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Existence of Weak Solutions for a Class of (p,q)-biharmonic Equations with Critical Exponent and Discontinuous Nonlinearity

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Abstract. We are concerned with a class of (p,q)-Laplace type biharmonic Kirchhoff equations

$$\begin{cases} M\left(\int_\Omega \mathcal{A}(|\Delta u|^p)\,dx\right)\Delta(a(|\Delta u|^p)|\Delta u|^{p-2}\Delta u) = \lambda f(u) + |u|^{q_2^*-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases}$$

where Ω is a bounded open set in $\mathbb{R}^N$ with smooth boundary, λ is a positive real parameter, $2 \le p < q < q_2^*$, $q_2^* = \frac{Nq}{N-2q}$ is the critical exponent, $N > 2q$ and $\mathcal{A}(t) = \int_0^t a(s)\,ds$ for $t \in \mathbb{R}^+$. Here, $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a Kirchhoff function, $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous function satisfying some properties and $f\colon \mathbb{R} \to \mathbb{R}$ is a function which can have an uncountable set of discontinuity points. In this article, we study the existence of a positive weak solution for the problem above involving critical growth and a discontinuous nonlinearity via mountain pass theorem.

1. Introduction

The study of nonlinear differential equations involving double phase operators has been paid to a great deal of attention in the recent decades; see [6, 10, 14–16, 23, 27, 28]. Such operators can be corroborated as a model for many physical phenomena which arise in the research of elasticity, strongly anisotropic materials and Lavrentiev's phenomenon; see [29–32] for more details. In particular, Zhikov investigated the behavior of strongly anisotropic materials and found that their hardening properties varied sharply with the point. This phenomenon is described the following functional

$$\int_\Omega (|\nabla u|^p + \upsilon(x)|\nabla u|^q)\,dx, \tag{1.1}$$

where the function $\upsilon(\cdot)$ was used as an aid to regulating the mixture between two different materials. The functional (1.1) belongs to the class of the integral functionals with

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nonstandard growth conditions; see also [17, 20–22] for (p,q) elliptic problems involving critical growth.

On the other hands, for the problems involving p-biharmonic operators, Kratochvíl and Nečas [25] considered fourth-order differential equations which arise in the study of beam deflection problems on the nonlinear elastic foundation; see also [1, 19, 24] and the references therein. In particular, understanding the fourth-order differential equations is significantly important in physics or other science and engineering fields.

In the present paper, motivated by Zhikov and Kratochvíl–Nečas' works above, we are concerned with a class of (p,q)-quasilinear equations involving p-biharmonic operators when f has an uncountable set of discontinuity points:

$$\begin{cases} M\left(\int_\Omega \mathcal{A}(|\Delta u|^p)\,dx\right)\Delta(a(|\Delta u|^p)|\Delta u|^{p-2}\Delta u) = \lambda f(u) + |u|^{q_2^*-2}u & \text{in } \Omega, \\ u = \Delta u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $\lambda > 0$, $2 \le p < q < q_2^*$, $N > 2q$, $f\colon \mathbb{R} \to \mathbb{R}$ is a function that can have an uncountable set of discontinuity points and $a\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a function of C^1 class.

We are going to explore the above problem (1.2). For this, let us introduce the critical exponent q_2^* defined by

$$q_2^* = \begin{cases} \frac{Nq}{N-2q} & \text{if } N > 2q, \\ \infty & \text{if } N \le 2q. \end{cases}$$

Assume that the Kirchhoff function $M\colon \mathbb{R}^+ \to \mathbb{R}^+$ satisfies the following condition:

(M) $M \in C(\mathbb{R}^+, \mathbb{R}^+)$ is increasing and satisfies $\inf_{t\in\mathbb{R}^+} M(t) \ge m_0 > 0$, where m_0 is a constant.

A typical example for M is given by $M(t) = b_0 + b_1 t^n$ with $n > 0$, $b_0 > 0$ and $b_1 \ge 0$.

Next, we suppose that functions a and f satisfy the following conditions.

(A1) The function $a \in C^1(\mathbb{R}^+, \mathbb{R}^+)$ and there exist constants $a_0, a_1 > 0$ such that

$$a_0\big(1 + t^{\frac{q-p}{p}}\big) \le a(t) \le a_1\big(1 + t^{\frac{q-p}{p}}\big) \quad \text{for all } t > 0.$$

(A2) There exists a constant $\alpha \in (0,1]$ such that

$$\mathcal{A}(t) \ge \alpha a(t) t \quad \text{for all } t \ge 0,$$

where $\mathcal{A}(t) := \int_0^t a(s)\,ds$.

(f1) There exist a constant $b > 0$ and r with $q < r < q_2^*$ such that

$$|f(z)| \le b(1 + |z|^{r-1}) \quad \text{for all } z \in \mathbb{R}.$$

(f2) There exists $\mu \in \left(\frac{p}{\alpha}, q_2^*\right)$ such that

$$0 \le \mu F(z) \le z\underline{f}(z) \quad \text{for all } z \in \mathbb{R},$$

where

$$\underline{f}(z) := \lim_{\delta\to 0^+} \operatorname*{ess\,inf}_{|\xi - z|<\delta} f(\xi) \quad \text{and} \quad \overline{f}(z) := \lim_{\delta\to 0^+} \operatorname*{ess\,sup}_{|\xi - z|<\delta} f(\xi)$$

which are N-measurable and $F(z) := \int_0^z f(s)\,ds$.

(f3) There is $\beta > 0$ (to be specified later) such that

$$H(z-\beta) \le f(z) \quad \text{for all } z \in \mathbb{R},$$

where H is the Heaviside function, i.e.,

$$H(s) = \begin{cases} 0 & \text{if } s \le 0, \\ 1 & \text{if } s > 0. \end{cases}$$

(f4) $f(z) = 0$ if $z \le 0$ and

$$\limsup_{z\to 0^+} \frac{f(z)}{z^{q-1}} = 0.$$

The problems involving discontinuous nonlinearities appear in various physical situations such as electrical phenomena, plasma physics, etc. For the readers interested in these problems, we refer to [2–5] and the references therein.

As mentioned in [21], a typical example of a function satisfying the conditions (f1)–(f4) is as follows. Note that the function f in this example has an uncountable set of discontinuity points:

$$f(z) = \begin{cases} 0 & \text{if } z \in (-\infty, \beta/2), \\ 1 & \text{if } z \in \mathbb{Q} \cap [\beta/2, \beta], \\ 0 & \text{if } z \in (\mathbb{R}\setminus\mathbb{Q}) \cap [0,\beta], \\ \sum_{k=1}^{\ell} \frac{|z|^{q_k-1}}{\beta^{q_k-1}} & \text{if } z > \beta,\ \ell \ge 1 \text{ and } q_k \in (q, q_2^*). \end{cases}$$

Definition 1.1. We say that $u \in W_0^{2,q}(\Omega)$ with $u \ge 0$ is a weak solution of the problem (1.2) if

$$M\left(\int_\Omega \mathcal{A}(|\Delta u|^p)\,dx\right)\int_\Omega a(|\Delta u|^p)|\Delta u|^{p-2}\Delta u \cdot \Delta v\,dx = \lambda\int_\Omega \rho v\,dx + \int_\Omega |u|^{q_2^*-2}uv\,dx$$

for any $v \in W_0^{2,q}(\Omega)$ and

$$\rho(x) \in \left[\underline{f}(u(x)), \overline{f}(u(x))\right] \quad \text{a.e. in } \Omega.$$

In this regard, our aim is to show that (1.2) admits at least one positive weak solution to a class of (p,q)-biharmonic Kirchhoff equations with the critical exponent and a discontinuous nonlinearity. Our result extends Figueiredo and Nascimento's result [21] for a class of (p,q)-Laplace equations to a class of (p,q)-biharmonic equations. To overcome the lack of compactness in the study of p-biharmonic equations with the critical exponent, we adapt the concentration-compactness principle for the Sobolev space introduced by Chung and Ho [12]. Moreover, using a truncation method, we deal with the Kirchhoff function related to a class of p-biharmonic operator; see also [7, 12]. As far as we know, this is the first attempt for (p,q)-biharmonic operators. It is remarkable that we obtain the existence result for a class of (p,q)-biharmonic equations involving a discontinuous superlinear term provided that λ is suitable.

