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Evolution Families of (X, Y, φ)-type and Periodic Solutions to Nonautonomous Evolution Equations

Ngoc Huy Nguyen, Thieu Huy Nguyen* and Thi Ngoc Ha Vu

Abstract. Consider nonautonomous evolution equation $\dot{u} = A(t)u(t) + Bg(u)(t)$ in which the family of operators $(A(t))_{t\geq 0}$ generates the evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ of (X, Y, φ)-type, i.e., $\|\mathcal{U}(t,0)x\|_Y \leq \varphi(t)\|x\|_X$, $t > 0$, for certain couple of Banach spaces (X, Y) and real-valued, positive function φ satisfying $\lim_{t\to\infty} \varphi(t) = 0$. Inspired by Serrin's technique, we develop a unified approach toward the problems on the existence of periodic solutions to above equation. As illustrations of our abstract results, we give applications to the existence and uniqueness of periodic solutions to Oseen–Navier–Stokes and damped wave equations, as well as the existence of local stable manifolds nearby the periodic solution to the damped wave equations.

1. Introduction and preliminaries

Consider the semilinear equation

$$u'(t) - A(t)u(t) = Bg(u)(t) \tag{1.1}$$

where the family of operators $(A(t))_{t\geq 0}$ is T-periodic and generates an evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$, the nonlinear operator g maps T-periodic functions to T-periodic functions, and the linear operator B is the "connection" operator between relevant spaces under consideration. The investigation of existence and uniqueness of a T-periodic solution to (1.1) is an important research direction related to dynamics of such evolution equations. Here, we want to mentioned some approaches, such as Massera methodology [23,39], Tikhonov's fixed-point principle [30] or the Lyapunov functionals [38]. The most popular approaches for proving the existence of a periodic solution are the use of ultimate boundedness of solutions and the fixed-point of Poincaré map realized through some compact embeddings (see [4,21,30–32,38]). As a well-known methodology, the so-called Massera-type theorem can be roughly explained that if a differential equation posses a time-bounded solution then it has a time-periodic one. Practically, to invoke the Massera's principle we have to

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*Corresponding author.

combine somehow the boundedness of solutions with some compactness at least at the level of weak-* topology (e.g., Alaoglu's theorem). At this point, it is worth noting that Serrin [32] was the first one who proved the stability of solutions to Navier–Stokes equations (NSE) in L^2-spaces (on bounded domains) implies the existence of periodic solutions. The ideas and methods of Seerin have been extended by many others such as Kaniel and Shinbrot [17], Miyakawa and Teramoto [24], Maremonti, Kozono and Nakao [20]. The existence results of such solutions to NSE on certain exterior domains were shown by Maremonti and Padula [22] under conditions related to symmetry and small complements. Furthermore, Galdi and Sohr [8] proved the existence of periodic solutions to NSE in arbitrary exterior domain using the spaces featuring the decay of the solutions at spatial infinity. Yamazaki [37] used interpolation spaces and the iteration scheme introduced by [9, 18] to obtain the existence and uniqueness of periodic weak mild solutions in weak L^n spaces on exterior domains. For further results in unbounded domain, we refer to Taniuchi [34] and van Baalen and Wittwer [36], Galdi and Silvestre [7].

In the present paper, inspired by the Serrin's methodology, we introduce a general method to obtain the existence and uniqueness of the periodic solution to the abstract evolution equation (1.1). Namely, we combine the boundedness and (X, Y, φ)-type of the corresponding evolution family (see Definition 2.4 below) to construct a Cauchy sequence which converges to the initial vector from which emanates a periodic solution to nonhomogeneous linearized equations. Then we pass to the semilinear equations by using fixed-point arguments. This can be considered as a generalization of our previous approach (see [15, 27]) which corresponds to the hyperbolic semigroups and wave equations. In the present paper we extend such an approach so that we can widen the range of applications not only to hyperbolic PDE but also to parabolic PDE in unbounded domains related to $L^p - L^q$ smoothing properties of the evolution family. On the other hand, in the present paper, we also extend the results achieved in [28] to the case of nonautonomous equations and evolution family. Another advantage of our approach here is lying in the fact that we do not use any compactness arguments. Consequently, we can prove the existence and uniqueness of general linear inhomogeneous evolution equations in a direct and elegant manner. Our main result is contained in Theorems 2.5 and 2.6. Then, in Section 3, we apply the abstract results to Oseen–Navier–Stokes equations in exterior domains as well as exponentially dichotomic evolution families and damped wave equations. These applications related to two kinds of the function φ, namely, $\varphi(t) = \frac{C}{t^\alpha}$, $\alpha > 0$, (polynomial decaying) or $\varphi(t) = Ce^{-\alpha t}$ (exponential decaying).

2. Periodic solutions to evolution equations

We first consider Banach spaces X and Y which are continuously embedded in a Hausdorff topological vector space. Suppose that there is a family of operators $\mathcal{U}(t, s) \in \mathcal{L}(X + Y)$,

$t \geq 0$, such that $\mathcal{U}(t,s)|_X$ and $\mathcal{U}(t,s)|_Y$ are strongly continuous, exponentially bounded on X and Y, respectively. Then, let us consider the following linear evolution equation

$$u' - A(t)u = Bf(t) \text{ for } t \geq 0, \quad u(0) = u_0, \tag{2.1}$$

where the family of operators $(A(t))_{t\geq 0}$ is T-periodic, f belongs to $\mathcal{F}_\tau(\mathbb{R}_+, Z) := \{h\colon \mathbb{R}_+ \to Z \mid h$ is continuous in topology τ of Z and $\sup_{t\geq 0} \|h(t)\|_Z < \infty\}$ endowed with norm $\|h\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} := \sup_{t\geq 0} \|h(t)\|_Z$. The operator B is the "connection" operator, which is a linear operator from Z to Y. Note that in the case of fluid flow equations drove from Navier–Stokes equations, $B = \mathbb{P}\,\mathrm{div}$, the composition of Helmholtz and divergent operators, meanwhile, as in other cases, one may take $B = \mathrm{Id}$, the identity operator on Y.

The family of partial differential operators $(A(t))_{t\geq 0}$ is given such that the homogeneous Cauchy problem

$$u' = A(t)u \text{ for } t > s \geq 0, \quad u(s) = x \in X \tag{2.2}$$

is well-posed in the sense, roughly speaking, that there is an evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ such that the solution to (2.2) is given by $u(t) = \mathcal{U}(t,s)x$ for $t > s$ and $x \in X$. The reader is referred to [25,29] for details on the concept of evolution families, well-posedness for non-autonomous abstract Cauchy problems as well as their applications to partial differential equations. The precise concept of an evolution family is given in the following definition.

Definition 2.1. A family of bounded linear operators $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ on a Banach space X is a (*strongly continuous, exponentially bounded*) evolution family if

(i) $\mathcal{U}(t,t) = \mathrm{Id}$ and $\mathcal{U}(t,r)\mathcal{U}(r,s) = \mathcal{U}(t,s)$ for all $t \geq r \geq s \geq 0$,

(ii) the map $(t,s) \mapsto \mathcal{U}(t,s)x$ is continuous for every $x \in X$, where $(t,s) \in \{(t,s) \in \mathbb{R}^2 : t \geq s \geq 0\}$,

(iii) there are constants K, α such that $\|\mathcal{U}(t,s)x\|_X \leq Ke^{\alpha(t-s)}\|x\|_X$ for all $t \geq s \geq 0$ and $x \in X$.

The existence of the evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ allows us to define a notion of mild solutions as follows. By the mild solution to (2.1), we mean a function $u\colon \mathbb{R}_+ \to Y$ satisfying the following integral equation

$$u(t) = \mathcal{U}(t,0)u_0 + \int_0^t \mathcal{U}(t,s)Bf(s)\,ds. \tag{2.3}$$

We now assume the following assumption that will be used in the rest of the paper.

