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Existence and Asymptotic Behaviors to a Nonlinear Fourth-order Parabolic Equation with a General Source

Bo Liang, Qingchun Li, Yongbo Zhu and Yongzheng Zhu*

Abstract. The existence and asymptotic behavior of solutions a fourth-order partial differential equation with a p-Laplacian diffusion and a nonlinear source are studied by using potential well theory. When the initial functionals satisfy $\mathcal{F}(w_0) < d$, $\mathcal{D}(w_0) > 0$ or $\mathcal{F}(w_0) = d$, $\mathcal{D}(w_0) \geq 0$, the existence and exponential decay result of weak solutions are given. For $\mathcal{F}(w_0) < d$, $\mathcal{D}(w_0) < 0$ or $\mathcal{F}(w_0) = d$, $\mathcal{D}(w_0) < 0$, we obtain the blow-up behavior at a finite time for weak solutions. For $\mathcal{F}(w_0) > d$, we show the global existence for small initial datum and blow-up for big initial datum. Moreover, the uniqueness holds for bounded solutions. In addition, we show that the p-Laplacian term has an essential effect to the source function so that we add some growth conditions to $g(w)$.

1. Introduction

In 1968, Sattinger introduces the potential well approach for the first time in [13]. A potential well is defined as the area in physics that has the lowest potential energy within a given range of space. The potential well can be considered as an adequate energy functional in applicable Sobolev spaces in mathematics. For the research works, Sattinger (see [13]) investigated the existence of global solutions to a hyperbolic equation. Payne and Sattinger utilized the similar method to determine the existence and blow-up behaviors of a second-order diffusion equation with a general source function $g(w)$ (see [11]). Lin [10] improved the related results for the same equation and achieved the finite blow-up behavior for $0 < \mathcal{F}(w_0) < d$, $\mathcal{D}(w_0) < 0$, and the global existence for critical case $\mathcal{D}(w_0) \geq 0$, $\mathcal{E}(0) = d$ or $\mathcal{F}(w_0) = d$. Recently the semi-linear parabolic equations and pseudo-parabolic equations with singular potential term have been considered by some authors (may refer to [2, 7, 17]). The initial boundary value problem of a class of coupled parabolic systems with nonlinear coupled source terms has been investigated in [20]. For different initial data, the global existence, finite time blowup behavior, and long time decay of solutions are obtained. Furthermore, in [16], a time-fractional pseudo-parabolic problem is addressed.

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*Corresponding author.

Recently, there have been some research results about the applications of potential well theory in fourth-order parabolic equations. Various study results on the applications of potential well theory in fourth-order parabolic equations have been published. Xu, Chen and Liu [19] considered a fourth-order semi-linear parabolic problem with a general source. By employing an improved potential well theory, they demonstrated that the global existence and the blow-up behavior are affected by the initial energy. They also obtained a global attractor for global solutions by employing an iterative technique. Qu and Zhou [12] addressed the nonlocal source problem for a 4th-order PDE in one-dimensional space, and the related global existence and nonexistence were derived for weak solutions. They also studied the asymptotic behavior and extinction features of global weak solutions. Li, Gao, and Han [6] used the modified potential well method to establish the existence, uniqueness, and asymptotic behavior of solutions for an analogous issue. Han [5] also applied the same method to give the blow-up behaviors and global existence for the fourth-order parabolic equation with a p-Laplacian diffusion and the source $|w|^{q-1}w$. Zhou extended Han's findings by providing specific values for each asymptotic parameter. We can quote $[1, 3, 4, 8, 9, 18, 21, 23]$ for more related references.

The paper considers the initial-boundary value problem for the fourth-order parabolic equation with a general source:

$$
\begin{cases} w_t + \Delta^2 w - \operatorname{div}(|\nabla w|^{p-2}\nabla w) = g(w), & (x,t) \in U \times (0,T), \\ w = \frac{\partial w}{\partial \nu} = 0, & (x,t) \in \partial U \times (0,T), \\ w(x,0) = w_0(x), & x \in U. \end{cases} \tag{1.1}
$$

Let N be the spatial dimension and ν be the boundary's outward normal vector. we assume that U is a bounded domain in R^N and ∂U is sufficiently smooth. This model can reflect the epitaxial manufacturing process for nano-scale films (see $[14, 22]$). The potential well theory will be utilized to assess the existence and asymptotic behavior of weak solutions.

The following is how the paper is organized. We introduce certain fundamental concepts, notations, conditions, and lemmas in Section 2. Section 3 is devoted to a summary of the main findings. In Sections 4 and 5, we exhibit the technique to evaluate the existence and uniqueness of solutions, as well as the blow-up behavior of solutions for $\mathcal{F}(w_0) < d$ and $\mathcal{F}(w_0) = d$. Finally, Section 6 establishes the global existence and blow-up in finite time for the case $\mathcal{F}(w_0) > d$.

2. Preliminaries

We define the norm of $winL^p(U)$ as

$$\|w\|_{L^p} = \left(\int_U |w|^p \,\mathrm{d}x \right)^{1/p}$$

for $p \geq 1$. The inner product for $w, v \in L^2(U)$ is given by

$$(w, v) = \int_U wv \,\mathrm{d}x.$$

For $H_0^2(U) = \left\{ w \in H^2(U) \mid w = \frac{\partial w}{\partial \nu} = 0 \text{ on } \partial U \right\}$, Poincaré's inequality allows us to define the equivalent norm of $H_0^2(U)$ as

$$\|w\|_{H_0^2(U)} = \|\Delta w\|_{L^2(U)}.$$

The following are the conditions for the source function $g(\cdot)$ as well as the variables p and q:

(H1) $1 < p < \frac{2N}{N-2}$ when $N > 2$; $1 < p < \infty$ when $N \leq 2$;

(H2) $q > 1$ if $p \leq 2$ and $q > p - 1$ if $p > 2$;

(H3) $g \in C^1$, $g(0) = g'(0) = 0$, $g'(s) > 0$ for $s \neq 0$;

(H4) $g(s)$ is an increasing function. It is convex if $s > 0$ and concave if $s < 0$;

(H5) When $p > 2$, $sg(\theta s) \geq \theta^{p-1} sg(s)$ for $s \neq 0$ and $\theta > 1$;

(H6) Define $G(s) = \int_0^s g(\sigma)\,\mathrm{d}\sigma$ and assume that $(q+1)G(s) \leq sg(s)$ and $sg(s) \leq \gamma G(s)$, where the constant γ satisfies

(a) $\max\left\{q+1, \frac{p}{2}(q+1)\right\} \leq \gamma < \infty$ if $N \leq 4$;

(b) $\max\left\{q+1, \frac{p}{2}(q+1)\right\} \leq \gamma < \frac{2N}{N-4}$ if $N > 4$.

A typical example of the above requirements is $g(w) = |w|^{q-1}w$ and more general forms for g can be given. In addition, getting (H4) from (H5) is impossible. However, we could infer from (H4) that $sg(\theta s) > \theta sg(s)$ for $s \neq 0$. Our assumptions (H2) and (H5) differ from the references [11] and [10] due to the p-Laplacian term plays an important impact on the source g.

Lemma 2.1. [11]

(i) *For $w \in R$, there is some positive constant A that fulfills $G(w) \leq A|w|^\gamma$.*

(ii) *There is some positive constant B satisfying $G(w) \geq B|w|^{q+1}$ for $|w| \geq 1$.*

(iii) *For $w \in R$, $w(wg'(w) - g(w)) \geq 0$. In addition, $w(wg'(w) - g(w)) = 0$ if and only if $w = 0$.*

Corollary 2.2. [10]

(i) $|wg(w)| \leq A\gamma|w|^{\gamma}$, $|g(w)| \leq \gamma A|w|^{\gamma-1}$ *for* $w \in R$.

(ii) $wg(w) \geq B(p+1)|w|^{p+1}$ *for* $|w| \geq 1$.