The main result of this paper is as follows:

Theorem 1.2. *Assume that* (M)*,* (A1)–(A2) *and* (f1)–(f4) *hold. Then the following holds:*

(1) *There exists $\Lambda^* > 0$ such that the problem* (1.2) *admits a positive weak solution $u_\lambda \in W_0^{2,q}(\Omega)$ for all $\lambda \geq \Lambda^*$.*

(2) *There exists $\beta^* > 0$ such that $\{x \in \Omega : u_\lambda(x) > \beta^*\}$ has positive measure for all $\lambda \geq \Lambda^*$.*

2. Preliminaries

In this section, we briefly introduce some definitions and basic results on the critical point theory for locally Lipschitz continuous functionals; see [11, 13].

Let $(X, \|\cdot\|_X)$ be a real reflexive Banach space. We denote the dual space of X by X^*, while $\langle\,\cdot\,,\cdot\,\rangle$ stands for the duality pairing between X^* and X. A functional $J\colon X \to \mathbb{R}$ is called locally Lipschitz when, for every $u \in X$, there corresponds a neighborhood U of u and a constant $L \geq 0$ such that

$$|J(v_1) - J(v_2)| \leq L\|v_1 - v_2\|_X \quad \text{for all } v_1, v_2 \in U.$$

If $u, v \in X$, the symbol $J^0(u;v)$ indicates the generalized directional derivative of J at a point u along direction v, namely

$$J^0(u;v) := \limsup_{h\to 0,\, t\to 0^+} \frac{J(u+h+tv) - J(u+h)}{t}.$$

The generalized gradient of J at $u \in X$ denoted by $\partial J(u)$, is defined as being the subset of X^* such that

$$\partial J(u) = \big\{u^* \in X^* : \langle u^*, v\rangle \leq J^0(u;v) \text{ for all } v \in X\big\}.$$

Since $J^0(u;0) = 0$, $\partial J(u)$ is the subdifferential of $J^0(u;0)$. The subset $\partial J(u) \subset X^*$ is nonempty, convex and weak*-compact. Moreover, $\partial J(u) = \{J'(u)\}$ if $J \in C^1(X,\mathbb{R})$.

A critical point of J is an element $u_0 \in X$ such that $0 \in \partial J(u_0)$ and a critical value of J is a real number c such that $J(u_0) = c$ for some critical point $u_0 \in X$.

A sequence $\{u_n\} \subset X$ is said to be a Palais–Smale sequence for J ($(\mathrm{PS})_c$-sequence for short), if for $c \in \mathbb{R}$,

$$J(u_n) \to c \quad \text{and} \quad \omega^*(u_n) \to 0 \quad \text{as } n \to \infty,$$

where $\omega^*(u) = \min\{\|u^*\|_{X^*} : u^* \in \partial J(u)\}$. A functional J satisfies the $(\mathrm{PS})_c$-condition if any Palais–Smale sequence at level c has a convergent subsequence.

Lemma 2.1. [21, Theorem 2.1] *Let J be a locally Lipschitz functional with $J(0) = 0$ satisfying*

(1) *there exist two constants ζ, $R > 0$ such that $J(u) \geq \zeta$ with $\|u\|_X = R$ for $u \in X$;*

(2) *there exists $e \in X \setminus \{0\}$ with $\|u\|_X \geq R$ such that $J(e) < 0$.*

If

$$c = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} J(\gamma(t))$$

with

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } J(\gamma(1)) \leq 0\}$$

and J satisfies the $(PS)_c$-condition, then $c \geq \zeta$ is a critical point of J such that there is $u \in X$ verifying

$$J(u) = c \quad \text{and} \quad 0 \in \partial J(u).$$

Lemma 2.2 (Riesz representation theorem)**.** [21, Proposition 2.2] [8] *Let $\mathcal{B}$ be a bounded linear functional on $L^r(\Omega)$ for $1 < r < \infty$. Then, there is a unique function $u \in L^{r'}(\Omega)$, $r' = \frac{r}{r-1}$ such that*

$$\langle \mathcal{B}, v\rangle = \int_\Omega uv\, dx \quad \text{for all } v \in L^r(\Omega).$$

Moreover,

$$\|u\|_{L^{r'}(\Omega)} = \|\mathcal{B}\|_{(L^r(\Omega))^*}.$$

The Sobolev space $W^{2,q}(\Omega)$ is defined by

$$W^{2,q}(\Omega) := \big\{u \in L^q(\Omega) : |D^i u| \in L^q(\Omega) \text{ for all } i \text{ with } |i| \leq 2\big\}$$

endowed with the standard norm $\|\cdot\|_{W^{2,q}(\Omega)}$. Let the space $W_0^{2,q}(\Omega)$ be the completion of $C_0^\infty(\Omega)$. By the Poincaré inequality, we endow the space $W_0^{2,q}(\Omega)$ with equivalent norm given by

$$\|u\| = \left(\int_\Omega |\Delta u|^q\, dx\right)^{1/q}.$$

Moreover, $(W_0^{2,q}(\Omega), \|\cdot\|)$ is a reflexive Banach space; see [18, Theorem 8.1.13].

Lemma 2.3. [12, Proposition 3.4] *Let $k \in \mathbb{N}$ be such that $km < N$. Let h satisfy*

$$1 \le h \le m_k^* = \frac{Nm}{N - km}.$$

Then we have continuous embedding

$$W^{k,m}(\Omega) \hookrightarrow L^h(\Omega).$$

If, in addition, $h < m_k^*$, then the above embedding is compact.

Remark 2.4. If $2q < N$ and $1 \le h \le q_2^*$, the embedding

$$W_0^{2,q}(\Omega) \hookrightarrow L^h(\Omega)$$

is continuous, that is, there exists $\mathcal{S}_h = \mathcal{S}_h(N, q, \Omega) > 0$ such that

$$\|u\|_{L^h(\Omega)} \le \mathcal{S}_h \|u\| \quad \text{for all } u \in W_0^{2,q}(\Omega).$$

Throughout this paper, we denote by $X := W_0^{2,q}(\Omega)$. Let $X^* := W_0^{-2,q}(\Omega)$ denote the dual space of X and let $\|\cdot\|_{X^*}$ be its norm. Subsequently, C denotes a universal positive constant.

3. Existence of nontrivial weak solutions

Let us define the functional $\Phi \colon X \to \mathbb{R}$ by

$$\Phi(u) = \frac{1}{p}\mathcal{M}\left(\int_\Omega \mathcal{A}(|\Delta u|^p)\,dx\right) - \frac{1}{q_2^*}\int_\Omega |u|^{q_2^*}\,dx.$$

It is obvious that the functional Φ is well defined on X, $\Phi \in C^1(X, \mathbb{R})$ and its Fréchet derivative is given by

$$\langle \Phi'(u), v\rangle = M\left(\int_\Omega \mathcal{A}(|\Delta u|^p)\,dx\right)\int_\Omega a(|\Delta u|^p)|\Delta u|^{p-2}\Delta u \cdot \Delta v\,dx - \int_\Omega |u|^{q_2^*-2}uv\,dx$$

for any $u, v \in X$.