Assumption 2.2. *We assume that $A(t)$ is T-periodic, i.e., $A(t+T) = A(t)$ for a fixed constant $T > 0$ and all $t \in \mathbb{R}_+$. Then $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ becomes T-periodic in the sense that*

$$\mathcal{U}(t+T, s+T) = \mathcal{U}(t,s) \quad \text{for all } t \geq s \geq 0. \tag{2.4}$$

Remark 2.3. We would like to note that the relation (2.4) is a consequence of the assumption that $A(t)$ is T -periodic in combination with the well-posedness (existence and uniqueness) of the solution of the Cauchy problem (2.2). Indeed, for any $x \in X$, we have that $u(t) = U(t,s)x$ is a solution to (2.2). Put $\widetilde{u}(t) = \mathcal{U}(t+T, s+T)x$. Then, by Pazy [29, Chapter 5, Theorem 6.1] we have

$$\frac{\partial \widetilde{u}}{\partial t} = \frac{\partial \mathcal{U}(t+T,s+T)x}{\partial t} = A(t+T)\mathcal{U}(t+T,s+T)x = A(t)\widetilde{u} \quad (\text{since } A(t+T) = A(t)).$$

Moreover, $\widetilde{u}(s) = \mathcal{U}(s+T, s+T)x = x$. Therefore, $\widetilde{u}(t) = \mathcal{U}(t+T, s+T)x$ is also a solution to (2.2) with the same initial value $\widetilde{u}(s) = x$. The uniqueness of the solution implies that $u(t) = \widetilde{u}(t)$ for all $t \geq s$, and hence, $U(t,s)x = U(t+T, s+T)x$ for all $x \in X$. Thus, relation (2.4) follows.

Also, we assume the following condition of (X, Y, φ)-*type* on the evolution family.

Definition 2.4. Let X, Y be the Banach spaces as above, and $\varphi\colon (0,\infty) \to (0,\infty)$ be a continuous function such that $\lim_{t\to\infty} \varphi(t) = 0$. The evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ is said to be of (X, Y, φ)-*type* if for $t > 0$ we have

$$\|\mathcal{U}(t,0)x\|_Y \leq \varphi(t)\|x\|_X \quad \text{for all } x \in X. \tag{2.5}$$

For a Banach space Z with norm $\|\cdot\|_Z$, we consider the following abstract function space

$$\mathcal{F}_\tau(\mathbb{R}_+, Z) := \left\{ h\colon \mathbb{R}_+ \to Z \;\middle|\; h \text{ is continuous in topology } \tau \text{ of } Z \text{ and } \sup_{t\geq 0} \|h(t)\|_Z < \infty \right\}$$

endowed with norm $\|h\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} := \sup_{t\geq 0} \|h(t)\|_Z$. Moreover, we suppose that the topology τ in Z is chosen such that $\mathcal{F}_\tau(\mathbb{R}_+, Z)$ is a Banach space with the above norm. The two typical examples of topology τ of Z that we will use later are the norm-topology (i.e., the topology generated by norm of Z) and the weak*-topology on Z. In those cases, we have

$$\mathcal{F}_\tau(\mathbb{R}_+, Z) = C_b(\mathbb{R}_+, Z) := \left\{ v\colon \mathbb{R}_+ \to Z \;\middle|\; v \text{ is continuous and } \sup_{t\in\mathbb{R}_+} \|v(t)\|_Z < \infty \right\}$$

in case τ is the norm-topology on Z, or

$$\begin{aligned}\mathcal{F}_\tau(\mathbb{R}_+, Z) &= C_{w*,b}(\mathbb{R}_+, Z)\\ &:= \left\{ v\colon \mathbb{R}_+ \to Z \;\middle|\; v \text{ is weak}^* \text{ continuous and } \sup_{t\in\mathbb{R}_+} \|v(t)\|_Z < \infty \right\}\end{aligned}$$

in case τ is the weak*-topology on Z.

Furthermore, we denote the norm on the space $X \cap Y$ by $\|x\|_{X\cap Y} = \max\{\|x\|_X, \|y\|_Y\}$ making $X \cap Y$ to be a Banach space. Our first result for linear equation is stated in the following theorem.

Theorem 2.5. *Consider the evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ of (X,Y,φ)-type as in above Definition. Let $f \in \mathcal{F}_\tau(\mathbb{R}_+, Z)$ and suppose that there exists $x_0 \in X$ such that the mild solution $u(t) = \mathcal{U}(t,0)x_0 + \int_0^t \mathcal{U}(t,s)Bf(s)\,ds$, $t \geq 0$, belongs to $\mathcal{F}_\tau(\mathbb{R}_+, X\cap Y)$ and satisfies $\|u\|_{\mathcal{F}_\tau(\mathbb{R}_+,X\cap Y)} \leq M\|f\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)}$. Finally, suppose that*

$$\sup_{0\leq t\leq T}\left\|\int_0^t \mathcal{U}(t,s)Bf(s)\,ds\right\|_Y \leq N\|f\|_{C_b(\mathbb{R}_+,Z)}.$$

Then, if f is T-periodic in time, there exists a unique T-periodic mild solution $\widehat{u}$ of (2.1) with

$$\|\widehat{u}\|_{\mathcal{F}_\tau(\mathbb{R}_+,Y)} \leq \widetilde{M}\|f\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} \quad \text{for } \widetilde{M} := M\sup_{0\leq t\leq T}\|\mathcal{U}(t,0)\| + N. \tag{2.6}$$

Proof. By the hypothesis of the theorem, we have that the mild solution u of (2.1) with $u(0) = x_0 \in X$ (i.e., $u(t) = \mathcal{U}(t,0)x_0 + \int_0^t \mathcal{U}(t,s)Bf(s)\,ds$, $t \geq 0$) belongs to $C_b(\mathbb{R}_+, X)$. We next prove that $\{u(nT)\}_{n\in\mathbb{N}}$ is a Cauchy sequence in Y. Indeed, putting $w(t) = u(t+(m-n)T)$ for arbitrary fixed natural numbers $m > n \in \mathbb{N}$, using the periodicity of f we now prove that w can be rewritten as

$$w(t) = \mathcal{U}(t,0)u((m-n)T) + \int_0^t \mathcal{U}(t,s)Bf(s)\,ds \quad \text{for all } t \geq 0. \tag{2.7}$$

Indeed,

$$\begin{aligned}
w(t) &= u(t+(m-n)T)\\
&= \mathcal{U}(t+(m-n)T,0)u(0) + \int_0^{t+(m-n)T} \mathcal{U}(t+(m-n)T,s)Bf(s)\,ds\\
&= \mathcal{U}(t+(m-n)T,(m-n)T)\mathcal{U}((m-n)T,0)u(0)\\
&\quad + \int_0^{(m-n)T} \mathcal{U}(t+(m-n)T,(m-n)T)\mathcal{U}((m-n)T,s)Bf(s)\,ds\\
&\quad + \int_{(m-n)T}^{t+(m-n)T} \mathcal{U}(t+(m-n)T,s)Bf(s)\,ds\\
&= \mathcal{U}(t+(m-n)T,(m-n)T)\\
&\quad \times \left(\mathcal{U}((m-n)T,0)u(0) + \int_0^{(m-n)T} \mathcal{U}((m-n)T,s)Bf(s)\,ds\right)\\
&\quad + \int_{(m-n)T}^{t+(m-n)T} \mathcal{U}(t+(m-n)T,s)Bf(s)\,ds\\
&= \mathcal{U}(t,0)u((m-n)T) + \int_0^t \mathcal{U}(t,s)Bf(s)\,ds.
\end{aligned}$$

Therefore, (2.7) follows. Now, the relation in (2.5) yields

$$\|u(t)-w(t)\|_Y = \|\mathcal{U}(t,0)(u(0)-w(0))\|_Y \leq \varphi(t)\|u(0)-w(0)\|_X \leq C\varphi(t), \quad t > 0$$

for $C := 2\|u\|_{\mathcal{F}_\tau(\mathbb{R}_+, X\cap Y)}$ independent of m, n.

