TAIWANESE JOURNAL OF MATHEMATICS
Vol. 28, No. 6, pp. 1245–1266, December 2024
DOI: 10.11650/tjm/240705

Approximate Optimality Conditions for Nonsmooth Optimization Problems

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Abstract. In this research article, a concept of ε-quasi subdifferential for locally Lipschitz functions is proposed. Calculuses of scalar product rule and sum rule for ε-quasi subdifferentials are investigated. A notion of ε-quasi normal set is introduced and its properties are presented. Based on the obtained results, optimality conditions for ε-quasi solutions in Karush–Kuhn–Tucker type of some classes of nonsmooth optimization problems are established. Several illustrative examples are also given.

1. Introduction

In optimization theory, optimality conditions in Karush–Kuhn–Tucker (KKT) type are important because they can be used to solve problems. It is well known that KKT conditions for exact solutions to non-differentiable problems are given in terms of subdifferentials of the involved functions. When approximate solutions are considered, approximate KKT conditions can be established via approximate subdifferentials of the involved functions. There are several papers dealing with approximate solutions to convex/nonconvex optimization problems, in which KKT conditions are given in terms of subdifferentials/approximate subdifferentials of involved functions [1,2,4,6,8,19,22,27–31,33].

To the best of our knowledge, although approximate subdifferential can be used for dealing with approximate solutions to optimization problems, there exist rarely few papers that used approximate subdifferentials to give approximate optimality conditions for nonconvex problems. Concretely, approximate optimality conditions of such nonconvex problems are usually given in terms of subdifferentials of the involved functions plus a set of tolerant vectors [4, 6, 9, 16, 27–29]. Meanwhile, approximate optimality conditions of convex problems are always established via approximate subdifferentials of involved convex functions.

Our aim in this paper is to establish approximate optimality conditions in KKT type for a class of nonconvex optimization problems by using a new kind of ε-quasi subdifferential. Our research is divided into two parts. The first one is devoted to studying

Received October 2, 2023; Accepted July 15, 2024.
Communicated by Jein-Shan Chen.
2020 *Mathematics Subject Classification.* 41A29, 41A65, 90C26, 90C46.
Key words and phrases. ε-quasi solution, ε-quasi subdifferential, ε-quasi normal set, approximate optimality condition.
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ε-quasi subdifferentials for locally Lipschitz functions. The second one is to give approximate optimality conditions in KKT type for approximate quasi solutions of nonsmooth optimization problems based on such approximate subdifferentials without adding a set of tolerant vectors.

There exist several types of approximate subdifferentials proposed over the years. For convex functions, we can deal with ε-subdifferential [12]. For nonconvex functions, there are several notions of approximate subdifferentials introduced in several papers [3,6,13–15, 17,18,20,21,23,32]. We note that in [21,32], there are some comparisons between several subdifferentials. We are interested in the following two notions. These are the inspiration for us to propose new concept. One is the Dini ε-subdifferential [14,15], proposed by Ioffe. It is related to Dini directional derivatives. The second one is the ε-generalized subdifferential, introduced by Bustos [3]. It is given upon Clarke subdifferentials. These notions are defined as follows:

$$\partial_\varepsilon^- f(z) := \{u \in \mathbb{R}^n \mid f^-(z;d) + \varepsilon\|d\| \geq \langle u, d\rangle, \forall\, d \in \mathbb{R}^n\} \tag{1.1}$$

where $f^-(z;d)$ is Dini directional derivative and

$$\partial_\varepsilon^{\mathrm{loc}} f(z) := \bigcup_{\|x-z\| \leq \sqrt{\varepsilon}} (\partial^c f(x) + \sqrt{\varepsilon} B), \tag{1.2}$$

where B is a closed unit ball in $\mathbb{R}^n$ and $\partial^c f(x)$ is the Clarke subdifferential of the locally Lipschitz function f at x. It is obvious that, $\partial_\varepsilon^- f(z) \subseteq \partial_\varepsilon^{\mathrm{loc}} f(z)$ for all $\varepsilon \in [0,1]$.

Although attracted by several researchers, the Dini directional derivative $f^-(z;d)$ is not usually sublinear in the direction d. Hence, in several cases, the Clarke directional derivative is preferred. Motivated by the Dini ε-subdifferential given by (1.1), the construction of ε-local subgradients given by (1.2), and their inclusion relation, we propose a notion of ε-quasi subdifferential for locally Lipschitz functions by using Clarke directional derivative instead of Dini directional derivative. We denote it by $\partial_\varepsilon^c f(z)$ and expect that $\partial_\varepsilon^- f(z) \subseteq \partial_\varepsilon^c f(z) \subseteq \partial_\varepsilon^{\mathrm{loc}} f(z)$ with $\varepsilon > 0$ small enough. The key to obtaining approximate optimality conditions in this research is the establishment of the formulas for sum rule and scalar product calculus of ε-quasi subdifferentials. Several properties of this notion are studied. Furthermore, according to the notion of ε-quasi subdifferential, we also propose the concept of ε-quasi normal set. It generalizes the concept of the normal set used in nonsmooth analysis. The results are applied to establish approximate optimality conditions for a class of nonsmooth optimization problems.

The paper is organized as follows. The next section is devoted to preliminaries and basic results. In the third section, we present new results related to ε-quasi subdifferential for locally Lipschitz functions. Concretely, the properties of convexity, closedness, and boundedness of ε-quasi subdifferential for a locally Lipschitz function are proved. The calculuses of the sum rule and of the scalar product rule for ε-quasi subdifferentials are established. The properties of ε-quasi normal sets are investigated. In the last section, approximate optimality conditions in the Karush–Kuhn–Tucker type for nonsmooth optimization problems in terms of ε-quasi subdifferentials are introduced. Examples are also given.

2. Preliminaries and basic results

Throughout this paper, the notations $\langle \cdot, \cdot \rangle$, B, and $B(x,\delta)$ stand for an inner product on $\mathbb{R}^n$, a closed unit ball in $\mathbb{R}^n$, and a closed ball with center x and radius δ, respectively. For $A \subset \mathbb{R}^n$, the convex hull of A and the cone generated by A are denoted and defined, respectively, by

$$\operatorname{co}(A) := \left\{ x \in \mathbb{R}^n \;\middle|\; x = \sum_{i=1}^{k} \lambda_i x_i, \sum_{i=1}^{k} \lambda_i = 1, x_i \in A, \lambda_i \geq 0, k \in \mathbb{N} \right\},$$
$$\operatorname{cone}(A) := \{ x \in \mathbb{R}^n \mid x = \lambda a, a \in A, \lambda \geq 0 \}.$$

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be a locally Lipschitz function at z if there exist a neighborhood of the point z, denoted by $U(z)$, and a real number $K > 0$ such that

$$|f(x) - f(y)| \leq K\|x - y\|, \quad \forall\, x, y \in U(z).$$

If the inequality above is valid for all $x, y \in \mathbb{R}^n$, the function f is called Lipschitz on $\mathbb{R}^n$ and K is said to be a Lipschitz constant of f.

The one-side directional derivative of the function f at z in direction d is denoted and defined by $f'(z;d) := \lim_{t\downarrow 0} \frac{f(z+td)-f(z)}{t}$. If f is a convex function on $\mathbb{R}^n$, the convex subdifferential of f at z is given by $\partial f(z) := \{u \in \mathbb{R}^n \mid f'(z;d) \geq \langle u, d\rangle, \forall\, d \in \mathbb{R}^n\}$. The Clarke directional derivative of a locally Lipschitz function f at z is denoted and defined by $f^c(z;d) := \lim_{t\downarrow 0,\, x\to z} \frac{f(x+td)-f(x)}{t}$. Its subdifferential is $\partial^c f(z) := \{u \in \mathbb{R}^n \mid f^c(z;d) \geq \langle u, d\rangle, \forall\, d \in \mathbb{R}^n\}$. The function f is said to be regular at z if $f^c(z;d) = f'(z;d)$ for all $d \in \mathbb{R}^n$. When f is convex, we get $f^c(z;d) = f'(z;d)$ and hence, $\partial^c f(z) = \partial f(z)$.