Next we define the functional $\Psi \colon X \to \mathbb{R}$ by

$$\Psi(u) = \int_\Omega F(u)\,dx.$$

Then Ψ is locally Lipschitz continuous on X and $\partial\Psi(u) \subset X^*$. Moreover, if $\rho \in \partial\Psi(u)$, it satisfies

$$\rho(x) \in \big[\underline{f}(u(x)), \overline{f}(u(x))\big] \quad \text{a.e. in } \Omega. \tag{3.1}$$

Also we define the functional $J_\lambda\colon X \to \mathbb{R}$ related to the problem (1.2)

$$J_\lambda(u) = \Phi(u) - \lambda\Psi(u).$$

Then J_λ is locally Lipschitz continuous and

$$\partial J_\lambda(u) = \{\Phi'(u)\} - \lambda\partial\Psi(u) \quad \text{for any } u \in X.$$

In order to prove Theorem 1.2, we apply a truncation technique used in [7, 12], as follows: Fix $t_0 > 0$ to be specified later and a truncation of $M(t)$ defined by

$$M_0(t) := \begin{cases} M(t) & \text{for } 0 \le t \le t_0, \\ M(t_0) & \text{for } t > t_0. \end{cases} \tag{3.2}$$

It is clear that $M_0 \in C(\mathbb{R}^+, \mathbb{R}^+)$,

$$m_0 \le M_0(t) \le M(t_0) \quad \text{for all } t \in \mathbb{R}_0^+ \tag{3.3}$$

and

$$m_0 t \le \mathcal{M}_0(t) \le M(t_0)t \quad \text{for all } t \in \mathbb{R}_0^+ \tag{3.4}$$

due to (M), where $\mathcal{M}_0(t) := \int_0^t M_0(s)\,ds$ for $t \in \mathbb{R}_0^+$. Then we define $\Phi_0\colon X \to \mathbb{R}$ by

$$\Phi_0(u) = \frac{1}{p}\mathcal{M}_0\left(\int_\Omega \mathcal{A}(|\Delta u|^p)\,dx\right) - \frac{1}{q_2^*}\int_\Omega |u|^{q_2^*}\,dx.$$

It is obvious that the functional Φ_0 is well defined on X, $\Phi_0 \in C^1(X, \mathbb{R})$ and its Fréchet derivative is given by

$$\langle \Phi_0'(u), v\rangle = M_0\left(\int_\Omega \mathcal{A}(|\Delta u|^p)\,dx\right)\int_\Omega a(|\Delta u|^p)|\Delta u|^{p-2}\Delta u \cdot \Delta v\,dx - \int_\Omega |u|^{q_2^*-2}uv\,dx$$

for any $u, v \in X$. Then the functional $\widetilde{J}_\lambda\colon X \to \mathbb{R}$ is given by

$$\widetilde{J}_\lambda(u) = \frac{1}{p}\mathcal{M}_0\left(\int_\Omega \mathcal{A}(|\Delta u|^p)\,dx\right) - \frac{1}{q_2^*}\int_\Omega |u|^{q_2^*}\,dx - \lambda\int_\Omega F(u)\,dx.$$

The modified functional $\widetilde{J}_\lambda$ is also locally Lipschitz continuous and

$$\partial\widetilde{J}_\lambda(u) = \{\Phi_0'(u)\} - \lambda\partial\Psi(u) \quad \text{for any } u \in X.$$

The following result is to show that the modified energy functional $\widetilde{J}_\lambda$ satisfies the $(\mathrm{PS})_c$-condition. With the aid of the Concentration-Compactness Principle; see [12], we prove that the functional $\widetilde{J}_\lambda$ satisfies the Palais–Smale condition. This plays a key role in obtaining the existence of a nontrivial weak solution for the given problem.

Now, we define the truncation $M_0(t)$ of $M(t)$ given in (3.2) and the truncated energy functional $\widetilde{J}_\lambda$ by fixing $t_0 \in (0,1)$ such that

$$m_0 < M(t_0) < \frac{m_0 \alpha \mu}{p}, \tag{3.5}$$

where α is given in the assumption (A2) and μ is given in (f2).

Lemma 3.1. *Assume that* (M), (A1)–(A2), (f1)–(f2) *and* (f4) *hold. Then the functional* $\widetilde{J}_\lambda$ *satisfies the (PS)$_c$-condition for*

$$c < \left(\frac{1}{\mu} - \frac{1}{q_2^*}\right)(m_0 a_0 S)^{N/q}, \tag{3.6}$$

where $S := \inf_{\phi \in X\setminus\{0\}} \frac{\|\phi\|}{\|\phi\|_{L^q(\Omega)}} > 0$.

Proof. Given $c \in \mathbb{R}$, let $\{u_n\} \subset X$ be a (PS)$_c$-sequence of the functional $\widetilde{J}_\lambda$, that is,

$$\widetilde{J}_\lambda(u_n) \to c \quad \text{and} \quad \omega^*(u_n) \to 0 \quad \text{as } n \to \infty,$$

which shows that

$$c = \widetilde{J}_\lambda(u_n) + o_n(1) \quad \text{and} \quad \omega^*(u_n) = o_n(1),$$

where $o_n(1) \to 0$ as $n \to \infty$. Then there exists a sequence $\{w_n\} \subset \partial \widetilde{J}_\lambda(u_n)$ such that

$$\|w_n\|_{X^*} = \omega^*(u_n) = o_n(1) \quad \text{and} \quad w_n = \Phi'(u_n) - \lambda \rho_n,$$

where $\rho_n \in \partial\Psi(u_n)$.

First we claim that the sequence $\{u_n\}$ in X is bounded. According to the assumptions (M), (A2) and (f2), we have for all n large enough,

$$\begin{aligned}
&c + 1 + \|u_n\| \\
&\geq \widetilde{J}_\lambda(u_n) - \frac{1}{\mu}\langle w_n, u_n\rangle + o_n(1) \\
&= \frac{1}{p}\mathcal{M}_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right) - \lambda \int_\Omega F(u_n)\,dx - \frac{1}{q_2^*}\int_\Omega |u_n|^{q_2^*}\,dx \\
&\quad - \frac{1}{\mu} M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\,dx + \frac{\lambda}{\mu}\int_\Omega \rho_n u_n\,dx + \frac{1}{\mu}\int_\Omega |u_n|^{q_2^*}\,dx \\
&\geq \frac{m_0}{p}\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx - \frac{1}{\mu}M(t_0)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\,dx \\
&\quad + \lambda \int_\Omega \left(\frac{1}{\mu}\rho_n u_n - F(u_n)\right)dx + \left(\frac{1}{\mu} - \frac{1}{q_2^*}\right)\int_\Omega |u_n|^{q_2^*}\,dx \\
&\geq \left(\frac{m_0\alpha}{p} - \frac{M(t_0)}{\mu}\right)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\,dx + \lambda\int_\Omega \left(\frac{1}{\mu}\rho_n u_n - F(u_n)\right)dx.
\end{aligned} \tag{3.7}$$

The assumption (f2) implies that

$$\frac{1}{\mu}\rho_n(x)u_n(x) \geq \frac{1}{\mu}\underline{f}(u_n(x))u_n(x) \geq F(u_n(x)) \quad \text{a.e. in } \Omega. \tag{3.8}$$

Combining this with (3.7) again, we have

$$c + 1 + \|u_n\| \geq \left(\frac{m_0\alpha}{p} - \frac{M(t_0)}{\mu}\right)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\,dx.$$

From (M), (A1) and (3.2), it follows that

$$c + 1 + \|u_n\| \geq a_0\left(\frac{m_0\alpha}{p} - \frac{M(t_0)}{\mu}\right)\int_\Omega |\Delta u_n|^q\,dx = a_0\left(\frac{m_0\alpha}{p} - \frac{M(t_0)}{\mu}\right)\|u_n\|^q.$$

Note that if $\{u_n\}$ is unbounded in X, then we derive a contradiction because $M(t_0) < m_0\alpha\mu/p$ due to (3.5). Therefore, we conclude that $\{u_n\}$ is bounded in X. Then there exists $C > 0$ such that

$$\sup\int_\Omega |\Delta u_n|^p\,dx \leq C \quad \text{and} \quad \sup\int_\Omega |\Delta u_n|^q\,dx \leq C. \tag{3.9}$$

Also, there exists a subsequence of $\{u_n\}$, denoted again by $\{u_n\}$, such that $u_n \rightharpoonup u$ in X as $n \to \infty$. By Lemma 2.3 and the compact embedding, we have

$$u_n(x) \to u(x) \text{ a.e. in } \Omega, \quad u_n \to u \text{ in } L^s(\Omega) \quad \text{and} \quad |u_n(x)| \leq v(x) \tag{3.10}$$

for some $1 \leq s < q_2^*$ and $v \in L^s(\Omega)$ as $n \to \infty$.