Taking $t := nT$ on the above inequality we obtain

$$\|u(nT) - u(mT)\|_Y \leq C\varphi(nT)$$

for all $m > n \in \mathbb{N}$. From the fact $\lim_{t\to\infty}\varphi(t) = 0$, it follows that $\{u(nT)\}_{n\in\mathbb{N}}$ is Cauchy sequence in Y. Since Y is a Banach space, the sequence $\{u(nT)\}_{n\in\mathbb{N}}$ is convergent in Y, and we put

$$u^* := \lim_{n\to\infty} u(nT) \in Y.$$

Taking now u^* as initial value, we then prove that the mild solution $\widehat{u}(t) = \mathcal{U}(t,0)u^* + \int_0^t \mathcal{U}(t,s)Bf(s)\,ds$ is T-periodic. To do this, we put $v(t) := \mathcal{U}(t+nT,0)x_0 + \int_0^{t+nT}\mathcal{U}(t+nT,s)Bf(s)\,ds$ for every fixed $n \in \mathbb{N}$ and all $t \geq 0$, i.e., $v(t) = u(t+nT)$ for

$$u(t) = \mathcal{U}(t,0)x_0 + \int_0^t \mathcal{U}(t,s)Bf(s)\,ds \tag{2.8}$$

as in previous step.

Again, by the periodicity of f we obtain that v satisfies

$$v(t) = \mathcal{U}(t,0)u(nT) + \int_0^t \mathcal{U}(t,s)Bf(s)\,ds$$

for u being defined as in (2.8).

We then have

$$\|\widehat{u}(T) - v(T)\|_Y = \|\mathcal{U}(T,0)(\widehat{u}(0) - v(0))\|_Y \leq \|\mathcal{U}(T,0)\|\|\widehat{u}(0) - v(0)\|_Y.$$

This means

$$\|\widehat{u}(T) - u((n+1)T)\|_Y \leq \|\mathcal{U}(T,0)\|\|u^* - u(nT)\|_Y.$$

Letting now $n \to \infty$ and using the fact that $\lim_{n\to\infty} u(nT) = u^* = \widehat{u}(0)$ in Y (see above) we obtain

$$\widehat{u}(T) = \widehat{u}(0).$$

Therefore, $\widehat{u}(t)$ is $T-$periodic. The inequality (2.6) follows from the facts that $\|u^*\|_Y \leq \|u\|_{\mathcal{F}_\tau(\mathbb{R}_+, X\cap Y)} \leq M\|f\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)}$ and $\|\widehat{u}\|_{\mathcal{F}_\tau(\mathbb{R}_+,Y)} = \sup_{0\leq t\leq T}\|\widehat{u}(t)\|_Y$ thanks to the periodicity of $\widehat{u}$. For the reader's convenience, we present the estimates

$$\begin{aligned}\|\widehat{u}\|_{\mathcal{F}_\tau(\mathbb{R}_+,Y)} &= \sup_{0\leq t\leq T}\|\widehat{u}(t)\|_Y \leq \sup_{0\leq t\leq T}\|\mathcal{U}(t,0)\|_Y\|u^*\|_Y + N\|f\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} \\ &\leq \left(M\sup_{0\leq t\leq T}\|\mathcal{U}(t,0)\|_Y + N\right)\|f\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)}.\end{aligned}$$

This yields the inequality (2.6).

The uniqueness of the T-periodic solution follows from (2.5). Namely, if u and v are two T-periodic solutions of (2.3) with initial values u_0 and v_0, respectively, then $u(t) - v(t) = \mathcal{U}(t,0)(u_0 - v_0)$, and from the fact that $u(t) - v(t)$ is bounded it follows from (2.5) that $\|u(t) - v(t)\|_Y = \|\mathcal{U}(t,0)(u_0 - v_0)\|_Y \le \varphi(t)\|u_0 - v_0\|_Y$.

Therefore, $\lim_{t\to\infty} \|u(t) - v(t)\|_Y = 0$. This, together with periodicity and continuity of u and v, follows that $u(t) = v(t)$ for all $t \in \mathbb{R}_+$. □

We now consider the following semi-linear evolution equation

$$u'(t) = A(t)u(t) + Bg(u)(t), \quad u(0) = u_0 \in X, \tag{2.9}$$

where the operators $A(t)$ satisfy the above hypotheses for linear equations, and the non-linear operator $g\colon \mathcal{F}_\tau(\mathbb{R}_+, Y) \to \mathcal{F}_\tau(\mathbb{R}_+, Z)$ satisfies

(2.10)
(1) $\|g(0)\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} \le \gamma$ where γ is a non-negative constant,
(2) g maps T-periodic functions to T-periodic functions,
(3) there exist positive constants ρ and L such that
$$\|g(v_1) - g(v_2)\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} \le L\|v_1 - v_2\|_{\mathcal{F}_\tau(\mathbb{R}_+,Y)}$$
for all $v_1, v_2 \in \mathcal{F}_\tau(\mathbb{R}_+, Y)$ with $\|v_1\|_{\mathcal{F}_\tau(\mathbb{R}_+,Y)}, \|v_2\|_{\mathcal{F}_\tau(\mathbb{R}_+,Y)} \le \rho$.

Furthermore, by the *mild solution* to (2.9) we mean the function u satisfying the following equation

$$u(t) = \mathcal{U}(t,0)u_0 + \int_0^t \mathcal{U}(t,s)Bg(u)(s)\,ds \quad \text{for all } t \ge 0.$$

We then come to our next result on the existence and uniqueness of the periodic mild solution to (2.9).

Theorem 2.6. *Let the hypotheses of Theorem* 2.5 *be satisfied, and let g satisfy the conditions in* (2.10)*. Then, if L and γ are small enough,* (2.9) *has one and only one mild T-periodic solution $\widehat{u}$ on a small ball of $\mathcal{F}_\tau(\mathbb{R}_+, Y)$.*

Proof. Consider the following ball $\mathcal{B}_\rho^T$ defined by

$$\mathcal{B}_\rho^T := \left\{ v \in \mathcal{F}_\tau(\mathbb{R}_+, Y) : v \text{ is } T\text{-periodic and } \|v\|_{\mathcal{F}_\tau(\mathbb{R}_+,Y)} \le \rho \right\}.$$

We then define the following transformation Φ given as follows: Consider the equation

$$u'(t) = A(t)u(t) + Bg(v)(t). \tag{2.11}$$

Then, for $v \in \mathcal{B}_\rho^T$ we set

$$\Phi(v) = u \text{ where } u \in \mathcal{F}_\tau(\mathbb{R}_+, Y) \text{ is the unique } T\text{-periodic mild solution to (2.11).}$$

We now show that for sufficiently small L and γ, the transformation Φ is self-map on $\mathcal{B}_\rho^T$, and a contraction. To do this, taking any $v \in \mathcal{B}_\rho^T$, by the properties of g given in (2.10) we have that

$$\|g(v)\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} \leq \|g(v) - g(0)\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} + \|g(0)\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} \leq L\rho + \gamma.$$

Applying Theorem 2.5 for the right-hand side $Bg(v)(t)$ instead of $Bf(t)$ and using inequality (2.6) we obtain that for $v \in \mathcal{B}_\rho^T$ there exists a unique T-periodic mild solution u to (2.11) satisfying

$$\|u\|_{\mathcal{F}_\tau(\mathbb{R}_+,Y)} \leq \widetilde{M}\|g(v)\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} \leq \widetilde{M}(L\rho + \gamma).$$

Therefore, if L and γ are sufficiently small, the map Φ is a self-map on $\mathcal{B}_\rho^T$. Then, by (2.3) with $g(v)$ instead of f, we have the following representation of Φ:

$$\Phi(v)(t) = \mathcal{U}(t,0)u(0) + \int_0^t \mathcal{U}(t,\tau)Bg(v)(\tau)\,d\tau \quad \text{for } \Phi(v) = u. \tag{2.12}$$

Furthermore, for $v_1, v_2 \in \mathcal{B}_\rho^T$ by the representation (2.12) we obtain that the function $u := \Phi(v_1) - \Phi(v_2)$ is the unique T-periodic mild solution to the equation

$$u'(t) = A(t)u(t) + B(g(v_1) - g(v_2))(t).$$

Thus, again by Theorem 2.5 we arrive at

$$\|\Phi(v_1) - \Phi(v_2)\|_{\mathcal{F}_\tau(\mathbb{R}_+,Y)} \leq \widetilde{M}\|g(v_1) - g(v_2)\|_{\mathcal{F}_\tau(\mathbb{R}_+,Z)} \leq \widetilde{M}L\|v_1 - v_2\|_{\mathcal{F}_\tau(\mathbb{R}_+,Y)}.$$

We hence obtain that for sufficiently small L, and γ, the map $\Phi\colon \mathcal{B}_\rho^T \to \mathcal{B}_\rho^T$ is a contraction, thus, there exists a unique fixed point $\widehat{u}$ of Φ, and by the definition of Φ, this function $\widehat{u}$ is the unique T-periodic mild solution to (2.9). □

3. Applications

In this section we apply our abstract results obtained in the previous section to various equations including Oseen–Navier–Stokes equations and the nonautonomous damped wave equations.

