\|V\|^2_{V^{\beta_1}} \\ &\leq |b(U, u_2, U)| + |b(V, v_2, U)| + |b(U, v_1, V)| + |b(V, u_2, V)|. \end{aligned} \tag{3.19}$$

Our intention is using the Gronwall inequality. Thus, we need to estimate the nonlinear terms appearing on the right-hand side of (3.19). For the clarity of presentation, we can split the domain $\bigcup_{i=1}^3 \mathfrak{M}_i$ as follows:

$$\mathfrak{M}_1 := M_1 \cup M_2 \cup M_4 \cup M_5 \cup M_7 \cup M_8, \quad \mathfrak{M}_2 := M_3 \cup M_6, \quad \mathfrak{M}_3 := M_9 \cup M_{10},$$

where

$$M_1 := \left\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \;\middle|\; \alpha_1 = 1, \beta_1 = 1, \alpha_2 \geq \frac{3}{2}, \beta_2 \geq \frac{3}{2}\right\},$$
$$M_2 := \left\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \;\middle|\; \alpha_1 = 1, 1 < \beta_1 < \frac{3}{2}, \alpha_2 \geq \frac{3}{2}, \beta_2 \geq \frac{3}{2\beta_1}\right\},$$
$$M_3 := \left\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \;\middle|\; \alpha_1 = 1, \beta_1 \geq \frac{3}{2}, \alpha_2 \geq \frac{3}{2}, \beta_2 \geq 0\right\},$$
$$M_4 := \left\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \;\middle|\; 1 < \alpha_1 < \frac{5}{4}, \beta_1 = 1, \alpha_2 \geq \frac{3}{2}, \beta_2 \geq \frac{3}{2\alpha_1}\right\},$$
$$M_5 := \left\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \;\middle|\; 1 < \alpha_1 < \frac{5}{4}, 1 < \beta_1 < \frac{5}{2} - \alpha_1, \alpha_2 \geq \max\left\{\frac{5-2\alpha_1}{4\alpha_1 - 2}, \frac{5-2\beta_1}{4\beta_1-2}\right\},\right.$$
$$\left. \beta_2 \geq \max\left\{\frac{5-2\alpha_1}{2(\alpha_1+\beta_1-1)}, \frac{5-2\beta_1}{2(\alpha_1+\beta_1-1)}\right\}\right\},$$
$$M_6 := \left\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \;\middle|\; 1 < \alpha_1 < \frac{5}{4}, \beta_1 \geq \frac{5}{2} - \alpha_1, \alpha_2 \geq \frac{5-2\alpha_1}{4\alpha_1-2}, \beta_2 \geq 0\right\},$$
$$M_7 := \left\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \;\middle|\; \frac{5}{4} \leq \alpha_1 < \frac{3}{2}, \beta_1 = 1, \alpha_2 \geq \frac{3}{2}, \beta_2 \geq \frac{3}{2\alpha_1}\right\},$$
$$M_8 := \left\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \;\middle|\; \frac{5}{4} \leq \alpha_1 < \frac{3}{2}, 1 < \beta_1 < \frac{5}{2} - \alpha_1, \alpha_2 \geq \max\left\{\frac{5-2\alpha_1}{4\alpha_1 - 2}, \frac{5-2\beta_1}{4\beta_1-2}\right\},\right.$$
$$\left. \beta_2 \geq \max\left\{\frac{5-2\alpha_1}{2(\alpha_1+\beta_1-1)}, \frac{5-2\beta_1}{2(\alpha_1+\beta_1-1)}\right\}\right\},$$
$$M_9 := \left\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \;\middle|\; \frac{5}{4} \leq \alpha_1 < \frac{3}{2}, \beta_1 \geq \frac{5}{2} - \alpha_1, \alpha_2 \geq 0, \beta_2 \geq 0\right\},$$
$$M_{10} := \left\{(\alpha_1,\beta_1,\alpha_2,\beta_2) \;\middle|\; \alpha_1 \geq \frac{3}{2}, \beta_1 \geq 1, \alpha_2 \geq 0, \beta_2 \geq 0\right\}.$$

By employing inequalities, interpolation, and embeddings in the appropriate situation. We now estimate the nonlinear terms in following cases.

Case 1*:* $(\alpha_1,\beta_1,\alpha_2,\beta_2) \in M_1$. Estimation of the nonlinear terms in (3.19) for this case can be done as follows:

$$\begin{aligned}
&|b(U,u_2,U)| \\
&= |b(U,U,u_2)| \lesssim \|U\|_{L^{\frac{2\alpha_2+2}{\alpha_2}}(\mathbb{T})} \|\nabla U\|_{L^2(\mathbb{T})} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \\
&\lesssim \|U\|_{V^1} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|U\|_{L^2(\mathbb{T})}^{\frac{2\alpha_2-1}{2\alpha_2+2}} \|U\|_{L^6(\mathbb{T})}^{\frac{3}{2\alpha_2+2}} \lesssim \|U\|_{V^1}^{\frac{2\alpha_2+5}{2\alpha_2+2}} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|U\|_{V^0}^{\frac{2\alpha_2-1}{2\alpha_2+2}} \\
&\lesssim \frac{\nu}{6}\|U\|_{V^1}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{\frac{4\alpha_2+4}{2\alpha_2-1}} \|U\|_{V^0}^2 \lesssim \frac{\nu}{6}\|U\|_{V^1}^2 + \left(1 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\right)\|U\|_{V^0}^2,
\end{aligned} \tag{3.20}$$

$$\begin{aligned}
|b(V,v_2,U)| = |b(V,U,v_2)| &\lesssim \|V\|_{L^{\frac{2\beta_2+2}{\beta_2}}(\mathbb{T})} \|\nabla U\|_{L^2(\mathbb{T})} \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|U\|_{V^1} \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \|V\|_{L^2(\mathbb{T})}^{\frac{1}{\beta_2+1}} \|V\|_{L^{\frac{2\beta_2}{\beta_2-1}}(\mathbb{T})}^{\frac{\beta_2}{\beta_2+1}}
\end{aligned} \tag{3.21}$$

$$\begin{aligned}
&\lesssim \frac{\nu}{6}\|U\|_{V^1}^2 + \frac{\kappa}{6}\|V\|^2_{L^{\frac{2\beta_2}{\beta_2-1}}(\mathbb{T})} + \|v_2\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\|V\|^2_{L^2(\mathbb{T})}\\
&\lesssim \frac{\nu}{6}\|U\|_{V^1}^2 + \frac{\kappa}{6}\|V\|^2_{V^1} + \|v_2\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\|V\|^2_{V^0},
\end{aligned}$$

$$\begin{aligned}
|b(U,v_1,V)| &= |b(U,V,v_1)| \lesssim \|U\|_{L^{\frac{2\beta_2+2}{\beta_2}}(\mathbb{T})}\|\nabla V\|_{L^2(\mathbb{T})}\|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}\\
&\lesssim \|V\|_{V^1}\|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}\|U\|^{\frac{1}{\beta_2+1}}_{L^2(\mathbb{T})}\|U\|^{\frac{\beta_2}{\beta_2+1}}_{L^{\frac{2\beta_2}{\beta_2-1}}(\mathbb{T})}\\
&\le \frac{\kappa}{6}\|V\|^2_{V^1} + \frac{\nu}{6}\|U\|^2_{L^{\frac{2\beta_2}{\beta_2-1}}(\mathbb{T})} + \|v_1\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\|U\|^2_{L^2(\mathbb{T})}\\
&\lesssim \frac{\kappa}{6}\|V\|^2_{V^1} + \frac{\nu}{6}\|U\|^2_{V^1} + \|v_1\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\|U\|^2_{V^0},
\end{aligned} \tag{3.22}$$

$$\begin{aligned}
&|b(V,u_2,V)|\\
&= |b(V,V,u_2)| \lesssim \|V\|_{L^{\frac{2\alpha_2+2}{\alpha_2}}(\mathbb{T})}\|\nabla V\|_{L^2(\mathbb{T})}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\\
&\lesssim \|V\|_{V^1}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|^{\frac{2\alpha_2-1}{2\alpha_2+2}}_{L^2(\mathbb{T})}\|V\|^{\frac{3}{2\alpha_2+2}}_{L^6(\mathbb{T})} \lesssim \|V\|^{\frac{2\alpha_2+5}{2\alpha_2+2}}_{V^1}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|^{\frac{2\alpha_2-1}{2\alpha_2+2}}_{V^0}\\
&\lesssim \frac{\kappa}{6}\|V\|^2_{V^1} + \|u_2\|^{\frac{4\alpha_2+4}{2\alpha_2-1}}_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|^2_{V^0} \lesssim \frac{\kappa}{6}\|V\|^2_{V^1} + \big(1+\|u_2\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})}\big)\|V\|^2_{V^0}.
\end{aligned} \tag{3.23}$$

It follows from (3.19) and (3.20)–(3.23) that

$$\begin{aligned}
&\frac{d}{dt}\big\{\|U\|^2_{V^0} + \|V\|^2_{V^0}\big\} + \nu\|U\|^2_{V^1} + \kappa\|V\|^2_{V^1}\\
&\lesssim \big(1+\|u_2\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})} + \|v_1\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})} + \|v_2\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\big)\big\{\|U\|^2_{V^0} + \|V\|^2_{V^0}\big\}.
\end{aligned} \tag{3.24}$$

Case 2: $(\alpha_1,\beta_1,\alpha_2,\beta_2) \in M_2$. Estimation of the nonlinear terms in (3.19) for this case can be done as follows:

$$\begin{aligned}
&|b(U,u_2,U)|\\
&= |b(U,U,u_2)| \lesssim \|U\|_{L^{\frac{2\alpha_2+2}{\alpha_2}}(\mathbb{T})}\|\nabla U\|_{L^2(\mathbb{T})}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\\
&\lesssim \|U\|_{V^{\frac{3}{2\alpha_2+2}}}\|U\|_{V^1}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \lesssim \|U\|^{\frac{2\alpha_2+5}{2\alpha_2+2}}_{V^1}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|^{\frac{2\alpha_2-1}{2\alpha_2+2}}_{V^0}\\
&\lesssim \frac{\nu}{6}\|U\|^2_{V^1} + \|u_2\|^{\frac{4\alpha_2+4}{2\alpha_2-1}}_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|^2_{V^0} \lesssim \frac{\nu}{6}\|U\|^2_{V^1} + \big(1+\|u_2\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})}\big)\|U\|^2_{V^0},
\end{aligned} \tag{3.25}$$

$$\begin{aligned}
&|b(V,v_2,U)|\\
&= |b(V,U,v_2)| \lesssim \|V\|_{L^{\frac{2\beta_2+2}{\beta_2}}(\mathbb{T})}\|\nabla U\|_{L^2(\mathbb{T})}\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}\\
&\lesssim \|U\|_{V^1}\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}\|V\|_{V^{\frac{3}{2\beta_2+2}}} \lesssim \|U\|_{V^1}\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}\|V\|^{1-\delta_2}_{V^0}\|V\|^{\delta_2}_{V^{\beta_1}}\\
&\lesssim \frac{\nu}{6}\|U\|^2_{V^1} + \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \|v_2\|^{\frac{2}{1-\delta_2}}_{L^{2\beta_2+2}(\mathbb{T})}\|V\|^2_{V^0}\\
&\lesssim \frac{\nu}{6}\|U\|^2_{V^1} + \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \big(1+\|v_2\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\big)\|V\|^2_{V^0}.
\end{aligned} \tag{3.26}$$

We see that $3/(2\beta_1) < (5-2\beta_1)/(2\beta_1-2)$ as $1 < \beta_1 < 3/2$. We could estimate the nonlinear term $b(U, v_1, V)$ as follows. If $3/(2\beta_1) \le \beta_2 < (5-2\beta_1)/(2\beta_1-2)$, we have

$$\begin{aligned}
&|b(U,v_1,V)| \\
&= |b(U,V,v_1)| \lesssim \|U\|_{L^{\frac{6\beta_2+6}{(2\beta_1+1)(\beta_2+1)-3}}(\mathbb{T})} \|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})} \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|U\|_{V^{\frac{3+(2-2\beta_1)(\beta_2+1)}{2\beta_2+2}}} \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \|V\|_{V^{\beta_1}} \lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \|U\|_{V^0}^{1-\sigma_2} \|U\|_{V^1}^{\sigma_2} \|V\|_{V^{\beta_1}} \\
&\lesssim \frac{\nu}{6}\|U\|_{V^1}^2 + \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{\frac{2}{1-\sigma_2}} \|U\|_{V^0}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^1}^2 + \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \big(1 + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big) \|U\|_{V^0}^2.
\end{aligned} \tag{3.27}$$

If $\beta_2 = (5-2\beta_1)/(2\beta_1-2)$, we have

$$\begin{aligned}
&|b(U,v_1,V)| \\
&= |b(U,V,v_1)| \lesssim \|U\|_{L^2(\mathbb{T})} \|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})} \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|U\|_{V^0} \|V\|_{V^{\beta_1}} \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \big(1 + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big)\|U\|_{V^0}^2.
\end{aligned} \tag{3.28}$$

If $\beta_2 > (5-2\beta_1)/(2\beta_1-2)$, we have

$$\begin{aligned}
&|b(U,v_1,V)| \\
&= |b(U,V,v_1)| \lesssim \|U\|_{L^2(\mathbb{T})} \|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})} \|v_1\|_{L^{\frac{3}{\beta_1-1}}(\mathbb{T})} \\
&\lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \|U\|_{V^0} \|V\|_{V^{\beta_1}} \lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \big(1 + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big)\|U\|_{V^0}^2,
\end{aligned} \tag{3.29}$$

$$\begin{aligned}
&|b(V,u_2,V)| \\
&= |b(V,V,u_2)| \lesssim \|V\|_{L^{\frac{2\alpha_2+2}{\alpha_2}}(\mathbb{T})} \|\nabla V\|_{L^2(\mathbb{T})} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \\
&\lesssim \|V\|_{V^{\frac{3}{2\alpha_2+2}}} \|V\|_{V^1} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \lesssim \|V\|_{V^1}^{\frac{2\alpha_2+5}{2\alpha_2+2}} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|V\|_{V^0}^{\frac{2\alpha_2-1}{2\alpha_2+2}} \\
&\lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{\frac{4\alpha_2+4}{2\alpha_2-1}} \|V\|_{V^0}^2 \lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \big(1 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|V\|_{V^0}^2.
\end{aligned} \tag{3.30}$$