Moreover, using the Concentration-Compactness Principle due to Chung and Ho [12] (see also Lions [26]), we obtain an at most countable index set Λ and sequences $\{\mu_j\}, \{\nu_j\} \subset (0, \infty)$ such that

$$|\Delta u_n|^q \to \mu \quad \text{and} \quad |u_n|^{q_2^*} \to \nu$$

in a weak*-sense of measure as $n \to \infty$. Then the limit measures are of the form

$$\mu \geq |\Delta u|^q + \sum_{j\in\Lambda}\mu_j\delta_{x_j}, \quad \nu = |u|^{q_2^*} + \sum_{j\in\Lambda}\nu_j\delta_{x_j} \quad \text{and} \quad S\nu_j^{\frac{q}{q_2^*}} \leq \mu_j \tag{3.11}$$

for all $j \in \Lambda$ where δ_{x_j} is the Dirac mass at $x_j \in \Omega$.

Next, we will claim that $\Lambda = \emptyset$. Arguing by contradiction that $\Lambda \neq \emptyset$, we fix $j \in \Lambda$. Without loss of generality, we may suppose $B_2(0) \subset \Omega$. Considering $\psi \in C_0^\infty(\Omega)$ such that $\psi = 1$ in $B_1(0)$, $\psi \equiv 0$ in $\Omega \setminus B_2(0)$ and $|\Delta\psi|_\infty \leq 2$, we define $\psi_\varrho(x) := \psi((x - x_j)/\varrho)$, where $\varrho > 0$. Hence, $\{\psi_\varrho u_n\}$ is bounded in X and

$$\begin{aligned} o_n(1) = \langle w_n, \psi_\varrho u_n\rangle = M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)&\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^{p-2}\Delta u_n\Delta(\psi_\varrho u_n)\,dx \\ - \int_\Omega \rho_n\psi_\varrho u_n\,dx - &\int_\Omega |u_n|^{q_2^*}\psi_\varrho\,dx. \end{aligned}$$

Then we have for sufficiently large n that

$$
\begin{aligned}
&M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_\Omega \psi_\varrho a(|\Delta u_n|^p)|\Delta u_n|^{p-2}\Delta u_n\Delta u_n\,dx\\
&= -2M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^{p-2}\Delta u_n\nabla\psi_\varrho\cdot\nabla u_n\,dx\\
&\quad - M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_\Omega u_n a(|\Delta u_n|^p)|\Delta u_n|^{p-2}\Delta u_n\Delta\psi_\varrho\,dx\\
&\quad + \int_\Omega \rho_n\psi_\varrho u_n\,dx + \int_\Omega |u_n|^{q_2^*}\psi_\varrho\,dx.
\end{aligned}
\tag{3.12}
$$

Now, we will estimate terms in the right-hand side of (3.12). Since $\operatorname{supp}(\psi_\varrho)$ is compact and it is contained in $B_{2\varrho}(x_j)$, it follows from (A1) that

$$
\begin{aligned}
&\left|M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^{p-2}\Delta u_n\nabla\psi_\varrho\cdot\nabla u_n\,dx\right|\\
&\le M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_{B_{2\varrho}(0)} |a(|\Delta u_n|^p)||\Delta u_n|^{p-1}|\nabla u_n||\nabla\psi_\varrho|\,dx\\
&\le M(t_0)a_1\int_{B_{2\varrho}(0)} |\Delta u_n|^{p-1}|\nabla u_n||\nabla\psi_\varrho| + |\Delta u_n|^{q-1}|\nabla u_n||\nabla\psi_\varrho|\,dx.
\end{aligned}
$$

Then let $\delta > 0$ be arbitrary and fixed. By Young's inequality and (3.9), we observe that

$$
\begin{aligned}
\int_{B_{2\varrho}(0)} |\Delta u_n|^{q-1}|\nabla u_n||\nabla\psi_\varrho|\,dx &\le \delta\int_{B_{2\varrho}(0)} |\Delta u_n|^q\,dx + C(\delta)\int_{B_{2\varrho}(0)} |\nabla u_n|^q|\nabla\psi_\varrho|^q\,dx\\
&\le C\delta + C(\delta)\int_{B_{2\varrho}(0)} |\nabla u_n|^q|\nabla\psi_\varrho|^q\,dx
\end{aligned}
$$

and

$$
\int_{B_{2\varrho}(0)} |\Delta u_n|^{p-1}|\nabla u_n||\nabla\psi_\varrho|\,dx \le C\delta + C(\delta)\int_{B_{2\varrho}(0)} |\nabla u_n|^p|\nabla\psi_\varrho|^p\,dx,
$$

where $C(\delta)$ denotes a positive constant depending on δ but independent of n and ϱ. Combining this with (3.10) gives

$$
\limsup_{n\to\infty}\int_{B_{2\varrho}(0)} |\Delta u_n|^{q-1}|\nabla u_n||\nabla\psi_\varrho|\,dx \le C\delta + C(\delta)\int_{B_{2\varrho}(0)} |\nabla u|^q|\nabla\psi_\varrho|^q\,dx
$$

and

$$
\limsup_{n\to\infty}\int_{B_{2\varrho}(0)} |\Delta u_n|^{p-1}|\nabla u_n||\nabla\psi_\varrho|\,dx \le C\delta + C(\delta)\int_{B_{2\varrho}(0)} |\nabla u|^p|\nabla\psi_\varrho|^p\,dx.
$$

Note that $|\nabla u| \in L^{q_1^*}(\Omega)$ because $|\nabla u| \in W^{1,q}(\Omega)$ and $u \in L^{q_2^*}(\Omega)$, where q_1^* is given in

Lemma 2.3. By Hölder's inequality, we observe that

$$\begin{aligned}\int_{B_{2\varrho}(0)} |\nabla u \nabla \psi_\varrho|^q \, dx &\leq \||\nabla u|^q\|_{L^{\frac{q_1^*}{q}}(B_{2\varrho}(0))} \||\nabla \psi_\varrho|^q\|_{L^{\frac{N}{q}}(B_{2\varrho}(0))} \\ &= \||\nabla u|^q\|_{L^{\frac{q_1^*}{q}}(B_{2\varrho}(0))} \left(\int_{B_{2\varrho}(0)} |\nabla \psi_\varrho|^N \, dx\right)^{q/N} \\ &= \||\nabla u|^q\|_{L^{\frac{q_1^*}{q}}(B_{2\varrho}(0))} \left(\int_{B_2(0)} |\nabla \psi|^N \, dx\right)^{q/N}\end{aligned}$$

and

$$\int_{B_{2\varrho}(0)} |\nabla u \nabla \psi_\varrho|^p \, dx \leq \||\nabla u|^p\|_{L^{\frac{p_1^*}{p}}(B_{2\varrho}(0))} \left(\int_{B_2(0)} |\nabla \psi|^N \, dx\right)^{p/N}.$$

It follows that

$$\int_{B_{2\varrho}(0)} |\nabla u \nabla \psi_\varrho|^q \, dx \to 0 \quad \text{as } \varrho \to 0+$$

and

$$\int_{B_{2\varrho}(0)} |\nabla u \nabla \psi_\varrho|^p \, dx \to 0 \quad \text{as } \varrho \to 0+.$$

Thus, we derive

$$\begin{aligned}&\limsup_{\varrho \to 0+} \limsup_{n \to \infty} \left| M_0 \left(\int_\Omega \mathcal{A}(|\Delta u_n|^p) \, dx\right) \int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^{p-2} \Delta u_n \nabla \psi_\varrho \cdot \nabla u_n \, dx \right| \\ &\leq 2a_1 M(t_0) C\delta.\end{aligned}$$