3.1. Oseen–Navier–Stokes equations

We consider the flow of an imcompressible, viscous fluid in the exterior of a rotating obstacle that is translating with a time-dependent velocity. Here the angular velocity of the obstacle also depends on time and the axis of rotation may change. The equations describing this problem are the Oseen–Navier–Stokes equations in a time-dependent exterior domain with a prescribed velocity field at infinity. After rewriting the problem on a fixed exterior domain $\Omega \subset \mathbb{R}^3$, the system is reduced to

$$(3.1)\quad \begin{cases} u_t + (u\cdot\nabla)u - \Delta u + \nabla p = (\eta + \omega\times x)\cdot\nabla u - \omega\times u + \operatorname{div} F & \text{in } \Omega\times(0,\infty),\\ \nabla\cdot u = 0 & \text{in } \Omega\times(0,\infty),\\ u = \eta + \omega\times x & \text{on } \partial\Omega\times(0,\infty),\\ u|_{t=0} = u_0 & \text{in } \Omega,\\ \lim_{|x|\to\infty} u = 0 \end{cases}$$

where $u = (u_1(x,t), u_2(x,t), u_3(x,t))^T$ is supposed to be the velocity of the fluid; $p = p(x,t)$ the pressure; and $\operatorname{div} F$ is the external force for a 2^{nd}-order tensor $F = F(x,t)$. Meanwhile, $\eta = (0,0,a(t))^T$ and $\omega = (0,0,k(t))^T$ stand for the translational and angular velocities respectively of the obstacle. Here $\Omega = \mathbb{R}^3 \setminus D(0)$ with $D(0)$ being the position of $D \subset \mathbb{R}^3$ at $t = 0$. Galdi [6] proved the existence of the periodic solution of such equations in L^2 space using Galerkin method. In the present paper, we will consider the periodic solution on weak L^3 space over Ω (see details in Theorem 3.5).

Here, we recall some preliminaries on function and interpolation spaces for latter use.

Given an exterior domain Ω of class $C^{1,1}$ in $\mathbb{R}^3$, we denote by $C_0^\infty(\Omega)$ the space of all smooth functions with compact supports in Ω. Then, we consider the following spaces:

$$\begin{aligned} C_0^\infty(\Omega)^3 &:= \{(v_1,v_2,v_3) : v_j \in C_0^\infty(\Omega), j = 1,2,3\},\\ C_{0,\sigma}^\infty(\Omega)^3 &:= \{v \in C_0^\infty(\Omega)^3 : \operatorname{div} v = 0 \text{ in } \Omega\},\\ L_\sigma^p(\Omega)^3 &:= \overline{C_{0,\sigma}^\infty(\Omega)}^{\|\cdot\|_{L^p}}. \end{aligned}$$

We note that, as in the works [10, 12], the regularity of boundary is needed for the well-posedness and $L^p - L^q$-smoothing property of the linearized problem, and $C^{1,1}$-regularity was enough for such properties. The Lorentz space $L^{r,q}(\Omega)^3$ $(1 < r < \infty,\ 1 \le q \le \infty)$ was defined in [1, 19, 35], and here $L^{r,r}(\Omega)^3 = L^r(\Omega)^3$. Moreover, $L^{r,\infty}(\Omega)^3$ is called the weak-L^r space denoted by $L_w^r(\Omega)^3 := L^{r,\infty}(\Omega)^3$.

Denote by $\|\cdot\|_{r,w}$ the norm in $L_w^r(\Omega)^3$. We take the following inequality from [3, Lemma 2.1] which is known as weak Hölder inequality.

Lemma 3.1. *Consider indices p, q, r satisfying $1 < p \le \infty$, $1 < q < \infty$, $1 < r < \infty$, and $\frac{1}{p} + \frac{1}{q} = \frac{1}{r}$. Then,*

$$(3.2)\quad \|fg\|_{r,w} \le C\|f\|_{p,w}\|g\|_{q,w} \quad \text{for } f \in L_w^p(\Omega)^3,\ g \in L_w^q(\Omega)^3,$$

where C is a positive constant depending only on p and q. Note that $L_w^\infty(\Omega)^3 = L^\infty(\Omega)^3$.

Denote $\mathbb{P} = \mathbb{P}_r$ the Helmholtz projection on $L^r(\Omega)$ $(1 < r < \infty)$, i.e., the projection onto $L^r_\sigma(\Omega)^3$ relative to the Leray–Helmholtz decomposition (see [3]):

$$L^r(\Omega)^3 = L^r_\sigma(\Omega)^3 \oplus \{\nabla p \in L^r(\Omega)^3 : p \in L^r_{\mathrm{loc}}(\overline{\Omega})\}.$$

Next, for each $t \geq 0$ we define the operator $\mathcal{L}(t)$ as follows:

$$\begin{aligned} D(\mathcal{L}(t)) &:= \left\{u \in L^r_\sigma(\Omega)^3 \cap W^{1,r}_0(\Omega)^3 \cap W^{2,r}(\Omega)^3 : (\omega(t) \times x) \cdot \nabla u \in L^r(\Omega)^3\right\} \\ \mathcal{L}(t)u &:= -\mathbb{P}\big[\Delta u + (\eta(t) + \omega(t) \times x) \cdot \nabla u - \omega(t) \times u\big] \quad \text{for } u \in D(\mathcal{L}(t)). \end{aligned} \tag{3.3}$$

It is known that the family of operators $\{\mathcal{L}(t)\}_{t\geq 0}$ generates a bounded evolution family $\{U(t,s)\}_{t\geq s\geq 0}$ on $L^r_\sigma(\Omega)^3$ for each $1 < r < \infty$ under the conditions that $\eta, \omega \in C^\theta_{\mathrm{loc}}([0,\infty);\mathbb{R}^3)$ for some $\theta \in (0,1)$ (see [10]). Furthermore, the solenoidal Lorentz spaces are identified (see [3]) by

$$L^{r,q}_\sigma(\Omega)^3 := (L^{r_1}_\sigma(\Omega)^3, L^{r_2}_\sigma(\Omega)^3)_{\theta,q},$$

where $1 < r_0 < r < r_1 < \infty$, $1 \leq q \leq \infty$ and $\frac{1}{r} = \frac{1-\theta}{r_0} + \frac{\theta}{r_1}$. Then $\{U(t,s)\}_{t\geq s\geq 0}$ are extended to strongly continuous, bounded evolution operators on $L^{r,q}_\sigma(\Omega)^3$. Denote also by $L^r_{\sigma,w}(\Omega)^3 := L^{r,\infty}_\sigma(\Omega)^3$.