It follows from (3.19) and (3.25)–(3.30) that

$$\begin{aligned}
&\frac{d}{dt}\big\{\|U\|_{V^0}^2 + \|V\|_{V^0}^2\big\} + \nu\|U\|_{V^1}^2 + \kappa\|V\|_{V^{\beta_1}}^2 \\
&\lesssim \big(1 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2} + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2} + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big)\big\{\|U\|_{V^0}^2 + \|V\|_{V^0}^2\big\}.
\end{aligned} \tag{3.31}$$

Case 3: $(\alpha_1, \beta_1, \alpha_2, \beta_2) \in M_3$. Estimation of the nonlinear terms in (3.19) for this case can be done as follows:

$$\begin{aligned}
&|b(U,u_2,U)|\\
&=|b(U,U,u_2)|\lesssim\|U\|_{L^{\frac{2\alpha_2+2}{\alpha_2}}(\mathbb{T})}\|\nabla U\|_{L^2(\mathbb{T})}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\\
&\lesssim\|U\|_{V^{\frac{3}{2\alpha_2+2}}}\|U\|_{V^1}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\lesssim\|U\|_{V^1}^{\frac{2\alpha_2+5}{2\alpha_2+2}}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|_{V^0}^{\frac{2\alpha_2-1}{2\alpha_2+2}}\\
&\lesssim\frac{\nu}{6}\|U\|_{V^1}^2+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{\frac{4\alpha_2+4}{2\alpha_2-1}}\|U\|_{V^0}^2\lesssim\frac{\nu}{6}\|U\|_{V^1}^2+\big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|U\|_{V^0}^2,
\end{aligned}\tag{3.32}$$

$$|b(V,v_2,U)|\lesssim\|V\|_{V^0}\|U\|_{V^1}\|v_2\|_{V^{\beta_1}}\lesssim\frac{\nu}{6}\|U\|_{V^1}^2+\|v_2\|_{V^{\beta_1}}^2\|V\|_{V^0}^2,\tag{3.33}$$

$$|b(U,v_1,V)|\lesssim\|U\|_{V^1}\|v_1\|_{V^{\beta_1}}\|V\|_{V^0}\lesssim\frac{\nu}{6}\|U\|_{V^1}^2+\|v_1\|_{V^{\beta_1}}^2\|V\|_{V^0}^2,\tag{3.34}$$

$$|b(V,u_2,V)|\lesssim\|V\|_{V^0}\|u_2\|_{V^1}\|V\|_{V^{\beta_1}}\lesssim\frac{\kappa}{2}\|V\|_{V^{\beta_1}}^2+\|u_2\|_{V^1}^2\|V\|_{V^0}^2.\tag{3.35}$$

It follows from (3.19) and (3.32)–(3.35) that

$$\begin{aligned}
&\frac{d}{dt}\big\{\|U\|_{V^0}^2+\|V\|_{V^0}^2\big\}+\nu\|U\|_{V^1}^2+\kappa\|V\|_{V^{\beta_1}}^2\\
&\lesssim\big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}+\|u_2\|_{V^1}^2+\|v_1\|_{V^{\beta_1}}^2+\|v_2\|_{V^{\beta_1}}^2\big)\big\{\|U\|_{V^0}^2+\|V\|_{V^0}^2\big\}.
\end{aligned}\tag{3.36}$$

Case 4: $(\alpha_1,\beta_1,\alpha_2,\beta_2)\in M_4$. We first estimate the nonlinear term $b(U,u_2,U)$ in (3.19) as follows. If $3/2\le\alpha_2<(5-2\alpha_1)/(2\alpha_1-2)$, we have

$$\begin{aligned}
&|b(U,u_2,U)|\\
&=|b(U,U,u_2)|\lesssim\|U\|_{L^{\frac{6\alpha_2+6}{(2\alpha_1+1)\alpha_2+2(\alpha_1-1)}}(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\\
&\lesssim\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|_{V^{\alpha_1}}\|U\|_{V^{\frac{2\alpha_2+5}{2\alpha_2+2}-\alpha_1}}\lesssim\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|_{V^0}^{\theta_4}\|U\|_{V^{\alpha_1}}^{2-\theta_4}\\
&\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{\frac{2}{\theta_4}}\|U\|_{V^0}^2\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|U\|_{V^0}^2.
\end{aligned}\tag{3.37}$$

If $\alpha_2=(5-2\alpha_1)/(2\alpha_1-2)$, we have

$$\begin{aligned}
|b(U,u_2,U)|&=|b(U,U,u_2)|\lesssim\|U\|_{L^2(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\\
&\lesssim\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|_{V^0}\|U\|_{V^{\alpha_1}}\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^2\|U\|_{V^0}^2\\
&\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|U\|_{V^0}^2.
\end{aligned}\tag{3.38}$$

If $\alpha_2>(5-2\alpha_1)/(2\alpha_1-2)$, we have

$$\begin{aligned}
|b(U,u_2,U)|&=|b(U,U,u_2)|\lesssim\|U\|_{L^2(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|u_2\|_{L^{\frac{3}{\alpha_1-1}}(\mathbb{T})}\\
&\lesssim\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|_{V^{\alpha_1}}\|U\|_{V^0}\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^2\|U\|_{V^0}^2\\
&\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|U\|_{V^0}^2.
\end{aligned}\tag{3.39}$$

Next, we estimate the nonlinear term $b(V, v_2, U)$ in (3.19) as follows. Notice that $(3-2\alpha_1)/(2\alpha_1) < (5-2\alpha_1)/(2\alpha_1) < 3/(2\alpha_1)$ for all $\alpha_1 \in (1, 5/4)$. If $3/(2\alpha_1) \le \beta_2 < (5-2\alpha_1)/(2\alpha_1 - 2)$, we have

$$
\begin{aligned}
&|b(V, v_2, U)| \\
&= |b(V, U, v_2)| \lesssim \|V\|_{L^{\frac{6\beta_2+6}{(2\alpha_1+1)\beta_2+2(\alpha_1-1)}}(\mathbb{T})} \|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})} \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \|U\|_{V^{\alpha_1}} \|V\|_{V^{\frac{2\beta_2+5}{2\beta_2+2}-\alpha_1}} \lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \|V\|_{V^0}^{1-\sigma_4} \|V\|_{V^1}^{\sigma_4} \|U\|_{V^{\alpha_1}} \\
&\lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{\frac{2}{1-\sigma_4}} \|V\|_{V^0}^2 + \frac{\kappa}{6}\|V\|_{V^1}^2 + \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \frac{\kappa}{6}\|V\|_{V^1}^2 + \left(1 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\right)\|V\|_{V^0}^2.
\end{aligned} \tag{3.40}
$$

If $\beta_2 = (5-2\alpha_1)/(2\alpha_1 - 2)$, we have

$$
\begin{aligned}
|b(V, v_2, U)| &= |b(V, U, v_2)| \lesssim \|V\|_{L^2(\mathbb{T})} \|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})} \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \|V\|_{V^0} \|U\|_{V^{\alpha_1}} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^2 \|V\|_{V^0}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \left(1 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\right)\|V\|_{V^0}^2.
\end{aligned} \tag{3.41}
$$

If $\beta_2 > (5-2\alpha_1)/(2\alpha_1 - 2)$, we have

$$
\begin{aligned}
|b(V, v_2, U)| &= |b(V, U, v_2)| \lesssim \|V\|_{L^2(\mathbb{T})} \|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})} \|v_2\|_{L^{\frac{3}{\alpha_1-1}}(\mathbb{T})} \\
&\lesssim \|V\|_{V^0} \|U\|_{V^{\alpha_1}} \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^2 \|V\|_{V^0}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \left(1 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\right)\|V\|_{V^0}^2.
\end{aligned} \tag{3.42}
$$

We continue estimating the nonlinear term $b(U, v_1, V)$ in (3.19). We get

$$
\begin{aligned}
&|b(U, v_1, V)| \\
&= |b(U, V, v_1)| \lesssim \|U\|_{L^{\frac{2\beta_2+2}{\beta_2}}(\mathbb{T})} \|\nabla V\|_{L^2(\mathbb{T})} \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|U\|_{V^{\frac{3}{2\beta_2+2}}} \|V\|_{V^1} \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \|U\|_{V^0}^{1-\delta_4} \|U\|_{V^{\alpha_1}}^{\delta_4} \|V\|_{V^1} \\
&\lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{\frac{2}{1-\delta_4}} \|U\|_{V^0}^2 + \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \frac{\kappa}{6}\|V\|_{V^1}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \frac{\kappa}{6}\|V\|_{V^1}^2 + \left(1 + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\right)\|U\|_{V^0}^2.
\end{aligned} \tag{3.43}
$$

Finally, we estimate the nonlinear term $b(V, u_2, V)$ in (3.19) as follows:

$$
\begin{aligned}
&|b(V, u_2, V)| \\
&= |b(V, V, u_2)| \lesssim \|V\|_{L^{\frac{2\alpha_2+2}{\alpha_2}}(\mathbb{T})} \|\nabla V\|_{L^2(\mathbb{T})} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \\
&\lesssim \|V\|_{V^{\frac{3}{2\alpha_2+2}}} \|V\|_{V^1} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \lesssim \|V\|_{V^1}^{\frac{2\alpha_2+5}{2\alpha_2+2}} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|V\|_{V^0}^{\frac{2\alpha_2-1}{2\alpha_2+2}} \\
&\lesssim \frac{\kappa}{6}\|V\|_{V^1}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{\frac{4\alpha_2+4}{2\alpha_2-1}} \|V\|_{V^0}^2 \lesssim \frac{\kappa}{6}\|V\|_{V^1}^2 + \left(1 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\right)\|V\|_{V^0}^2.
\end{aligned} \tag{3.44}
$$

It follows from (3.19) and (3.37)–(3.44) that

$$\begin{aligned}&\frac{d}{dt}\big\{\|U\|_{V^0}^2+\|V\|_{V^0}^2\big\}+\nu\|U\|_{V^{\alpha_1}}^2+\kappa\|V\|_{V^1}^2\\&\lesssim\big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}+\|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}+\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big)\big\{\|U\|_{V^0}^2+\|V\|_{V^0}^2\big\}.\end{aligned}\tag{3.45}$$

Case 5*:* $(\alpha_1,\beta_1,\alpha_2,\beta_2)\in M_5$. The nonlinear term $b(U,u_2,U)$ is estimated as follows. If $(5-2\alpha_1)/(4\alpha_1-2)\le\alpha_2<(5-2\alpha_1)/(2\alpha_1-2)$, we have

$$\begin{aligned}&|b(U,u_2,U)|\\&=|b(U,U,u_2)|\lesssim\|U\|_{L^{\frac{6\alpha_2+6}{(2\alpha_1+1)\alpha_2+2(\alpha_1-1)}}(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\\&\lesssim\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|_{V^{\alpha_1}}\|U\|_{V^{\frac{2\alpha_2+5}{2\alpha_2+2}-\alpha_1}}\lesssim\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|_{V^0}^{\theta_5}\|U\|_{V^{\alpha_1}}^{2-\theta_5}\\&\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{\frac{2}{\theta_5}}\|U\|_{V^0}^2\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|U\|_{V^0}^2.\end{aligned}\tag{3.46}$$

If $\alpha_2=(5-2\alpha_1)/(2\alpha_1-2)$, we have

$$\begin{aligned}|b(U,u_2,U)|&=|b(U,U,u_2)|\lesssim\|U\|_{L^2(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\\&\lesssim\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|_{V^0}\|U\|_{V^{\alpha_1}}\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^2\|U\|_{V^0}^2\\&\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|U\|_{V^0}^2.\end{aligned}\tag{3.47}$$

If $\alpha_2>(5-2\alpha_1)/(2\alpha_1-2)$, we have

$$\begin{aligned}|b(U,u_2,U)|&=|b(U,U,u_2)|\lesssim\|U\|_{L^2(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|u_2\|_{L^{\frac{3}{\alpha_1-1}}(\mathbb{T})}\\&\lesssim\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|_{V^{\alpha_1}}\|U\|_{V^0}\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^2\|U\|_{V^0}^2\\&\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|U\|_{V^0}^2.\end{aligned}\tag{3.48}$$

The nonlinear term $b(V,v_2,U)$ is estimated as follows. If $(5-2\alpha_1)/[2(\alpha_1+\beta_1-1)]\le\beta_2<(5-2\alpha_1)/(2\alpha_1-2)$, we have

$$\begin{aligned}&|b(V,v_2,U)|\\&=|b(V,U,v_2)|\lesssim\|V\|_{L^{\frac{6\beta_2+6}{(2\alpha_1+1)\beta_2+2(\alpha_1-1)}}(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}\\&\lesssim\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}\|U\|_{V^{\alpha_1}}\|V\|_{V^{\frac{2\beta_2+5}{2\beta_2+2}-\alpha_1}}\lesssim\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}\|V\|_{V^0}^{1-\sigma_5}\|V\|_{V^{\beta_1}}^{\sigma_5}\|U\|_{V^{\alpha_1}}\\&\lesssim\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{\frac{2}{1-\sigma_5}}\|V\|_{V^0}^2+\frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2+\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2\\&\lesssim\frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2+\frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2+\big(1+\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big)\|V\|_{V^0}^2.\end{aligned}\tag{3.49}$$