Since $\delta > 0$ was taken arbitrarily, we arrive at

$$\limsup_{\varrho \to 0+} \limsup_{n \to \infty} M_0 \left(\int_\Omega \mathcal{A}(|\Delta u_n|^p) \, dx\right) \int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^{p-2} \Delta u_n \nabla \psi_\varrho \cdot \nabla u_n| \, dx = 0.$$

By a similar argument, we have

$$\begin{aligned}&\left| M_0 \left(\int_\Omega \mathcal{A}(|\Delta u_n|^p) \, dx\right) \int_\Omega u_n a(|\Delta u_n|^p)|\Delta u_n|^{p-2} \Delta u_n \Delta \psi_\varrho \, dx \right| \\ &\leq M_0(t_0) \int_{B_{2\varrho}(0)} a(|\Delta u_n|^p)|\Delta u_n|^{p-1} |u_n \Delta \psi_\varrho| \, dx \\ &\leq M_0(t_0) \int_{B_{2\varrho}(0)} a_1(|\Delta u_n|^{p-1} |u_n \Delta \psi_\varrho| + |\Delta u_n|^{q-1} |u_n \Delta \psi_\varrho|) \, dx \\ &\leq 2a_1 M(t_0) \left[C\delta + C(\delta) \left(\int_{B_{2\varrho}(0)} |u_n \Delta \psi_\varrho|^p \, dx + \int_{B_{2\varrho}(0)} |u_n \Delta \psi_\varrho|^q \, dx\right)\right].\end{aligned}$$

Note that

$$\begin{aligned}\int_{B_{2\varrho}(0)} |u_n \Delta\psi_\varrho|^p\,dx &\le \||u_n|^p\|_{L^{\frac{p_2^*}{p}}(B_{2\varrho}(0))} \||\Delta\psi_\varrho|^p\|_{L^{\frac{N}{2p}}(B_{2\varrho}(0))} \\ &= \||u_n|^p\|_{L^{\frac{p_2^*}{p}}(B_{2\varrho}(0))} \left(\int_{B_{2\varrho}(0)} |\Delta\psi_\varrho|^{\frac{N}{2}}\,dx\right)^{p/(2N)} \\ &= \||u_n|^p\|_{L^{\frac{p_2^*}{p}}(B_{2\varrho}(0))} \left(\int_{B_2(0)} |\Delta\psi|^{\frac{N}{2}}\,dx\right)^{2p/N}.\end{aligned}$$

As above, it follows from (3.10) that

$$\limsup_{\varrho\to 0+}\left[\limsup_{n\to\infty}\left|M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_\Omega u_n a(|\Delta u_n|^p)|\Delta u_n|^{p-2}\Delta u_n \Delta\psi_\varrho\,dx\right|\right] = 0.$$

Owing to (3.1) and (f1),

$$0 \le \rho_n(x) \le b(1+|u_n(x)|^{r-1}) \quad \text{a.e. in } \Omega.$$

This implies

$$\int_{B_{2\varrho}(0)} \rho_n\psi_\varrho u_n\,dx \le b\left(\int_{B_{2\varrho}(0)} \psi_\varrho|u_n|\,dx + \int_{B_{2\varrho}(0)} \psi_\varrho|u_n|^r\,dx\right)$$

and so we deduce

$$\lim_{\varrho\to 0+}\left(\lim_{n\to\infty}\int_\Omega \rho_n\psi_\varrho u_n\,dx\right) = 0.$$

Therefore,

$$M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\psi_\varrho\,dx = \int_\Omega |u_n|^{q_2^*}\psi_\varrho\,dx + o_n(1).$$

Using (M) and (A1), we obtain

$$m_0 a_0 \int_\Omega |\Delta u_n|^q\psi_\varrho\,dx \le m_0 a_0\int_\Omega (|\Delta u_n|^p + |\Delta u_n|^q)\psi_\varrho\,dx \le \int_\Omega |u_n|^{q_2^*}\psi_\varrho\,dx + o_n(1).$$

By taking the limit as $n\to\infty$, we have

$$m_0 a_0\int_\Omega \psi_\varrho\,d\mu \le \int_\Omega \psi_\varrho\,d\nu.$$

Letting $\varrho\to 0+$, we assert $m_0a_0\mu_j \le \nu_j$. From (3.11), we conclude that

$$\nu_j \ge (m_0 a_0 S)^{\frac{N}{2q}} \quad \text{for some } j\in\Lambda. \tag{3.13}$$

In view of (3.3), (3.8) and the assumption (A2), we have

$$
\begin{aligned}
c+o_n(1) &= \widetilde{J}_\lambda(u_n) - \frac{1}{\mu}\langle w_n, u_n\rangle \\
&\geq \frac{1}{p}\mathcal{M}_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right) - \frac{1}{\mu}M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\,dx \\
&\quad + \lambda\int_\Omega\left(\frac{1}{\mu}\rho_n u_n - F(u_n)\right)\,dx + \left(\frac{1}{\mu}-\frac{1}{q_2^*}\right)\int_\Omega |u_n|^{q_2^*}\,dx \\
&\geq \frac{1}{p}M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx \\
&\quad - \frac{1}{\mu}M_0\left(\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx\right)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\,dx + \left(\frac{1}{\mu}-\frac{1}{q_2^*}\right)\int_\Omega |u_n|^{q_2^*}\,dx \\
&\geq \frac{1}{p}m_0\alpha\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\,dx - \frac{1}{\mu}M(t_0)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\,dx \\
&\quad + \left(\frac{1}{\mu}-\frac{1}{q_2^*}\right)\int_\Omega |u_n|^{q_2^*}\,dx \\
&\geq \left(\frac{m_0\alpha}{p}-\frac{M(t_0)}{\mu}\right)\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\,dx + \left(\frac{1}{\mu}-\frac{1}{q_2^*}\right)\int_\Omega |u_n|^{q_2^*}\,dx.
\end{aligned}
$$

By the choice of t_0 in (3.5), we obtain that

$$
c+o_n(1) \geq \left(\frac{1}{\mu}-\frac{1}{q_2^*}\right)\int_\Omega |u_n|^{q_2^*}\,dx \geq \left(\frac{1}{\mu}-\frac{1}{q_2^*}\right)\int_{B_{2\varrho(0)}} |u_n|^{q_2^*}\psi_\varrho\,dx.
$$

Taking the limits as $n\to\infty$, we have

$$
c \geq \left(\frac{1}{\mu}-\frac{1}{q_2^*}\right)\int_{B_{2\varrho(0)}} \psi_\varrho\,d\nu.
$$

By taking the limits as $\varrho\to 0+$ and (3.13), we conclude

$$
c \geq \left(\frac{1}{\mu}-\frac{1}{q_2^*}\right)\nu_j \geq \left(\frac{1}{\mu}-\frac{1}{q_2^*}\right)(m_0a_0S)^{\frac{N}{2q}},
$$

which contradicts to our assumption (3.6). Therefore, Λ is empty and it follows that

$$
\int_\Omega |u_n|^{q_2^*}\,dx \to \int_\Omega |u|^{q_2^*}\,dx. \tag{3.14}
$$

Now we will prove that

$$
u_n \to u \quad \text{in } X \text{ as } n\to\infty.
$$

Taking (3.14) and Brezis and Lieb [9] into account

$$
\lim_{n\to\infty}\int_\Omega (|u_n|^{q_2^*-2}u_n)(u_n-u)\,dx = 0. \tag{3.15}
$$

Under the assumption (f1), we have

$$0 \le \rho_n \le b(1 + |u_n|^{r-1}) \quad \text{a.e. in } \Omega,$$

which implies that

$$\int_\Omega |\rho_n|^{\frac{r}{r-1}}\, dx \le C(|\Omega| + \|u_n\|^r_{L^r(\Omega)}).$$