Now we need to introduce several associated functionals in order to effectively utilize the potential well method. For $w \in H_0^2(U)$, define

$$\mathcal{F}(w) = \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{p}\|\nabla w\|_{L^p}^p - \int_U G(w)\,\mathrm{d}x,$$
$$\mathcal{D}(w) = \|\Delta w\|_{L^2}^2 + \|\nabla w\|_{L^p}^p - \int_U wg(w)\,\mathrm{d}x.$$

The Nahari manifold is given by

$$\mathbb{K} = \big\{w \in H_0^2(U) \mid \mathcal{D}(w) = 0, \|\Delta w\|_{L^2} \neq 0\big\}.$$

The related sets are expressed by

$$\mathbb{W} = \big\{w \in H_0^2(U) \mid \mathcal{F}(w) < d, \mathcal{D}(w) > 0\big\} \cup \{0\},$$
$$\overline{\mathbb{W}} = \big\{w \in H_0^2(U) \mid \mathcal{F}(w) \leq d, \mathcal{D}(w) \geq 0\big\} \cup \{0\},$$
$$\mathbb{V} = \big\{w \in H_0^2(U) \mid \mathcal{F}(w) < d, \mathcal{D}(w) < 0\big\}.$$

The depth of potential well is defined by

$$d = \inf_{w \in \mathbb{K}} \mathcal{F}(w).$$

The improved functional is given as

$$\mathcal{D}_\vartheta(w) = \vartheta\big(\|\Delta w\|_{L^2}^2 + \|\nabla w\|_{L^p}^p\big) - \int_U wg(w)\,\mathrm{d}x$$

with $\vartheta > 0$. The corresponding Nehari manifold is

$$\mathbb{K}_\vartheta = \big\{w \in H_0^2(U) \mid \mathcal{D}_\vartheta(w) = 0, \|\Delta w\|_{L^2} \neq 0\big\},$$
$$\mathbb{W}_\vartheta = \big\{w \in H_0^2(U) \mid \mathcal{D}_\vartheta(w) > 0, \mathcal{F}(w) < d(\vartheta)\big\} \cup \{0\},$$
$$\mathbb{V}_\vartheta = \big\{w \in H_0^2(U) \mid \mathcal{D}_\vartheta(w) < 0, \mathcal{F}(w) < d(\vartheta)\big\}.$$

The corresponding depth of the potential well is

$$d(\vartheta) = \inf_{w \in \mathbb{K}_\vartheta} \mathcal{F}(w).$$

For $s > d$, define

$$\mathbb{K}_+ = \big\{w \in H_0^2(U) \mid \mathcal{D}(w) > 0\big\}, \qquad \mathbb{K}_- = \big\{w \in H_0^2(U) \mid \mathcal{D}(w) < 0\big\},$$
$$\mathcal{F}^s = \big\{w \in H_0^2(U) \mid \mathcal{F}(w) < s\big\}, \qquad \mathbb{K}^s = \mathbb{K} \cap \mathcal{F}^s,$$
$$\theta_s = \inf\big\{\|w\|_{L^2} \mid w \in \mathbb{K}^s\big\}, \qquad \Theta_s = \sup\big\{\|w\|_{L^2} \mid w \in \mathbb{K}^s\big\}.$$

The following are some basic lemmas. The proofs of Lemmas 2.8, 2.9 and 2.11 follows a similar procedure to that of [5,6,10,21] and we leave out the details. Each proof will be given for the remaining lemmas.

Lemma 2.3. $d > 0$.

Proof. Let M denote the optimal embedding constant such that $\|w\|_{L^\gamma} \leq M\|\Delta w\|_{L^2}$. For each $w \in \mathbb{K}$, use Lemma 2.1 to have

$$\frac{1}{M^2}\|w\|_{L^\gamma}^2 \leq \|\Delta w\|_{L^2}^2 + \|\nabla w\|_{L^p}^p = \int_U wg(w)\,\mathrm{d}x \leq \gamma A\|w\|_{L^\gamma}^\gamma$$

which gives $\|w\|_{L^\gamma}^{\gamma-2} \geq \frac{1}{\gamma AM^2}$ and $\|\Delta w\|_{L^2}^2 \geq \frac{1}{M^2}\|w\|_{L^\gamma}^2 \geq \frac{1}{M^2}\left(\frac{1}{\gamma AK^2}\right)^{\frac{2}{\gamma-2}}$. Employ (H5) to obtain

$$\begin{aligned}
\mathcal{F}(w) &= \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{p}\|\nabla w\|_{L^p}^p - \int_U G(w)\,\mathrm{d}x \\
&\geq \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{p}\|\nabla w\|_{L^p}^p - \frac{1}{q+1}\int_U wg(w)\,\mathrm{d}x \\
&= \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{p}\|\nabla w\|_{L^p}^p - \frac{1}{q+1}\big(\|\Delta w\|_{L^2}^2 + \|\nabla w\|_{L^p}^p\big) \\
&= \left(\frac{1}{2} - \frac{1}{q+1}\right)\|\Delta w\|_{L^2}^2 + \left(\frac{1}{p} - \frac{1}{q+1}\right)\|\nabla w\|_{L^p}^p \\
&\geq \frac{q-1}{2(q+1)}\frac{1}{M^2}\left(\frac{1}{\gamma AM^2}\right)^{\frac{2}{\gamma-2}} \\
&> 0.
\end{aligned}$$

□

Lemma 2.4. *If $w \in H_0^2(U)$ with $\|\Delta w\|_{L^2} \neq 0$, then*

(i) $\lim_{\theta\to 0^+}\mathcal{F}(\theta w) = 0$, $\lim_{\theta\to+\infty}\mathcal{F}(\theta w) = -\infty$;

(ii) *$\mathcal{F}(\theta w)$ has a unique critical point $\theta^* = \theta^*(w) > 0$ (i.e., $\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{F}(\theta w)\big|_{\theta=\theta^*} = 0$), is decreasing on $(\theta^*, +\infty)$, is increasing on $(0, \theta^*)$ and has the maximum at $\theta = \theta^*$;*

(iii) *$\mathcal{D}(\theta w) > 0$ on $(0, \theta^*)$, $\mathcal{D}(\theta w) < 0$ on $(\theta^*, +\infty)$ and $\mathcal{D}(\theta^* w) = 0$.*

Proof. (i) It follows from Lemma 2.1 that

$$\begin{aligned}
|\mathcal{F}(\theta w)| &\leq \frac{\theta^2}{2}\|\Delta w\|_{L^2}^2 + \frac{\theta^p}{p}\|\nabla w\|_{L^p}^p + \left|\int_U G(\theta w)\,\mathrm{d}x\right| \\
&\leq \frac{\theta^2}{2}\|\Delta w\|_{L^2}^2 + \frac{\theta^p}{p}\|\nabla w\|_{L^p}^p + A\theta^\gamma\int_U |w|^\gamma\,\mathrm{d}x.
\end{aligned}$$

By (H3) and (H4), we pass to $\theta \to 0^+$ to obtain $\lim_{\theta\to 0^+} \mathcal{F}(\theta w) = 0$. Furthermore, for $|\theta w| \geq 1$, one has

$$\begin{aligned}\mathcal{F}(\theta w) &= \frac{\theta^2}{2}\|\Delta w\|_{L^2}^2 + \frac{\theta^p}{p}\|\nabla w\|_{L^p}^p - \int_U G(\theta w)\,\mathrm{d}x \\ &\leq \frac{\theta^2}{2}\|\Delta w\|_{L^2}^2 + \frac{\theta^p}{p}\|\nabla w\|_{L^p}^p - B|\theta|^{q+1}\int_U |w|^{q+1}\,\mathrm{d}x,\end{aligned}$$

which implies $\mathcal{F}(\theta w) \to -\infty$ as $\theta \to +\infty$.

(ii) A direct calculation gives

$$j(\theta) \equiv \frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{F}(\theta w) = \theta\|\Delta w\|_{L^2}^2 + \theta^{p-1}\|\nabla w\|_{L^p}^p - \int_U wg(\theta w)\,\mathrm{d}x.$$

Similar to the argument of (i), $j(\theta)$ is positive for small $\theta > 0$ and is negative for large θ. Thus, it can ensure the existence of θ^*. It remains to show the uniqueness of θ^*. Now suppose that there are two constants θ_1^* and θ_2^* ($\theta_1^* < \theta_2^*$) such that $j(\theta_1^*) = j(\theta_2^*) = 0$. This says that

$$\theta_1^*\|\Delta w\|_{L^2}^2 + \theta_1^{*p-1}\|\nabla w\|_{L^p}^p - \int_U wg(\theta_1^* w)\,\mathrm{d}x = 0, \tag{2.1}$$

$$\theta_2^*\|\Delta w\|_{L^2}^2 + \theta_2^{*p-1}\|\nabla w\|_{L^p}^p - \int_U wg(\theta_2^* w)\,\mathrm{d}x = 0. \tag{2.2}$$

By eliminating the term $\|\Delta w\|_{L^2}^2$ and putting $\overline{w} = \theta_1^* w$, we have

$$\begin{aligned}\theta_1^*\theta_2^*(\theta_1^{*p-2} - \theta_2^{*p-2})\|\nabla w\|_{L^p}^p &= \theta_2^*\int_U wg(\theta_1^* w)\,\mathrm{d}x - \theta_1^*\int_U wg(\theta_2^* w)\,\mathrm{d}x \\ &= \theta\int_U \overline{w}g(\overline{w})\,\mathrm{d}x - \int_U \overline{w}g(\theta\overline{w})\,\mathrm{d}x\end{aligned}$$

for $\theta = \frac{\theta_2^*}{\theta_1^*} > 1$.