For a given nonempty set $S \subset \mathbb{R}^n$, the indicator function and the support function of S are denoted and defined, respectively, by

$$\delta_S(x) := \begin{cases} 0 & \text{if } x \in S, \\ +\infty & \text{if } x \notin S \end{cases} \quad \text{and} \quad \sigma_S(x) := \sup_{s\in S}\{\langle s, x\rangle, x \in \mathbb{R}^n\}.$$

If S is a nonempty convex subset of $\mathbb{R}^n$, it is easy to check that $\delta_S(\cdot)$ and $\sigma_S(\cdot)$ are convex functions. In addition, if the sets $C, D \subset \mathbb{R}^n$ are convex, then we have $C \subseteq D$ if and only if $\sigma_C(x) \leq \sigma_D(x)$ for all $x \in \mathbb{R}^n$.

Let C be a nonempty subset of $\mathbb{R}^n$. The Clarke tangent cone to C at $z \in C$ is defined by

$$T(C, z) := \{v \in \mathbb{R}^n \mid d_C^c(z; v) = 0\},$$

where d_C is the distance function from a point to C [5, p. 51]. The Clarke normal cone at $z \in C$ is defined by

$$N(C, z) := \{u \in \mathbb{R}^n \mid \langle u, v \rangle \leq 0, \forall\, v \in T(C, z)\}.$$

If C is a convex set, its Clarke normal cone at a point $z \in C$ coincides with the one in the sense of Convex Analysis [5, Proposition 2.4.4], i.e.,

$$N(C, z) = \{u \in \mathbb{R}^n \mid \langle u, x - z \rangle \leq 0, \forall\, x \in C\}.$$

Furthermore, we have $N(C, z) = \partial \delta_C(z)$. The ε-normal set of the convex set C at a point $z \in C$ in the sense of Convex Analysis [6, Definition 2.110] is denoted and defined by

$$N_\varepsilon(C, z) := \{u \in \mathbb{R}^n \mid \langle u, x - z \rangle \leq \varepsilon, \forall\, x \in C\}.$$

We also note that there exists another notion ε-normal set of C at a point $z \in C$. That is given by Mordukhovich [24, Proposition 1.3] and defined by

$$\widehat{N}_\varepsilon(C, z) := \{u \in \mathbb{R}^n \mid \langle u, x - z \rangle \leq \varepsilon \|x - z\|, \forall\, x \in C\}. \tag{2.1}$$

There are two kinds of generalized convex functions used in this research. A locally Lipschitz function f is called Clarke pseudoconvex or Clarke quasiconvex at z if the following implications hold, respectively:

$$f^c(z; d) \geq 0 \quad \Longrightarrow \quad f(z + d) \geq f(z), \quad \forall\, d \in \mathbb{R}^n,$$

or

$$f(z + d) \leq f(z) \quad \Longrightarrow \quad f^c(z; d) \leq 0, \quad \forall\, d \in \mathbb{R}^n.$$

The function f is called pseudoconvex function or quasiconvex function if it is pseudoconvex or quasiconvex at every point of $\mathbb{R}^n$, respectively. We also note that a quasiconvex function is characterized by the convexity of its lower-level sets [7].

Definition 2.1. [27] Let $f \colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The function f is said to be ε-pseudoconvex at z if

$$f^c(z; d) + \sqrt{\varepsilon}\|d\| \geq 0 \quad \Longrightarrow \quad f(z + d) + \sqrt{\varepsilon}\|d\| \geq f(z), \quad \forall\, d \in \mathbb{R}^n.$$

The function f is called ε-pseudoconvex if f is ε-pseudoconvex at every point $z \in \mathbb{R}^n$. If it is regular at every point z, then it is called ε-semiconvex function [28].

We now recall some concepts of approximate solutions for optimization problems mentioned in [22, 28]. Let us consider the following problem

$$\text{Minimize } f(x) \text{ subject to } x \in A, \tag{2.2}$$

where $f\colon \mathbb{R}^n \to \mathbb{R}$ is a locally Lipschitz function and A is a nonempty, closed subset of $\mathbb{R}^n$.

Definition 2.2. Given $\varepsilon \geq 0$, a point $z \in A$ is said to be an

(i) ε-solution for (2.2) if $f(z) \leq f(x) + \varepsilon$ for all $x \in A$;

(ii) ε-quasi solution for (2.2) if $f(z) \leq f(x) + \sqrt{\varepsilon}\|x - z\|$ for all $x \in A$;

(iii) ε-regular solution if it is an ε-solution and an ε-quasi solution for (2.2).

When $A = \mathbb{R}^n$, the point z above is said to be an ε-minimum, an ε-quasi minimum, and an ε-regular minimum, respectively.

Remark 2.3. From (ii), if x belongs to a neighborhood $U_{\sqrt{\varepsilon}}(z)$, we get $f(z) \leq f(x) + \varepsilon$. Hence, an ε-quasi minimum of f is a local ε-minimum of f over $\mathbb{R}^n$.

We need some more results below.

Theorem 2.4. [26, Theorem 2.3.1]

(i) *Let M be a nonempty, closed, and convex subset of $\mathbb{R}^n$. Then, the support function σ_M is proper, sublinear, lower semi-continuous, and one has $M = \{u \in \mathbb{R}^n \mid \langle u, x\rangle \leq \sigma_M(x), \forall\, x \in \mathbb{R}^n\}$.*

(ii) *Let $p\colon \mathbb{R}^n \to \mathbb{R} \cup \{+\infty\}$ be a proper, sublinear, and lower semi-continuous (lsc) function. Then, the set $M_p := \{v \in \mathbb{R}^n \mid \langle v, x\rangle \leq p(x), \forall\, x \in \mathbb{R}^n\}$ is nonempty, closed and convex, and one has $\sigma_{M_p} = p$ on $\mathbb{R}^n$.*

(iii) *Let M_1, M_2 be nonempty, closed, and convex subsets of $\mathbb{R}^n$. Then, $M_1 \subseteq M_2$ if and only if $\sigma_{M_1}(x) \leq \sigma_{M_2}(x)$ for all $x \in \mathbb{R}^n$.*

Theorem 2.5. [26, Proposition 7.4.7] *Let $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i \in I := \{1, 2, \ldots, m\}$ be locally Lipschitz functions at z. Set $g(x) := \max_{i \in I} g_i(x)$ and $I(z) := \{i \in I \mid g(z) = g_i(z)\}$.*

(i) *The function g is a locally Lipschitz function at z;*

(ii) $\partial^c g(z) \subseteq \operatorname{co}\{\bigcup_{i \in I(z)} \partial^c g_i(z)\}$;

(iii) *If the functions g_i, $i \in I$, are regular at z, then $\partial^c g(z) = \operatorname{co}\{\bigcup_{i \in I(z)} \partial^c g_i(z)\}$.*

The proof of the following lemma is omitted.

Lemma 2.6. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Given $z \in \mathbb{R}^n$ and $\varepsilon \geq 0$, let*

$$p(d) := f^c(z; d) + \sqrt{\varepsilon}\|d\|.$$

Then, the function p is proper, sublinear and lsc function.

3. Approximate quasi subdifferentials for locally Lipschitz functions

3.1. ε-quasi subdifferentials

For compatibility with the notion of ε-quasi solutions given in Definition 2.2, we use $\sqrt{\varepsilon}$ instead of ε and propose the following notion.