Thus $\{\rho_n\}$ is bounded in $L^{\frac{r}{r-1}}(\Omega)$. Using Hölder's inequality, we have

$$\int_\Omega \rho_n(u_n - u)\, dx \le \|\rho_n\|_{L^{\frac{r}{r-1}}(\Omega)} \|u_n - u\|_{L^r(\Omega)}.$$

It follows from the boundedness of $\{\rho_n\}$ and (3.14) that

$$\lim_{n\to\infty} \int_\Omega \rho_n(u_n - u)\, dx = 0. \tag{3.16}$$

Note that $a(t) \ge a_0 t^{\frac{q-p}{p}}$ for every $t \ge 0$ due to (A1). Then by a similar argument in [20, Lemma 2.4(ii)], we have the well-known inequalities:

$$C|x-y|^q \le \big\langle a(|x|^p)|x|^{p-2}x - a(|y|^p)|y|^{p-2}y,\, x-y \big\rangle \quad \text{for all } x, y \in \Omega,$$

where $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^N$. Since $\{u_n - u\}$ is bounded in X and $\|w_n\|_{X^*} = o_n(1)$, we derive

$$\lim_{n\to\infty} \langle w_n, u_n - u \rangle = 0. \tag{3.17}$$

According to (3.15), (3.16) and (3.17), we get

$$\begin{aligned}
0 &= \lim_{n\to\infty} \langle w_n, u_n - u \rangle \\
&\ge \lim_{n\to\infty} \left[m_0 \int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^{p-2}\Delta u_n(\Delta u_n - \Delta u)\, dx \right] \\
&\quad - \lim_{n\to\infty} \left[\int_\Omega |u_n|^{q_2^*-2}u_n(u_n - u)\, dx + \int_\Omega \rho_n(u_n - u)\, dx \right] \\
&= \lim_{n\to\infty} m_0 \int_\Omega \left[a(|\Delta u_n|^p)|\Delta u_n|^{p-2}\Delta u_n - a(|\Delta u|^p)|\Delta u|^{p-2}\Delta u \right] (\Delta u_n - \Delta u)\, dx \\
&\ge \lim_{n\to\infty} C\|u_n - u\|^q.
\end{aligned}$$

Therefore we conclude that $u_n \to u$ in X as $n \to \infty$. This completes the proof. □

Next, to apply Lemma 2.1 we prove that $\widetilde{J}_\lambda$ satisfies mountain pass geometry; see also [12, Lemma 5.6].

Lemma 3.2. *Assume that* (M), (A1)–(A2) *and* (f1)–(f4) *hold. Then the functional* $\widetilde{J}_\lambda$ *satisfies the following properties:*

(i) *There exist* $v \in X$ *and* $T > 0$ *such that*

$$\max_{t\in[0,T]} \widetilde{J}_\lambda(tv) < c,$$

where c *is defined in Lemma* 2.1.

(ii) *There exist constants* $\zeta, R > 0$ *such that* $\widetilde{J}_\lambda(u) \geq \zeta$ *for all* $\|u\| = R$.

(iii) *There exists* $e \in X \setminus \{0\}$ *with* $\|e\| > R$ *such that* $\widetilde{J}_\lambda(e) < 0$.

Proof. Consider $v \in C_0^\infty(\Omega)$ such that $\|v\| = 1$, $\Upsilon = \{x \in \Omega : Tv(x) > \beta\}$ with $|\Upsilon| > 0$, T to be fixed later and the function $j\colon \mathbb{R} \to \mathbb{R}$ given by

$$j(t) = \frac{a_1 t^p}{p} M(t_0)\|\Delta v\|^p_{L^p(\Omega)} + \frac{a_1 t^q}{q} M(t_0) - \frac{t^{q_2^*}}{q_2^*}\|v\|^{q_2^*}_{L^{q_2^*}(\Omega)}.$$

So, there is t^* such that

$$j(t^*) = \max_{t\geq 0} j(t).$$

Note that j is increasing in $(0, t^*)$ and decreasing in (t^*, ∞). We can choose $T > 0$ such that

$$T < t^*, \quad j(T) < j(t^*) \quad \text{and} \quad j(T) < c.$$

First, we will prove that (i) is true. Taking into account the assumption (A1) and (3.4), the continuous embedding and $\|v\| = 1$, we have

$$\begin{aligned}
\widetilde{J}_\lambda(tv) &= \frac{1}{p}\mathcal{M}_0\left(\int_\Omega \mathcal{A}(|\Delta tv|^p)\,dx\right) - \frac{1}{q_2^*}\int_\Omega |tv|^{q_2^*}\,dx - \lambda\int_\Omega F(tv)\,dx \\
&\leq \frac{1}{p}M(t_0)\left(\int_\Omega a_1\left(|\Delta tv|^p + \frac{p}{q}|\Delta tv|^q\right)dx\right) - \frac{1}{q^*}\int_\Omega |tv|^{q_2^*}\,dx \\
&\leq \frac{a_1|t|^p}{p}M(t_0)\left(\int_\Omega |\Delta v|^p\,dx\right) + \frac{a_1 t^q}{q}M(t_0)\left(\int_\Omega |\Delta v|^q\,dx\right) - \frac{t^{q_2^*}}{q_2^*}\int_\Omega |v|^{q_2^*}\,dx \\
&= j(t).
\end{aligned}$$

This implies

$$\max_{t\in[0,T]} \widetilde{J}_\lambda(tv) \leq \max_{t\in[0,T]} j(t) \leq j(T) < c.$$

Next, we will claim (ii). In view of (f4), for all $\varepsilon > 0$ there exists $\delta = \delta(\varepsilon)$ such that

$$|F(z)| < \varepsilon|z|^q \quad \text{for } |z| < \delta.$$

By the assumption (f1), we have

$$|F(z)| \leq b\left(|z| + \frac{|z|^r}{r}\right) \leq C(\varepsilon)|z|^r \quad \text{for } |z| > \delta,$$

where $C(\varepsilon) := 2b\max\{\delta^{1-r}, r^{-1}\}$ and hence we obtain

$$|F(z)| < \varepsilon|z|^q + C(\varepsilon)|z|^r \quad \text{for all } z \in \mathbb{R}. \tag{3.18}$$

Let $\lambda > 0$ be arbitrary, but fixed. According to the assumptions (A1)–(A2), (3.18) and the continuous embedding in Remark 2.4, it follows that

$$\begin{aligned}
\widetilde{J}_\lambda(u) &\geq \frac{m_0}{p}\int_\Omega \mathcal{A}(|\Delta u|^p)\,dx - \lambda\int_\Omega |F(u)|\,dx - \int_\Omega |u|^{q_2^*}\,dx \\
&\geq \frac{\alpha a_0 m_0}{p}\int_\Omega (|\Delta u|^p + |\Delta u|^q)\,dx - \lambda\int_\Omega (\varepsilon|u|^q + C(\varepsilon)|u|^r)\,dx - \|u\|^{q_2^*}_{L^{q_2^*}(\Omega)} \\
&\geq \frac{\alpha a_0 m_0}{p}(\|u\|^p + \|u\|^q) - \lambda\varepsilon\mathcal{S}_q\|u\|^q - \lambda C(\varepsilon)\mathcal{S}_r\|u\|^r - \mathcal{S}_{q_2^*}\|u\|^{q_2^*}.
\end{aligned}$$

For each $\lambda > 0$ we take $\varepsilon < 2\alpha a_0 m_0/p\lambda\mathcal{S}_q$ and choose $0 < R < 1$ sufficient small with

$$R^{r-q} < \frac{2\alpha a_0 m_0 - \varepsilon p\lambda\mathcal{S}_q}{p(\lambda C(\varepsilon)\mathcal{S}_r + \mathcal{S}_{q_2^*})}.$$