For the case $p \leq 2$, the left-hand side is nonnegative and so

$$\theta\int_U \overline{w}g(\overline{w})\,\mathrm{d}x \geq \int_U \overline{w}g(\theta\overline{w})\,\mathrm{d}x.$$

Since (H4) implies $\int_U \overline{w}g(\theta\overline{w})\,\mathrm{d}x > \theta\int_U \overline{w}g(\overline{w})\,\mathrm{d}x$ for $\overline{w} \neq 0$ and $\theta > 1$, it yields a contradiction and so $\theta_1^* = \theta_2^*$.

For the case $p > 2$, we eliminate the term $\|\nabla w\|_{L^p}^p$ from (2.1) and (2.2) to obtain

$$\begin{aligned}\theta_1^*\theta_2^*(\theta_2^{*p-2} - \theta_1^{*p-2})\|\Delta w\|_{L^2}^2 &= \theta_2^{*p-1}\int_U wg(\theta_1^* w)\,\mathrm{d}x - \theta_1^{*p-1}\int_U wg(\theta_2^* w)\,\mathrm{d}x \\ &= \theta_2^{*p-1}\int_U \overline{w}g(\overline{w})\,\mathrm{d}x - \theta_1^{*p-1}\int_U \overline{w}g(\theta\overline{w})\,\mathrm{d}x.\end{aligned}$$

Since the left-hand side is positive, we can use (H5) to have

$$\theta^{p-1}\int_U \overline{w}g(\overline{w})\,\mathrm{d}x > \int_U \overline{w}g(\theta\overline{w})\,\mathrm{d}x \geq \theta^{p-1}\int_U \overline{w}g(\overline{w})\,\mathrm{d}x$$

which yields a contradiction and so $\theta_1^* = \theta_2^*$.

(iii) The proof is from

$$\mathcal{D}(\theta w) = \theta^2\|\Delta w\|_{L^2}^2 + \theta^p\|\nabla w\|_{L^p}^p - \theta\int_U wg(\theta w)\,\mathrm{d}x = \theta\frac{\mathrm{d}}{\mathrm{d}\theta}\mathcal{F}(\theta w). \qquad \square$$

Lemma 2.5. *Let $w \in H_0^2(U)$ and $r(\vartheta) = \left(\frac{\vartheta}{aM^\gamma}\right)^{\frac{1}{\gamma-2}}$ with $a = \sup_{s\in R}\frac{sg(s)}{|s|^\gamma}$. Then*

(i) *$\mathcal{D}_\vartheta(w) > 0$ if $0 < \|\Delta w\|_{L^2} \leq r(\vartheta)$;*

(ii) *$\|\Delta w\|_{L^2} > r(\vartheta)$ if $\mathcal{D}_\vartheta(w) < 0$;*

(iii) *$\|\Delta w\|_{L^2} = 0$ or $\|\Delta w\|_{L^2} \geq r(\vartheta)$ if $\mathcal{D}_\vartheta(w) = 0$.*

Proof. (i) From

$$\begin{aligned}\int_U wg(w)\,\mathrm{d}x &\leq a\int_U |w|^\gamma\,\mathrm{d}x = a\|w\|_{L^\gamma}^\gamma \leq aM^\gamma\|\Delta w\|_{L^2}^\gamma \\ &= aM^\gamma\|\Delta w\|_{L^2}^{\gamma-2}\|\Delta w\|_{L^2}^2 \leq \vartheta\|\Delta w\|_{L^2}^2,\end{aligned}$$

we have $\mathcal{D}_\vartheta(w) > 0$.

(ii) If $\mathcal{D}_\vartheta(w) < 0$, then

$$\vartheta\|\Delta w\|_{L^2}^2 < \int_U wg(w)\,\mathrm{d}x \leq a\|w\|_{L^\gamma}^\gamma \leq aM^\gamma\|\Delta w\|_{L^2}^{\gamma-2}\|\Delta w\|_{L^2}^2,$$

and so $\|\Delta w\|_{L^2} > r(\vartheta)$.

(iii) If $\|\Delta w\|_{L^2} = 0$, then $\mathcal{D}_\vartheta(w) = 0$. Otherwise, from $\mathcal{D}_\vartheta(w) = 0$ with $\|\Delta w\|_{L^2} \neq 0$, we obtain

$$\vartheta\|\Delta w\|_{L^2}^2 = \int_U wg(w)\,\mathrm{d}x - \vartheta\|\nabla w\|_{L^p}^p \leq a\|w\|_{L^\gamma}^\gamma \leq aM^\gamma\|\Delta w\|_{L^2}^{\gamma-2}\|\Delta w\|_{L^2}^2$$

and then $\|\Delta w\|_{L^2} \geq r(\vartheta)$. $\square$

Lemma 2.6. *For $w \in H_0^2(U)$ with $\|\Delta w\|_{L^2} \neq 0$ and $\vartheta > 0$, the equation*

$$\vartheta\left(\|\Delta(\theta w)\|_{L^2}^2 + \|\nabla(\theta w)\|_{L^p}^p\right) = \int_U \theta wg(\theta w)\,\mathrm{d}x \tag{2.3}$$

can determine a unique solution $\theta = \theta(\vartheta) > 0$. Moreover, $\theta(\vartheta)$ is strictly increasing.

Proof. The proof for the existence of $\theta(\vartheta)$ is similar to Lemma 2.4(ii) and we do not show the process again. Next we prove the monotonicity. Now define $\theta_1 = \theta(\vartheta')$ and $\theta_2 = \theta(\vartheta'')$ for $0 < \vartheta' < \vartheta''$ and we want to show $\theta_1 < \theta_2$. If it is false, then $\theta_1 = \theta_2$ or $\theta_1 > \theta_2$. By (2.3), θ_1 and θ_2 satisfy the equations

$$\vartheta'\theta_1\|\Delta w\|_{L^2}^2 + \vartheta'\theta_1^{p-1}\|\nabla w\|_{L^p}^p = \int_U wg(\theta_1 w)\,\mathrm{d}x, \tag{2.4}$$

$$\vartheta''\theta_2\|\Delta w\|_{L^2}^2 + \vartheta''\theta_2^{p-1}\|\nabla w\|_{L^p}^p = \int_U wg(\theta_2 w)\,\mathrm{d}x. \tag{2.5}$$

For the case $p \le 2$, by eliminating the term $\|\Delta w\|_{L^2}^2$, we have

$$\vartheta'\vartheta''\|\nabla w\|_{L^p}^p(\theta_2\theta_1^{p-1} - \theta_1\theta_2^{p-1}) = \vartheta''\theta_2\int_U wg(\theta_1 w)\,\mathrm{d}x - \vartheta'\theta_1\int_U wg(\theta_2 w)\,\mathrm{d}x.$$

If $\theta_1 = \theta_2$, then the left-hand side is equal to zero and the right-hand side is positive. So here is a contradiction. If $\theta_1 > \theta_2$, then we use the change $\overline{w} = \theta_2 w$ with $\theta = \frac{\theta_1}{\theta_2} > 1$ to have

$$\begin{aligned}\vartheta'\vartheta''\|\nabla w\|_{L^p}^p\theta_1\theta_2(\theta_1^{p-2} - \theta_2^{p-2}) &= \vartheta''\int_U \overline{w}g(\theta\overline{w})\,\mathrm{d}x - \vartheta'\theta\int_U \overline{w}g(\overline{w})\,\mathrm{d}x\\ &> (\vartheta'' - \vartheta')\theta\int_U \overline{w}g(\overline{w})\,\mathrm{d}x\\ &> 0\end{aligned}$$

which implies $\theta_2 > \theta_1$ for $p < 2$ and it contradicts to $\theta_1 > \theta_2$. If $p = 2$, it still has a contradiction again and we do not show the details.

For $p > 2$, we can eliminative $\|\nabla w\|_{L^p}^p$ from (2.4) and (2.5) to have

$$\begin{aligned}\vartheta'\vartheta''\|\Delta w\|_{L^2}^2\theta_1\theta_2(\theta_2^{p-2} - \theta_1^{p-2}) &= \vartheta''\theta_2^{p-1}\int_U wg(\theta_1 w)\,\mathrm{d}x - \vartheta'\theta_1^{p-1}\int_U wg(\theta_2 w)\,\mathrm{d}x\\ &= \vartheta''\theta_2^{p-1}\int_U wg(\theta_1 w)\,\mathrm{d}x - \vartheta'\theta_1^{p-1}\int_U wg(\theta_2 w)\,\mathrm{d}x\\ &= \frac{\theta_1^{p-1}}{\theta_2}\int_U \overline{w}g(\overline{w})\,\mathrm{d}x(\vartheta'' - \vartheta')\\ &> 0.\end{aligned}$$

This contradicts to $\theta_1 > \theta_2$. □

Lemma 2.7. (i) *$d(\vartheta) > a(\vartheta)r^2(\vartheta)$ for $\vartheta \in \left(0, \frac{q+1}{2}\right)$ with $a(\vartheta) = \frac{1}{2} - \frac{\vartheta}{q+1}$.*

(ii) $\lim_{\vartheta\to 0^+} d(\vartheta) = 0$, $\lim_{\vartheta\to+\infty} d(\vartheta) = -\infty$.