Definition 3.1. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function at z. Given $\varepsilon \geq 0$, a vector $u \in \mathbb{R}^n$ is said to be an ε-quasi subgradient of f at z if the following inequality is satisfied:

$$f^c(z;d) + \sqrt{\varepsilon}\|d\| \geq \langle u, d\rangle, \quad \forall\, d \in \mathbb{R}^n.$$

The set of all vectors u satisfying the inequality above is called ε-quasi subdifferential of f at z and is denoted by $\partial^c_\varepsilon f(z)$, i.e.,

$$\partial^c_\varepsilon f(z) := \{u \in \mathbb{R}^n \mid f^c(z;d) + \sqrt{\varepsilon}\|d\| \geq \langle u, d\rangle, \forall\, d \in \mathbb{R}^n\}.$$

It is should be noted that if $f^c(z;d)$ is replaced by $f^-(z;d)$ then the formulation above becomes the Dini ε-subdifferential [14, 15]. If $\varepsilon = 0$, we obtain the notion of the Clarke subdifferential of f.

Remark 3.2. If $\varepsilon \in [0,1]$, it is easy to check the following relationships:

$$\partial^-_\varepsilon f(z) \subseteq \partial^c_\varepsilon f(z) \subseteq \partial^{\mathrm{loc}}_\varepsilon f(z).$$

Remark 3.3. If $0 \leq \varepsilon_1 \leq \varepsilon_2$ then $\partial^c_{\varepsilon_1} f(z) \subseteq \partial^c_{\varepsilon_2} f(z)$.

The following results are crucial for us to establish approximate optimality conditions in the next section.

Proposition 3.4. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz function at z and let $\varepsilon \geq 0$. Then,*

$$\partial^c_\varepsilon(\lambda f)(z) = \lambda \partial^c_{\frac{\varepsilon}{\lambda^2}} f(z), \quad \forall\, \lambda \neq 0.$$

Proof. Let $u \in \partial^c_\varepsilon(\lambda f)(z)$ for $\lambda > 0$. By Definition 3.1, we get

$$(\lambda f)^c(z;d) + \sqrt{\varepsilon}\|d\| \geq \langle u, d\rangle, \quad \forall\, d \in \mathbb{R}^n.$$

The inequality above is equivalent to

$$f^c(z;d) + \frac{\sqrt{\varepsilon}}{\lambda}\|d\| \geq \langle \frac{u}{\lambda}, d\rangle, \quad \forall\, d \in \mathbb{R}^n.$$

Thus, $\frac{u}{\lambda} \in \partial^c_{\frac{\varepsilon}{\lambda^2}} f(z;d)$, i.e., $u \in \lambda\partial^c_{\frac{\varepsilon}{\lambda^2}} f(z)$. We get $\partial^c_\varepsilon(\lambda f)(z) = \lambda\partial^c_{\frac{\varepsilon}{\lambda^2}} f(z)$.

We now prove the formula above for $\lambda < 0$. It suffices to prove for $\lambda = -1$. A vector $u \in \partial^c_\varepsilon(-f)(z)$ if and only if

$$(-f)^c(z;d) + \sqrt{\varepsilon}\|d\| \geq \langle u, d\rangle, \quad \forall\, d \in \mathbb{R}^n.$$

By [5, Proposition 2.1.1(c)], this is equivalent to

$$f^c(z;-d)+\sqrt{\varepsilon}\|d\|\geq\langle u,d\rangle,\quad\forall\, d\in\mathbb{R}^n.$$

Hence,

$$f^c(z;d)+\sqrt{\varepsilon}\|d\|\geq\langle u,-d\rangle=\langle -u,d\rangle,\quad\forall\, d\in\mathbb{R}^n.$$

By Definition 3.1, we get $-u\in\partial^c_\varepsilon f(z)$. Thus, $u\in\partial^c_\varepsilon(-f)(z)$ if and only if $u\in-\partial^c_\varepsilon f(z)$. The desired result follows. □

Proposition 3.5. *Let $f\colon\mathbb{R}^n\to\mathbb{R}$ be a locally Lipschitz function at z. Given $\varepsilon\geq 0$, one has*

(3.1) $$\partial^c_\varepsilon f(z)=\partial^c f(z)+\sqrt{\varepsilon}B,$$

where B is a closed unit ball in $\mathbb{R}^n$.

Proof. Let M_1 and M_2 stand for $\partial^c_\varepsilon f(z)$ and $\partial^c f(z)+\sqrt{\varepsilon}B$, respectively. First, we check $M_2\subseteq M_1$. Let $u\in M_2$. There exist $w\in\partial^c f(z)$ and $b\in B$ such that $u=w+\sqrt{\varepsilon}b$. Hence,

$$\langle u,d\rangle=\langle w,d\rangle+\sqrt{\varepsilon}\langle b,d\rangle\leq f^c(z;d)+\sqrt{\varepsilon}\|d\|,\quad\forall\, d\in\mathbb{R}^n.$$

This means that $u\in M_1$, i.e., $M_2\subseteq M_1$.

Next, we check the opposite inclusion by showing that $\sigma_{M_1}(d)\leq\sigma_{M_2}(d)$ for each $d\in\mathbb{R}^n$. We have

$$\sigma_{M_1}(d)=\sup_{u\in M_1}\langle u,d\rangle\leq f^c(z;d)+\sqrt{\varepsilon}\|d\|,\quad\forall\, d\in\mathbb{R}^n.$$

For $M_2=\partial^c f(z)+\sqrt{\varepsilon}B$, if $w\in M_2$, then there exist $u\in\partial^c f(z)$, $b\in B$ such that $w=u+\sqrt{\varepsilon}b$. Hence,

$$\sigma_{M_2}(d)=\sup_{w\in M_2}\langle w,d\rangle\leq\sup_{u\in\partial^c f(z)}\langle u,d\rangle+\sup_{b\in\sqrt{\varepsilon}B}\langle b,d\rangle\leq f^c(z;d)+\sqrt{\varepsilon}\|d\|,\quad\forall\, d\in\mathbb{R}^n.$$

Note that, for each $d\in\mathbb{R}^n$, $f^c(z;d)=\max_{v\in\partial^c f(z)}\langle v,d\rangle$ and $\sqrt{\varepsilon}\|d\|=\sqrt{\varepsilon}\max_{b\in B}\langle b,d\rangle$. Since $\partial^c f(z)$ and B are closed and bounded sets, there exist $\overline{v}\in\partial^c f(z)$ and $\overline{b}\in B$ such that $\langle\overline{v};d\rangle=f^c(z;d)$ and $\sqrt{\varepsilon}\langle\overline{b},d\rangle=\sqrt{\varepsilon}\|d\|$. This means that, for each $d\in\mathbb{R}^n$, we can have $\overline{w}=\overline{v}+\sqrt{\varepsilon}\overline{b}$ such that $\overline{w}\in M_2$ and

$$\langle\overline{w},d\rangle=f^c(z;d)+\sqrt{\varepsilon}\|d\|,\quad\forall\, d\in\mathbb{R}^n.$$

This implies that $\sigma_{M_1}(d)\leq\sigma_{M_2}(d)$ for each $d\in\mathbb{R}^n$. By Theorem 2.4, we obtain $M_1\subseteq M_2$. The proof is complete. □

Proposition 3.6. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function at z and let $\varepsilon \geq 0$. Then, $\partial^c_\varepsilon f(z)$ is a closed, convex, and bounded set.*

Proof. Let $p(d) := f^c(z;d) + \sqrt{\varepsilon}\|d\|$. It is a proper, sublinear and lsc function due to Lemma 2.6. For any $u \in \partial^c_\varepsilon f(z)$, we get $\langle u, d\rangle \leq p(d)$ for all $d \in \mathbb{R}^n$. Hence, by Theorem 2.4, we deduce that $\partial^c_\varepsilon f(z)$ is a closed and convex set. The bounded property of $\partial^c_\varepsilon f(z)$ can be checked based on Proposition 3.5. Since $\sqrt{\varepsilon}B$ and $\partial^c f(z)$ are bounded sets, the bounded property of $\partial^c_\varepsilon f(z)$ follows. The proof is complete. □

Corollary 3.7. *Let $f, g\colon \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz functions at z. Given $\varepsilon \geq 0$, one has*

$$\partial^c_\varepsilon(f+g)(z) \subseteq \partial^c_{\varepsilon_1} f(z) + \partial^c_{\varepsilon_2} g(z)$$

for any $\varepsilon_1, \varepsilon_2 \geq 0$ such that $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} \geq \sqrt{\varepsilon}$. If f and g are regular at z and $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} = \sqrt{\varepsilon}$, then the equality holds.