Then for all $u \in X$ with $\|u\| = R$, we get

$$\begin{aligned}
\widetilde{J}_\lambda(u) &\geq \frac{2\alpha a_0 m_0}{p}\|u\|^q - \varepsilon\lambda\mathcal{S}_q\|u\|^q - \lambda C(\varepsilon)\mathcal{S}_r\|u\|^r - \mathcal{S}_{q_2^*}\|u\|^r \\
&= R^q\left(\frac{2\alpha a_0 m_0}{p} - \varepsilon\lambda\mathcal{S}_q - (\lambda C(\varepsilon)\mathcal{S}_r + \mathcal{S}_{q_2^*})R^{r-q}\right).
\end{aligned}$$

Therefore, we obtain $\zeta > 0$ such that

$$\widetilde{J}_\lambda(u) \geq \zeta \quad \text{with } \|u\| = R \text{ for all } u \in X.$$

Finally, to prove (iii) fix $\beta = \frac{T}{2}$. Then by (f3), we give

$$\begin{aligned}
\widetilde{J}_\lambda(Tv) &= \frac{1}{p}\mathcal{M}_0\left(\int_\Omega \mathcal{A}(|\Delta Tv|^p)\,dx\right) - \frac{1}{q_2^*}\int_\Omega |Tv|^{q_2^*}\,dx - \lambda\int_\Omega F(Tv)\,dx \\
&\leq j(T) - \lambda\int_\Omega\int_0^{Tv} H(z-\beta)\,dzdx \\
&= j(T) - \lambda\int_\Upsilon |Tv-\beta|\,dx.
\end{aligned}$$

This implies that for each $\lambda > 0$ there exists $\widetilde{T}(\lambda) > R$ such that

$$\widetilde{J}_\lambda(Tv) < 0 \quad \text{for sufficiently large } T \geq \widetilde{T}(\lambda). \tag{3.19}$$

Thus, we conclude that for each $\lambda > 0$ there exists $e_\lambda := \widetilde{T}(\lambda)v$ satisfying $\|e_\lambda\| > R$ and $\widetilde{J}(e_\lambda) < 0$. This completes the proof. □

For each $\lambda > 0$, let e_λ be as in the preceding lemma and define

$$c_\lambda = \inf_{\gamma\in\Gamma} \max_{t\in[0,1]} \widetilde{J}_\lambda(\gamma(t)), \tag{3.20}$$

where

$$\Gamma = \{\gamma \in C([0,1], X) : \gamma(0) = 0 \text{ and } \gamma(1) = e_\lambda\}.$$

As a consequence of Lemma 3.2 we have

Lemma 3.3. *The number c_λ is positive and there exists a sequence $\{u_n\}$ in X such that*

$$\widetilde{J}_\lambda(u_n) \to c_\lambda \quad \textit{and} \quad \omega^*(u_n) \to 0 \quad \textit{as } n \to \infty.$$

Furthermore, we have the following property for c_λ.

Lemma 3.4. *It holds that*

$$\lim_{\lambda\to\infty} c_\lambda = 0,$$

where c_λ is given in (3.20).

Proof. Let $\{\lambda_n\}$ be an arbitrary sequence of real positive numbers such that $\lambda_n \to \infty$ as $n \to \infty$. By the proof of Lemma 3.2, for each $n \in \mathbb{N}$, there exists $T_{\lambda_n} > 0$ such that

$$\widetilde{J}_{\lambda_n}(T_{\lambda_n} v) = \max_{0<T} \widetilde{J}_{\lambda_n}(Tv).$$

For this reason, $T_{\lambda_n} \frac{d}{dT} \widetilde{J}_{\lambda_n}(Tv)\big|_{T=T_{\lambda_n}} = \langle w_n, T_{\lambda_n} v\rangle = 0$, where $w_n = \Phi'(T_{\lambda_n} v) - \lambda_n \rho_{\lambda_n}$ and $\rho_{\lambda_n} \in \partial\Psi(T_{\lambda_n} v)$, namely,

$$\begin{aligned} &M_0\left(\int_\Omega \mathcal{A}(|\Delta T_{\lambda_n} v|^p)\,dx\right)\int_\Omega a(|\Delta T_{\lambda_n} v|^p)|\Delta T_{\lambda_n} v|^p\,dx \\ &= \lambda_n \int_\Omega \rho_{\lambda_n} T_{\lambda_n} v\,dx + \int_\Omega |T_{\lambda_n} v|^{q_2^*}\,dx. \end{aligned} \tag{3.21}$$

It follows from (f2) that

$$M_0\left(\int_\Omega \mathcal{A}(|\Delta T_{\lambda_n} v|^p)\,dx\right)\int_\Omega a(|\Delta T_{\lambda_n} v|^p)|\Delta T_{\lambda_n} v|^p\,dx \ge T_{\lambda_n}^{q_2^*} \int_\Omega |v|^{q_2^*}\,dx. \tag{3.22}$$

On the other hand, taking into account (3.4) and $\|v\| = 1$, we get

$$\begin{aligned} &M_0\left(\int_\Omega \mathcal{A}(|\Delta T_{\lambda_n} v|^p)\,dx\right)\int_\Omega a(|\Delta T_{\lambda_n} v|^p)|\Delta T_{\lambda_n} v|^p\,dx \\ &\le M(t_0)\int_\Omega a_1(|\Delta T_{\lambda_n} v|^p + |\Delta T_{\lambda_n} v|^q)\,dx \\ &\le a_1 M(t_0) T_{\lambda_n}^p \left(\int_\Omega |\Delta v|^p\,dx\right) + a_1 M(t_0) T_{\lambda_n}^q \left(\int_\Omega |\Delta v|^q\,dx\right) \\ &\le a_1 M(t_0) T_{\lambda_n}^p \|v\|^q |\Omega|^{\frac{q-p}{q}} + a_1 M(t_0) T_{\lambda_n}^q \|v\|^q \\ &\le a_1 M(t_0)\big(1 + |\Omega|^{\frac{q-p}{q}}\big) \max\{T_{\lambda_n}^p, T_{\lambda_n}^q\}. \end{aligned} \tag{3.23}$$

Using (3.22) and (3.23), we deduce that the sequence $\{T_{\lambda_n}\}$ is bounded because $p < q < q_2^*$. Up to a subsequence, we may assume that $T_{\lambda_n} \to T_0$ as $n \to \infty$. Moreover, by (3.21) and (3.23), we have

$$\lambda_n \int_\Omega \rho_{\lambda_n} T_{\lambda_n} v \, dx + \int_\Omega |T_{\lambda_n} v|^{q_2^*} \, dx < C \tag{3.24}$$

for all $n \in \mathbb{N}$. Note that $T_0 = 0$. Indeed, if $T_0 > 0$, then it follows from the assumption (f2) that

$$\lambda_n \int_\Omega \rho_{\lambda_n} T_{\lambda_n} v \, dx + \int_\Omega |T_{\lambda_n} v|^{q_2^*} \, dx \to \infty \quad \text{as } n \to \infty$$

which contradicts (3.24). So, we get $T_0 = 0$.