(iii) *$d(\vartheta)$ is decreasing strictly on $\vartheta \in [1, +\infty)$, is increasing strictly on $\vartheta \in [0, 1]$, and has the maximum at $\vartheta = 1$.*

(iv) *There is a unique point* $b \in \left(\frac{q+1}{2}, \max\left\{\frac{\gamma}{2}, \frac{\gamma}{p}\right\}\right)$ *such that* $d(b) = 0$ *and* $d(\vartheta) > 0$ *if* $\vartheta \in (0, b)$.

Proof. (i) If $\mathcal{D}_\vartheta(w) = 0$ and $\|\Delta w\|_{L^2} \neq 0$, then Lemma 2.5 means $\|\Delta w\|_{L^2} \geq r(\vartheta)$. (H6) gives

$$\begin{aligned}
\mathcal{F}(w) &= \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{p}\|\nabla w\|_{L^p}^p - \int_U G(w)\,\mathrm{d}x \\
&\geq \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{p}\|\nabla w\|_{L^p}^p - \frac{1}{q+1}\int_U wg(w)\,\mathrm{d}x \\
&\geq \left(\frac{1}{2} - \frac{\vartheta}{q+1}\right)\|\Delta w\|_{L^2}^2 \\
&\geq a(\vartheta)r^2(\vartheta)
\end{aligned}$$

for $0 < \vartheta < \frac{q+1}{2}$.

(ii) By Corollary 2.2, we have

$$\vartheta\|\Delta w\|_{L^2}^2 \leq \frac{1}{\theta}\int_U wg(\theta w)\,\mathrm{d}x \leq \gamma A\theta^{\gamma-2}\int_U |w|^\gamma\,\mathrm{d}x,$$

which implies $\lim_{\vartheta\to+\infty}\theta(\vartheta) = +\infty$.

Next we want to prove $\lim_{\vartheta\to 0}\theta(\vartheta) = 0$ and it is easy to obtain this result when $p = 2$. For $p < 2$, (2.3) and (H4) give

$$\vartheta\theta^{2-p}\|\Delta w\|_{L^2}^2 + \vartheta\|\nabla w\|_{L^p}^p = \theta^{1-p}\int_U wg(\theta w)\,\mathrm{d}x > \theta^{2-p}\int_U wg(w)\,\mathrm{d}x.$$

We rewrite it as the form

$$\vartheta\|\nabla w\|_{L^p}^p > \theta^{2-p}\left(\int_U wg(w)\,\mathrm{d}x - \vartheta\|\Delta w\|_{L^2}^2\right).$$

It can give $\lim_{\vartheta\to 0}\theta(\vartheta) = 0$. For the final case $p > 2$, by using (H5), a similar process can give

$$\vartheta\|\Delta w\|_{L^2}^2 + \vartheta\theta^{p-2}\|\nabla w\|_{L^p}^p = \frac{1}{\theta}\int_U wg(\theta w)\,\mathrm{d}x > \theta^{p-2}\int_U wg(w)\,\mathrm{d}x,$$

and then

$$\vartheta\|\Delta w\|_{L^2}^2 > \theta^{p-2}\left(\int_U wg(w)\,\mathrm{d}x - \vartheta\|\nabla w\|_{L^p}^p\right).$$

That gives $\lim_{\vartheta\to 0}\theta(\vartheta) = 0$. Therefore, we can employ Lemma 2.4 to obtain

$$\begin{aligned}
&\lim_{\vartheta\to 0}\mathcal{F}(\theta w) = \lim_{\theta\to 0}\mathcal{F}(\theta w) = 0, \qquad && \lim_{\vartheta\to 0} d(\vartheta) = 0, \\
&\lim_{\vartheta\to+\infty}\mathcal{F}(\theta w) = \lim_{\theta\to+\infty}\mathcal{F}(\theta w) = -\infty, \qquad && \lim_{\vartheta\to+\infty} d(\vartheta) = -\infty.
\end{aligned}$$

(iii) For $0 < \vartheta' < \vartheta'' < 1$ (or the case $1 < \vartheta'' < \vartheta'$), we want to prove $d(\vartheta') < d(\vartheta'')$. For this purpose, it is enough for us to show that for any $w \in H_0^2(U)$ with $\mathcal{D}_{\vartheta''}(w) = 0$

and $\|\Delta w\|_{L^2} \neq 0$, there exists a function $v \in H_0^2(U)$ with $\mathcal{D}_{\vartheta'}(v) = 0$ and $\|\Delta v\|_{L^2} \neq 0$ such that $\mathcal{F}(v) < \mathcal{F}(w) - \varepsilon(\vartheta', \vartheta'')$ for $\varepsilon(\vartheta', \vartheta'') > 0$.

For w, (2.3) can determine a real number $\theta(\vartheta)$ so that $\mathcal{D}_\vartheta(\theta(\vartheta)w) = 0$. We deduce from $\mathcal{D}_{\vartheta''}(w) = 0$ that $\theta(\vartheta'') = 1$. Moreover, by defining $v = \theta(\vartheta')w$, we have $\mathcal{D}_{\vartheta'}(v) = 0$ and $\|\Delta v\|_{L^2} \neq 0$.

By letting $h(\theta) = \mathcal{F}(\theta w)$, one has

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}\theta}h(\theta) &= \frac{1}{\theta}\big((1-\vartheta)\|\Delta\theta w\|_{L^2}^2 + (1-\vartheta)\|\nabla\theta w\|_{L^p}^p + \mathcal{D}_\vartheta(\theta w)\big)\\ &= (1-\vartheta)\theta\|\Delta w\|_{L^2}^2 + (1-\vartheta)\theta^{p-1}\|\nabla w\|_{L^p}^p.\end{aligned}$$

If $\vartheta', \vartheta'' \in (0,1)$ $(\vartheta' < \vartheta'')$, then

$$\mathcal{F}(w) - \mathcal{F}(v) = h(1) - h(\theta(\vartheta')) > (1-\vartheta'')r^2(\vartheta'')\theta(\vartheta')(1-\theta(\vartheta')) \doteq \varepsilon(\vartheta', \vartheta'').$$

If $\vartheta', \vartheta'' \in (1, +\infty)$ $(\vartheta'' < \vartheta')$, then

$$\mathcal{F}(w) - \mathcal{F}(v) = h(1) - h(\theta(\vartheta')) > (\vartheta''-1)r^2(\vartheta'')\theta(\vartheta'')(\theta(\vartheta')-1) \doteq \varepsilon(\vartheta', \vartheta'').$$

Thus we have (iii).

(iv) From (i)–(iii), there exists a positive constant $b \geq \frac{q+1}{2}$ such that $d(b) = 0$ and $d(\vartheta) > 0$ for $\vartheta \in (0, b)$. Moreover, by (H6), we have

$$\begin{aligned}\mathcal{F}(w) &= \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{p}\|\nabla w\|_{L^p}^p - \int_U G(w)\,\mathrm{d}x\\ &\leq \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{p}\|\nabla w\|_{L^p}^p - \frac{1}{\gamma}\int_U wg(w)\,\mathrm{d}x\\ &= \left(\frac{1}{2} - \frac{\vartheta}{\gamma}\right)\|\Delta w\|_{L^2}^2 + \left(\frac{1}{p} - \frac{\vartheta}{\gamma}\right)\|\nabla w\|_{L^p}^p + \frac{1}{\gamma}\mathcal{D}_\vartheta(w)\\ &< 0,\end{aligned}$$

for $\mathcal{D}_\vartheta(w) = 0$ with $\|\Delta w\|_{L^2} \neq 0$ if $\vartheta > \max\left\{\frac{\gamma}{2}, \frac{\gamma}{p}\right\}$. Hence $b \leq \max\left\{\frac{\gamma}{2}, \frac{\gamma}{p}\right\}$. □

Lemma 2.8. [6, 10]

(i) $\inf_{w\in\mathbb{K}} \|\Delta w\|_{L^2} \geq c_1 > 0$, $\inf_{w\in\mathbb{K}_-} \|\Delta w\|_{L^2} \geq c_2 > 0$;

(ii) $\mathcal{F}^s \cap \mathbb{K}_+$ *is bounded in* $H_0^2(U)$*-norm for any* $s > 0$.

Lemma 2.9. [6,10] *Assume* $w \in H_0^2(U)$ *and* $0 < \mathcal{F}(w) < d$. *Let* ϑ_1 *and* ϑ_2 $(\vartheta_1 < 1 < \vartheta_2)$ *be two solutions to the equation* $d(\vartheta) = \mathcal{F}(w)$. *Then* $\mathcal{D}_\vartheta(w)$ *does not change the sign for* $\vartheta \in (\vartheta_1, \vartheta_2)$.

We define the weak solutions as follows.