Proof. By Proposition 3.5, we get

$$\partial^c_\varepsilon(f+g)(z) = \partial^c(f+g)(z) + \sqrt{\varepsilon}B.$$

Since $\partial^c(f+g)(z) \subset \partial^c f(z) + \partial^c g(z)$ and $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} \geq \sqrt{\varepsilon}$, we deduce

$$\partial^c(f+g)(z) + \sqrt{\varepsilon}B \subseteq \partial^c f(z) + \partial^c g(z) + \sqrt{\varepsilon_1}B + \sqrt{\varepsilon_2}B = \partial^c_{\varepsilon_1} f(z) + \partial^c_{\varepsilon_2} g(z).$$

Note that, if f and g are regular at z, we get $\partial^c(f+g)(z) = \partial^c f(z) + \partial^c g(z)$ (see [26, Proposition 7.4.3]). Hence, the equality holds if $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} = \sqrt{\varepsilon}$. □

Remark 3.8. When $\varepsilon = 0$, Proposition 3.4 and Corollary 3.7 reduce to the ones introduced in the Clarke's book [5].

Example 3.9. Let $f(x) = -|x|$. Find the ε-quasi subdifferential of the function f at $z = 0$ with $\varepsilon = 1/4$.

First, we compute ε-quasi subdifferential of f at $z = 0$ due to Definition 3.1. Let $g(x) = |x|$. It is easy to see that $g^c(0;d) = |d|$. For $u \in \partial^c_\varepsilon g(0)$, we have $u \cdot d \leq g^c(0;d) + \sqrt{\varepsilon}|d|$ for all $d \in \mathbb{R}$. Hence,

$$|u \cdot d| \leq |d| + \frac{1}{2}|d| = \frac{3}{2}|d|, \quad \forall\, d \in \mathbb{R}^n.$$

Thus, $|u| \leq 3/2$. So, $\partial^c_\varepsilon g(0) = [-3/2, 3/2]$. Furthermore,

$$\partial^c_\varepsilon f(0) = \partial^c_\varepsilon(-g)(0) = -\partial^c_\varepsilon g(0) = [-3/2, 3/2].$$

Next, we check the result above by using Proposition 3.5. It is easy to see that $\partial^c f(0) = \partial^c(-g)(0) = -\partial^c g(0) = [-1, 1]$. Hence, for $\varepsilon = 1/4$, we get

$$\partial^c_\varepsilon f(0) = \partial^c f(0) + \sqrt{\varepsilon}B = [-1, 1] + [-1/2, 1/2] = [-3/2, 3/2].$$

Remark 3.10. A similar formula of (3.1) was given in [17] by using Frechét ε-subdifferential for convex function defined on a Banach space. Another one is given in [25] by using Limiting ε-subdifferential for a lower-semicontinuous function defined on an Asplund space.

We now consider the relation between the ε-quasi subdifferential and the ε-subdifferential of a convex function. Recall that for a convex function f, the ε-subdifferential of f at z (see [11]) is defined by

$$\partial_\varepsilon f(z) := \{u \in \mathbb{R}^n \mid f(x) - f(z) \geq \langle u, x - z\rangle - \varepsilon, \forall\, x \in \mathbb{R}^n\}.$$

A natural question arises: Could we compare ε-subdifferential with ε-quasi subdifferential of a convex function? The following example shows that, in general, ε-subdifferential and ε-quasi subdifferential of a convex function may have no inclusion.

Example 3.11. Let $f(x) = |x|$ and $g(x) = x^2$. Compute ε-subdifferential and ε-quasi subdifferential of f and g at $z = 0$ with $\varepsilon = 1/4$.

Obviously, the functions f and g are convex. First, we deal with f. A simple computation gives $\partial f(0) = [-1, 1] = \partial^c f(0)$. By Proposition 3.5, we have

$$\partial^c_\varepsilon f(0) = \partial^c f(0) + \sqrt{\varepsilon} B = [-1, 1] + [-1/2, 1/2] = [-3/2, 3/2].$$

On the other hand, since f is convex function, it is easy to check that $\partial_\varepsilon f(0) = [-1, 1]$. In this case, we get

$$\partial_\varepsilon f(0) \subseteq \partial^c_\varepsilon f(0).$$

Next, we consider the function g. By Proposition 3.5, we get

$$\partial^c_\varepsilon g(0) = \partial^c g(0) + \sqrt{\varepsilon} B = \{0\} + [-1/2, 1/2] = [-1/2, 1/2].$$

On the other hand,

$$\partial_\varepsilon g(0) = \{u \in \mathbb{R} \mid x^2 \geq ux - 1/4, \forall\, x \in \mathbb{R}\} = [-1, 1].$$

Hence,

$$\partial^c_\varepsilon g(0) \subseteq \partial_\varepsilon g(0).$$

Proposition 3.12. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function at z. Given $\varepsilon \geq 0$, let $g(x) := f(x) + \sqrt{\varepsilon}\|x - z\|$. One has*

(i) *$\partial^c g(z) \subseteq \partial^c_\varepsilon f(z)$, the equality is valid if f is regular at z.*

(ii) *For $v \in \partial^c g(z)$, $\|v\| \leq \max_{u \in \partial^c f(z)} \|u\| + \sqrt{\varepsilon}$.*

Proof. (i) The conclusion $\partial^c g(z) \subseteq \partial^c_\varepsilon f(z)$ follows by the sum rule of the Clarke subdifferential. If f is regular at z, we get $f^c(z;d) = f'(z;d)$, for all $d \in \mathbb{R}^n$. In this case if we take $u \in \partial^c_\varepsilon f(z)$ then is easy to check that $u \in \partial^c g(z)$.

(ii) Let $v \in \partial^c g(z)$. We get $v \in \partial^c f(z) + \sqrt{\varepsilon}\partial^c \| \cdot - z\|(z)$. There exist $u \in \partial^c f(z)$ and $b \in B$ such that $v = u + \sqrt{\varepsilon} b$. Hence, $\|v\| \le \|u\| + \sqrt{\varepsilon}$. The proof is complete. □

The following simple example illustrates that the equality in Proposition 3.12(i) may not true in general.

Example 3.13. Set $f(x) = -\sqrt{\varepsilon}|x|$ and $g(x) = f(x) + \sqrt{\varepsilon}|x|$. We can check that $g(x) = 0$. Hence, $\partial^c g(0) = \{0\}$. On the other hand, it is easy to compute $\partial^c_\varepsilon f(0) = \partial^c f(0) + \sqrt{\varepsilon}[-1, 1]$. Hence, $\partial^c g(0) \subset \partial^c_\varepsilon f(0)$ and $\partial^c g(0) \ne \partial^c_\varepsilon f(0)$.

3.2. ε-quasi subdifferentials for Max-function

The following propositions help us to establish approximate optimality conditions in the next section.

Proposition 3.14. *Let $g_i \colon \mathbb{R}^n \to \mathbb{R}$, $i \in I := \{1, 2, \dots, m\}$, be locally Lipschitz functions at z. Given $\varepsilon_i \ge 0$, $i \in I$, set $g(x) := \max_{i \in I}\{h_i(x) := g_i(x) + \sqrt{\varepsilon_i}\|x - z\|\}$, and $I(z) := \{i \in I \mid g(z) = h_i(z)\}$. Then,*

$$\partial^c g(z) \subseteq \operatorname{co}\Big\{ \bigcup_{i \in I(z)} \partial^c_{\varepsilon_i} g_i(z) \Big\}.$$

The equality holds if the functions g_i, $i \in I$, are regular at z.