Also, for $1 \leq q < \infty$, we have the dual space

$$(L^{r,q}_\sigma(\Omega)^3)' = L^{r',q'}_\sigma(\Omega)^3 \quad \text{where } r' = \frac{r}{r-1},\ q' = \frac{q}{q-1} \text{ and } q' = \infty \text{ if } q = 1.$$

Moreover, for $0 < \theta < 1$ we consider the space of Hölder continuous functions

$$C^\theta([0,\infty);\mathbb{R}^3) := \left\{ f \in C([0,\infty);\mathbb{R}^3) : \sup_{t>s\geq 0} \frac{|f(t)-f(s)|}{(t-s)^\theta} < \infty \right\}.$$

We analyze the case in which both $\eta(t)$ and $\omega(t)$ are prescribed T-periodic functions such that

$$\eta, \omega \in C^\theta([0,\infty);\mathbb{R}^3) \cap C^1([0,\infty);\mathbb{R}^3) \cap L^\infty([0,\infty);\mathbb{R}^3) \quad \text{with some } \theta \in (0,1). \tag{3.4}$$

Let us introduce the following notations:

$$\begin{aligned} |(\eta,\omega)|_0 &:= \sup_{T\geq t\geq 0} (|\eta(t)| + |\omega(t)|), \\ |(\eta,\omega)|_1 &:= \sup_{T\geq t\geq 0} (|\eta'(t)| + |\omega'(t)|), \\ |(\eta,\omega)|_\theta &:= \sup_{T\geq t>s\geq 0} \frac{|\eta(t)-\eta(s)| + |\omega(t)-\omega(s)|}{(t-s)^\theta}. \end{aligned}$$

There is a constant $m \in (0,\infty)$ such that

$$|(\eta,\omega)|_0 + |(\eta,\omega)|_1 + |(\eta,\omega)|_\theta \leq m. \tag{3.5}$$

We recall the following $L^{r,q} - L^{p,q}$ estimates taken from [12, Theorems 2.1, 2.2].

Proposition 3.2. *Suppose that η and ω fulfill (3.4) and (3.5) for an $m\in(0,\infty)$. Denote by $\|\cdot\|_{r,q}$ the norm in $L^{r,q}$ (here $1<r<\infty$, $1\le q\le\infty$). Then,*

$$\|U(t,s)x\|_{r,q}\le M(t-s)^{-\frac{3}{2}\left(\frac{1}{p}-\frac{1}{r}\right)}\|x\|_{p,q} \quad \text{for all } t>s\ge 0 \text{ (here } 1<p\le r<\infty). \tag{3.6}$$

We take $R_0>0$ satisfying

$$\mathbb{R}^3\setminus\Omega\subset B_{R_0}:=\{x\in\mathbb{R}^3:|x|<R_0\}.$$

Then, we fix a cut-off function $\phi\in C_0^\infty(B_{3R_0})$ such that $\phi=1$ on B_{2R_0} and set

$$b(x,t)=\frac{1}{2}\operatorname{rot}\{\phi(x)(\eta(t)\times x-|x|^2\omega(t))\}$$

which fulfills

$$\operatorname{div}b=0,\quad b|_{\partial\Omega}=\eta+\omega\times x,\quad b(t)\in C_0^\infty(B_{3R_0}).$$

Moreover, an elementary calculation shows that

$$\omega\times b=\operatorname{div}\begin{pmatrix}\frac{-(a(t))^2|x|^2\phi(x)}{2} & 0 & a(t)k(t)x_2\phi(x)\\ 0 & \frac{-(a(t))^2|x|^2\phi(x)}{2} & -a(t)k(t)x_1\phi(x)\\ 0 & 0 & 0\end{pmatrix}=\operatorname{div}(-F_1),$$

$$b_t=\operatorname{div}\begin{pmatrix}0 & \frac{-a'(t)|x|^2\phi(x)}{2} & \frac{-k'(t)x_1\phi(x)}{2}\\ \frac{a'(t)|x|^2\phi(x)}{2} & 0 & \frac{-k'(t)x_2\phi(x)}{2}\\ k'(t)x_1\phi(x) & k'(t)x_2\phi(x) & 0\end{pmatrix}=\operatorname{div}(-F_2).$$

If we set $z(x,t)=u(x,t)-b(x,t)$ then the fact that u fulfills (3.1) is equivalent to z satisfies

$$\begin{cases} z_t-\Delta z-(\eta+\omega\times x)\cdot\nabla z+\omega\times z+\nabla p & \\ \quad =\operatorname{div}G-(z\cdot\nabla)z-(b\cdot\nabla)z-(z\cdot\nabla)b-(b\cdot\nabla)b & \text{in } \Omega\times(0,\infty),\\ \nabla\cdot z=0 & \text{in } \Omega\times(0,\infty),\\ z=0 & \text{on } \partial\Omega\times(0,\infty),\\ z|_{t=0}=z_0 & \text{in } \Omega,\\ \lim_{|x|\to\infty}u=0 & \end{cases} \tag{3.7}$$

where $z_0(x):=u_0(x)-b(x,0)$ and

$$G:=F+F_1+F_2+\nabla b+(\eta+\omega\times x)\otimes\nabla b.$$

To eliminate the pressure term we apply Helmholtz operator $\mathbb{P}$ to (3.7). Then, we may rewrite the equation as a non-autonomous abstract Cauchy problem

$$z_t+\mathcal{L}(t)z=\mathbb{P}\operatorname{div}(G-z\otimes z-b\otimes z-z\otimes b-b\otimes b),\quad z|_{t=0}=z_0\in L^3_{\sigma,w}(\Omega)^3, \tag{3.8}$$

where $\mathcal{L}(t)$ is defined as in (3.3).

As proved in [10], the family of operators $(\mathcal{L}(t))_{t\geq 0}$ generates an evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ in the sense, roughly speaking, that $z(t) = \mathcal{U}(t,0)z_0$ is the solution to homogeneous equation $z_t + \mathcal{L}(t)z = 0$; $z(0) = z_0$. Therefore, we can define a *mild solution* of (3.8) as the function $z(t)$ fulfilling the following integral equation in which the integral is understood in weak sense as in [37, Remark 1.2]:

$$z(t) = \mathcal{U}(t,0)z(0) + \int_0^t \mathcal{U}(t,\tau)\mathbb{P}\,\mathrm{div}(-z\otimes z - b\otimes z - z\otimes b - b\otimes b + G(\tau))\,d\tau \quad \text{for } t\geq 0.$$

Denote by $\mathbb{R}_+ := (0,\infty)$ and write $\|\cdot\|_{s,w}$ for the norm in $L^s_{\sigma,w}(\Omega)^3$. In this situation, we choose the topology τ to be the weak*-topology. Concretely, we choose

$$\begin{aligned}
&\mathcal{F}_\tau(\mathbb{R}_+, L^s_{\sigma,w}(\Omega)^3)\\
&= C_{w*,b}(\mathbb{R}_+, L^s_{\sigma,w}(\Omega)^3)\\
&:= \left\{ v\colon \mathbb{R}_+ \to L^s_{\sigma,w}(\Omega)^3 \;\middle|\; v \text{ is weak* continuous and } \sup_{t\in\mathbb{R}_+} \|v(t)\|_{s,w} < \infty \right\}
\end{aligned}$$

endowed with the norm

$$\|v\|_{\infty,s,w} := \sup_{t\in\mathbb{R}_+} \|v(t)\|_{s,w}.$$

Remark 3.3. Let η and ω be T-periodic functions satisfying (3.4) and (3.5). Let the external force F fulfill that F belongs to $C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$ and is T-periodic. Then G is T-periodic and belonging to $C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$, moreover

$$\|G\|_{\infty,3/2,w} \leq \|F\|_{\infty,3/2,w} + C(m). \tag{3.9}$$

We know that $\mathcal{L}(t)$ is T-periodic, i.e., $\mathcal{L}(t+T) = \mathcal{L}(t)$ for a fixed constant $T>0$ and all $t\in\mathbb{R}_+$. Therefore, the corresponding evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ becomes T-periodic in the sense that

$$\mathcal{U}(t+T, s+T) = \mathcal{U}(t,s) \quad \text{for all } t\geq s\geq 0.$$

We rewrite (3.8) in the form

$$z_t + \mathcal{L}(t)z = \mathbb{P}\,\mathrm{div}\, g(z)(t), \quad z|_{t=0} = z_0 \in L^3_{\sigma,w}(\Omega), \tag{3.10}$$

where $g(z) = G - z\otimes z - b\otimes z - z\otimes b - b\otimes b$.