Definition 2.10. If a function w with $w \in L^\infty(0,T;H_0^2(U))$ and $w_t \in L^2(0,T;L^2(U))$ satisfies

$$(w_t,\phi)+(\Delta w,\Delta\phi)+(|\nabla w|^{p-2}\nabla w,\nabla\phi)=(g(w),\phi), \tag{2.6}$$

$$\int_0^t \|w_\tau\|_{L^2}^2\,\mathrm{d}\tau+\mathcal{F}(w)=\mathcal{F}(w_0), \tag{2.7}$$

and $w(x,0)=w_0$ for $t\in(0,T)$ and $\phi\in H_0^2(U)$, then it is said to be a weak solution of (1.1). $w(x,t)$ is said to be a global weak solution if it is a weak solution for each $T>0$.

Lemma 2.11. [6,10] *Assume that $0<\mathcal{F}(w_0)<d$ and w is a weak solution of* (1.1). *Let ϑ_1 and ϑ_2 ($\vartheta_1<1<\vartheta_2$) be two solutions of $d(\vartheta)=\mathcal{F}(w_0)$.*

(i) *$w\in\mathbb{W}_\vartheta$ for $\vartheta\in(\vartheta_1,\vartheta_2)$ if $\mathcal{D}(w_0)>0$;*

(ii) *$w\in\mathbb{V}_\vartheta$ for $\vartheta\in(\vartheta_1,\vartheta_2)$ if $\mathcal{D}(w_0)<0$.*

Lemma 2.12. *For fixed constant $s>d$,*

$$0<\theta_s\le\Theta_s<+\infty.$$

Proof. For $w\in H_0^2(U)$, the Gagliardo–Nirenberg inequality gives

$$\|w\|_{L^\gamma}\le C\|\Delta w\|_{L^2}^{\alpha}\|w\|_{L^2}^{(1-\alpha)}$$

with $\alpha=\frac{N(\gamma-2)}{4\gamma}$ (The condition (H6) can ensure $\alpha\in(0,1)$).

Now for $s>d$ and $w\in\mathbb{K}^s$, Corollary 2.2 means

$$\|\Delta w\|_{L^2}^2<\int_U wg(w)\,\mathrm{d}x\le\gamma A\|w\|_{L^\gamma}^\gamma\le C\|\Delta w\|_{L^2}^{\alpha\gamma}\|w\|_{L^2}^{(1-\alpha)\gamma},$$

and then

$$\|\Delta w\|_{L^2}^{2-\alpha\gamma}\le C\|w\|_{L^2}^{(1-\alpha)\gamma}.$$

Lemma 2.8(i) implies $\theta_s>0$ and the Sobolev embedding theorem $\|w\|_{L^2}\le M_*\|\Delta w\|_{L^2}$ gives $\Theta_s<\infty$. □

3. Main results

We list the main results in this section.

Theorem 3.1. *For $w_0\in H_0^2(U)$, $\mathcal{F}(w_0)<d$ and $\mathcal{D}(w_0)>0$, the problem* (1.1) *owns a global solution w satisfying Definition* 2.10 *and $w(t)\in\mathbb{W}$ for each t. It is unique for bounded weak solutions. Moreover, $\|w\|_{L^2}^2\le\|w_0\|_{L^2}^2e^{-\mu t}$ for some constant $\mu>0$.*

Theorem 3.2. *If $w_0 \in H_0^2(U)$, $\mathcal{F}(w_0) < d$ and $\mathcal{D}(w_0) < 0$, then $\lim_{t \to T^-} \int_0^t \|w\|_{L^2}^2 \,\mathrm{d}\tau = +\infty$ for any weak solution w and some constant $T > 0$, i.e., w blows up at $t = T$.*

Theorem 3.3. *If $w_0 \in H_0^2(U)$, $\mathcal{F}(w_0) = d$ and $\mathcal{D}(w_0) \geq 0$, then* (1.1) *owns a weak solution satisfying Definition* 2.10 *and $w(t) \in \overline{\mathbb{W}}$ for each t. Besides, it is unique for bounded weak solutions.*

Moreover, $\|w\|_{L^2}^2 \leq C_1 e^{-C_2 t}$ for constants C_1 and C_2 if $\mathcal{D}(w(x,t)) > 0$ for $t > 0$. Otherwise, w will vanish in a finite time.

Theorem 3.4. *If $w_0 \in H_0^2(U)$, $\mathcal{F}(w_0) = d$ and $\mathcal{D}(w_0) < 0$, then $\lim_{t \to T^-} \int_0^t \|w\|_{L^2}^2 \,\mathrm{d}\tau = +\infty$ for any weak solution w and some constant $T > 0$, i.e., w blows up at $t = T$.*

Theorem 3.5. *Assume that w is a weak solution, $w_0 \in H_0^2(U)$ and $\mathcal{F}(w_0) > d$.*

(i) *If $w_0 \in \mathbb{K}_+$ and $\|w_0\|_{L^2} \leq \theta_{\mathcal{F}(w_0)}$, then $w(t) \to 0$ in $H_0^2(U)$ as $t \to \infty$.*

(ii) *If $w_0 \in \mathbb{K}_-$ and $\|w_0\|_{L^2} \geq \Theta_{\mathcal{F}(w_0)}$, then w blows up at some point $t = T$.*

4. The case $\mathcal{F}(w_0) < d$

In this section, we are going to show the proof for the global existence, uniqueness and time decay rate if $\mathcal{F}(w_0) < d$ and $\mathcal{D}(w_0) > 0$, as well as the blow-up behavior in finite time if $\mathcal{F}(w_0) < d$ and $\mathcal{D}(w_0) < 0$.

Proof of Theorem 3.1. Let $\{\phi_j(x)\}$ $(j = 1, 2, \ldots)$ be a basis of $H_0^2(U)$ and we introduce the approximate solutions of (1.1) as

$$w^m = \sum_{j=1}^{m} a_j^m(t)\phi_j(x), \quad m = 1, 2, \ldots$$

which solve

$$(w_t^m, \phi_j) + (\Delta w^m, \Delta\phi_j) + \big(|\nabla w^m|^{p-2}\nabla w^m, \nabla\phi_j\big) = (g(w^m), \phi_j), \tag{4.1}$$

$$w^m(x,0) = \sum_{j=1}^{m} b_j^m \phi_j(x) \tag{4.2}$$

with $w^m(x,0) \to w_0(x)$ in $H_0^2(U)$ as $m \to \infty$. Peano's theorem ensures the local existence of (4.1)–(4.2) and it can become global from the following uniform estimates (4.6)–(4.9).

For this purpose, we take $\frac{\mathrm{d}}{\mathrm{d}t} a_j^m(t)$ as a multiplier of (4.1) to have

$$\int_0^t \|w_\tau^m\|_{L^2}^2 \,\mathrm{d}\tau + \mathcal{F}(w^m) = \mathcal{F}(w^m(0)) \tag{4.3}$$

for $t \in (0, \infty)$. Moreover, it is easy to check that

$$\mathcal{F}(w^m(0)) \to \mathcal{F}(w_0) < d, \quad \mathcal{D}(w^m(0)) \to \mathcal{D}(w_0) > 0$$

as $m \to \infty$, which implies $w^m(x, 0) \in \mathbb{W}$ and for big m. Furthermore, one has

$$\int_0^t \|w_\tau^m\|_{L^2}^2 \,\mathrm{d}\tau + \mathcal{F}(w^m) = \mathcal{F}(w^m(0)) < d, \quad \mathcal{D}(w^m(0)) > 0. \tag{4.4}$$

Next we prove $w^m \in \mathbb{W}$ for big m and each t. If it is false, then we can seek a constant $t_0 > 0$ such that $\mathcal{D}(w^m(t_0)) = 0$ with $\|\Delta w^m(t_0)\|_{L^2} \neq 0$ or $\mathcal{F}(w^m(t_0)) = d$ by applying the continuity of $\mathcal{D}(w^m)$ and $\mathcal{F}(w^m)$ with respect to t. By (4.4), we deduce that $\mathcal{F}(w^m(t_0)) = d$ does not hold. Thus, $\mathcal{D}(w^m(t_0)) = 0$ with $\|\Delta w^m(t_0)\|_{L^2} \neq 0$ which implies $\mathcal{F}(w^m(t_0)) \geq d$ by the definition of d. It contradicts to (4.4). Therefore, we obtain $w^m(x, t) \in \mathbb{W}$.