If $\beta_2 = (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$
\begin{aligned}
|b(V,v_2,U)| &= |b(V,U,v_2)| \lesssim \|V\|_{L^2(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}\|V\|_{V^0}\|U\|_{V^{\alpha_1}} \lesssim \frac{\nu}{6}\|U\|^2_{V^{\alpha_1}} + \|v_2\|^2_{L^{2\beta_2+2}(\mathbb{T})}\|V\|^2_{V^0} \\
&\lesssim \frac{\nu}{6}\|U\|^2_{V^{\alpha_1}} + \left(1+\|v_2\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\right)\|V\|^2_{V^0}.
\end{aligned} \tag{3.50}
$$

If $\beta_2 > (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$
\begin{aligned}
|b(V,v_2,U)| &= |b(V,U,v_2)| \lesssim \|V\|_{L^2(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|v_2\|_{L^{\frac{3}{\alpha_1-1}}(\mathbb{T})} \\
&\lesssim \|V\|_{V^0}\|U\|_{V^{\alpha_1}}\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \lesssim \frac{\nu}{6}\|U\|^2_{V^{\alpha_1}} + \|v_2\|^2_{L^{2\beta_2+2}(\mathbb{T})}\|V\|^2_{V^0} \\
&\lesssim \frac{\nu}{6}\|U\|^2_{V^{\alpha_1}} + \left(1+\|v_2\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\right)\|V\|^2_{V^0}.
\end{aligned} \tag{3.51}
$$

The nonlinear term $b(U,v_1,V)$ is estimated as follows. If $(5-2\beta_1)/[2(\alpha_1+\beta_1-1)] \le \beta_2 < (5-2\beta_1)/(2\beta_1-2)$, we have

$$
\begin{aligned}
&|b(U,v_1,V)| \\
&= |b(U,V,v_1)| \lesssim \|U\|_{L^{\frac{6\beta_2+6}{(2\beta_1+1)\beta_2+2(\beta_1-1)}}(\mathbb{T})}\|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})}\|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}\|V\|_{V^{\beta_1}}\|U\|_{V^{\frac{2\beta_2+5}{2\beta_2+2}-\beta_1}} \lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}\|U\|^{1-\delta_5}_{V^0}\|U\|^{\delta_5}_{V^{\alpha_1}}\|V\|_{V^{\beta_1}} \\
&\lesssim \|v_1\|^{\frac{2}{1-\delta_5}}_{L^{2\beta_2+2}(\mathbb{T})}\|U\|^2_{V^0} + \frac{\nu}{6}\|U\|^2_{V^{\alpha_1}} + \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} \\
&\lesssim \frac{\nu}{6}\|U\|^2_{V^{\alpha_1}} + \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \left(1+\|v_1\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\right)\|U\|^2_{V^0}.
\end{aligned} \tag{3.52}
$$

If $\beta_2 = (5-2\beta_1)/(2\beta_1-2)$, we have

$$
\begin{aligned}
|b(U,v_1,V)| &= |b(U,V,v_1)| \lesssim \|U\|_{L^2(\mathbb{T})}\|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})}\|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}\|U\|_{V^0}\|V\|_{V^{\beta_1}} \lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \|v_1\|^2_{L^{2\beta_2+2}(\mathbb{T})}\|U\|^2_{V^0} \\
&\lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \left(1+\|v_1\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\right)\|U\|^2_{V^0}.
\end{aligned} \tag{3.53}
$$

If $\beta_2 > (5-2\beta_1)/(2\beta_1-2)$, we have

$$
\begin{aligned}
|b(U,v_1,V)| &= |b(U,V,v_1)| \lesssim \|U\|_{L^2(\mathbb{T})}\|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})}\|v_1\|_{L^{\frac{3}{\beta_1-1}}(\mathbb{T})} \\
&\lesssim \|U\|_{V^0}\|V\|_{V^{\beta_1}}\|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \|v_1\|^2_{L^{2\beta_2+2}(\mathbb{T})}\|U\|^2_{V^0} \\
&\lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \left(1+\|v_1\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\right)\|U\|^2_{V^0}.
\end{aligned} \tag{3.54}
$$

The nonlinear term $b(V,u_2,V)$ is estimated as follows. If $(5-2\beta_1)/(4\beta_1-2) \le \alpha_2 <$

$(5-2\beta_1)/(2\beta_1-2)$, we have

$$
\begin{aligned}
&|b(V,u_2,V)| \\
&= |b(V,V,u_2)| \lesssim \|V\|_{L^{\frac{6\alpha_2+6}{(2\beta_1+1)\alpha_2+2(\beta_1-1)}}(\mathbb{T})} \|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \\
&\lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|V\|_{V^{\beta_1}} \|V\|_{V^{\frac{2\alpha_2+5}{2\alpha_2+2}-\beta_1}} \lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|V\|_{V^0}^{\gamma_5} \|V\|_{V^{\beta_1}}^{2-\gamma_5} \\
&\lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{\frac{2}{\gamma_5}} \|V\|_{V^0}^2 \lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|V\|_{V^0}^2.
\end{aligned} \tag{3.55}
$$

If $\alpha_2 = (5-2\beta_1)/(2\beta_1-2)$, we have

$$
\begin{aligned}
|b(V,u_2,V)| = |b(V,V,u_2)| &\lesssim \|V\|_{L^2(\mathbb{T})} \|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \\
&\lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|V\|_{V^0} \|V\|_{V^{\beta_1}} \lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^2 \|V\|_{V^0}^2 \\
&\lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|V\|_{V^0}^2.
\end{aligned} \tag{3.56}
$$

If $\alpha_2 > (5-2\beta_1)/(2\beta_1-2)$, we have

$$
\begin{aligned}
|b(V,u_2,V)| = |b(V,V,u_2)| &\lesssim \|V\|_{L^2(\mathbb{T})} \|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})} \|u_2\|_{L^{\frac{3}{\beta_1-1}}(\mathbb{T})} \\
&\lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|V\|_{V^{\beta_1}} \|V\|_{V^0} \lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^2 \|V\|_{V^0}^2 \\
&\lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|V\|_{V^0}^2.
\end{aligned} \tag{3.57}
$$

It follows from (3.19) and (3.46)–(3.57) that

$$
\begin{aligned}
&\frac{d}{dt}\big\{\|U\|_{V^0}^2 + \|V\|_{V^0}^2\big\} + \nu\|U\|_{V^{\alpha_1}}^2 + \kappa\|V\|_{V^{\beta_1}}^2 \\
&\lesssim \big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2} + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2} + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big)\big\{\|U\|_{V^0}^2 + \|V\|_{V^0}^2\big\}.
\end{aligned} \tag{3.58}
$$

Case 6*:* $(\alpha_1,\beta_1,\alpha_2,\beta_2) \in M_6$. The nonlinear term $b(U,u_2,U)$ is estimated as follows. If $(5-2\alpha_1)/(4\alpha_1-2) \le \alpha_2 < (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$
\begin{aligned}
&|b(U,u_2,U)| \\
&= |b(U,U,u_2)| \lesssim \|U\|_{L^{\frac{6\alpha_2+6}{(2\alpha_1+1)\alpha_2+2(\alpha_1-1)}}(\mathbb{T})} \|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \\
&\lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|U\|_{V^{\alpha_1}} \|U\|_{V^{\frac{2\alpha_2+5}{2\alpha_2+2}-\alpha_1}} \lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|U\|_{V^0}^{\theta_6} \|U\|_{V^{\alpha_1}}^{2-\theta_6} \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{\frac{2}{\theta_6}} \|U\|_{V^0}^2 \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|U\|_{V^0}^2.
\end{aligned} \tag{3.59}
$$

If $\alpha_2 = (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$
\begin{aligned}
|b(U,u_2,U)| = |b(U,U,u_2)| &\lesssim \|U\|_{L^2(\mathbb{T})} \|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})} \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \\
&\lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \|U\|_{V^0} \|U\|_{V^{\alpha_1}} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^2 \|U\|_{V^0}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|U\|_{V^0}^2.
\end{aligned} \tag{3.60}
$$

If $\alpha_2 > (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$
\begin{aligned}
|b(U,u_2,U)| = |b(U,U,u_2)| &\lesssim \|U\|_{L^2(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|u_2\|_{L^{\frac{3}{\alpha_1-1}}(\mathbb{T})} \\
&\lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|U\|_{V^{\alpha_1}}\|U\|_{V^0} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^2\|U\|_{V^0}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|U\|_{V^0}^2.
\end{aligned}
\tag{3.61}
$$

The nonlinear term $b(V,v_2,U)$ can be estimated as follows:

$$
|b(V,v_2,U)| \lesssim \|V\|_{V^0}\|v_2\|_{V^{\beta_1}}\|U\|_{V^{\alpha_1}} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|v_2\|_{V^{\beta_1}}^2\|V\|_{V^0}^2. \tag{3.62}
$$

The nonlinear term $b(U,v_1,V)$ can be estimated as follows:

$$
|b(U,v_1,V)| \lesssim \|U\|_{V^{\alpha_1}}\|v_1\|_{V^{\beta_1}}\|V\|_{V^0} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|v_1\|_{V^{\beta_1}}^2\|V\|_{V^0}^2. \tag{3.63}
$$

The nonlinear term $b(V,u_2,V)$ can be estimated as follows:

$$
|b(V,u_2,V)| \lesssim \|V\|_{V^0}\|u_2\|_{V^{\alpha_1}}\|V\|_{V^{\beta_1}} \lesssim \frac{\kappa}{2}\|V\|_{V^{\beta_1}}^2 + \|u_2\|_{V^{\alpha_1}}^2\|V\|_{V^0}^2. \tag{3.64}
$$

It follows from (3.19) and (3.59)–(3.64) that

$$
\begin{aligned}
&\frac{d}{dt}\big\{\|U\|_{V^0}^2 + \|V\|_{V^0}^2\big\} + \nu\|U\|_{V^{\alpha_1}}^2 + \kappa\|V\|_{V^{\beta_1}}^2 \\
&\lesssim \big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2} + \|u_2\|_{V^{\alpha_1}}^2 + \|v_1\|_{V^{\beta_1}}^2 + \|v_2\|_{V^{\beta_1}}^2\big)\big\{\|U\|_{V^0}^2 + \|V\|_{V^0}^2\big\}.
\end{aligned}
\tag{3.65}
$$

Case 7: $(\alpha_1,\beta_1,\alpha_2,\beta_2) \in M_7$. The nonlinear term $b(U,u_2,U)$ can be estimated as follows:

$$
|b(U,u_2,U)| \lesssim \|U\|_{V^0}\|u_2\|_{V^{\alpha_1}}\|U\|_{V^{\alpha_1}} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|u_2\|_{V^{\alpha_1}}^2\|U\|_{V^0}^2. \tag{3.66}
$$

The nonlinear term $b(V,v_2,U)$ can be estimated as follows. Notice that $(3-2\alpha_1)/(2\alpha_1) < (5-2\alpha_1)/(2\alpha_1) < 3/(2\alpha_1)$ for all $\alpha_1 \in [5/4,3/2)$. If $3/(2\alpha_1) \le \beta_2 < (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$
\begin{aligned}
&|b(V,v_2,U)| \\
&= |b(V,U,v_2)| \lesssim \|V\|_{L^{\frac{6\beta_2+6}{(2\alpha_1+1)\beta_2+2(\alpha_1-1)}}(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}\|U\|_{V^{\alpha_1}}\|V\|_{V^{\frac{2\beta_2+5}{2\beta_2+2}-\alpha_1}} \lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}\|V\|_{V^0}^{1-\sigma_7}\|V\|_{V^1}^{\sigma_7}\|U\|_{V^{\alpha_1}} \\
&\lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{\frac{2}{1-\sigma_7}}\|V\|_{V^0}^2 + \frac{\kappa}{6}\|V\|_{V^1}^2 + \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \frac{\kappa}{6}\|V\|_{V^1}^2 + \big(1+\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big)\|V\|_{V^0}^2.
\end{aligned}
\tag{3.67}
$$

If $\beta_2 = (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$
\begin{aligned}
|b(V,v_2,U)| &= |b(V,U,v_2)| \lesssim \|V\|_{L^2(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}\|V\|_{V^0}\|U\|_{V^{\alpha_1}} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^2\|V\|_{V^0}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \big(1+\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big)\|V\|_{V^0}^2.
\end{aligned} \tag{3.68}
$$

If $\beta_2 > (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$
\begin{aligned}
|b(V,v_2,U)| &= |b(V,U,v_2)| \lesssim \|V\|_{L^2(\mathbb{T})}\|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})}\|v_2\|_{L^{\frac{3}{\alpha_1-1}}(\mathbb{T})} \\
&\lesssim \|V\|_{V^0}\|U\|_{V^{\alpha_1}}\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^2\|V\|_{V^0}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \big(1+\|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big)\|V\|_{V^0}^2.
\end{aligned} \tag{3.69}
$$

We continue estimating the nonlinear term $b(U,v_1,V)$ in (3.19). We get

$$
\begin{aligned}
&|b(U,v_1,V)| \\
&= |b(U,V,v_1)| \lesssim \|U\|_{L^{\frac{2\beta_2+2}{\beta_2}}(\mathbb{T})}\|\nabla V\|_{L^2(\mathbb{T})}\|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|U\|_{V^{\frac{3}{2\beta_2+2}}}\|V\|_{V^1}\|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}\|U\|_{V^0}^{1-\delta_7}\|U\|_{V^{\alpha_1}}^{\delta_7}\|V\|_{V^1} \\
&\lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{\frac{2}{1-\delta_7}}\|U\|_{V^0}^2 + \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \frac{\kappa}{6}\|V\|_{V^1}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \frac{\kappa}{6}\|V\|_{V^1}^2 + \big(1+\|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big)\|U\|_{V^0}^2.
\end{aligned} \tag{3.70}
$$