The linearized equation of (3.10) is

$$v_t + \mathcal{L}(t)v = \mathbb{P}\,\mathrm{div}\, G(t), \quad v|_{t=0} = v_0 \in L^3_{\sigma,w}(\Omega)^3. \tag{3.11}$$

Using the evolution family $(\mathcal{U}(t,s))_{t\geq \mathbb{S}\geq 0}$ generated by $(\mathcal{L}(t))_{t\geq 0}$, we can defined *mild solution* to (3.11) which is the function $v(t)$ satisfying the following equation in which the integral is understood in distribution sense as in [37, Remark 1.2]:

$$v(t) = \mathcal{U}(t,0)v(0) + \int_0^t \mathcal{U}(t,\tau)\mathbb{P}\operatorname{div} G(\tau)\, d\tau. \tag{3.12}$$

In this situation, from [16, Theorem 2.2] we obtain the boundedness of the function $v(t)$ on $\mathbb{R}_+$ which is stated in the following lemma.

Lemma 3.4. *Let Ω be an exterior domain Ω in $\mathbb{R}^3$ with a $C^{1,1}$-boundary and $z_0 \in L^3_{\sigma,w}(\Omega)^3$. Suppose that $G \in C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$. Then,* (3.11) *has a unique mild solution $v \in C_{w*,b}(\mathbb{R}_+, L^3_{\sigma,w}(\Omega)^3)$ which is represented in* (3.12) *with $v(0) = v_0$. Also,*

$$\|v\|_{\infty,3,w} \leq M\|v_0\|_{3,w} + \widetilde{M}\|G\|_{\infty,3/2,w}$$

where M, M_1 and $\widetilde{M}$ are positive constants which are independent of v_0, v, and F.

Proof. See [16, Theorem 2.2]. □

The following theorem contains our results on periodicity of solutions to nonautonomous Oseen–Navier–Stokes flows.

Theorem 3.5. *Consider an exterior domain Ω in $\mathbb{R}^3$ with a $C^{1,1}$-boundary and $z_0 \in L^3_{\sigma,w}(\Omega)^3$. Suppose that $F \in C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$ is T-periodic. Let η and ω be T-periodic functions fulfilling* (3.4) *and* (3.5)*. If $\|F\|_{\infty,3/2,w}$ and m are sufficiently small, Problem* (3.8) *possesses a unique T-periodic mild solution $\widehat{z}$ on a small ball of $C_{w*,b}(\mathbb{R}_+, L^3_{\sigma,w}(\Omega)^3)$.*

Proof. We first choose $X := L^r_{\sigma,w}(\Omega)^3$, $Y := L^3_{\sigma,w}(\Omega)^3$, and $Z := L^{3/2}_{\sigma,w}(\Omega)^3$ with $3/2 \leq r < 3$. Next, we derive from (3.6) the estimate

$$\|\mathcal{U}(t,0)x\|_{3,w} \leq Mt^{-\left(\frac{3}{2r}-\frac{1}{2}\right)}\|x\|_{r,w}.$$

Then it is obvious that $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ is of (X,Y,φ)*-type* with the function $\varphi(t) = Mt^{-\left(\frac{3}{2r}-\frac{1}{2}\right)}$ for all $t > 0$. Concerning the estimates for g, we first have

$$\|g(0)\|_{C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})} = \|G - b\otimes b\|_{\infty,3/2,w} \leq \|G\|_{\infty,3/2,w} + \|b\otimes b\|_{\infty,3/2,w}.$$

It follows from the weak Hölder inequality (3.1) that

$$\|b\otimes b\|_{\infty,3/2,w} \leq C\|b\|^2_{\infty,3,w} \leq Cm^2. \tag{3.13}$$

Combining (3.13) and (3.9) we obtain

$$\|g(0)\|_{C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})} \leq \|F\|_{\infty,3/2,w} + C(m) + Cm^2 := \gamma.$$

Thus, the sufficient smallness of m and $\|F\|_{\infty,3/2,w}$ implies that γ is small enough. Again, for $v_1, v_2 \in \mathcal{B}^T_\rho$ by the weak Hölder's inequality (3.2), we have that

$$\begin{aligned}
&\|g(v_1) - g(v_2)\|_{C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})} \\
&= \| - v_1 \otimes v_1 + v_2 \otimes v_2 - b \otimes v_1 - v_1 \otimes b + b \otimes v_2 + v_2 \otimes b\|_{\infty,3/2,w} \\
&\le \| - (v_1 - v_2) \otimes v_1 - v_2 \otimes (v_1 - v_2) - b \otimes (v_1 - v_2) - (v_1 - v_2) \otimes b\|_{\infty,3/2,w} \\
&\le (2C\rho + 2Cm)\|v_1 - v_2\|_{\infty,3,w}.
\end{aligned}$$

Theorem 2.6 now yields assertion of the theorem for the choice $\mathcal{F}_\tau(\mathbb{R}_+, Z) = C_{w*,b}(\mathbb{R}_+, L^{3/2}_{\sigma,w}(\Omega)^{3\times 3})$. □

Remark 3.6. Since $\mathbb{P}\nabla p = 0$ for the Helmholtz projection $\mathbb{P}$, we have already got rid of the pressure term ∇p in (3.8). After obtaining the T-periodic solution $\widehat{z}$ we can recover the pressure p (or its gradient ∇p) by

$$\nabla p = \widehat{z}_t - \Delta \widehat{z} - (\eta + \omega \times x) \cdot \nabla \widehat{z} + \omega \times \widehat{z} + \operatorname{div}(G - \widehat{z} \otimes \widehat{z} - b \otimes \widehat{z} - \widehat{z} \otimes b - b \otimes b).$$

Furthermore, since $\widehat{z}$, η and ω are all T-periodic, we obtain that ∇p and hence p are T-periodic as well. Moreover, the pressure p can then be recovered from ∇p in a similar ways as in Sorh [33, Chapter II, Lemma 2.2.1] (see also Bogovskii [2]).

3.2. Nonautonomous damped wave equations: Periodic solutions and stable manifolds nearby

In this section, we apply our abstract results in Section 2 to the nonautonomous damped wave equations. It is worth noting that the evolution family $(\mathcal{U}(t,s))_{t\ge s\ge 0}$ related to the nonautonomous damped wave equation has an exponential dichotomy. Precisely, in that case we will prove that $(\mathcal{U}(t,s))_{t\ge s\ge 0}$ is conditionally (X, Y, φ)-stable for certain Banach spaces X, Y, and $\varphi(t) = Me^{-\nu t}$, $t \ge 0$, $\nu > 0$, which is an exponential decaying function. To that purpose, suppose $\mathcal{A}$ is a selfadjoint, positive definite operator with compact resolvent in a Hilbert space H and $r\colon D(\mathcal{A}^{1/2}) \to H$ is of class C^1 with $r(0) = 0$, $r'(0) = 0$. We consider the following abstract damped wave equation

$$\begin{cases} \ddot{u} + a(t)(\alpha \dot{u} + \mathcal{A}u + \omega u) = a(t)r(u) + f(t), & t > 0, \\ u(0) = u_0, \quad \dot{u}(0) = u_1, & u_0, u_1 \in H, \end{cases} \tag{3.14}$$

where $\alpha > 0$, $\omega \in \mathbb{R}$ are constants; the function $a(\cdot) \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ is T -periodic and satisfies the condition $0 < \gamma_0 \le a(t) \le \gamma_1$ for fixed constants γ_0, γ_1 and for a.e. $t \ge 0$. Here, $f \in C_b(\mathbb{R}_+, H)$ is the external force.

To transform this equation to the first-order problem, we set $v = \dot{u}$ and handle with the variable $U = \binom{u}{v}$ which belongs to the space $Y = D(\mathcal{A}^{1/2}) \times H$. Then, we obtain the following equations

$$\partial_t U = A(t)U + g(U)(t), \ \ t > 0, \quad U(0) = \begin{pmatrix} u_0 \\ u_1 \end{pmatrix} := U_0 \in Y, \tag{3.15}$$

where the operator matrices $A(t)$ are defined for $t > 0$ as $A(t) = \begin{pmatrix} 0 & I \\ -a(t)(\mathcal{A}+\omega) & -\alpha a(t) \end{pmatrix}$ with the same domains $D(A(t)) = D(\mathcal{A}) \times H$ for all $t > 0$, and $g(U)(t) = \begin{pmatrix} 0 \\ a(t)r(u)+f(t) \end{pmatrix}$.