Proof. For each $i \in I$, the function h_i is locally Lipschitz at z. Since

$$g(x) = \max_{i \in I} \big\{ h_i(x) := g_i(x) + \sqrt{\varepsilon_i}\|x - z\| \big\},$$

by [5, Proposition 2.3.12], we get

$$\partial^c g(z) \subseteq \operatorname{co}\Big\{ \bigcup_{i \in I(z)} \partial^c h_i(z) \Big\}.$$

The inclusion follows by noting that $\partial^c h_i(z) \subseteq \partial^c_{\varepsilon_i} g_i(z)$. Moreover, if the functions g_i, $i \in I$, are regular at z then $\partial^c h_i(z) = \partial^c_{\varepsilon_i} g_i(z)$. By the second conclusion of [5, Proposition 2.3.12], we get

$$\partial^c g(z) = \operatorname{co}\Big\{ \bigcup_{i \in I(z)} \partial^c h_i(z) \Big\} = \operatorname{co}\Big\{ \bigcup_{i \in I(z)} \partial^c_{\varepsilon_i} g_i(z) \Big\}. \qquad \square$$

Proposition 3.15. *Let $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i \in I := \{1, 2, \dots, m\}$, be locally Lipschitz functions at z. Let $g(x) := \max_{i\in I} g_i(x)$ and $I(z) := \{i \in I \mid g(z) = g_i(z)\}$. Given $\varepsilon \geq 0$, for any $\varepsilon_i \geq 0$, $i \in I(z)$, such that $\sqrt{\varepsilon}$ is a convex combination of $\sqrt{\varepsilon_i}$, $i \in I(z)$, one has*

$$\partial^c_\varepsilon g(z) \subseteq \operatorname{co}\Big\{ \bigcup_{i\in I(z)} \partial^c_{\varepsilon_i} g_i(z) \Big\}.$$

Proof. For $z \in \mathbb{R}^n$, let $u \in \partial^c_\varepsilon g(z)$. By Proposition 3.5, we get $u \in \partial^c g(z) + \sqrt{\varepsilon}B$. By [5, Proposition 2.3.12], we deduce that $u \in \operatorname{co}\big\{ \bigcup_{i\in I(z)} \partial^c g_i(z) \big\} + \sqrt{\varepsilon}B$. Then, there exist $u_i \in \partial^c g_i(z)$, $b \in B$, and $\lambda_i \geq 0$, $\sum_{i\in I(z)} \lambda_i = 1$, such that $u = \sum_{i\in I(z)} \lambda_i u_i + \sqrt{\varepsilon} b$. For any ε_i, $i \in I(z)$ such that $\sqrt{\varepsilon} = \sum_{i\in I(z)} \lambda_i \sqrt{\varepsilon_i}$, we get

$$u = \sum_{i\in I(z)} \lambda_i (u_i + \sqrt{\varepsilon_i} b) \in \operatorname{co}\Big\{ \bigcup_{i\in I(z)} \partial^c_{\varepsilon_i} g_i(z) \Big\}.$$

The proof is complete. □

3.3. ε-quasi normal set

In this subsection, we propose a notion of ε-quasi normal set. Some properties of this notion are also investigated.

Definition 3.16. Let K be a nonempty subset of $\mathbb{R}^n$. Let $z \in K$, and $\varepsilon \geq 0$. A vector $u \in \mathbb{R}^n$ is said to be an ε-quasi normal vector to K at z if

$$\langle u, d\rangle \leq \sqrt{\varepsilon}\|d\|, \quad \forall\, d \in T(K, z).$$

The collection of all ε-quasi normal vectors to K at z is called the ε-quasi normal set of K at z and is denoted by $N^Q_\varepsilon(K, z)$. Obviously, $N(K, z) \subset N^Q_\varepsilon(K, z)$.

Remark 3.17. If $\varepsilon = 0$, the ε-quasi normal set $N^Q_\varepsilon(K, z)$ reduces to the normal cone of K at z in the sense of Clarke. Furthermore, if K is convex, it reduces to the normal cone in the sense of Convex Analysis.

Proposition 3.18. *The ε-quasi normal set $N^Q_\varepsilon(K, z)$ is a closed and convex set.*

Proof. (i) $N^Q_\varepsilon(K, z)$ is a convex set: Let $u_1, u_2 \in N^Q_\varepsilon(K, z)$. We have

$$\langle u_1, d\rangle \leq \sqrt{\varepsilon}\|d\| \quad \text{and} \quad \langle u_2, d\rangle \leq \sqrt{\varepsilon}\|d\|, \quad \forall\, d \in T(K, z).$$

By multiplying the first inequality to $\lambda \in [0, 1]$, the second one to $(1-\lambda)$, and summarizing the inequalities side by side, we obtain

$$\langle \lambda u_1 + (1-\lambda) u_2, d\rangle \leq \sqrt{\varepsilon}\|d\|, \quad \forall\, d \in T(K, z).$$

Hence, $\lambda u_1 + (1-\lambda)u_2 \in N^Q_\varepsilon(K,z)$. Thus, $N^Q_\varepsilon(K,z)$ is a convex set.

(ii) $N^Q_\varepsilon(K,z)$ is closed: Let $\{u_n\} \subset N^Q_\varepsilon(K,z)$ be an arbitrary sequence such that $\{u_n\} \to u_0$. We claim that $u_0 \in N^Q_\varepsilon(K,z)$. Indeed, we have that

$$\langle u_0, d\rangle = \langle u_0 - u_n + u_n, d\rangle = \langle u_0 - u_n, d\rangle + \langle u_n, d\rangle.$$

Since $\{u_n\} \to u_0$ and $u_n \in N^Q_\varepsilon(K,z)$ for all n, we deduce that

$$\langle u_0, d\rangle \le \sqrt{\varepsilon}\|d\|, \quad \forall\, d \in T(K,z),$$

i.e., $u_0 \in N^Q_\varepsilon(K,z)$. The proof is complete. □

Proposition 3.19. *If K is a nonempty convex subset of $\mathbb{R}^n$ then*

$$N^Q_\varepsilon(K,z) = \partial^c_\varepsilon \delta_K(z). \tag{3.2}$$

Proof. Note that

$$\partial^c_\varepsilon \delta_K(z) = \{u \in \mathbb{R}^n \mid \delta^c_K(z; x-z) + \sqrt{\varepsilon}\|x-z\| \ge \langle u, x-z\rangle, \forall\, x \in K\}$$

and

$$\begin{aligned} N^Q_\varepsilon(K,z) &= \{u \in \mathbb{R}^n \mid \langle u, d\rangle \le \sqrt{\varepsilon}\|d\|, \forall\, d \in T(K,z)\} \\ &= \{u \in \mathbb{R}^n \mid \langle u, x-z\rangle \le \sqrt{\varepsilon}\|x-z\|, \forall\, x \in K\}. \end{aligned}$$

Since K is a convex set, δ_K is a convex function. Hence,

$$\delta^c_K(z; x-z) = \delta'_K(z; x-z) = 0, \quad \forall\, x \in K.$$

The desired result follows. □

Remark 3.20. If K is a convex subset of $\mathbb{R}^n$ and $\varepsilon \in [0,1]$, then, due to (2.1), we get $\widehat{N}_\varepsilon(K,z) \subseteq N^Q_\varepsilon(K,z)$.

Remark 3.21. If $\varepsilon = 0$ then the equality (3.2) reduces to $N(K,z) = \partial\delta_K(z)$.

Proposition 3.22. *Let K be a nonempty subset of $\mathbb{R}^n$ and $z \in K$. For $\varepsilon \ge 0$, one has*

$$N(K,z) + \sqrt{\varepsilon}B \subseteq N^Q_\varepsilon(K,z).$$

The equality holds if K is convex.