By (H6), one has

$$\begin{aligned}
\mathcal{F}(w^m) &= \frac{1}{2}\|\Delta w^m\|_{L^2}^2 + \frac{1}{p}\|\nabla w^m\|_{L^p}^p - \int_U G(w^m) \,\mathrm{d}x \\
&\geq \frac{1}{2}\|\Delta w^m\|_{L^2}^2 + \frac{1}{p}\|\nabla w^m\|_{L^p}^p - \frac{1}{q+1}\int_U w^m g(w^m) \,\mathrm{d}x \\
&= \left(\frac{1}{2} - \frac{1}{q+1}\right)\|\Delta w^m\|_{L^2}^2 + \left(\frac{1}{p} - \frac{1}{q+1}\right)\|\nabla w^m\|_{L^p}^p + \frac{1}{q+1}\mathcal{D}(w^m).
\end{aligned}$$

For large m, (4.4) implies

$$\left(\frac{1}{2} - \frac{1}{q+1}\right)\|\Delta w^m\|_{L^2}^2 + \left(\frac{1}{p} - \frac{1}{q+1}\right)\|\nabla w^m\|_{L^p}^p < d \tag{4.5}$$

with $0 \leq t < \infty$. Now combining (4.4) with (4.5) gives

$$\|w^m\|_{H_0^2(U)}^2 \leq \frac{2(q+1)d}{q-1}, \tag{4.6}$$

$$\|\nabla w^m\|_{L^p}^p \leq \frac{p(q+1)d}{q+1-p}, \tag{4.7}$$

$$\int_0^t \|w_\tau^m\|_{L^2}^2 \,\mathrm{d}\tau < d. \tag{4.8}$$

Besides, applying Corollary 2.2 and $H_0^2(U) \hookrightarrow L^\gamma(U)$ to have

$$\begin{aligned}
\|g(w^m)\|_{\frac{\gamma}{\gamma-1}}^{\frac{\gamma}{\gamma-1}} &\leq \int_U (\gamma A|w^m|^{\gamma-1})^{\frac{\gamma}{\gamma-1}} \,\mathrm{d}x = (\gamma A)^{\frac{\gamma}{\gamma-1}}\|w^m\|_{L^\gamma}^\gamma \\
&\leq (\gamma A)^{\frac{\gamma}{\gamma-1}} M^\gamma \|w^m\|_{H_0^2(U)}^\gamma \leq (\gamma A)^{\frac{\gamma}{\gamma-1}} M^\gamma \left(\frac{2(q+1)d}{q-1}\right)^{\frac{\gamma}{2}}.
\end{aligned} \tag{4.9}$$

The estimates (4.6)–(4.9) and Aubin's lemma (see [15]) allow us to find a function w and a subsequence of $\{w^m\}$ (still denoted by itself here and hereafter) such that for each $T > 0$,

$$\begin{aligned}
&w_t^m \rightharpoonup w_t && \text{in } L^2(0,T;L^2(U)), && (4.10)\\
&w^m \overset{*}{\rightharpoonup} w && \text{in } L^\infty(0,T;H_0^2(U)), && (4.11)\\
&w^m \to w && \text{strongly in } W_0^{1,p}(U) \text{ and } L^\gamma(U) \text{ for each } t\in(0,T), && (4.12)\\
&w^m \to w && \text{a.e. in } U\times(0,T), &&\\
&g(w^m) \rightharpoonup g(w) && \text{in } L^{\frac{\gamma}{\gamma-1}}(U\times(0,T)), &&\\
&g(w^m) \to g(w) && \text{a.e. in } U\times(0,T) &&
\end{aligned}$$

as $m \to \infty$, where we the notation $\rightharpoonup$ denotes the weak convergence and $\overset{*}{\rightharpoonup}$ denotes the weak-star convergence respectively.

For any $v \in L^2(0,T;H_0^2(U))$ (may need an approximate process), we employ (4.1)–(4.2) to have

$$\int_0^T \big((w_t^m, v) + (\Delta w^m, \Delta v) + (|\nabla w^m|^{p-2}\nabla w^m, \nabla v)\big)\,\mathrm{d}t = \int_0^T (g(w^m), v)\,\mathrm{d}t.$$

Take the limit $m \to \infty$ to give

$$\int_0^T \big((w_t, v) + (\Delta w, \Delta v) + (|\nabla w|^{p-2}\nabla w, \nabla v)\big)\,\mathrm{d}t = \int_0^T (g(w), v)\,\mathrm{d}t.$$

The arbitrariness of T ensures

$$(w_t, \phi) + (\Delta w, \Delta\phi) + (|\nabla w|^{p-2}\nabla w, \nabla\phi) = (g(w), \phi) \tag{4.13}$$

for each $\phi \in H_0^2(U)$ and $t > 0$.

Now we want to prove (2.7). By mean value theorem, one has

$$\left|\int_U (G(w^m) - G(w))\,\mathrm{d}x\right| \le \int_U |g(\xi_m)(w^m - w)|\,\mathrm{d}x \le \|g(\xi_m)\|_{L^{\frac{\gamma}{\gamma-1}}} \|w^m - w\|_{L^\gamma}$$

where $\xi_m = (1-\delta_m)w^m + \delta_m w$ for some $0 < \delta_m < 1$. It follows that

$$\lim_{m\to\infty} \int_U G(w^m)\,\mathrm{d}x = \int_U G(w)\,\mathrm{d}x.$$

By (4.10)–(4.12), the weak lower semi-continuity of L^2 space allows us to pass to the limit $m \to \infty$ in (4.3). Hence, we can deduce (2.7).

Following that, we establish the uniqueness of bounded weak solutions. Assume for this reason that w and v are two bounded weak solutions that fulfill (4.13). In the difference of the corresponding equalities for w and v, we choose $\varphi = w - v$ as the test function to obtain

$$\begin{aligned}&\int_0^t\int_U \big(\varphi_t\varphi + |\Delta\varphi|^2 + (|\nabla w|^{p-2}\nabla w - |\nabla v|^{p-2}\nabla v, \nabla w - \nabla v)\big)\,\mathrm{d}x\mathrm{d}t\\ =&\int_0^t\int_U (g(w) - g(v))(w - v)\,\mathrm{d}x\mathrm{d}t.\end{aligned}$$

From $\varphi(x,0) = 0$, we employ the monotonicity of $|s|^{p-2}s$ for $s \in R$ or R^N and the boundedness of two solutions to have

$$\int_U \varphi^2\,\mathrm{d}x \le C\int_0^t\int_U \varphi^2\,\mathrm{d}x\mathrm{d}t.$$

Gronwall's inequality gives

$$\int_U \varphi^2(x,t)\,\mathrm{d}x = 0,$$

and then $\varphi = 0$ in $U \times (0,\infty)$.

In order to show the decay behavior, taking $\varphi = w$ in (4.1) to have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 = (w_t, w) = -\|\Delta w\|_{L^2}^2 - \|\nabla w\|_{L^p}^p + \int_U wg(w)\,\mathrm{d}x = -\mathcal{D}(w). \tag{4.14}$$

According to Lemma 2.11, we conclude that $w(x,t) \in \mathbb{W}_\vartheta$ for $t \in (0,\infty)$ and $\vartheta_1 < \vartheta < \vartheta_2$ if $\mathcal{F}(w_0) < d$ and $\mathcal{D}(w_0) > 0$. This implies $\mathcal{D}_{\vartheta_1}(w) \ge 0$ for $0 < t < \infty$. Therefore, one has

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 &= -\mathcal{D}(w) = (\vartheta_1 - 1)\|\Delta w\|_{L^2}^2 + (\vartheta_1 - 1)\|\nabla w\|_{L^p}^p - \mathcal{D}_{\vartheta_1}(w)\\ &\le M_*^{-2}(\vartheta_1 - 1)\|w\|_{L^2}^2,\end{aligned}$$

where M_* is the best constant for the embedding $H_0^2(U) \hookrightarrow L^2(U)$. Consequently,

$$\|w\|_{L^2}^2 \le \|w_0\|_{L^2}^2 e^{-2M_*^{-2}(1-\vartheta_1)t}$$

with $C = 2M_*^{-2}(1 - \vartheta_1) > 0$. □

Proof of Theorem 3.2. Suppose that w is a global weak solution to (1.1) with $\mathcal{F}(w_0) < d, \mathcal{D}(w_0) < 0$. Define a function with respect to t:

$$\mathcal{L}(t) = \int_0^t \|w\|_{L^2}^2\,\mathrm{d}\tau, \quad t \ge 0.$$

One has

$$\mathcal{L}'(t) = \|w\|_{L^2}^2, \tag{4.15}$$

and

$$\mathcal{L}''(t) = 2(w_t, w) = -2\left(\|\Delta w\|_{L^2}^2 + \|\nabla w\|_{L^p}^p - \int_U wg(w)\,\mathrm{d}x\right) = -2\mathcal{D}(w). \tag{4.16}$$

It is easy to check

$$\begin{aligned}\mathcal{F}(w) &= \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{p}\|\nabla w\|_{L^p}^p - \int_U G(w)\,\mathrm{d}x \\ &\geq \frac{1}{2}\|\Delta w\|_{L^2}^2 + \frac{1}{p}\|\nabla w\|_{L^p}^p - \frac{1}{q+1}\int_U wg(w)\,\mathrm{d}x \\ &= \frac{q-1}{2(q+1)}\|\Delta w\|_{L^2}^2 + \frac{q+1-p}{p(q+1)}\|\nabla w\|_{L^p}^p + \frac{1}{q+1}\mathcal{D}(w).\end{aligned} \tag{4.17}$$