Finally, we estimate the nonlinear term $b(V,u_2,V)$ in (3.19) as follows:

$$
\begin{aligned}
&|b(V,u_2,V)| \\
&= |b(V,V,u_2)| \lesssim \|V\|_{L^{\frac{2\alpha_2+2}{\alpha_2}}(\mathbb{T})}\|\nabla V\|_{L^2(\mathbb{T})}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \\
&\lesssim \|V\|_{V^{\frac{3}{2\alpha_2+2}}}\|V\|_{V^1}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \lesssim \|V\|_{V^1}^{\frac{2\alpha_2+5}{2\alpha_2+2}}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|_{V^0}^{\frac{2\alpha_2-1}{2\alpha_2+2}} \\
&\lesssim \frac{\kappa}{6}\|V\|_{V^1}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{\frac{4\alpha_2+4}{2\alpha_2-1}}\|V\|_{V^0}^2 \lesssim \frac{\kappa}{6}\|V\|_{V^1}^2 + \big(1+\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2}\big)\|V\|_{V^0}^2.
\end{aligned} \tag{3.71}
$$

It follows from (3.19) and (3.66)–(3.71) that

$$
\begin{aligned}
&\frac{d}{dt}\big\{\|U\|_{V^0}^2 + \|V\|_{V^0}^2\big\} + \nu\|U\|_{V^{\alpha_1}}^2 + \kappa\|V\|_{V^{\beta_1}}^2 \\
&\lesssim \big(1+\|u_2\|_{V^{\alpha_1}}^2 + \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2} + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2} + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\big) \\
&\quad\times \big\{\|U\|_{V^0}^2 + \|V\|_{V^0}^2\big\}.
\end{aligned} \tag{3.72}
$$

Case 8: $(\alpha_1,\beta_1,\alpha_2,\beta_2) \in M_8$. The nonlinear term $b(U,u_2,U)$ can be estimated as follows:

$$
|b(U,u_2,U)| \lesssim \|U\|_{V^0}\|u_2\|_{V^{\alpha_1}}\|U\|_{V^{\alpha_1}} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|u_2\|_{V^{\alpha_1}}^2\|U\|_{V^0}^2. \tag{3.73}
$$

The nonlinear term $b(V, v_2, U)$ is estimated as follows. If $(5-2\alpha_1)/[2(\alpha_1+\beta_1-1)] \le \beta_2 < (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$\begin{aligned}
&|b(V, v_2, U)| \\
&= |b(V, U, v_2)| \lesssim \|V\|_{L^{\frac{6\beta_2+6}{(2\alpha_1+1)\beta_2+2(\alpha_1-1)}}(\mathbb{T})} \|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})} \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \|U\|_{V^{\alpha_1}} \|V\|_{V^{\frac{2\beta_2+5}{2\beta_2+2}-\alpha_1}} \lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \|V\|_{V^0}^{1-\sigma_8} \|V\|_{V^{\beta_1}}^{\sigma_8} \|U\|_{V^{\alpha_1}} \\
&\lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{\frac{2}{1-\sigma_8}} \|V\|_{V^0}^2 + \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \left(1 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\right)\|V\|_{V^0}^2.
\end{aligned} \tag{3.74}$$

If $\beta_2 = (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$\begin{aligned}
|b(V, v_2, U)| = |b(V, U, v_2)| &\lesssim \|V\|_{L^2(\mathbb{T})} \|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})} \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \|V\|_{V^0} \|U\|_{V^{\alpha_1}} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^2 \|V\|_{V^0}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \left(1 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\right)\|V\|_{V^0}^2.
\end{aligned} \tag{3.75}$$

If $\beta_2 > (5-2\alpha_1)/(2\alpha_1-2)$, we have

$$\begin{aligned}
|b(V, v_2, U)| = |b(V, U, v_2)| &\lesssim \|V\|_{L^2(\mathbb{T})} \|\nabla U\|_{L^{\frac{6}{5-2\alpha_1}}(\mathbb{T})} \|v_2\|_{L^{\frac{3}{\alpha_1-1}}(\mathbb{T})} \\
&\lesssim \|V\|_{V^0} \|U\|_{V^{\alpha_1}} \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})} \lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^2 \|V\|_{V^0}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \left(1 + \|v_2\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\right)\|V\|_{V^0}^2.
\end{aligned} \tag{3.76}$$

The nonlinear term $b(U, v_1, V)$ is estimated as follows. If $(5-2\beta_1)/[2(\alpha_1+\beta_1-1)] \le \beta_2 < (5-2\beta_1)/(2\beta_1-2)$, we have

$$\begin{aligned}
&|b(U, v_1, V)| \\
&= |b(U, V, v_1)| \lesssim \|U\|_{L^{\frac{6\beta_2+6}{(2\beta_1+1)\beta_2+2(\beta_1-1)}}(\mathbb{T})} \|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})} \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \|V\|_{V^{\beta_1}} \|U\|_{V^{\frac{2\beta_2+5}{2\beta_2+2}-\beta_1}} \lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \|U\|_{V^0}^{1-\delta_8} \|U\|_{V^{\alpha_1}}^{\delta_8} \|V\|_{V^{\beta_1}} \\
&\lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{\frac{2}{1-\delta_8}} \|U\|_{V^0}^2 + \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 \\
&\lesssim \frac{\nu}{6}\|U\|_{V^{\alpha_1}}^2 + \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \left(1 + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\right)\|U\|_{V^0}^2.
\end{aligned} \tag{3.77}$$

If $\beta_2 = (5-2\beta_1)/(2\beta_1-2)$, we have

$$\begin{aligned}
|b(U, v_1, V)| = |b(U, V, v_1)| &\lesssim \|U\|_{L^2(\mathbb{T})} \|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})} \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \\
&\lesssim \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \|U\|_{V^0} \|V\|_{V^{\beta_1}} \lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^2 \|U\|_{V^0}^2 \\
&\lesssim \frac{\kappa}{6}\|V\|_{V^{\beta_1}}^2 + \left(1 + \|v_1\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2}\right)\|U\|_{V^0}^2.
\end{aligned} \tag{3.78}$$

If $\beta_2 > (5-2\beta_1)/(2\beta_1-2)$, we have

$$
\begin{aligned}
|b(U,v_1,V)| = |b(U,V,v_1)| &\lesssim \|U\|_{L^2(\mathbb{T})}\|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})}\|v_1\|_{L^{\frac{3}{\beta_1-1}}(\mathbb{T})} \\
&\lesssim \|U\|_{V^0}\|V\|_{V^{\beta_1}}\|v_1\|_{L^{2\beta_2+2}(\mathbb{T})} \lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \|v_1\|^2_{L^{2\beta_2+2}(\mathbb{T})}\|U\|^2_{V^0} \\
&\lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \big(1+\|v_1\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\big)\|U\|^2_{V^0}.
\end{aligned}
\tag{3.79}
$$

We see that in this case $\beta_1 < \alpha_1$. This implies that $(5-2\alpha_1)/(4\alpha_1-2) < (5-2\beta_1)/(4\beta_1-2) \leq (5-2\beta_1)/(2\beta_1-2)$. The nonlinear term $b(V,u_2,V)$ is estimated as follows. If $(5-2\beta_1)/(4\beta_1-2) \leq \alpha_2 < (5-2\beta_1)/(2\beta_1-2)$, we have

$$
\begin{aligned}
&|b(V,u_2,V)| \\
&= |b(V,V,u_2)| \lesssim \|V\|_{L^{\frac{6\alpha_2+6}{(2\beta_1+1)\alpha_2+2(\beta_1-1)}}(\mathbb{T})}\|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \\
&\lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|_{V^{\beta_1}}\|V\|_{V^{\frac{2\alpha_2+5}{2\alpha_2+2}-\beta_1}} \lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|^{\gamma_8}_{V^0}\|V\|^{2-\gamma_8}_{V^{\beta_1}} \\
&\lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \|u_2\|^{\frac{2}{\gamma_8}}_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|^2_{V^0} \lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \big(1+\|u_2\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})}\big)\|V\|^2_{V^0}.
\end{aligned}
\tag{3.80}
$$

If $\alpha_2 = (5-2\beta_1)/(2\beta_1-2)$, we have

$$
\begin{aligned}
|b(V,u_2,V)| = |b(V,V,u_2)| &\lesssim \|V\|_{L^2(\mathbb{T})}\|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})}\|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})} \\
&\lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|_{V^0}\|V\|_{V^{\beta_1}} \lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \|u_2\|^2_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|^2_{V^0} \\
&\lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \big(1+\|u_2\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})}\big)\|V\|^2_{V^0}.
\end{aligned}
\tag{3.81}
$$

If $\alpha_2 > (5-2\beta_1)/(2\beta_1-2)$, we have

$$
\begin{aligned}
|b(V,u_2,V)| = |b(V,V,u_2)| &\lesssim \|V\|_{L^2(\mathbb{T})}\|\nabla V\|_{L^{\frac{6}{5-2\beta_1}}(\mathbb{T})}\|u_2\|_{L^{\frac{3}{\beta_1-1}}(\mathbb{T})} \\
&\lesssim \|u_2\|_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|_{V^{\beta_1}}\|V\|_{V^0} \lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \|u_2\|^2_{L^{2\alpha_2+2}(\mathbb{T})}\|V\|^2_{V^0} \\
&\lesssim \frac{\kappa}{6}\|V\|^2_{V^{\beta_1}} + \big(1+\|u_2\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})}\big)\|V\|^2_{V^0}.
\end{aligned}
\tag{3.82}
$$

It follows from (3.19) and (3.73)–(3.82) that

$$
\begin{aligned}
&\frac{d}{dt}\big\{\|U\|^2_{V^0} + \|V\|^2_{V^0}\big\} + \nu\|U\|^2_{V^{\alpha_1}} + \kappa\|V\|^2_{V^{\beta_1}} \\
&\lesssim \big(1+\|u_2\|^2_{V^{\alpha_1}} + \|u_2\|^{2\alpha_2+2}_{L^{2\alpha_2+2}(\mathbb{T})} + \|v_1\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})} + \|v_2\|^{2\beta_2+2}_{L^{2\beta_2+2}(\mathbb{T})}\big) \\
&\quad \times \big\{\|U\|^2_{V^0} + \|V\|^2_{V^0}\big\}.
\end{aligned}
\tag{3.83}
$$

Case 9*:* $(\alpha_1,\beta_1,\alpha_2,\beta_2) \in M_9$ *and Case* 10*:* $(\alpha_1,\beta_1,\alpha_2,\beta_2) \in M_{10}$. The nonlinear term $b(U,u_2,U)$ can be estimated as follows:

$$
|b(U,u_2,U)| \lesssim \|U\|_{V^0}\|u_2\|_{V^{\alpha_1}}\|U\|_{V^{\alpha_1}} \lesssim \frac{\nu}{6}\|U\|^2_{V^{\alpha_1}} + \|u_2\|^2_{V^{\alpha_1}}\|U\|^2_{V^0}.
\tag{3.84}
$$

The nonlinear term $b(V, v_2, U)$ can be estimated as follows:

$$|b(V, v_2, U)| \lesssim \|V\|_{V^0}\|v_2\|_{V^{\beta_1}}\|U\|_{V^{\alpha_1}} \lesssim \frac{\nu}{6}\|U\|^2_{V^{\alpha_1}} + \|v_2\|^2_{V^{\beta_1}}\|V\|^2_{V^0}. \tag{3.85}$$

The nonlinear term $b(U, v_1, V)$ can be estimated as follows:

$$|b(U, v_1, V)| \lesssim \|U\|_{V^{\alpha_1}}\|v_1\|_{V^{\beta_1}}\|V\|_{V^0} \lesssim \frac{\nu}{6}\|U\|^2_{V^{\alpha_1}} + \|v_1\|^2_{V^{\beta_1}}\|V\|^2_{V^0}. \tag{3.86}$$

The nonlinear term $b(V, u_2, V)$ can be estimated as follows:

$$|b(V, u_2, V)| \lesssim \|V\|_{V^0}\|u_2\|_{V^{\alpha_1}}\|V\|_{V^{\beta_1}} \lesssim \frac{\kappa}{2}\|V\|^2_{V^{\beta_1}} + \|u_2\|^2_{V^{\alpha_1}}\|V\|^2_{V^0}. \tag{3.87}$$

It follows from (3.19) and (3.84)–(3.87) that

$$\begin{aligned} &\frac{d}{dt}\big\{\|U\|^2_{V^0} + \|V\|^2_{V^0}\big\} + \nu\|U\|^2_{V^{\alpha_1}} + \kappa\|V\|^2_{V^{\beta_1}} \\ &\lesssim \big(\|u_2\|^2_{V^{\alpha_1}} + \|v_1\|^2_{V^{\beta_1}} + \|v_2\|^2_{V^{\beta_1}}\big)\big\{\|U\|^2_{V^0} + \|V\|^2_{V^0}\big\}. \end{aligned} \tag{3.88}$$

We now can infer from the Gronwall's inequality, (3.24), (3.31), (3.36), (3.45), (3.58), (3.65), (3.72), (3.83) and (3.88) the uniqueness of the global weak solution of the system (2.2). Moreover, using Lemma 2.7 implies that $u \in C_{\text{loc}}([\tau, \infty); V^0)$ for any $\tau \in \mathbb{R}$. □

Remark 3.2. This theorem illustrates how the strength of nonlinearities and the degree of dissipation can work together to yield the global existence and uniqueness of the weak solution of (2.2). These results also extend and improve the previous results such as Wu [47,48], Ye [52,53], Wang and Liu [46] (see also [18,19,21,30,31,38,50,51,56,59,60] and references therein). Although some of the estimations overlap, we still keep them for the sake of completeness of the proof.

The global well-posedness of (2.3) reads as follows.