Proof. Let $u \in N(K,z) + \sqrt{\varepsilon}B$. There exists $v \in N(K,z)$ and $b \in B$ such that $u = v + \sqrt{\varepsilon}b$. Since $v \in N(K,z)$, $\langle v, d\rangle \le 0$ for all $d \in T(K,z)$. Hence, $\langle u, d\rangle = \langle v, d\rangle + \langle \sqrt{\varepsilon}b, d\rangle \le \langle \sqrt{\varepsilon}b, d\rangle \le \sqrt{\varepsilon}\|d\|$ for all $d \in T(K,z)$, i.e., $u \in N^Q_\varepsilon(K,z)$. Thus, the inclusion holds.

Now suppose that K is a convex subset of $\mathbb{R}^n$. It is easy to check that $N^Q_\varepsilon(K,z) = \partial^c(\delta_K + h)(z)$, where $h(x) := \sqrt{\varepsilon}\|x-z\|$. Since δ_K and h are convex functions and $z \in K$, we can deduce that $\partial^c(\delta_K + h)(z) = \partial\delta_K(z) + \partial h(z) = N(K,z) + \sqrt{\varepsilon}B$. The desired result follows. □

4. Approximate optimality conditions

In this section, we establish approximate optimality conditions for nonsmooth optimization problems in terms of approximate quasi subdifferentials of locally Lipschitz functions.

Definition 4.1. [27] Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. The function f is said to be ε-pseudoconvex at z if

$$f^c(z;d) + \sqrt{\varepsilon}\|d\| \geq 0 \quad \Longrightarrow \quad f(z+d) + \sqrt{\varepsilon}\|d\| \geq f(z), \quad \forall\, d \in \mathbb{R}^n. \tag{4.1}$$

The function f is called ε-pseudoconvex if f is ε-pseudoconvex at every point $z \in \mathbb{R}^n$.

Theorem 4.2. *Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a locally Lipschitz function. Then, the following assertions hold:*

(i) *If z is an ε-quasi-minimum of f then $0 \in \partial^c_\varepsilon f(z)$.*

(ii) *If f is ε-pseudoconvex at z and $0 \in \partial^c_\varepsilon f(z)$ then z is an ε-quasi-minimum of f.*

Proof. (i) Let z be an ε-quasi minimum of f, i.e., $f(z) \leq f(x) + \sqrt{\varepsilon}\|x - z\|$, $\forall\, x \in \mathbb{R}^n$. Set $g(x) = f(x) + \sqrt{\varepsilon}\|x - z\|$. Then, g is also a locally Lipschitz function. Note that $g(z) = f(z)$. Hence, $g(z) \leq g(x)$ for all $x \in \mathbb{R}^n$. Thus, z is a minimum of g on $\mathbb{R}^n$. So, we obtain $0 \in \partial^c g(z)$. Let $h(x) := \|x - z\|$. The function h is a convex function and $\partial^c h(z) = B$, where B is a closed unit ball in $\mathbb{R}^n$. Hence, we get

$$0 \in \partial^c g(z) = \partial^c\{f(\cdot) + \sqrt{\varepsilon}\| \cdot - z\|\}(z) \subseteq \partial^c f(z) + \sqrt{\varepsilon}B = \partial^c_\varepsilon f(z).$$

(ii) If $0 \in \partial^c_\varepsilon f(z)$ then $f^c(z;d) + \sqrt{\varepsilon}\|d\| \geq 0$ for all $d \in \mathbb{R}^n$. Since f is ε-pseudoconvex, the inequality above deduces $f(z+d) + \sqrt{\varepsilon}\|d\| \geq f(z)$, $\forall\, d \in \mathbb{R}^n$. Set $d := x - z$. It is easy to see that z is an ε-quasi minimum of f. □

Theorem 4.3. *For the problem* (2.2)*, the following assertions hold:*

(i) *If $z \in A$ is an ε-quasi solution of* (2.2) *then*

$$0 \in \partial^c_{\varepsilon_1} f(z) + N^Q_{\varepsilon_2}(A, z)$$

for any $\varepsilon_1, \varepsilon_2 \geq 0$ such that $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} = \sqrt{\varepsilon}$.

(ii) *Suppose that f is ε-pseudoconvex function and the feasible set A is a closed convex subset of $\mathbb{R}^n$. Then, a point $z \in A$ is an ε-quasi solution of* (2.2) *if*

$$0 \in \partial^c_{\varepsilon_1} f(z) + N^Q_{\varepsilon_2}(A, z),$$

where $\varepsilon_1, \varepsilon_2 \geq 0$ with $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} = \sqrt{\varepsilon}$.

Proof. (i) Suppose that z is an ε-quasi solution of (2.2). Then, $f(z) \leq f(x) + \sqrt{\varepsilon}\|x - z\|$ for all $x \in A$. Hence, it is a solution of (4.2), formulated by

$$\text{(4.2)} \qquad \text{Minimize } \overline{f}(x) := f(x) + \sqrt{\varepsilon}\|x - z\| \text{ subject to } x \in A.$$

Thus, $0 \in \partial^c \overline{f}(z) + N(A, z)$, where $N(A, z)$ is the normal cone to A at z [5, Corollary, p. 52]. For any $\varepsilon_1, \varepsilon_2 \geq 0$ such that $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} = \sqrt{\varepsilon}$ and noting that $\partial^c \overline{f}(z) \subseteq \partial^c f(z) + \sqrt{\varepsilon}B$, we get $0 \in \partial^c f(z) + \sqrt{\varepsilon_1}B + N(A, z) + \sqrt{\varepsilon_2}B$. By Proposition 3.5 and Proposition 3.22, we obtain

$$0 \in \partial^c_{\varepsilon_1} f(z) + N^Q_{\varepsilon_2}(A, z).$$

(ii) Suppose that $0 \in \partial^c_{\varepsilon_1} f(z) + N^Q_{\varepsilon_2}(A, z)$, with $\varepsilon_1, \varepsilon_2 \geq 0$ such that $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} = \sqrt{\varepsilon}$. Then, there exists $u \in \partial^c_{\varepsilon_1} f(z)$ such that $-u \in N^Q_{\varepsilon_2}(A, z)$. Since A is convex, we can deduce that $\langle u, x - z\rangle \geq -\sqrt{\varepsilon_2}\|x - z\|$ for all $x \in A$. Since $u \in \partial^c_{\varepsilon_1} f(z)$,

$$f^c(z; x - z) + \sqrt{\varepsilon_1}\|x - z\| \geq \langle u, x - z\rangle \geq -\sqrt{\varepsilon_2}\|x - z\|, \quad \forall\, x \in A,$$

i.e.,

$$f^c(z; x - z) + (\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2})\|x - z\| \geq 0, \quad \forall\, x \in A.$$

By (4.1) and noting that $\sqrt{\varepsilon_1} + \sqrt{\varepsilon_2} = \sqrt{\varepsilon}$, we deduce $f(x) + \sqrt{\varepsilon}\|x - z\| \geq f(z)$, i.e., the point z is an ε-quasi solution for (2.2). □

The last part is devoted to presenting approximate optimality conditions for the problems with constraint functions. By a new approach, the ε-optimality necessary condition in the Karush–Kuhn–Tucker type is given via approximate quasi subdifferentials of the involved functions without plus a set of tolerant vectors.

Let us consider the following problem

$$\text{(4.3)} \qquad \text{Minimize } f(x) \text{ subject to } g_i(x) \leq 0,\ i \in I := \{1, 2, \ldots, m\},$$

where f and g_i, $i \in I$, are locally Lipschitz functions on $\mathbb{R}^n$. The feasible set of (4.3) is $A := \{x \in \mathbb{R}^n \mid g_i(x) \leq 0, i \in I\}$. We need the following constraint qualification condition

$$\text{(4.4)} \qquad \exists\, d \in \mathbb{R}^n : g_i^c(x; d) < 0 \text{ for all } i \in I(x) := \{i \in I \mid g_i(x) = 0\}.$$

By letting $g(x) := \max_{i \in I} g_i(x)$, we formulate a problem associated to (4.3) as follows:

$$\text{(4.5)} \qquad \text{Minimize } \overline{f}(x) = f(x) + \sqrt{\varepsilon}\|x - z\| \text{ subject to } g(x) \leq 0.$$

Note that A is also the feasible set of (4.5). By [10, Theorem 6], if there exists $d \in \mathbb{R}^n$ such that $g^c(z; d) < 0$ and $g(z) = 0$, then $N(A, z) \subset \text{cone}[\partial^c g(z)]$. Furthermore, if z is a local solution to the problem (4.5), then, by [5, Proposition, p. 52], we get

$$0 \in \partial^c \overline{f}(z) + N(A, z).$$

Hence, we get

$$0 \in \partial^c \overline{f}(z) + \text{cone}[\partial^c g(z)]. \tag{4.6}$$

Thus, there exists $\lambda \geq 0$ such that

$$0 \in \partial^c \overline{f}(z) + \lambda \partial^c g(z).$$

We deduce the necessary approximate optimality condition for (4.3) as follows.