We can employ (2.7), (4.2), (4.16) and (4.17) to obtain

$$\begin{aligned}\mathcal{L}''(t) &\geq (q-1)\|\Delta w\|_{L^2}^2 + \frac{2(q+1-p)}{p}\|\nabla w\|_{L^p}^p - 2(q+1)\mathcal{F}(w) \\ &\geq (q-1)\|\Delta w\|_{L^2}^2 + 2(q+1)\int_0^t \|w_\tau\|_{L^2}^2\,\mathrm{d}\tau - 2(q+1)\mathcal{F}(w_0) \\ &\geq \frac{q-1}{M_*^2}\mathcal{L}'(t) + 2(q+1)\int_0^t \|w_\tau\|_{L^2}^2\,\mathrm{d}\tau - 2(q+1)\mathcal{F}(w_0).\end{aligned}$$

In view of

$$(\mathcal{L}'(t))^2 = 4\left(\int_0^t\int_U w_\tau w\,\mathrm{d}x\mathrm{d}\tau\right)^2 + 2\|w_0\|_{L^2}^2\mathcal{L}'(t) - \|w_0\|_{L^2}^4,$$

one has

$$\begin{aligned}\mathcal{L}''(t)\mathcal{L}(t) - \frac{q+1}{2}\mathcal{L}'(t)^2 \geq{}& 2(q+1)\int_0^t \|w_\tau\|_{L^2}^2\,\mathrm{d}\tau\int_0^t \|w\|_{L^2}^2\,\mathrm{d}\tau - 2(q+1)\mathcal{F}(w_0)\mathcal{L}(t) \\ &+ \frac{q-1}{M_*^2}\mathcal{L}'(t)\mathcal{L}(t) - 2(q+1)\left(\int_0^t\int_U w_\tau w\,\mathrm{d}x\mathrm{d}\tau\right)^2 \\ &- (q+1)\|w_0\|_{L^2}^2\mathcal{L}'(t) + \frac{q+1}{2}\|w_0\|_{L^2}^4.\end{aligned}$$

By Hölder inequality, we deduce that

$$\begin{aligned}&\mathcal{L}''(t)\mathcal{L}(t) - \frac{q+1}{2}\mathcal{L}'(t)^2 \\ &\geq \frac{q-1}{M_*^2}\mathcal{L}'(t)\mathcal{L}(t) - (q+1)\|w_0\|_{L^2}^2\mathcal{L}'(t) + \frac{q+1}{2}\|w_0\|_{L^2}^4 - 2(q+1)\mathcal{F}(w_0)\mathcal{L}(t) \\ &\geq \frac{q-1}{M_*^2}\mathcal{L}'(t)\mathcal{L}(t) - (q+1)\|w_0\|_{L^2}^2\mathcal{L}'(t) - 2(q+1)\mathcal{F}(w_0)\mathcal{L}(t).\end{aligned} \tag{4.18}$$

Next we consider two cases $\mathcal{F}(w_0) \leq 0$ and $0 < \mathcal{F}(w_0) < d$ respectively. If $\mathcal{F}(w_0) \leq 0$, then (4.18) implies

$$\mathcal{L}''(t)\mathcal{L}(t) - \frac{q+1}{2}\mathcal{L}'(t)^2 \geq \frac{q-1}{M_*^2}\mathcal{L}'(t)\mathcal{L}(t) - (q+1)\|w_0\|_{L^2}^2\mathcal{L}'(t).$$

Here we need to show $\mathcal{D}(w) < 0$ for $t > 0$ firstly. If it does not hold, then there exists a constant t_0 such that $\mathcal{D}(w) < 0$ for $0 \geq t < t_0$ and $\mathcal{D}(w(t_0)) = 0$. For $0 \leq t < t_0$, Lemma 2.5 implies $\|\Delta w\|_{L^2} > r(1)$ and $\|\Delta w(t_0)\|_{L^2} \geq r(1)$. Hence, $w(t_0) \in \mathbb{K}$ and $\mathcal{F}(w(t_0)) \geq d$. That contradicts to (2.7) and so we have $\mathcal{D}(w) < 0$ $(t > 0)$.

Now apply (4.16) to get $\mathcal{L}''(t) > 0$ for $t \geq 0$, and so $\mathcal{L}'(t)$ is increasing with respect to t. Besides, (4.15) means $\mathcal{L}'(0) \geq 0$ and there is a constant $t_1 \geq 0$ such that $\mathcal{L}'(t_1) > 0$ and

$$\mathcal{L}(t) \geq \mathcal{L}'(t_1)(t - t_1)$$

for $t > t_1$. For big enough t,

$$\frac{q-1}{M_*^2}\mathcal{L}(t) > (q+1)\|w_0\|_{L^2}^2,$$

and

$$\mathcal{L}''(t)\mathcal{L}(t) - \frac{q+1}{2}\mathcal{L}'(t)^2 > 0. \tag{4.19}$$

For the case $0 < \mathcal{F}(w_0) < d$, we still want to obtain (4.19). We apply Lemma 2.11 to give $w(t) \in \mathbb{V}_\vartheta$ with $\vartheta_1 < \vartheta < \vartheta_2$, here ϑ_1 and ϑ_2 $(\vartheta_1 < 1 < \vartheta_2)$ are two roots to $d(\vartheta) = \mathcal{F}(w_0)$. As a result, one has $\mathcal{D}_{\vartheta_2}(w) \leq 0$ and $\|\Delta w\|_{L^2} \geq r(\vartheta_2)$. (4.16) yields

$$\begin{aligned}
\mathcal{L}''(t) &= -2\mathcal{D}(w) \\
&= 2(\vartheta_2 - 1)\|\Delta w\|_{L^2}^2 + 2(\vartheta_2 - 1)\|w\|_{L^p}^p - 2\mathcal{D}_{\vartheta_2}(w) \\
&\geq 2(\vartheta_2 - 1)r^2(\vartheta_2).
\end{aligned}$$

Then

$$\mathcal{L}'(t) \geq 2r^2(\vartheta_2)(\vartheta_2 - 1)t \quad \text{and} \quad \mathcal{L}(t) \geq r^2(\vartheta_2)(\vartheta_2 - 1)t^2.$$

From it, we deduce that

$$\frac{q-1}{2M_*^2}\mathcal{L}(t) > (q+1)\|w_0\|_{L^2}^2, \quad \frac{q-1}{2M_*^2}\mathcal{L}'(t) > 2(q+1)\mathcal{F}(w_0)$$

for sufficiently large t. Hence (4.18) is positive and (4.19) holds again.

According to (4.19), we can seek a constant $\widetilde{t}$ such that $\big(\frac{\mathcal{L}'(t)}{\mathcal{L}^{\frac{1+q}{2}}}\big)' > 0$ for $t > \widetilde{t}$ and

$$\frac{\mathcal{L}'(t)}{\mathcal{L}^{\frac{1+q}{2}}(t)} > \frac{\mathcal{L}'(\widetilde{t})}{\mathcal{L}^{\frac{1+q}{2}}(\widetilde{t})}.$$

By solving this equation, there exists two constants $C_3, C_4 > 0$ such that

$$\mathcal{L}^{\frac{q-1}{2}}(t) > \frac{C_3}{C_4 - t},$$

which gives Theorem 3.2. $\square$

5. The case $\mathcal{F}(w_0) = d$

The critical case $\mathcal{F}(w_0) = d$ is considered in this section and the proofs of Theorems 3.3 and 3.4 will be shown.

Proof of Theorem 3.3. Let $\theta_s = 1 - \frac{1}{s}$ $(s = 1, 2, \ldots)$ and $w_0^s = \theta_s w_0(x)$. We introduce the following approximation problem

$$\tag{5.1} \begin{cases} w_t^s + \Delta^2 w^s - \operatorname{div}(|\nabla w^s|^{p-2}\nabla w^s) = g(w^s), & (x,t) \in U \times (0,T), \\ w^s = \frac{\partial w^s}{\partial \nu} = 0, & (x,t) \in \partial U \times (0,T), \\ w^s(x,0) = \theta_s w_0^s(x), & x \in U. \end{cases}$$

According to $\mathcal{D}(w_0) \geq 0$ and Lemma 2.4, there exists a unique constant $\theta^* = \theta^*(w_0) \geq 1$ such that $\mathcal{D}(\theta^* w_0) = 0$. The condition $\theta_s < 1 \leq \theta^*$ implies $\mathcal{D}(w_0^s) = \mathcal{D}(\theta_s w_0) > 0$ and $\mathcal{F}(w_0^s) = \mathcal{F}(\theta_s w_0) < \mathcal{F}(w_0) = d$. Theorem 3.1 allow us to deduce that (5.1) has a solution $w^s \in L^\infty(0,\infty; H_0(U))$, $w_t^s \in L^2(0,\infty; L^2(U))$, $w^s \in \mathbb{W}$ with

$$\int_0^t \|w_\tau^s\|_{L^2}^2 \, \mathrm{d}\tau + \mathcal{F}(w^s) = \mathcal{F}(w_0^s) < d.$$

Thus we only take a similar process of Theorem 3.1 to seek a subsequence of $\{w^s\}$ and a function w such that w is a solution of (1.1) with $\mathcal{D}(w) \geq 0$ and $\mathcal{F}(w) \leq d$. The uniqueness is also similar to Theorem 3.1 and we do not give the details here.