Theorem 3.3. *Let ν, κ, μ, η, α_1, β_1, a, b be positive real parameters and let $r \geq 1$ and $s \geq 1$. Assume that $f \in L^2_{\text{loc}}(\mathbb{R}; V^0)$, $g \in L^2_{\text{loc}}(\mathbb{R}; V^0)$, $u_\tau \in V^0$ and $v_\tau \in V^0$. Then, the system* (2.3) *possesses a global weak solution obeying Definition* 2.4 *with initial condition (u_τ, v_τ). Furthermore, the global weak solution is unique and depends continuity on the initial data if $\alpha_1 \geq 1$ and $\beta_1 \geq 1$.*

Proof. It follows from (2.14), (2.15), Definitions 2.3 and 2.4 that rk and sk play the role as $2\alpha_2$ and $2\beta_2$, respectively. Therefore, we can take k large enough such that all constraints of rk and sk like α_2 and β_2 in $\bigcup_{i=1}^3 \mathfrak{M}_i$ are satisfied. Thus, the proof of Theorem 3.3 can be done by some modification of that of Theorem 3.1. We will omit it here because of the length of paper and similarity (see also [29,44]). □

4. Attractors

In this section, we require the external forcing terms satisfying a specific condition in the following definition (see, e.g., [12, 33, 36]).

Definition 4.1. Let $\mathcal{B}$ be a Banach space.

(i) A function $\varphi(s) \in L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$ is said to be translation bounded in $L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$, i.e., $\varphi(s) \in L^2_b(\mathbb{R};\mathcal{B})$ if
$$\sup_{t\in\mathbb{R}} \int_t^{t+1} \|\varphi(s)\|^2_{\mathcal{B}}\, ds < \infty.$$

(ii) A function $\varphi(s) \in L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$ is said to be normal in $L^2_{\text{loc}}(\mathbb{R};\mathcal{B})$, i.e., $\varphi(s) \in L^2_c(\mathbb{R};\mathcal{B})$ if for any $\epsilon > 0$, there exists $\delta > 0$, such that
$$\sup_{t\in\mathbb{R}} \int_t^{t+\delta} \|\varphi(s)\|^2_{\mathcal{B}}\, ds \le \epsilon.$$

We see that the class of normal functions is a proper closed subspace of the class of translation bounded functions (see [33, 36]).

Taking a fixed pair of forces f_0 and g_0 such that $\sigma_0 := (f_0, g_0)$ is translation bounded in $L^2_{\text{loc}}(\mathbb{R};V^0) \times L^2_{\text{loc}}(\mathbb{R};V^0)$, i.e.,
$$\|f_0\|_b^2 := \|f_0\|^2_{L^2_b(\mathbb{R};V^0)} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|f_0(s)\|^2_{V^0}\, ds < \infty$$
and
$$\|g_0\|_b^2 := \|g_0\|^2_{L^2_b(\mathbb{R};V^0)} := \sup_{t\in\mathbb{R}} \int_t^{t+1} \|g_0(s)\|^2_{V^0}\, ds < \infty.$$
We denote $L^{2,w}_{\text{loc}}(\mathbb{R};V^0) \times L^{2,w}_{\text{loc}}(\mathbb{R};V^0)$ the space $L^2_{\text{loc}}(\mathbb{R};V^0) \times L^2_{\text{loc}}(\mathbb{R};V^0)$ endowed with the local weak convergence topology. Then σ_0 is translation compact in $L^{2,w}_{\text{loc}}(\mathbb{R};V^0) \times L^{2,w}_{\text{loc}}(\mathbb{R};V^0)$, i.e., the translation family of σ_0
$$\Sigma := \{\sigma_0(\,\cdot + h) := (f_0(\,\cdot + h), g_0(\,\cdot + h)) \mid h \in \mathbb{R}\}$$
is precompact in $L^{2,w}_{\text{loc}}(\mathbb{R};V^0) \times L^{2,w}_{\text{loc}}(\mathbb{R};V^0)$ (see [12]). Moreover,
$$\|f\|_b^2 \le \|f_0\|_b^2 \quad \text{and} \quad \|g\|_b^2 \le \|g_0\|_b^2 \tag{4.1}$$
for every $\sigma = (f, g) \in \Sigma$.

We will exploit the theory of the evolutionary system in [2, Section 4], [35, Section 4], [13, Section 8], [16, Sections 5–6] and [14]. All results for (2.2) below can be adapted to (2.3). To avoid repetition and cumbersomeness in presentation, we select (2.2) to investigate and state the main results for (2.3). Assume that $(u(t), v(t))$, $t \in [\tau,\infty)$, be a weak solution of (2.2) with the initial data $(u(\tau), \theta(\tau)) \in V^0 \times V^0$ and $\sigma = (f, g) \in \Sigma$ guaranteed by Theorem 3.1. We first define the strong and weak distances by

$$d_s((u_1, v_1); (u_2, v_2)) := \|u_1 - u_2\|_{V^0} + \|v_1 - v_2\|_{V^0}, \quad \forall\, u_1, u_2, v_1, v_2 \in V^0$$

and

$$d_w((u_1, v_1); (u_2, v_2)) := \sum_{k \in J} \frac{1}{2^{|k|}} \left(\frac{|u_{1k} - u_{2k}|}{1 + |u_{1k} - u_{2k}|} + \frac{|v_{1k} - v_{2k}|}{1 + |v_{1k} - v_{2k}|} \right),$$

where u_{ik} and v_{ik} are Fourier coefficients of u_i and v_i, $i = 1, 2$, respectively.

We take L^2-scalar product of the first equation in (2.2) with u and the second equation in (2.2) with v; bearing in mind Lemma 2.1, we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|_{V^0}^2 + \nu\|u(t)\|_{V^{\alpha_1}}^2 + \mu\|u(t)\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2} = b(v(t), v(t), u(t)) + \langle f(t), u(t)\rangle, \tag{4.2}$$

and

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|_{V^0}^2 + \kappa\|v(t)\|_{V^{\beta_1}}^2 + \eta\|v(t)\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2} = b(v(t), u(t), v(t)) + \langle g(t), v(t)\rangle. \tag{4.3}$$

Summing-up (4.2) and (4.3) and using Lemma 2.1 again, we get

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\big\{\|u(t)\|_{V^0}^2 + \|v(t)\|_{V^0}^2\big\} + \nu\|u(t)\|_{V^{\alpha_1}}^2 + \kappa\|v(t)\|_{V^{\beta_1}}^2 \\ &+ \mu\|u(t)\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2} + \eta\|v(t)\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2} \\ =\ &\langle f(t), u(t)\rangle + \langle g(t), v(t)\rangle. \end{aligned} \tag{4.4}$$

By using (2.1) and the Cauchy–Schwarz inequality, we deduce from (4.4) that

$$\begin{aligned} &\frac{d}{dt}\big\{\|u(t)\|_{V^0}^2 + \|v(t)\|_{V^0}^2\big\} + \theta\big\{\|u(t)\|_{V^{\alpha_1}}^2 + \|v(t)\|_{V^{\beta_1}}^2\big\} \\ &+ 2\mu\|u(t)\|_{L^{2\alpha_2+2}(\mathbb{T})}^{2\alpha_2+2} + 2\eta\|v(t)\|_{L^{2\beta_2+2}(\mathbb{T})}^{2\beta_2+2} \\ \lesssim\ &\frac{1}{\nu}\|f(t)\|_{V^0}^2 + \frac{1}{\kappa}\|g(t)\|_{V^0}^2, \end{aligned} \tag{4.5}$$

where $\theta = \min\{\nu, \kappa\}$. Therefore,

$$\frac{d}{dt}\big(\big\{\|u(t)\|_{V^0}^2 + \|v(t)\|_{V^0}^2\big\}e^{\theta t}\big) \lesssim \frac{e^{\theta t}}{\nu}\|f(t)\|_{V^0}^2 + \frac{e^{\theta t}}{\kappa}\|g(t)\|_{V^0}^2.$$

Integrating in time from τ to t, we obtain

$$\begin{aligned} &\big\{\|u(t)\|_{V^0}^2 + \|v(t)\|_{V^0}^2\big\}e^{\theta t} - \big\{\|u(\tau)\|_{V^0}^2 + \|v(\tau)\|_{V^0}^2\big\}e^{\theta\tau} \\ \lesssim\ &\frac{1}{\nu}\int_\tau^t \|f(s)\|_{V^0}^2 e^{\theta s}\, ds + \frac{1}{\kappa}\int_\tau^t \|g(s)\|_{V^0}^2 e^{\theta s}\, ds \end{aligned}$$

for all $t \geq \tau$. Estimating the last integrals

$$\begin{aligned}\int_\tau^t \|f(s)\|_{V^0}^2 e^{\theta s}\, ds &\leq \int_{t-1}^t \|f(s)\|_{V^0}^2 e^{\theta s}\, ds + \int_{t-2}^{t-1} \|f(s)\|_{V^0}^2 e^{\theta s}\, ds + \cdots \\ &\leq \|f\|_b^2(1 + e^{-\theta} + \cdots)e^{\theta t} \leq \frac{e^\theta}{e^\theta - 1}\|f\|_b^2 e^{\theta t} \leq \frac{e^\theta}{e^\theta - 1}\|f_0\|_b^2 e^{\theta t}\end{aligned}$$

and

$$\begin{aligned}\int_\tau^t \|g(s)\|_{V^0}^2 e^{\theta s}\, ds &\leq \int_{t-1}^t \|g(s)\|_{V^0}^2 e^{\theta s}\, ds + \int_{t-2}^{t-1} \|g(s)\|_{V^0}^2 e^{\theta s}\, ds + \cdots \\ &\leq \|g\|_b^2(1 + e^{-\theta} + \cdots)e^{\theta t} \leq \frac{e^\theta}{e^\theta - 1}\|g\|_b^2 e^{\theta t} \leq \frac{e^\theta}{e^\theta - 1}\|g_0\|_b^2 e^{\theta t}.\end{aligned}$$

Therefore

$$\|u(t)\|_{V^0}^2 + \|v(t)\|_{V^0}^2 \lesssim \{\|u(\tau)\|_{V^0}^2 + \|v(\tau)\|_{V^0}^2\}e^{-\theta(t-\tau)} + \frac{e^\theta}{\nu(e^\theta - 1)}\|f_0\|_b^2 + \frac{e^\theta}{\kappa(e^\theta - 1)}\|g_0\|_b^2.$$

This implies that there exists a uniformly (w.r.t. $\tau \in \mathbb{R}$ and $\sigma \in \Sigma$) absorbing ball $B_s(0, R) \subset V^0 \times V^0$, where the radius R depends on ν, κ, $\|f_0\|_b^2$ and $\|g_0\|_b^2$. Let us denote X_{cab} a closed absorbing ball

$$X_{\text{cab}} := \big\{(u, \theta) \in V^0 \times V^0 : \|u\|_{V^0}^2 + \|v\|_{V^0}^2 \leq R^2\big\}.$$

This means that for any bounded set $B \subset V^0 \times V^0$, there exists a time $\bar{t} \geq 0$ independent of the initial time τ such that

$$(u(t), v(t)) \in X_{\text{cab}}, \quad \forall\, t \geq t_1 := \tau + \bar{t} \tag{4.6}$$

for every weak solutions $(u(t), v(t))$ with $\sigma = (f, g) \in \Sigma$ and the initial time $(u(\tau), v(\tau)) \in B$. It is known that X_{cab} is weakly compact in $V^0 \times V^0$ and metrizable with the weak metric d_w. The weak metric d_w induces the weak topology in X_{cab}. For any sequence of weak solutions (u_n, v_n) of (2.2) the following result holds

Lemma 4.2. *Assume that (u_n, v_n) is a sequence of weak solutions of (2.2) with $\sigma_n = (f_n, g_n) \in \Sigma$ satisfying $(u_n(t), v_n(t)) \in X_{\text{cab}}$ for all $t \geq t_1$. Then*

$$\begin{aligned} u_n &\text{ is bounded in } L^2(t_1, t_2; V^{\alpha_1}),\ L^{2\alpha_2+2}(t_1, t_2; L^{2\alpha_2+2}(\mathbb{T})) \text{ and } L^\infty(t_1, t_2; V^0), \\ \partial_t u_n &\text{ is bounded in } L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(t_1, t_2; V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T})), \\ v_n &\text{ is bounded in } L^2(t_1, t_2; V^{\beta_1}),\ L^{2\beta_2+2}(t_1, t_2; L^{2\beta_2+2}(\mathbb{T})) \text{ and } L^\infty(t_1, t_2; V^0), \\ \partial_t v_n &\text{ is bounded in } L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(t_1, t_2; V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T}))\end{aligned}$$

for all $t_2 \geq t_1$, $\gamma_1 = \max\{5/2-\alpha_1, \alpha_1, 5/2-\beta_1, \beta_1\}$ *and* $\gamma_2 = \max\{\alpha_2, \beta_2\}$. *Moreover, there exists a subsequence* (u_{n_j}, v_{n_j}) *converging to some solution* (u, v) *in* $C_w([t_1, t_2]; V^0) \times C_w([t_1, t_2]; V^0)$, *i.e.,*

$$\langle u_{n_j}, \psi\rangle \to \langle u, \psi\rangle \text{ uniformly on } [t_1, t_2] \text{ as } n_j \to \infty \text{ for all } \psi \in V^0,$$
$$\langle v_{n_j}, \psi\rangle \to \langle v, \psi\rangle \text{ uniformly on } [t_1, t_2] \text{ as } n_j \to \infty \text{ for all } \psi \in V^0.$$

Proof. The proof is analogous with the results of [16, Lemma 5.4], [35, Lemma 5.3], [34, Lemma 3.2] and [41, Lemma 2.1]. Therefore, we only sketch the main steps. Let (u_n, v_n) be a sequence satisfying (2.2) with forces $\sigma_n = (f_n, g_n) \in \Sigma$. Thus

$$\begin{cases} \partial_t u_n + \nu\Lambda^{2\alpha_1} u_n + B(u_n, u_n) - B(v_n, v_n) + \mu P_\sigma(|u_n|^{2\alpha_2} u_n) = P_\sigma f_n, \\ \partial_t v_n + \kappa\Lambda^{2\beta_1} v_n + B(u_n, v_n) - B(v_n, u_n) + \eta P_\sigma(|v_n|^{2\beta_2} v_n) = P_\sigma g_n. \end{cases} \tag{4.7}$$