Equation (3.15) is a special case of (2.9) with $B = \mathrm{Id}$ which is the identity operator on Y. Hence, in this case we can choose $Z = Y$. Correspondingly, a mild solution to (3.15) is defined as the function u satisfying the following equation

$$U(t) = \mathcal{U}(t,0)U_0 + \int_0^t \mathcal{U}(t,s)g(U)(s)\,ds \quad \text{for all } t \geq 0. \tag{3.16}$$

Furthermore, the linearized equation of (3.15) is represented as

$$U' - A(t)U = f(t), \quad U(0) = U_0. \tag{3.17}$$

Respectively, a mild solution of (3.17) is understood as a function $U\colon \mathbb{R}_+ \to Y$ satisfying

$$U(t) = \mathcal{U}(t,0)U_0 + \int_0^t \mathcal{U}(t,s)f(s)\,ds. \tag{3.18}$$

It was proved in [11, p. 4724] that the operator $\mathbb{A} := \begin{pmatrix} 0 & I \\ -\mathcal{A}-\omega & -\alpha \end{pmatrix}$ generates a hyperbolic C_0 semigroup $(\mathcal{T}(t))_{t\geq 0}$ if $-\omega \notin \sigma(A)$. Then, we can decompose $A(t) = \begin{pmatrix} I & 0 \\ 0 & a(t) \end{pmatrix} \mathbb{A}$, and similarly as in [27], we have that the family of operators $(A(t))$ generates an evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ with $\mathcal{U}(t,s) = e^{t-s}\mathcal{T}\big(\int_s^t a(\tau)\,d\tau\big)$ for all $t \geq s \geq 0$, which has an exponential dichotomy, that means that the evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ satisfies that there exist bounded linear projections $P(t)$, $t \geq 0$, on Y and positive constants N, ν such that

(a) $\mathcal{U}(t,s)P(s) = P(t)\mathcal{U}(t,s)$, $t \geq s \geq 0$,

(b) the restriction $\mathcal{U}(t,s)_|\colon \operatorname{Ker} P(s) \to \operatorname{Ker} P(t)$, $t \geq s \geq 0$, is an isomorphism and its inverse is denoted by $(\mathcal{U}(t,s)_|)^{-1} = U(s,t)_|$ for $t \geq s \geq 0$,

(c) $\|\mathcal{U}(t,s)x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in P(s)Y$, $t \geq s \geq 0$,

(d) $\|\mathcal{U}(s,t)_|x\| \leq Ne^{-\nu(t-s)}\|x\|$ for $x \in \operatorname{Ker} P(t)$, $t \geq s \geq 0$.

The projections $P(t)$ and constants N, ν are called the *dichotomy projections*, and *dichotomy constants*, respectively. We refer to [27] for detail discussions on exponentially dichotomic evolution family related to nonautonomous damped wave equations. The reader is also referred to [13] for characterizations of exponential dichotomies of evolution families in general admissible spaces.

Moreover, the space $P(0)Y$ can be characterized by $P(0)Y = \{x \in Y : \sup_{t\geq 0} \|\mathcal{U}(t,0)x\| < \infty\}$ (see [13]).

To apply our results obtained in Section 2, we choose the topology τ to be the norm-topology. Concretely, we choose

$$\mathcal{F}_\tau(\mathbb{R}_+, Y) = C_b(\mathbb{R}_+, Y) := \left\{ v\colon \mathbb{R}_+ \to Y \;\middle|\; v \text{ is continuous and } \sup_{t\in\mathbb{R}_+} \|v(t)\|_{s,w} < \infty \right\}$$

endowed with the norm

$$\|v\|_{\infty,Y} := \sup_{t\in\mathbb{R}_+} \|v(t)\|_Y.$$

Also, the following remark is crucial for later use.

Remark 3.7. If the evolution family $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ on Banach space Y has an exponential dichotomy then it is obvious that $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ is of (X,Y,φ)-type with $X = P(0)Y$ and $\varphi(t) = Me^{-\nu t}$ for all $t \geq 0$, where $(P(t))_{t\geq 0}$ is the dichotomy projection, and M, ν are the dichotomy constants.

Moreover, since the operator r is of C^1 and $r(0) = r'(0) = 0$, it follows that r is locally Lipschitz with a small Lipschitz constant in a small neighborhood of 0. Therefore, the operator g satisfies condition in (2.10) with $Y = Z$, $g(0) = f$ and with the Lipschitz constant being small if the radius ρ is small. Thus, applying Theorem 2.6 we obtain the following results for the damped wave equation (3.15).

Theorem 3.8. *Let $\mathcal{A}$ be a selfadjoint, positive definite operator with compact resolvent in a Hilbert space H, $\alpha > 0$, and $\omega \in \mathbb{R}$ such that $-\omega \notin \sigma(\mathcal{A})$. Let the function $a(\cdot) \in L_{1,\mathrm{loc}}(\mathbb{R}_+)$ be T-periodic and satisfies the condition $0 < \gamma_0 \leq a(t) \leq \gamma_1$ for fixed positive constants γ_0, γ_1. Suppose $r\colon D(\mathcal{A}^{1/2}) \to H$ is of class C^1 with $r(0) = r'(0) = 0$. Let $f \in C_b(\mathbb{R}_+, H)$ be T-periodic. Then, if $\|f\|_{C_b(\mathbb{R}_+,H)}$ is small enough, the equation* (3.14) *has a unique T-periodic mild solution $\widehat{u}$ in a small neighborhood of* 0.

Proof. We first choose $X := P(0)Y$, $Z := Y$. Then, from Remark 3.7, we obtain that $(\mathcal{U}(t,s))_{t\geq s\geq 0}$ is of (X,Y,φ)-type with $\varphi(t) = Me^{-\nu t}$ for all $t \geq 0$.

For a given $f \in C_b(\mathbb{R}_+, Y)$, by [27, Lemma 3.3] we have that (3.18) has a bounded solution

$$u(t) = \int_0^\infty \mathcal{G}(t,\tau) f(\tau)\, d\tau,$$

where

$$\mathcal{G}(t,\tau) = \begin{cases} P(t)\mathcal{U}(t,\tau) & \text{for } t > \tau \geq 0, \\ -\mathcal{U}(t,\tau)_|(I - P(\tau)) & \text{for } 0 \leq t < \tau. \end{cases} \tag{3.19}$$

Moreover, using the fact that $\|\mathcal{G}(t,\tau)\| \le N(1+H)e^{-\nu|t-\tau|}$ for $t \ne \tau$ and $t,\tau \ge 0$, this solution can be estimated by

$$\|u\|_{C_b} \le (1+H)M\|f\|_{C_b} \int_0^\infty e^{-\nu|t-\tau|}\, d\tau \le \frac{2M(H+1)}{\nu}\|f\|_{C_b}.$$

Then, applying Theorem 2.5 we obtain that for T-periodic function $f \in C_b(\mathbb{R}_+, Y)$ there exists a unique T-periodic solution $\widehat{u}$ of (3.18) (i.e., a T-periodic mild solution of (2.1)) satisfying

$$\|\widehat{u}\|_{C_b} \le \widetilde{M}\|f\|_{C_b} \tag{3.20}$$

where $\widetilde{M} := \left(\frac{2M(H+1)}{\nu} + T\right) \sup_{0\le t\le T} \|\mathcal{U}(t,0)\|$. Altogether, the linear problem (3.18) has a unique T-periodic solution $\widehat{u}$ satisfying inequality (3.20) for each T-periodic input function f. Therefore, the assertion of the theorem then follows from Theorem 2.6. □

Lastly, we will prove the existence of a local stable manifold nearby the mild periodic solution to the damped wave equation that is nearby the periodic solution (3.16). As previously, we denote by $B_r(x)$ the ball in Y centered at x with radius r. We recall the definition of a local stable manifold for (3.16) nearby its periodic solution (as in [14]).