Now we prove the large time behavior under the condition $\mathcal{D}(w) > 0$ for $0 < t < \infty$. This means that w can not vanish in a finite time. We treat $\varphi = w$ as a test function in (2.6) to have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|w\|_{L^2}^2 = \int_U w_t w \, \mathrm{d}x = -\mathcal{D}(w) < 0.$$

This implies $w_t \not\equiv 0$. From (2.7),

$$0 < \mathcal{F}(w(t_0)) = d - \int_0^{t_0} \|w_\tau\|_{L^2}^2 \, \mathrm{d}\tau = d_1 < d$$

for a small $t_0 > 0$. It allows us to treat t_0 as the initial time. In view of Lemma 2.11, one has $w \in \mathbb{W}_\vartheta$ for $t > t_0$ and $\vartheta_1 < \vartheta < \vartheta_2$. Here ϑ_1 and ϑ_2 $(\vartheta_1 < 1 < \vartheta_2)$ are two roots to $d(\vartheta) = d_1$. Thus $\mathcal{D}_{\vartheta_1}(w) \geq 0$ $(t > t_0)$ and

$$\frac{1}{2}\frac{d}{dt}\|w\|_{L^2}^2 = -\mathcal{D}(w) \leq (\vartheta_1 - 1)\|\Delta w\|_{L^2}^2 - \mathcal{D}_{\vartheta_1}(w) \leq M_*^{-2}(\vartheta_1 - 1)\|w\|_{L^2}^2.$$

Gronwall's inequality gives

$$\|w\|_{L^2}^2 \leq \|w(t_0)\|_{L^2}^2 C_1 e^{-C_2 t}$$

with $C_1 = \|w(t_0)\|_{L^2}^2 e^{2M_*^{-2}(1-\vartheta_1)t_0}$ and $C_2 = 2M_*^{-2}(1-\vartheta_1)$.

On the other hand, if there is a point $t_0 > 0$ such that $\mathcal{D}(w) > 0$ for $0 < t < t_0$ with $\mathcal{D}(w(x,t_0)) = 0$, then apply (2.7) again and $w_t \not\equiv 0$ to obtain

$$\mathcal{F}(w(t_0)) = d - \int_0^{t_0} \|w_\tau\|_{L^2}^2 \, \mathrm{d}\tau < d.$$

The definition of d can ensure $\|\Delta w(t_0)\|_{L^2}^2 = 0$ and $w(t_0) = 0$. Since $\|w\|_{L^2}^2$ is decreasing with respect to t, we have $w(x,t) \equiv 0$ for $t \geq t_0$, which can complete the proof. □

Proof of Theorem 3.4. From $\mathcal{F}(w_0) = d$ and $\mathcal{D}(w_0) < 0$, the continuity of $\mathcal{F}(\cdot)$ and $\mathcal{D}(\cdot)$ ensures that there is a constant $t_0 > 0$ such that $\mathcal{F}(w) > 0$ and $\mathcal{D}(w) < 0$ for $0 < t \leq t_0$. $(w_t, w) = -\mathcal{D}(w)$ implies $w_t \not\equiv 0$ $(0 < t \leq t_0)$ and so

$$\mathcal{F}(w(t_0)) = d - \int_0^{t_0} \|w_\tau\|_{L^2}^2 \, \mathrm{d}\tau = d_1 < d.$$

The constant t_0 can be treated as the initial time. Lemma 2.11 implies $w \in \mathbb{V}_\vartheta$ with $\vartheta_1 < \vartheta < \vartheta_2$ and $t > t_0$ here ϑ_1 and ϑ_2 $(\vartheta_1 < 1 < \vartheta_2)$ are two roots to $d(\vartheta) = d_1$. Thus, one has $\mathcal{D}_\vartheta(w) < 0$ and $\|\Delta w\|_{L^2}^2 > r(\vartheta)$ $(\vartheta_1 < \vartheta < \vartheta_2,\ t > t_0)$, and then $\mathcal{D}_{\vartheta_2}(w) \leq 0$ and $\|\Delta w\|_{L^2} \geq r(\vartheta_2)$ $(t > t_0)$. The remaining proof is similar to that of Theorem 3.2 and we do not show the details again. □

6. The case $\mathcal{F}(w_0) > d$

We investigate the existence and nonexistence of solutions for the case $\mathcal{F}(w_0) > d$ in this section.

Proof of Theorem 3.5. For $w_0 \in H_0^2(U)$, use the notation $T(w_0)$ to represent the maximal time of existence and we define the ω-limit set of w_0 as

$$\omega(w_0) = \bigcap_{t \geq 0} \overline{\{w(s) : s \geq t\}}.$$

(i) Suppose that $w_0 \in \mathbb{K}_+$ with $\|w_0\|_{L^2} \leq \theta_{\mathcal{F}(w_0)}$. At first we need to show $w(t) \in \mathbb{K}_+$ for $t > 0$. If it is not true, then there exists a constant $t_0 > 0$ such that $w(t) \in \mathbb{K}_+$ for $t \in [0, t_0)$ and $w(t_0) \in \mathbb{K}$. From $\mathcal{D}(w(t)) = -(w_t, w)$, we have $w_t \not\equiv 0$ for $t \in (0, t_0)$. By (2.7), we can give $\mathcal{F}(w(t_0)) < \mathcal{F}(w_0)$. Hence $w(t_0) \in \mathcal{F}^{\mathcal{F}(w_0)}$, $w(t_0) \in \mathbb{K}^{\mathcal{F}(w_0)}$ and

$$\|w(t_0)\|_{L^2} \geq \theta_{\mathcal{F}(w_0)}. \tag{6.1}$$

However, by $w(t) \in \mathbb{K}_+$ and (4.14), one has

$$\|w(t_0)\|_{L^2} < \|w_0\|_{L^2} \leq \theta_{\mathcal{F}(w_0)}$$

which contradicts to (6.1). Therefore, we obtain $w(t) \in \mathbb{K}_+$ with $w(t) \in \mathcal{F}^{\mathcal{F}(w_0)}$ for $t \geq 0$. Lemma 2.8(ii) implies that $w(t)$ is bounded in H_0^2-norm for $t \geq 0$. For $\omega \in \omega(w_0)$, we have $\omega(w_0) \cap \mathbb{K} = \emptyset$, and then $\omega(w_0) = \{0\}$ (see [5] and [21] for details).

(ii) Let $w_0 \in \mathbb{K}_-$ and $\|w_0\|_{L^2} \geq \Theta_{\mathcal{F}(w_0)}$. For each $t \in [0, T(w_0))$, we need to show $w \in \mathbb{K}_-$. If it is false, then we can seek a constant $t^0 \in (0, T(w_0))$ such that $w(t) \in \mathbb{K}_-$ for $t \in [0, t^0)$ and $w(t^0) \in \mathbb{K}$. By a similar proof as part (i), one has $\mathcal{F}(w(t^0)) < \mathcal{F}(w_0)$ and $w(t^0) \in \mathcal{F}^{\mathcal{F}(w_0)}$. Thus $w(t^0) \in \mathbb{K}^{\mathcal{F}(w_0)}$ and

$$\|w(t^0)\|_{L^2} \leq \Theta_{\mathcal{F}(w_0)}. \tag{6.2}$$

According to $w(t) \in \mathbb{K}_-$ and (4.14), one has $\|w(t^0)\|_{L^2} > \|w_0\|_{L^2} \geq \Theta_{\mathcal{F}(w_0)}$. It contradicts to (6.2).

Now if we suppose $T(w_0) = \infty$, then for $\omega \in \omega(w_0)$ we have $\omega(w_0) \cap \mathbb{K} = \emptyset$ and so $\omega(w_0) = \{0\}$. It yields a contradiction to Lemma 2.8(i). Therefore, we finally have $T(w_0) < \infty$ (see [5] and [21] for details). □

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Bo Liang
School of Mathematics and Finance, Chuzhou University, Chuzhou 239004, China
and
School of Science, Dalian Jiaotong University, Dalian 116028, China
E-mail address: `cnliangbo@163.com`

Qingchun Li
School of Mathematical Science, Dalian University of Technology, Dalian 116024, China
E-mail address: `lqc192001@163.com`

Yongbo Zhu and Yongzheng Zhu
School of Science, Dalian Jiaotong University, Dalian 116028, China
E-mail addresses: `z18530253165@163.com`, `zhuyz07@djtu.edu.cn`