Standard estimates in [2, Theorem 2.1] show that, for all $t_2 > t_1$, the sequence $\{u_n\}$ is uniformly bounded in

$$L^\infty(t_1, t_2; V^0) \cap L^2(t_1, t_2; V^{\alpha_1}) \cap L^{2\alpha_2+2}(t_1, t_2; L^{2\alpha_2+2}(\mathbb{T})),$$

and the sequence $\{v_n\}$ is uniformly bounded in

$$L^\infty(t_1, t_2; V^0) \cap L^2(t_1, t_2; V^{\beta_1}) \cap L^{2\beta_2+2}(t_1, t_2; L^{2\beta_2+2}(\mathbb{T})),$$

and $\partial_t u_n$ and $\partial_t v_n$ are uniformly bounded in

$$L^2(t_1, t_2; V^{-\gamma_1}) + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(t_1, t_2; L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T})) \subset L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(t_1, t_2; V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T})).$$

Since

$$V^{\alpha_1} \cap L^{2\alpha_2+2}(\mathbb{T}) \hookrightarrow\hookrightarrow V^0 \hookrightarrow V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T})$$

and

$$V^{\beta_1} \cap L^{2\beta_2+2}(\mathbb{T}) \hookrightarrow\hookrightarrow V^0 \hookrightarrow V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T}),$$

we deduce from the Aubin–Lions lemma (see [43]) that the sequences $\{u_n\}$ and $\{v_n\}$ are compact in $L^2(\tau, T; V^0)$ and so we can extract subsequences, still denoted by u_n and v_n, respectively, such that

$$u_n \rightharpoonup u \text{ weakly in } L^2(t_1, t_2; V^{\alpha_1}), \tag{4.8}$$
$$v_n \rightharpoonup v \text{ weakly in } L^2(t_1, t_2; V^{\beta_1}), \tag{4.9}$$
$$u_n \rightharpoonup u \text{ weakly in } L^{2\alpha_2+2}(t_1, t_2; L^{2\alpha_2+2}(\mathbb{T})), \tag{4.10}$$
$$v_n \rightharpoonup v \text{ weakly in } L^{2\beta_2+2}(t_1, t_2; L^{2\beta_2+2}(\mathbb{T})), \tag{4.11}$$
$$u_n \rightharpoonup^* u \text{ weakly star in } L^\infty(t_1, t_2; V^0), \tag{4.12}$$
$$v_n \rightharpoonup^* v \text{ weakly star in } L^\infty(t_1, t_2; V^0), \tag{4.13}$$
$$u_n \to u \text{ strongly in } L^2(t_1, t_2; V^0), \tag{4.14}$$
$$v_n \to v \text{ strongly in } L^2(t_1, t_2; V^0). \tag{4.15}$$

Note that $\sigma_0 = (f_0, g_0)$ is translation compact in $L^{2,w}_{\mathrm{loc}}(\mathbb{R}; V^0) \times L^{2,w}_{\mathrm{loc}}(\mathbb{R}; V^0)$ (see [12, 40]). Thus, passing to a subsequence, we also get

$$f_n \to f \quad \text{weakly in } L^2(t_1, t_2; V^0), \tag{4.16}$$

$$g_n \to g \quad \text{weakly in } L^2(t_1, t_2; V^0) \tag{4.17}$$

with some $(f, g) \in L^2(t_1, t_2; V^0) \times L^2(t_1, t_2; V^0)$. Using all convergences (4.8)–(4.17) and passing the limits in (4.7) yields the following equations

$$\begin{cases} \partial_t u + \nu\Lambda^{2\alpha_1} u + B(u, u) - B(v, v) + \mu P_\sigma(|u|^{2\alpha_2} u) = P_\sigma f, \\ \partial_t v + \kappa\Lambda^{2\beta_1} v + B(u, v) - B(v, u) + \eta P_\sigma(|v|^{2\beta_2} v) = P_\sigma g. \end{cases}$$

Moreover, the pair of functions (u, v) is the solution of (2.2) and inherits all the regularity from (u_n, v_n), i.e.,

$$u \in L^\infty(t_1, t_2; V^0) \cap L^2(t_1, t_2; V^{\alpha_1}) \cap L^{2\alpha_2+2}(t_1, t_2; L^{2\alpha_2+2}(\mathbb{T})) \cap C_w([t_1, t_2]; V^0)$$

and

$$v \in L^\infty(t_1, t_2; V^0) \cap L^2(t_1, t_2; V^{\beta_1}) \cap L^{2\beta_2+2}(t_1, t_2; L^{2\beta_2+2}(\mathbb{T})) \cap C_w([t_1, t_2]; V^0).$$

It follows from (2.4) and (2.5) that there exists a subsequence (u_{n_j}, v_{n_j}) converges to (u, v) in $C_w([t_1, t_2]; V^0) \times C_w([t_1, t_2]; V^0)$. □

Following the ideas in [2, Section 4], [35, Section 4], [13, Section 8], [16, Sections 5–6] and [14], we consider the following evolutionary system of (2.2)

$$\begin{aligned} \mathcal{E}([\tau, \infty)) := \{(u(\cdot), v(\cdot)) : {} & (u(\cdot), v(\cdot)) \text{ is a weak solution on } [\tau, \infty) \text{ with } (f, g) \in \Sigma \\ & \text{and } (u(t), v(t)) \in X_{\mathrm{cab}}, \forall\, t \in [\tau, \infty)\}, \quad \tau \in \mathbb{R}, \\ \mathcal{E}((-\infty, \infty)) := \{(u(\cdot), v(\cdot)) : {} & (u(\cdot), v(\cdot)) \text{ is a weak solution on } (-\infty, \infty) \text{ with } (f, g) \in \Sigma \\ & \text{and } (u(t), v(t)) \in X_{\mathrm{cab}}, \forall\, t \in (-\infty, \infty)\}. \end{aligned}$$

Clearly, all conditions of an evolutionary system $\mathcal{E}$ hold because of the translation identity, i.e., a weak solution of (2.2) with $(f, g) \in \Sigma$ initiating at time $\tau + h$ is also a weak solution of (2.2) with $(f(\cdot + h), g(\cdot + h)) \in \Sigma$ initiating at time τ.

We need to check that the evolutionary system $\mathcal{E}$ of (2.2) satisfies the following properties

(A1) $\mathcal{E}([0, \infty))$ is a precompact set in $C([0, \infty); X_{\mathrm{cab},w})$.

(A2) (Energy inequality) For any $\varepsilon > 0$, there exists $\delta > 0$, such that for every $(u, v) \in \mathcal{E}([0, \infty))$ and $t > 0$,

$$\|u(t)\|^2_{V^0} + \|v(t)\|^2_{V^0} \le \|u(t_0)\|^2_{V^0} + \|v(t_0)\|^2_{V^0} + \varepsilon$$

for t_0 a.e. in $(t - \delta, t)$.

(A3) (Strong convergence a.e.) If $(u_n, v_n) \in \mathcal{E}([0,\infty))$ is $d_{C([0,T];X_{\mathrm{cab},w})}$-Cauchy sequence in $C([0,T]; X_{\mathrm{cab},w})$ for some $T > 0$, then $(u_n(t), v_n(t))$ is d_s-Cauchy sequence a.e. in $[0,T]$.

Let

$$\overline{\mathcal{E}}([\tau,\infty)) := \overline{\mathcal{E}([\tau,\infty))}^{C([\tau,\infty);X_{\mathrm{cab},w})}, \quad \forall\, \tau \in \mathbb{R}$$

and

$$\overline{\mathcal{E}}((-\infty,\infty)) := \{u(\cdot) : u(\cdot)|_{[\tau,\infty)} \in \overline{\mathcal{E}}([\tau,\infty)), \forall\, \tau \in \mathbb{R}\}.$$

It can be checked that $\overline{\mathcal{E}}$ is also an evolutionary system and it is called the closure of the evolutionary system $\mathcal{E}$.

Let $\mathcal{K} := \mathcal{E}((-\infty,\infty))$ and $\overline{\mathcal{K}} := \overline{\mathcal{E}}((-\infty,\infty))$, which are called the kernel of $\mathcal{E}$ and $\overline{\mathcal{E}}$, respectively. Let also

$$\Pi_+\mathcal{K} := \big\{(u(\cdot), v(\cdot))|_{[0,\infty)} : (u,v) \in \mathcal{K}\big\} \quad \text{and} \quad \Pi_+\overline{\mathcal{K}} := \big\{(u(\cdot), v(\cdot))|_{[0,\infty)} : (u,v) \in \overline{\mathcal{K}}\big\}.$$

Then, we have the following lemma.

Lemma 4.3. *The evolutionary system $\mathcal{E}$ of* (2.2) *with the forces f_0 and g_0 satisfies* (A1) *and* (A3)*. Moreover, if f_0 and g_0 are normal in $L^2_{\mathrm{loc}}(\mathbb{R}; V^0)$ and $L^2_{\mathrm{loc}}(\mathbb{R}; V^0)$, respectively, then $\mathcal{E}$ of* (2.2) *also satisfies* (A2).

Proof. The proof is standard. It is analogous to that of [2, Lemma 4.3] and [16, Lemma 5.7].

First, we verify that (A1) holds. Indeed, we deduce from Definition 2.3, Theorem 3.1 and (4.6) that $\mathcal{E}([0,\infty)) \subset C([0,\infty); X_{\mathrm{cab},w})$. Let $\{(u_n, v_n)\}$ be a sequence in $\mathcal{E}([0,\infty))$. It follows from Lemma 4.2 that there exists a subsequence, still denoted by $\{(u_n, v_n)\}$, which converges in $C([0,1]; X_{\mathrm{cab},w})$ to some $(u^1, v^1) \in C([0,1]; X_{\mathrm{cab},w})$ as $n \to \infty$. Passing to a subsequence and dropping a subindex once more, we have that this subsequence converges in $C([0,2]; X_{\mathrm{cab},w})$ to some $(u^2, v^2) \in C([0,2]; X_{\mathrm{cab},w})$ as $n \to \infty$. Note that $(u^1(t), v^1(t)) = (u^2(t), v^2(t))$ on $[0,1]$. Continuing this diagonalization process, we obtain a subsequence $\{(u_{n_j}, v_{n_j})\}$ of $\{(u_n, v_n)\}$ that converges in $C([0,\infty); X_{\mathrm{cab},w})$ to some $(u,v) \in C([0,\infty); X_{\mathrm{cab},w})$ as $n_j \to \infty$. Therefore, (A1) holds.

Next, we prove that (A3) is valid. Take a sequence $\{(u_n, v_n)\} \subset \mathcal{E}([0,\infty))$ be such that it is a $d_{C([0,T];X_{\mathrm{cab},w})}$-Cauchy sequence in $C([0,T]; X_{\mathrm{cab},w})$ for some $T > 0$. Thanks to Lemma 4.2 again, the sequence $\{(u_n, v_n)\}$ is bounded in $L^2(0,T; V^{\alpha_1}) \times L^2(0,T; V^{\beta_1})$. Hence, there exists some $(u(t), v(t)) \in C([0,T]; X_{\mathrm{cab},w})$ such that

$$\int_0^T \|u_n(s) - u(s)\|^2_{V^0}\, ds \to 0 \quad \text{as } n \to \infty,$$
$$\int_0^T \|v_n(s) - v(s)\|^2_{V^0}\, ds \to 0 \quad \text{as } n \to \infty.$$

In particular, by using the Radon-Riesz property, we get $\|u_n(t)\|_{V^0} \to \|u(t)\|_{V^0}$ and $\|v_n(t)\|_{V^0} \to \|v(t)\|_{V^0}$ as $n \to \infty$ a.e. on $[0,T]$, which means that $\{(u(t),v(t))\}$ is a d_s-Cauchy sequence a.e. on $[0,T]$. Thus, (A3) is valid.

Finally, for any $(u,v) \in \mathcal{E}([0,\infty))$ and $t > 0$, using the property of normal functions, we can infer from (4.1) and (4.5) that, for any $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|u(t)\|_{V^0}^2 + \|v(t)\|_{V^0}^2 \le \|u(t_0)\|_{V^0}^2 + \|v(t_0)\|_{V^0}^2 + \epsilon$$

for t_0 a.e. in $(t-\delta, t)$. This implies that (A2) holds. □

Using Lemma 4.3 and [2, Theorems 4.1, 4.2 and 4.3], we get the following result.

Theorem 4.4. (i) *The weak uniform global attractor $\mathcal{A}_w$ and the weak trajectory attractor $\mathfrak{A}_w$ for (2.2) with forces f_0 and g_0 exist, $\mathcal{A}_w$ is the maximal invariant and maximal quasi-invariant set w.r.t. the closure $\overline{\mathcal{E}}$ of the corresponding evolutionary system $\mathcal{E}$ and*

$$\begin{aligned} \mathcal{A}_w &= \omega_w(X_{\text{cab}}) = \omega_s(X_{\text{cab}}) = \{(u(0),v(0)) : (u,v) \in \overline{\mathcal{K}}\}, \\ \mathfrak{A}_w &= \Pi_+\overline{\mathcal{K}} = \{(u(\cdot),v(\cdot))|_{[0,\infty)} : (u,v) \in \overline{\mathcal{K}}\}, \\ \mathcal{A}_w &= \mathfrak{A}_w(t) = \{(u(t),v(t)) : (u,v) \in \mathfrak{A}_w\}, \quad \forall\, t \ge 0. \end{aligned}$$

Moreover, $\mathfrak{A}_w$ satisfies the finite weak uniform tracking property and is weakly equicontinuity on $[0,\infty)$.