For each $n \in \mathbb{N}$, we consider the path $\widetilde{\gamma}(t) = te_{\lambda_n}$ with $t \in [0, 1]$, where e_{λ_n} is taken from the proof of Lemma 3.2. Clearly, $\widetilde{\gamma} \in \Gamma$ and note that, by applying (3.19) for $\lambda = \lambda_n$,

$$\max_{t \geq 0} \widetilde{J}_{\lambda_n}(tv) = \max_{t \in \left[0, \widetilde{T}(\lambda)\right]} \widetilde{J}_{\lambda_n}(tv) = \max_{t \in [0,1]} \widetilde{J}_{\lambda_n}(te_{\lambda_n}) = \max_{t \in [0,1]} \widetilde{J}_{\lambda_n}(\widetilde{\gamma}(t)).$$

Thus, by (3.4) and (3.23), we have the following estimate

$$\begin{aligned}
0 < c_{\lambda_n} &= \inf_{\gamma \in \Gamma} \max_{t \in [0,1]} \widetilde{J}_{\lambda_n}(\gamma(t)) \\
&\leq \max_{t \in [0,1]} \widetilde{J}_{\lambda_n}(\widetilde{\gamma}(t)) = \max_{t \geq 0} \widetilde{J}_{\lambda_n}(tv) = \widetilde{J}_{\lambda_n}(T_{\lambda_n} v) \\
&\leq M(t_0) \int_\Omega a_1 (|\Delta T_{\lambda_n} v|^p + |\Delta T_{\lambda_n} v|^q) \, dx \\
&\leq a_1 M(t_0) T_{\lambda_n}^p \left(\int_\Omega |\Delta v|^p \, dx \right) + a_1 M(t_0) T_{\lambda_n}^q \left(\int_\Omega |\Delta v|^q \, dx \right) \\
&\leq 2a_1 M(t_0) \max\{T_{\lambda_n}^p, T_{\lambda_n}^q\}.
\end{aligned}$$

Combining this and the fact $T_{\lambda_n} \to 0$, we obtain $c_{\lambda_n} \to 0$ as $n \to \infty$. This completes the proof. □

4. Proof of Theorem 1.2

Proof of Theorem 1.2. (1) By Lemma 3.4, there exists $\Lambda^* > 0$ such that

$$c_\lambda < \left(\frac{1}{\mu} - \frac{1}{q_2^*} \right) (m_0 a_0 S)^{N/q} \quad \text{and} \quad c_\lambda < \frac{a_0 t_0 (m_0 \alpha \mu - pM(t_0))}{a_1 p \mu} \quad \text{for } \lambda \geq \Lambda^*, \tag{4.1}$$

where c_λ is given by (3.20) and S is in Lemma 3.1. By Lemma 3.3, we have a sequence $\{u_n\}$ in X such that

$$\widetilde{J}_\lambda(u_n) \to c_\lambda \quad \text{and} \quad \omega^*(u_n) \to 0 \quad \text{as } n \to \infty.$$

Then there exists a sequence $\{w_n\} \subset \partial \widetilde{J}_\lambda(u_n)$ such that

$$\|w_n\|_{X^*} = \omega^*(u_n) = o_n(1) \quad \text{and} \quad w_n = \Phi'(u_n) - \lambda \rho_n,$$

where $\rho_n \in \partial\Psi(u_n)$ and $o_n(1) \to 0$ as $n \to \infty$. Since $\widetilde{J}_\lambda$ satisfies the $(\mathrm{PS})_c$-condition due to Lemma 3.1, we have a convergent subsequence of $\{u_n\}$, denoted again by $\{u_n\}$ in X such that $u_n \to u_\lambda$ in X as $n \to \infty$. In view of Lemmas 3.1 and 3.2, it follows from Theorem 2.1 that the modified energy functional $\widetilde{J}_\lambda$ has a solution $u_\lambda \in X$. Moreover, u_λ is nontrivial because $\widetilde{J}_\lambda(u_\lambda) = c_\lambda > 0$.

In order to finish the proof of Theorem 1.2, we prove that u_λ is also a nontrivial solution to the problem (1.2) for all $\lambda \geq \Lambda^*$. According to (3.3), (A2) and (f2), we have

$$\begin{aligned}
c_\lambda + o_n(1) &= \widetilde{J}_\lambda(u_n) - \frac{1}{\mu}\langle w_n, u_n\rangle \\
&\geq \frac{m_0}{p}\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx - \frac{M(t_0)}{\mu}\int_\Omega a(|\Delta u_n|^p)|\Delta u_n|^p\,dx \\
&\quad + \lambda\int_\Omega \left(\frac{1}{\mu}\rho_n u_n - F(u_n)\right)dx + \left(\frac{1}{\mu} - \frac{1}{q_2^*}\right)\int_\Omega |u_n|^{q_2^*}\,dx \\
&\geq \frac{m_0}{p}\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx - \frac{M(t_0)}{\alpha\mu}\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx \\
&\geq \left(\frac{m_0}{p} - \frac{M(t_0)}{\alpha\mu}\right)\int_\Omega \mathcal{A}(|\Delta u_n|^p)\,dx \\
&\geq \alpha a_0\left(\frac{m_0}{p} - \frac{M(t_0)}{\alpha\mu}\right)\int_\Omega (|\Delta u_n|^p + |\Delta u_n|^q)\,dx.
\end{aligned}$$

Passing to the limit as $n \to \infty$ in the last inequality, it follows from the continuity of the function a that

$$c_\lambda \geq \alpha a_0\left(\frac{m_0}{p} - \frac{M(t_0)}{\alpha\mu}\right)\int_\Omega (|\Delta u_\lambda|^p + |\Delta u_\lambda|^q)\,dx,$$

and so

$$a_1\int_\Omega (|\Delta u_\lambda|^p + |\Delta u_\lambda|^q)\,dx \leq \frac{a_1}{a_0}\left(\frac{p\mu}{m_0\alpha\mu - pM(t_0)}\right)c_\lambda. \tag{4.2}$$

Combining (4.2) and the second inequality in (4.1), we conclude that

$$\int_\Omega \mathcal{A}(|\Delta u_\lambda|^p)\,dx \leq a_1\int_\Omega (|\Delta u_\lambda|^p + |\Delta u_\lambda|^q)\,dx \leq t_0 \quad \text{for all } \lambda \geq \Lambda^*,$$

which yields that $\widetilde{J}_\lambda = J_\lambda$ for all $\lambda \geq \Lambda^*$ in view of (3.2). Therefore, u_λ is also a nontrivial solution to the original problem (1.2) provided $\lambda \geq \Lambda^*$. Moreover, (4.2) also implies that

$$\lim_{\lambda\to\infty}\|u_\lambda\| = 0.$$

Choose u^- as a test function. Then it is obviously that $u_\lambda = u_\lambda^+ \geq 0$.

(2) Now, let u_λ be a solution of (1.2) and $\lambda \geq \Lambda^*$. Then, we prove that there exists $\beta^* > 0$ such that the set $\{x \in \Omega : u_\lambda(x) > \beta^*\}$ has positive measure. Assume to the contrary that $u_\lambda(x) \leq \beta$ a.e. in Ω for all $\beta > 0$. Since u_λ is a solution, we have

$$M\left(\int_\Omega \mathcal{A}(|\Delta u_\lambda|^p)\,dx\right)\int_\Omega a(|\Delta u_\lambda|^p)|\Delta u_\lambda|^p\,dx = \lambda\int_\Omega \rho u_\lambda\,dx + \int_\Omega |u_\lambda|^{q_2^*}\,dx.$$

According to the assumption (A1) and (f1), we obtain that

$$\begin{aligned}
m_0 a_0\|u_\lambda\|^q &\leq m_0 a_0\int_\Omega (|\Delta u_\lambda|^p + |\Delta u_\lambda|^q)\,dx \\
&\leq M\left(\int_\Omega \mathcal{A}(|\Delta u_\lambda|^p)\,dx\right)\int_\Omega a(|\Delta u_\lambda|^p)|\Delta u_\lambda|^p\,dx = \lambda\int_\Omega \rho u_\lambda\,dx + \int_\Omega |u_\lambda|^{q_2^*}\,dx \\
&\leq \int_\Omega \lambda b(u_\lambda + |u_\lambda|^r)\,dx + \int_\Omega |u_\lambda|^{q_2^*}\,dx \leq \lambda b(\beta+\beta^r)|\Omega| + \beta|\Omega| \\
&\leq 3(\lambda b + 1)\beta|\Omega|,
\end{aligned}$$

where $\beta < 1$. Since $\widetilde{J}_\lambda(u_\lambda) = c_\lambda > 0$, there exists $\sigma > 0$ such that $\|u_\lambda\| \geq \sigma$. Then,

$$a_0 m_0 \sigma^q \leq 3(\lambda b + 1)\beta|\Omega|.$$

But this inequality is impossible if we choose for each λ,

$$\beta = \min\left\{\frac{1}{2}, \frac{T}{2}, \frac{a_0 m_0 \sigma^q}{3(\lambda b+1)|\Omega|}\right\}.$$

This completes the proof. □

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