Definition 3.9. Let $\widehat{u}$ be a T-periodic solution to (3.16). A set $\mathbf{S} \subset \mathbb{R}_+ \times Y$ is said to be a *local stable manifold* for (3.16) nearby $\widehat{u}$ if and only if for every $t \in \mathbb{R}_+$ the space Y is decomposed through a direct sum $Y = Y_0(t) \oplus Y_1(t)$ such that

$$\inf_{t\in\mathbb{R}_+} Sn(Y_0(t), Y_1(t)) := \inf_{t\in\mathbb{R}_+} \inf_{i=0,1} \left\{\|x_0 + x_1\| : x_i \in Y_i(t), \|x_i\| = 1\right\} > 0$$

and there exist positive constants ρ, ρ_0, ρ_1 and a family of Lipschitz continuous maps

$$h_t \colon B_{\rho_0}(\widehat{u}(t)) \cap Y_0(t) \to B_{\rho_1}(\widehat{u}(t)) \cap Y_1(t), \quad t \in \mathbb{R}_+$$

with the Lipschitz constants independent of t such that

(i) $\mathbf{S} = \{(t, x + h_t(x)) \in \mathbb{R}_+ \times (Y_0(t) \oplus Y_1(t)) \mid t \in \mathbb{R}_+, x \in B_{\rho_0}(\widehat{u}(t)) \cap Y_0(t)\}$, and we denote by $\mathbf{S}_t := \{x + h_t(x) \mid (t, x + h_t(x)) \in \mathbf{S}\}$, $t \ge 0$,

(ii) $\mathbf{S}_t$ is homeomorphic to

$$B_{\rho_0}(\widehat{u}(t)) \cap Y_0(t) := \left\{x \in Y_0(t) : \|x - \widehat{u}(t)\| \le \rho_0\right\} \quad \text{for all } t \ge 0,$$

(iii) to each $x_0 \in S_{t_0}$ there exists a unique solution $u(t)$ to (3.16) on $[t_0, \infty)$ such that $u(t_0) = x_0$ and $\operatorname{ess\,sup}_{t\ge t_0} \|u(t)\| \le \rho$.

It is worth noting that, when identifying $Y_0(t) \oplus Y_1(t)$ with $Y_0(t) \times Y_1(t)$, we may write $S_t = \text{graph}(h_t)$ for $\text{graph}(h_t)$ being the graph of the map h_t.

We need the following lemma taken from [27], that gives the forms of bounded solutions to (3.16).

Lemma 3.10. *Let the evolution family $(\mathcal{U}(t,s))_{t\ge s\ge 0}$ have an exponential dichotomy with the corresponding dichotomy projections $(P(t))_{t\ge 0}$ and dichotomy constants $M, \nu > 0$. Let $\mathcal{G}(t,\tau)$ be the Green's function defined as in* (3.19)*. Let $g\colon C_b(\mathbb{R}_+, Y) \to C_b(\mathbb{R}_+, Y)$ satisfy conditions in* (2.10)*. Then, if $U \in C_b(\mathbb{R}_+, Y)$ is the solution to* (3.16) *such that $\sup_{t\ge 0} \|U(t)\|_Y \le \rho$ for a fixed $\rho > 0$, then for $t \ge 0$ this function U can be rewritten in the form*

$$U(t) = \mathcal{U}(t,0)V_0 + \int_0^\infty \mathcal{G}(t,\tau)g(U)(\tau)\,d\tau \quad \text{for some } V_0 \in P(0)Y. \tag{3.21}$$

Proof. See [27, Lemma 3.3]. □

We now state and prove our last result on the existence of an stable manifold for solutions to (3.16) nearby its T-periodic solution.

Theorem 3.11. *Let the assumptions of Theorem* 3.8 *be satisfied with given sufficiently small f. Let $\widehat{u}$ be the T-periodic solution of* (3.16) *obtained in Theorem* 3.8*. Then, there exists a local stable manifold $\mathbf{S}$ near the solution $\widehat{u}$. Moreover, every solution $u(t)$ on the manifold $\mathbf{S}$ is exponentially attracted to $\widehat{u}(t)$ in the sense that, there exist positive constants μ and C_μ independent of $t_0 \ge 0$ such that*

$$\|u(t) - \widehat{u}(t)\| \le C_\mu e^{-\mu(t-t_0)}\|P(t_0)u(t_0) - P(t_0)\widehat{u}(t_0)\| \tag{3.22}$$

for all $t \ge t_0$.

Proof. We will apply our result obtained in [26, Theorem 3.8]. To this purpose, let u be a solution to (3.16) and put $w = u - \widehat{u}$. Then, u satisfies (3.16) if and only if w satisfies

$$w(t) = \mathcal{U}(t,0)w(0) + \int_0^t \mathcal{U}(t,\tau)\big[g(\tau, w(\tau) + \widehat{u}(\tau)) - g(\tau, \widehat{u}(\tau))\big]\,d\tau \quad \text{for } t \ge 0. \tag{3.23}$$

Putting now $F(t,w) = g(t, w + \widehat{u}) - g(t, \widehat{u})$ we obtain that $F(t,0) = 0$, and since g satisfies (2.10), F satisfies the hypotheses in [26, Theorem 3.8]. Therefore, by [26, Theorem 3.8] we have that there exists a local stable manifold $\mathbf{S}$ (in neighborhood of 0) for (3.23). Returning to the solution u of (3.16) by replacing w by $u - \widehat{u}$, we obtain that, this manifold $\mathbf{S}$ is the local stable manifold for (3.16) nearby the solution $\widehat{u}$. Finally, we prove the inequality (3.22). To this end, since both $\widehat{u}$ and u are bounded on $\mathbb{R}_+$, we can use the formula (3.21) (with 0 being replaced by t_0) to write

$$u(t) - \widehat{u}(t) = \mathcal{U}(t,t_0)(P(t_0)u(t_0) - P(t_0)\widehat{u}(t_0)) + \int_{t_0}^\infty \mathcal{G}(t-\tau)(g(u)(\tau) - g(\widehat{u})(\tau))\,d\tau.$$

Therefore,

$$\begin{aligned}\|u(t) - \widehat{u}(t)\| &\leq Me^{-\nu t}\|P(t_0)u(t_0) - P(t_0)\widehat{u}(t_0)\| \\ &\quad + (1+H)M\int_{t_0}^{\infty} e^{-\nu|t-\tau|}\|g(u)(\tau) - g(\widehat{u})(\tau)\|\,d\tau \\ &\leq Me^{-\nu t}\|P(t_0)u(t_0) - P(t_0)\widehat{u}(t_0)\| \\ &\quad + (1+H)ML_1\int_{t_0}^{\infty} e^{-\nu|t-\tau|}\|u(\tau) - \widehat{u}(\tau)\|\,d\tau.\end{aligned}$$

Using now inequality of Gronwall-type [5, Corollary III.2.3] we obtain for $\beta := (1 + H)ML_1 < \nu/2$ and $t \geq t_0$ that

$$\|u(t) - \widehat{u}(t)\| \leq Ce^{-\mu t}\|P(t_0)u(t_0) - P(t_0)\widehat{u}(t_0)\|$$

for $\mu := \sqrt{\nu^2 - 2\nu\beta}$ and $C := \frac{2M\nu}{\nu + \sqrt{\nu^2 - 2\nu\beta}}$. The proof is complete. □

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Ngoc Huy Nguyen
Department of Mathematics, Thuyloi University, 175 Tay Son, Dong Da, Hanoi, Vietnam
and
Faculty of Mathematics and Informatics, Hanoi University of Science and Technology, Khoa Toan - Tin, Dai hoc Bach khoa Hanoi, 1 Dai Co Viet, Hai Ba Trung, Hanoi, Vietnam
E-mail address: huynn@tlu.edu.vn

Thieu Huy Nguyen and Thi Ngoc Ha Vu
Faculty of Mathematics and Informatics, Hanoi University of Science and Technology, Khoa Toan - Tin, Dai hoc Bach khoa Hanoi, 1 Dai Co Viet, Hai Ba Trung, Hanoi, Vietnam
E-mail addresses: huy.nguyenthieu@hust.edu.vn, ha.vuthingoc@hust.edu.vn