Theorem 4.4 (Necessary condition)**.** *For the problem* (4.3)*, let $z \in A$. Suppose that the condition* (4.4) *holds for z. Given $\varepsilon \geq 0$, if z is an ε-quasi solution for* (4.3)*, then there exist $\overline{\lambda}_i \geq 0$, $i \in I(z)$, such that*

$$0 \in \partial^c_{\overline{\varepsilon}} f(z) + \sum_{i \in I(z)} \overline{\lambda}_i \partial^c_{\varepsilon_i} g_i(z), \tag{4.7}$$

where $\overline{\varepsilon} \geq 0$ and $\varepsilon_i \geq 0$, $i \in I(z)$, satisfy the condition

$$\sqrt{\varepsilon} = \sqrt{\overline{\varepsilon}} + \sum_{i \in I(z)} \overline{\lambda}_i \sqrt{\varepsilon_i}. \tag{4.8}$$

Proof. Suppose that the constraint qualification condition (4.4) holds at z. If z is an ε-quasi solution for (4.3), it is a local solution for (4.5). Hence, by (4.6) and noting that $\partial^c \overline{f}(z) \subseteq \partial^c f(z) + \sqrt{\varepsilon} B$, there exists $\lambda \geq 0$ such that

$$0 \in \partial^c f(z) + \sqrt{\varepsilon} B + \lambda \partial^c g(z).$$

If $\lambda = 0$, choose $\overline{\varepsilon} = \varepsilon$ and $\overline{\lambda}_i = 0$, $i \in I(z)$. It is easy to see that (4.7) and (4.8) hold. When $\lambda > 0$, for any $\overline{\varepsilon}, \overline{\varepsilon}' \geq 0$ such that $\sqrt{\varepsilon} = \sqrt{\overline{\varepsilon}} + \sqrt{\overline{\varepsilon}'}$, we get

$$0 \in \partial^c f(z) + (\sqrt{\overline{\varepsilon}} + \sqrt{\overline{\varepsilon}'}) B + \lambda \partial^c g(z).$$

Hence,

$$0 \in \partial^c f(z) + \sqrt{\overline{\varepsilon}} B + \lambda \left[\partial^c g(z) + \frac{\sqrt{\overline{\varepsilon}'}}{\lambda} B \right].$$

Set $\varepsilon^* = \frac{\overline{\varepsilon}'}{\lambda^2}$. By Proposition 3.5, we obtain

$$0 \in \partial^c_{\overline{\varepsilon}} f(z) + \lambda \partial^c_{\varepsilon^*} g(z). \tag{4.9}$$

By Proposition 3.15 and from (4.9), there exist $\lambda_i \geq 0$, $i \in I(z)$, $\sum_{i \in I(z)} \lambda_i = 1$ such that

$$0 \in \partial^c_{\overline{\varepsilon}} f(z) + \lambda \sum_{i \in I(z)} \lambda_i \partial^c_{\varepsilon_i} g_i(z),$$

where $\varepsilon_i \geq 0$, $i \in I(z)$, such that $\sqrt{\varepsilon^*} = \sum_{i\in I(z)} \lambda_i \sqrt{\varepsilon_i}$. Set $\overline{\lambda}_i = \lambda\lambda_i$, $i \in I(z)$, we get

$$0 \in \partial^c_{\overline{\varepsilon}} f(z) + \sum_{i\in I(z)} \overline{\lambda}_i \partial^c_{\varepsilon_i} g_i(z).$$

Furthermore, we can see that

$$\sqrt{\varepsilon} = \sqrt{\overline{\varepsilon}} + \sqrt{\overline{\varepsilon}'} = \sqrt{\overline{\varepsilon}} + \lambda\sqrt{\varepsilon^*} = \sqrt{\overline{\varepsilon}} + \sum_{i\in I(z)} \overline{\lambda}_i \sqrt{\varepsilon_i}, \quad \overline{\lambda}_i = \lambda\lambda_i,\ i \in I(z).$$

The proof is complete. □

In some nonconvex problems, although constraint functions are not convex, their feasible sets can be convex. In the following result, we are concerned with the convexity property of the feasible set of the problem (4.3). We can obtain another version of approximate necessary optimality condition for (4.3) with a bit difference in the construction of KKT condition.

Lemma 4.5. *For the problem* (4.3), *let* $\varepsilon \geq 0$ *and* $z \in A$. *Suppose that the condition* (4.4) *holds for* z. *If the feasible set* A *is convex then*

$$N^Q_\varepsilon(A, z) \subseteq \operatorname{cone}\operatorname{co}\left\{ \bigcup_{i\in I(z)} \partial^c g_i(z) \right\} + \sqrt{\varepsilon}B.$$

Proof. Suppose that the condition (4.4) holds at z. Due to [10, Theorem 6], we can check that $N(A, z) \subseteq \operatorname{cone} \partial^c g(z)$, where $g := \max g_i$, $i \in I$. Combining this and the property of Max-function given by Theorem 2.5, we get

$$N(A, z) \subseteq \operatorname{cone}\operatorname{co}\left\{ \bigcup_{i\in I(z)} \partial^c g_i(z) \right\}.$$

Since A is convex, by Proposition 3.22, we get

$$N^Q_\varepsilon(A, z) = N(A, z) + \sqrt{\varepsilon}B \subseteq \operatorname{cone}\operatorname{co}\left\{ \bigcup_{i\in I(z)} \partial^c g_i(z) \right\} + \sqrt{\varepsilon}B.$$

The proof is complete. □

Theorem 4.6. *For the problem* (4.3), *assume that* g_i, $i \in I$, *are quasiconvex functions on* $\mathbb{R}^n$. *Let* $z \in A$. *Suppose that the constraint qualification condition* (4.4) *holds for* z. *Given* $\varepsilon \geq 0$, *if* z *is an* ε*-quasi-solution for* (4.3), *then there exist* $\overline{\lambda}_i \geq 0$, $i \in I(z)$, *such that*

$$0 \in \partial^c_{\overline{\varepsilon}} f(z) + \sum_{i\in I(z)} \overline{\lambda}_i \partial^c_{\varepsilon'} g_i(z), \tag{4.10}$$

where $\overline{\varepsilon}, \varepsilon' \geq 0$, *satisfy the following relation*

$$\sqrt{\overline{\varepsilon}} + \sum_{i\in I(z)} \overline{\lambda}_i \sqrt{\varepsilon'} = \sqrt{\varepsilon}. \tag{4.11}$$

Proof. Let $z \in A$ be an ε-quasi solution for (4.3). By Theorem 4.3(i), there exist $\overline{\varepsilon}, \widehat{\varepsilon} \geq 0$ and $\sqrt{\varepsilon} = \sqrt{\overline{\varepsilon}} + \sqrt{\widehat{\varepsilon}}$ such that

$$0 \in \partial^c_{\overline{\varepsilon}} f(z) + N^Q_{\widehat{\varepsilon}}(A, z).$$