(ii) *Furthermore, if f_0 and g_0 are normal in $L^2_{\text{loc}}(\mathbb{R};V^0)$ and $L^2_{\text{loc}}(\mathbb{R};V^0)$, respectively, and every complete trajectory of $\overline{\mathcal{E}}$ is strongly continuous, then the weak global attractor $\mathcal{A}_w$ is a strongly compact strong global attractor $\mathcal{A}_s$, and the weak trajectory attractor $\mathfrak{A}_w$ is a strongly compact strong trajectory attractor $\mathfrak{A}_s$. Moreover, $\mathfrak{A}_s = \Pi_+\overline{\mathcal{K}}$ satisfies the finite strong uniform tracking property and is strongly equicontinuous on $[0,\infty)$.*

We now require the external forcing terms satisfying a stronger condition. We assume that $\sigma_0 = (f_0, g_0)$ is translation compact in $L^2_{\text{loc}}(\mathbb{R};V^0) \times L^2_{\text{loc}}(\mathbb{R};V^0)$, i.e.,

$$\overline{\Sigma} := \overline{\{f_0(\,\cdot + h) : h \in \mathbb{R}\}}^{L^2_{\text{loc}}(\mathbb{R};V^0)} \times \overline{\{g_0(\,\cdot + h) : h \in \mathbb{R}\}}^{L^2_{\text{loc}}(\mathbb{R};V^0)}$$

is compact in $L^2_{\text{loc}}(\mathbb{R};V^0) \times L^2_{\text{loc}}(\mathbb{R};V^0)$. We notice that the class of translation compact functions is also a closed subspace of the class of translation bounded functions, but it is a proper subset of the class of normal functions (for more details, see [33, 36]). Moreover, $L^2_{\text{loc}}(\mathbb{R};V^0) \times L^2_{\text{loc}}(\mathbb{R};V^0)$ is metrizable and the corresponding metric space is complete (see, e.g., [12]). We have the following result.

Lemma 4.5. *Assume that* (u_n, v_n) *is a sequence of weak solutions of* (2.2) *with* $\sigma_n = (f_n, g_n) \in \overline{\Sigma}$ *satisfying* $(u_n(t), v_n(t)) \in X_{\text{cab}}$ *for all* $t \geq t_1$. *Then*

$$\begin{aligned}
u_n &\text{ is bounded in } L^2(t_1, t_2; V^{\alpha_1}),\ L^{2\alpha_2+2}(t_1, t_2; L^{2\alpha_2+2}(\mathbb{T})) \text{ and } L^\infty(t_1, t_2; V^0),\\
\partial_t u_n &\text{ is bounded in } L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(t_1, t_2; V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T})),\\
v_n &\text{ is bounded in } L^2(t_1, t_2; V^{\beta_1}),\ L^{2\beta_2+2}(t_1, t_2; L^{2\beta_2+2}(\mathbb{T})) \text{ and } L^\infty(t_1, t_2; V^0),\\
\partial_t v_n &\text{ is bounded in } L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(t_1, t_2; V^{-\gamma_1} + L^{\frac{2\gamma_2+2}{2\gamma_2+1}}(\mathbb{T}))
\end{aligned}$$

for all $t_2 \geq t_1$, $\gamma_1 = \max\{5/2 - \alpha_1, \alpha_1, 5/2 - \beta_1, \beta_1\}$ *and* $\gamma_2 = \max\{\alpha_2, \beta_2\}$.

Moreover, there exists a subsequence n_j *such that* $\sigma_n = (f_n, g_n) \in \overline{\Sigma}$ *converges in* $L^2_{\text{loc}}(\mathbb{R}; V^0) \times L^2_{\text{loc}}(\mathbb{R}; V^0)$ *to some* $\sigma = (f, g) \in \overline{\Sigma}$ *and* (u_{n_j}, v_{n_j}) *converges in* $C_w([t_1, t_2]; V^0) \times C_w([t_1, t_2]; V^0)$ *to some solution* $(u(t), v(t))$ *with the forces* $\sigma = (f, g) \in \overline{\Sigma}$, *i.e.,*

$$\begin{aligned}
\langle u_{n_j}, \psi\rangle \to \langle u, \psi\rangle &\text{ uniformly on } [t_1, t_2] \text{ as } n_j \to \infty \text{ for all } \psi \in V^0,\\
\langle v_{n_j}, \psi\rangle \to \langle v, \psi\rangle &\text{ uniformly on } [t_1, t_2] \text{ as } n_j \to \infty \text{ for all } \psi \in V^0.
\end{aligned}$$

Proof. The proof is adapted from the proof of Lemma 4.2 and using the property of the class of translation compact functions (see also [16, Lemma 6.1]). Therefore, we omit it here. □

Due to Lemma 4.5, we now can investigate another evolutionary system with $\overline{\Sigma}$ as a symbol space determining by

$$\begin{aligned}
\mathcal{E}_{\overline{\Sigma}}([\tau, \infty)) := \{(u(\cdot), v(\cdot)) : (u(\cdot), v(\cdot)) &\text{ is a weak solution on } [\tau, \infty) \text{ with}\\
&(f, g) \in \overline{\Sigma} \text{ and } (u(t), v(t)) \in X_{\text{cab}}, \forall\, t \in [\tau, \infty)\}, \quad \tau \in \mathbb{R},\\
\mathcal{E}_{\overline{\Sigma}}((-\infty, \infty)) := \{(u(\cdot), v(\cdot)) : (u(\cdot), v(\cdot)) &\text{ is a weak solution on } (-\infty, \infty) \text{ with}\\
&(f, g) \in \overline{\Sigma} \text{ and } (u(t), v(t)) \in X_{\text{cab}}, \forall\, t \in (-\infty, \infty)\}.
\end{aligned}$$

We will check that the evolutionary system $\mathcal{E}_{\overline{\Sigma}}$ of (2.2) satisfies the following properties

(B1) $\mathcal{E}_{\overline{\Sigma}}([0, \infty))$ is a compact set in $C([0, \infty); X_{\text{cab},w})$.

(B2) (Energy inequality) For any $\varepsilon > 0$, there exists $\delta > 0$, such that for every $(u, v) \in \mathcal{E}_{\overline{\Sigma}}([0, \infty))$ and $t > 0$,

$$\|u(t)\|^2_{V^0} + \|v(t)\|^2_{V^0} \leq \|u(t_0)\|^2_{V^0} + \|v(t_0)\|^2_{V^0} + \varepsilon$$

for t_0 a.e. in $(t - \delta, t)$.

(B3) (Strong convergence a.e.) Let $(u, v), (u_n, v_n) \in \mathcal{E}_{\overline{\Sigma}}([0, \infty))$ be such that $(u_n, v_n) \to (u, v)$ in $C([0, T]; X_{\text{cab},w})$ for some $T > 0$. Then $(u_n(t), v_n(t)) \to (u(t), v(t))$ strongly a.e. in $[0, T]$.

We then obtain the following results.

Lemma 4.6. *Assume that the external forcing terms are in $\overline{\Sigma}$. The evolutionary system $\mathcal{E}_{\overline{\Sigma}}$ of the family of the three dimensional generalized incompressible MHD equations with polynomial damping satisfies* (B1), (B2) *and* (B3).

Proof. The proof is similar to the arguments in Lemma 4.3 and [13, Lemma 8.6] (see also [16, Lemmas 5.7 and 6.2] and [2, 35]). The difference is that we must use Lemma 4.5 instead of Lemma 4.2; the external forcing terms satisfy the stronger condition; $\{(u_n, v_n)\}$ and (u, v) would now belong to $\mathcal{E}_{\overline{\Sigma}}([0,\infty))$. □

Lemma 4.7. *Assume that the external forcing terms are in $\overline{\Sigma}$. Let $\mathcal{E}_{\overline{\Sigma}}$ be the evolutionary system of the family of the three dimensional generalized incompressible MHD equations with polynomial damping. Then, $\mathcal{E}_{\overline{\Sigma}}$ is closed and $\overline{\mathcal{E}}_{\overline{\Sigma}} = \mathcal{E}_{\overline{\Sigma}}$.*

Proof. The proof is similar to the arguments in [35, Lemmas 4.12 and 4.13]. □

Theorem 4.8. (i) *Let $\sigma_0 = (f_0, g_0)$ be translation compact in $L^2_{\mathrm{loc}}(\mathbb{R}; V^0) \times L^2_{\mathrm{loc}}(\mathbb{R}; V^0)$. Then the weak uniform global attractor $\mathcal{A}_w^{\overline{\Sigma}}$ and the weak trajectory attractor $\mathfrak{A}_w^{\overline{\Sigma}}$ for* (2.2) *with forces $\sigma = (f, g) \in \overline{\Sigma}$ exist, $\mathcal{A}_w^{\overline{\Sigma}}$ is the maximal invariant and maximal quasi-invariant set w.r.t. the corresponding evolutionary system $\mathcal{E}_{\overline{\Sigma}}$ and*

$$\begin{aligned}
\mathcal{A}_w^{\overline{\Sigma}} &= \big\{(u(0), v(0)) : (u, v) \in \mathcal{E}_{\overline{\Sigma}}((-\infty, \infty))\big\} \\
&= \Big\{(u(0), v(0)) : (u, v) \in \bigcup_{\sigma \in \overline{\Sigma}} \mathcal{E}_\sigma((-\infty, \infty))\Big\}, \\
\mathfrak{A}_w^{\overline{\Sigma}} &= \Pi_+ \bigcup_{\sigma \in \overline{\Sigma}} \mathcal{E}_\sigma((-\infty, \infty)), \\
\mathcal{A}_w^{\overline{\Sigma}} &= \mathfrak{A}_w^{\overline{\Sigma}}(t) = \big\{(u(t), v(t)) : (u, v) \in \mathfrak{A}_w^{\overline{\Sigma}}\big\}, \quad \forall\, t \geq 0,
\end{aligned}$$

where $\mathcal{E}_\sigma((-\infty, \infty))$ is nonempty for any $\sigma = (f, g) \in \overline{\Sigma}$. Moreover, $\mathfrak{A}_w^{\overline{\Sigma}}$ satisfies the finite weak uniform tracking property and is weakly equicontinuous on $[0, \infty)$.

(ii) *Furthermore, if every complete trajectory of the generalized Boussinesq system with forces $\sigma = (f, g) \in \overline{\Sigma}$ is strongly continuous, then the weak uniform global attractor $\mathcal{A}_w^{\overline{\Sigma}}$ is a strongly compact strong global attractor $\mathcal{A}_s^{\overline{\Sigma}}$, and the weak trajectory attractor $\mathfrak{A}_w^{\overline{\Sigma}}$ is a strongly compact strong trajectory attractor $\mathfrak{A}_s^{\overline{\Sigma}}$. Moreover, $\mathfrak{A}_s^{\overline{\Sigma}}$ satisfies the finite strong uniform tracking property and is strongly equicontinuous on* $[0, \infty)$.

Proof. By Lemma 4.7, $\mathcal{E}_{\overline{\Sigma}}$ equals to its closure $\overline{\mathcal{E}}_{\overline{\Sigma}}$. Especially, we have

$$\mathcal{E}_{\overline{\Sigma}}((-\infty, \infty)) = \overline{\mathcal{E}}_{\overline{\Sigma}}((-\infty, \infty)).$$

It follows from [2, Lemma 4.1] that

$$\mathcal{E}_{\overline{\Sigma}}((-\infty,\infty)) = \bigcup_{\sigma\in\overline{\Sigma}} \mathcal{E}_\sigma((-\infty,\infty)),$$

and $\mathcal{E}_\sigma((-\infty,\infty))$ is nonempty for any $\sigma \in \overline{\Sigma}$. We know from Lemma 4.6 and Lemma 4.7 that $\mathcal{E}_{\overline{\Sigma}}$ is closed and satisfies (B1), (B2) and (B3). The rest part of the conclusions follow by applying [2, Theorems 4.1, 4.2 and 4.3], which is in fact [16, Theorems 2.7, 2.8 and 2.9]. □

We deduce from Lemma 4.7 that $\mathcal{E} \subset \overline{\mathcal{E}} \subset \mathcal{E}_{\overline{\Sigma}}$. Therefore, an interesting problem arises as follows: *Are the attractors $\mathcal{A}_\bullet$, $\mathfrak{A}_\bullet$ and $\mathcal{A}_\bullet^{\overline{\Sigma}}$, $\mathfrak{A}_\bullet^{\overline{\Sigma}}$ in Theorem 4.4 and Theorem 4.8 are identical?*

The answer is positive if (2.2) is well-posedness. Note that the answer may be negative if the weak solution of (2.2) is not unique. We have the following theorem.

Theorem 4.9. *Assume $\sigma_0 = (f_0, g_0)$ is translation compact in $L^2_{\text{loc}}(\mathbb{R}; V^0) \times L^2_{\text{loc}}(\mathbb{R}; V^0)$, $\mathcal{E}_\Sigma$ denotes the evolutionary system of (2.2) with forces in Σ and $\overline{\mathcal{E}}_\Sigma$ is its closure. Let $\mathcal{E}_{\overline{\Sigma}}$ be the evolutionary system of (2.2) with forces in $\overline{\Sigma}$. If $(\alpha_1, \beta_1, \alpha_2, \beta_2) \in \bigcup_{i=1}^3 \mathfrak{M}_i$, the following results then hold.*

(1) *The three weak uniform global attractors $\mathcal{A}_w^\Sigma$, $\overline{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\overline{\Sigma}}$ are the maximal invariant and maximal quasi-invariant set with respect to $\mathcal{E}_\Sigma$, $\overline{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\overline{\Sigma}}$ and satisfy*

$$\mathcal{A}_w^\Sigma = \overline{\mathcal{A}}_w^\Sigma = \mathcal{A}_w^{\overline{\Sigma}} = \big\{(u(0), v(0)) : (u,v) \in \mathcal{E}_{\overline{\Sigma}}((-\infty,\infty))\big\}.$$

(2) *The three weak trajectory attractors $\mathfrak{A}_w^\Sigma$, $\overline{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\overline{\Sigma}}$ for evolutionary systems $\mathcal{E}_\Sigma$, $\overline{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\overline{\Sigma}}$, respectively, exist and satisfy*

$$\mathfrak{A}_w^\Sigma = \overline{\mathfrak{A}}_w^\Sigma = \mathfrak{A}_w^{\overline{\Sigma}} = \Pi_+ \bigcup_{\sigma\in\overline{\Sigma}} \mathcal{E}_\sigma((-\infty,\infty)).$$

Hence, the three weak trajectory attractors satisfy the finite weak uniform tracking property for all the three evolutionary systems and are weakly equicontinuous on $[0,\infty)$.