Since g_i, $i \in I$, are quasiconvex functions, the feasible set A is a convex subset of $\mathbb{R}^n$. By applying Lemma 4.5, we deduce

$$0 \in \partial^c_{\overline{\varepsilon}} f(z) + \operatorname{cone}\operatorname{co}\left\{ \bigcup_{i \in I(z)} \partial^c g_i(z) \right\} + \sqrt{\widehat{\varepsilon}}B.$$

Hence, there exist $\lambda_i \geq 0$, $i \in I(z)$, $\sum_{i \in I(z)} \lambda_i = 1$, and $\lambda \geq 0$, such that

$$0 \in \partial^c_{\overline{\varepsilon}} f(z) + \lambda \sum_{i \in I(z)} \lambda_i \partial^c g_i(z) + \sqrt{\widehat{\varepsilon}}B. \tag{4.12}$$

If $\lambda = 0$, then (4.12) becomes $0 \in \partial^c_{\overline{\varepsilon}} f(z) + \sqrt{\widehat{\varepsilon}}B = \partial^c_{\varepsilon} f(z)$. On the other hand, if we choose $\overline{\lambda}_i = 0$, $i \in I(z)$, then by (4.11) we get $\varepsilon = \overline{\varepsilon}$. Consequently, the inclusion (4.10) holds. When $\lambda > 0$, (4.12) can be rewritten as

$$0 \in \partial^c_{\overline{\varepsilon}} f(z) + \lambda \left[\sum_{i \in I(z)} \lambda_i \left(\partial^c g_i(z) + \frac{\sqrt{\widehat{\varepsilon}}}{\lambda} B \right) \right].$$

Setting $\sqrt{\varepsilon'} := \frac{\sqrt{\widehat{\varepsilon}}}{\lambda}$ and $\overline{\lambda}_i = \lambda \lambda_i$, $i \in I(z)$, by Proposition 3.5, we obtain

$$0 \in \partial^c_{\overline{\varepsilon}} f(z) + \sum_{i \in I(z)} \overline{\lambda}_i \partial^c_{\varepsilon'} g_i(z).$$

Furthermore,

$$\sqrt{\varepsilon} = \sqrt{\overline{\varepsilon}} + \sqrt{\widehat{\varepsilon}} = \sqrt{\overline{\varepsilon}} + \lambda\sqrt{\varepsilon'} = \sqrt{\overline{\varepsilon}} + \sum_{i \in I(z)} \overline{\lambda}_i \sqrt{\varepsilon'}.$$

The necessity is proved. □

Theorem 4.7 (Sufficient condition)**.** *For the problem* (4.3)*, assume that* g_i*,* $i \in I$*, are quasiconvex functions on* $\mathbb{R}^n$*. Let* $z \in A$*. Assume further that* f *is an* ε*-pseudoconvex function. If the conditions* (4.7) *and* (4.8) *hold, then* z *is an* ε*-quasi solution for* (4.3)*.*

Proof. Suppose that (4.7) and (4.8) hold. There exist $u \in \partial^c_{\overline{\varepsilon}} f(z)$, $v_i \in \partial^c_{\varepsilon_i} g_i(z)$, $i \in I(z)$, such that

$$0 = u + \sum_{i \in I(z)} \overline{\lambda}_i v_i. \tag{4.13}$$

Since $v_i \in \partial^c_{\varepsilon_i} g_i(z)$, $i \in I(z)$,

$$\langle v_i, d\rangle \leq g_i^c(z; d) + \sqrt{\varepsilon_i}\|d\|, \quad i \in I(z),\ \forall\, d \in \mathbb{R}^n.$$

Note that the functions g_i, $i \in I$, are quasiconvex and $g_i(x) \leq g_i(z) = 0$ for all $i \in I(z)$. This implies that $g_i^c(z; x - z) \leq 0$, for all $i \in I(z)$. Hence, for all $x \in A$, we get

$$\sum_{i\in I(z)} \overline{\lambda}_i \langle v_i, x - z\rangle \leq \sum_{i\in I(z)} \overline{\lambda}_i g^c(z; x - z) + \sum_{i\in I(z)} \overline{\lambda}_i \sqrt{\varepsilon_i}\|x - z\| \leq \sum_{i\in I(z)} \overline{\lambda}_i \sqrt{\varepsilon_i}\|x - z\|.$$

This together with (4.13) imply that

$$\langle u, x - z\rangle \geq -\sum_{i\in I(z)} \overline{\lambda}_i \sqrt{\varepsilon_i}\|x - z\|, \quad \forall\, x \in A.$$

Since $u \in \partial^c_{\overline{\varepsilon}} f(z)$, $f^c(z; x - z) + \sqrt{\overline{\varepsilon}}\|x - z\| \geq \langle u, x - z\rangle$. Thus,

$$f^c(z; x - z) + \sqrt{\overline{\varepsilon}}\|x - z\| \geq -\sum_{i\in I(z)} \overline{\lambda}_i \sqrt{\varepsilon_i}\|x - z\|, \quad \forall\, x \in A.$$

Combining this and (4.8), we obtain

$$f^c(z; x - z) + \sqrt{\varepsilon}\|x - z\| \geq 0, \quad \forall\, x \in A.$$

Since f is ε-pseudoconvex,

$$f(x) + \sqrt{\varepsilon}\|x - z\| \geq f(z), \quad \forall\, x \in A,$$

i.e., z is an ε-quasi solution of (4.3). □

Example 4.8. Let us consider the following problem

$$\text{Min } \sin x \text{ subject to } x - \pi \leq 0,\ -x + \pi/3 \leq 0.$$

The feasible set A of the problem above is $[\pi/3, \pi]$. It is easy to see that $z = \pi/3$ is a local minimum of f. We will show that $z = \pi/3$ is a 1/4-quasi-solution of the problem. Set $f(x) = \sin x$ and $g(x) = -x + \pi/3$. We have $f(\pi/3) - f(x) = \sin(\pi/3) - \sin x$. It is easy to check that $|\sin(\pi/3) - \sin x| \leq \frac{1}{2}|\pi/3 - x|$, for all $x \in [\pi/3, \pi]$, i.e., $f(z) \leq f(x) + \sqrt{\varepsilon}|z - x|$ for all $x \in A$, where $\sqrt{\varepsilon} = 1/2$.

Next, we can verify Theorem 4.4. Indeed, for $\sqrt{\varepsilon} = 1/2$, by choosing $\sqrt{\varepsilon_1} = \sqrt{\varepsilon_2} = 1/4$, we have $\sqrt{\varepsilon} = \sqrt{\varepsilon_1} + \sqrt{\varepsilon_2}$. Then,

$$\partial^c_{\varepsilon_1} f(z) = \partial^c f(z) + \sqrt{\varepsilon_1} B = \{\sqrt{3}/2\} + 1/4[-1, 1]$$

and

$$\partial^c_{\varepsilon_2} g(z) = \partial^c g(z) + \sqrt{\varepsilon_2} B = \{-1\} + 1/4[-1, 1].$$

If we choose $\lambda = \frac{\sqrt{3}}{2}$, $u = \frac{\sqrt{3}}{2} + \frac{1}{4} \cdot \frac{\sqrt{3}}{2}$ and $v = -1 + \frac{1}{4} \cdot (-1)$ then $u \in \partial^c_{\varepsilon_1} f(z)$, $v \in \partial^c_{\varepsilon_2} g(z)$ and $u + \lambda v = 0$.

5. Conclusions

We proposed a new concept of ε-quasi subdifferential for locally Lipschitz functions defined on $\mathbb{R}^n$ and ε-quasi normal set for a closed subset in $\mathbb{R}^n$. Then, approximate optimality conditions for a class of nonsmooth optimization problems are given. Different from previous results, approximate optimality conditions of nonsmooth optimization problems in our result are given in terms of approximate quasi subdifferentials of the involved functions without plus a set of tolerant vectors.

Acknowledgments

The first author was supported partially by Saigon University, Project No. CSA2022-04. The third author was supported by the National Research Foundation of Korea (NRF) Grant funded by the Korean Government (NRF-2019R1A2C1008672).

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