(3) *$\mathcal{A}_w^\Sigma$, $\overline{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\overline{\Sigma}}$ are sections of $\mathfrak{A}_w^\Sigma$, $\overline{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\overline{\Sigma}}$:*

$$\mathcal{A}_w^\Sigma = \overline{\mathcal{A}}_w^\Sigma = \mathcal{A}_w^{\overline{\Sigma}} = \mathfrak{A}_w^\Sigma(t) = \overline{\mathfrak{A}}_w^\Sigma(t) = \mathfrak{A}_w^{\overline{\Sigma}}(t) = \big\{(u(t), v(t)) : (u,v) \in \mathfrak{A}_w^{\overline{\Sigma}}\big\}, \quad \forall\, t \ge 0.$$

(4) *The three weak uniform global attractors $\mathcal{A}_w^\Sigma$, $\overline{\mathcal{A}}_w^\Sigma$ and $\mathcal{A}_w^{\overline{\Sigma}}$ for evolutionary systems $\mathcal{E}_\Sigma$, $\overline{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\overline{\Sigma}}$, respectively, are strongly compact strong uniform global attractors and the three weak trajectory attractors $\mathfrak{A}_w^\Sigma$, $\overline{\mathfrak{A}}_w^\Sigma$ and $\mathfrak{A}_w^{\overline{\Sigma}}$ for evolutionary systems $\mathcal{E}_\Sigma$,*

$\overline{\mathcal{E}}_\Sigma$ and $\mathcal{E}_{\overline{\Sigma}}$, respectively, are strongly compact strong trajectory attractors. Moreover, the three trajectory attractors satisfy the finite strong uniform tracking property for all the three evolutionary systems and are strongly equicontinuous on $[0, \infty)$.

Proof. The proof is obtained directly by applying Theorem 3.1, [2, Theorems 4.5 and 4.6] and the above results. □

Remark 4.10. The content of Theorems 4.4 and 4.8 for the evolutionary system of (2.2) can be adapted to the evolutionary system of (2.3), respectively. Moreover, if $\alpha_1 \geq 1$ and $\beta_1 \geq 1$ in (2.3), then the content of Theorem 4.9 for the evolutionary system of (2.2) can be also adapted to the evolutionary system of (2.3) (see, e.g., [44]). Let us omit their proofs here.

References

[1] S. Abe and S. Thurner, *Anomalous diffusion in view of Einstein's 1905 theory of Brownian motion*, Phys. A **356** (2005), no. 2-4, 403–407.

[2] C. T. Anh and L. T. Tinh, *Regularity and attractors for the three-dimensional generalized Boussinesq system*, Math. Methods Appl. Sci. **46** (2023), no. 14, 15526–15556.

[3] J. W. Barrett and W. B. Liu, *Finite element approximation of the parabolic p-Laplacian*, SIAM J. Numer. Anal. **31** (1994), no. 2, 413–428.

[4] A. Behzadan and M. Holst, *Multiplication in Sobolev spaces, revisited*, Ark. Mat. **59** (2021), no. 2, 275–306.

[5] J. Benameur, *Global weak solution of 3D-NSE with exponential damping*, Open Math. **20** (2022), no. 1, 590–607.

[6] J. Benameur and M. Ltifi, *Strong solution of 3D-NSE with exponential damping*, arXiv:2103.16707.

[7] H. Bessaih and B. Ferrario, *The regularized 3D Boussinesq equations with fractional Laplacian and no diffusion*, J. Differential Equations **262** (2017), no. 3, 1822–1849.

[8] M. Blel and J. Benameur, *Long time decay of Leray solution of 3D-NSE with exponential damping*, arXiv:2201.08292.

[9] ______, *Asymptotic study of Leray solution of 3D-NSE with exponential damping*, arXiv:2206.03138.

[10] L. Caffarelli and L. Silvestre, *An extension problem related to the fractional Laplacian*, Comm. Partial Differential Equations **32** (2007), no. 7-9, 1245–1260.

[11] X. Cai and Q. Jiu, *Weak and strong solutions for the incompressible Navier–Stokes equations with damping*, J. Math. Anal. Appl. **343** (2008), no. 2, 799–809.

[12] V. V. Chepyzhov and M. I. Vishik, *Attractors for Equations of Mathematical Physics*, American Mathematical Society Colloquium Publications **49**, American Mathematical Society, Providence, RI, 2002.

[13] A. Cheskidov, *Global attractors of evolutionary systems*, J. Dynam. Differential Equations **21** (2009), no. 2, 249–268.

[14] A. Cheskidov and C. Foias, *On global attractors of the 3D Navier–Stokes equations*, J. Differential Equations **231** (2006), no. 2, 714–754.

[15] A. Cheskidov and S. Lu, *The existence and the structure of uniform global attractors for nonautonomous reaction-diffusion systems without uniqueness*, Discrete Contin. Dyn. Syst. Ser. S **2** (2009), no. 1, 55–66.

[16] ———, *Uniform global attractors for the nonautonomous 3D Navier–Stokes equations*, Adv. Math. **267** (2014), 277–306.

[17] P. Constantin and C. Foias, *Navier–Stokes Equations*, Chicago Lectures in Mathematics, University of Chicago Press, Chicago, IL, 1988.

[18] Y. Dai, R. Ji and J. Wu, *Unique weak solutions of the magnetohydrodynamic equations with fractional dissipation*, ZAMM Z. Angew. Math. Mech. **100** (2020), no. 7, e201900290, 20 pp.

[19] Y. Dai, Z. Tan and J. Wu, *A class of global large solutions to the magnetohydrodynamic equations with fractional dissipation*, Z. Angew. Math. Phys. **70** (2019), no. 5, Paper No. 153, 13 pp.

[20] N. Duan, *Well-posedness and decay of solutions for three-dimensional generalized Navier–Stokes equations*, Comput. Math. Appl. **76** (2018), no. 5, 1026–1033.

[21] J. Fan, A. Alsaedi, T. Hayat, G. Nakamura and Y. Zhou, *A regularity criterion for the 3D generalized MHD equations*, Math. Phys. Anal. Geom. **17** (2014), no. 3-4, 333–340.

[22] U. Frisch, S. Kurien, R. Pandit, W. Pauls, S. S. Ray, A. Wirth and J.-Z. Zhu, *Hyperviscosity, Galerkin truncation, and bottlenecks in turbulence*, Phys. Rev. Lett. **101** (2008), 144501, 4 pp.

[23] A. E. Gill, *Atmosphere—Ocean Dynamics*, Academic Press, London, 1982.

[24] M. Holst, E. Lunasin and G. Tsogtgerel, *Analysis of a general family of regularized Navier–Stokes and MHD models*, J. Nonlinear Sci. **20** (2010), no. 5, 523–567.

[25] M. Jara, *Nonequilibrium scaling limit for a tagged particle in the simple exclusion process with long jumps*, Comm. Pure Appl. Math. **62** (2009), no. 2, 198–214.

[26] Q. Jiu and H. Yu, *Decay of solutions to the three-dimensional generalized Navier–Stokes equations*, Asymptot. Anal. **94** (2015), no. 1-2, 105–124.

[27] O. V. Kapustyan, V. S. Melnik and J. Valero, *A weak attractor and properties of solutions for the three-dimensional Bénard problem*, Discrete Contin. Dyn. Syst. **18** (2007), no. 2-3, 449–481.

[28] M. Kaya and A. O. Çelebi, *Global attractor for the regularized Bénard problem*, Appl. Anal. **93** (2014), no. 9, 1989–2001.

[29] N. T. Le and L. T. Tinh, *On the three dimensional generalized Navier–Stokes equations with damping*, submitted.

[30] Y. Li, Z. Zeng and D. Zhang, *Non-uniqueness of weak solutions to 3D magnetohydrodynamic equations*, J. Math. Pures Appl. (9) **165** (2022), 232–285.

[31] ______, *Sharp non-uniqueness of weak solutions to 3D magnetohydrodynamic equations*, arXiv:2208.00624.

[32] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications: Vol. 1*, Travaux et Recherches Mathématiques **17**, Dunod, Paris, 1968.

[33] S. Lu, *Attractors for nonautonomous 2D Navier–Stokes equations with less regular normal forces*, J. Differential Equations **230** (2006), no. 1, 196–212.

[34] ______, *Attractors for nonautonomous reaction-diffusion systems with symbols without strong translation compactness*, Asymptot. Anal. **54** (2007), no. 3-4, 197–210; Erratum: Asymptot. Anal. **58** (2008), no. 3, 189–190.

[35] ______, *Strongly compact strong trajectory attractors for evolutionary systems and their applications*, Asymptot. Anal. **133** (2022), no. 1-2, 13–75.

[36] S. Lu, H. Wu and C. Zhong, *Attractors for nonautonomous 2D Navier–Stokes equations with normal external forces*, Discrete Contin. Dyn. Syst. **13** (2005), no. 3, 701–719.

[37] A. Mellet, S. Mischler and C. Mouhot, *Fractional diffusion limit for collisional kinetic equations*, Arch. Ration. Mech. Anal. **199** (2011), no. 2, 493–525.

[38] W. G. Melo, C. Perusato and N. F. Rocha, *On local existence, uniqueness and blow-up of solutions for the generalized MHD equations in Lei–Lin spaces*, Z. Angew. Math. Phys. **70** (2019), no. 3, Paper No. 74, 24 pp.

[39] R. Metzler and J. Klafter, *The random walk's guide to anomalous diffusion: A fractional dynamics approach*, Phys. Rep. **339** (2000), no. 1, 77 pp.

[40] J. C. Robinson, *Infinite-dimensional Dynamical Systems: An introduction to dissipative parabolic PDEs and the theory of global attractors*, Cambridge Texts in Applied Mathematics, Cambridge University Press, Cambridge, 2001.

[41] R. Rosa, *The global attractor for the 2D Navier–Stokes flow on some unbounded domains*, Nonlinear Anal. **32** (1998), no. 1, 71–85.

[42] M. Sermange and R. Temam, *Some mathematical questions related to the MHD equations*, Comm. Pure Appl. Math. **36** (1983), no. 5, 635–664.

[43] R. Temam, *Navier–Stokes Equations: Theory and numerical analysis*, Studies in Mathematics and its Applications **2**, North-Holland, Amsterdam, 1977.

[44] L. T. Tinh, *Asymptotic study of the three-dimensional generalized Navier–Stokes equations with exponential damping*, accepted.

[45] E. S. Titi and S. Trabelsi, *Global well-posedness of a 3D MHD model in porous media*, J. Geom. Mech. **11** (2019), no. 4, 621–637.

[46] Z. Wang and H. Liu, *Global well-posedness for the 3-D generalized MHD equations*, Appl. Math. Lett. **140** (2023), Paper No. 108585, 6 pp.

[47] J. Wu, *Generalized MHD equations*, J. Differential Equations **195** (2003), no. 2, 284–312.

[48] ______, *Global regularity for a class of generalized magnetohydrodynamic equations*, J. Math. Fluid Mech. **13** (2011), no. 2, 295–305.

[49] ______, *The 2D magnetohydrodynamic equations with partial or fractional dissipation*, in: *Lectures on the Analysis of Nonlinear Partial Differential Equations, Part 5*, 283–332, Morningside Lect. Math. **5**, Int. Press, Somerville, MA, 2018.

[50] K. Yamazaki, *Global regularity of the logarithmically supercritical MHD system with zero diffusivity*, Appl. Math. Lett. **29** (2014), 46–51.

[51] W. Yang, Q. Jiu and J. Wu, *The 3D incompressible magnetohydrodynamic equations with fractional partial dissipation*, J. Differential Equations **266** (2019), no. 1, 630–652.

[52] Z. Ye, *Regularity and decay of 3D incompressible MHD equations with nonlinear damping terms*, Colloq. Math. **139** (2015), no. 2, 185–203.

[53] ———, *Global well-posedness and decay results to 3D generalized viscous magnetohydrodynamic equations*, Ann. Mat. Pura Appl. (4) **195** (2016), no. 4, 1111–1121.

[54] ———, *Global regularity of the two-dimensional regularized MHD equations*, Dyn. Partial Differ. Equ. **16** (2019), no. 2, 185–223.

[55] Z. Zhang, C. Wu and Z. Yao, *Remarks on global regularity for the 3D MHD system with damping*, Appl. Math. Comput. **333** (2018), 1–7.

[56] X. Zhao, *Long time behavior of solutions to 3D generalized MHD equations*, Forum Math. **32** (2020), no. 4, 977–993.

[57] X. Zhao and H. Meng, *Asymptotic behavior of solutions to 3D incompressible Navier–Stokes equations with damping*, arXiv:1809.08394.

[58] X. Zhao and Y. Zhou, *Well-posedness and decay of solutions to 3D generalized Navier–Stokes equations*, Discrete Contin. Dyn. Syst. Ser. B **26** (2021), no. 2, 795–813.

[59] Y. Zhou, *Regularity criteria for the generalized viscous MHD equations*, Ann. Inst. H. Poincaré C Anal. Non Linéaire **24** (2007), no. 3, 491–505.

[60] M. Zhu, *Asymptotic behavior of solutions for the generalized MHD and Hall-MHD systems in* $\mathbb{R}^n$, Bull. Korean Math. Soc. **55** (2018), no. 3, 735–747.

Le Tran Tinh
Faculty of Natural Sciences, Hong Duc University, 565 Quang Trung, Dong Ve, Thanh Hoa, Vietnam
E-mail address: `letrantinh@hdu.edu.vn`