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Extremal Functions for Trudinger–Moser Inequalities Involving Various L^p-norms in High Dimension

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Abstract. In this paper, we deal with two Trudinger–Moser inequalities involving various L^p-norms on a smooth bounded domain of $\mathbb{R}^n$, $n \geq 3$. For any $p > 1$, we set

$$\lambda_p(\Omega) = \inf_{u \in H_0^{1,n}(\Omega),\, u \not\equiv 0} \frac{\|\nabla u\|_n^n}{\|u\|_p^n}$$

as an eigenvalue related to the n-Laplacian. Based on the method of blow-up analysis, if $p_j > 1$ for all $1 \leq j \leq l$, and satisfies

$$\max_{1 \leq j \leq l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1, \quad \sum_{j=1}^{l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1,$$

then we prove that

$$\sup_{u \in H_0^{1,n}(\Omega),\, \|\nabla u\|_n \leq 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}} \left(1 + \sum_{j=1}^{l} \alpha_j \|u\|_{p_j}^n\right)^{\frac{1}{n-1}}} \, dx$$

is attained, where $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, ω_{n-1} is the surface area of the unit ball in $\mathbb{R}^n$. Under the same assumptions as above, we conclude that

$$\sup_{u \in H_0^{1,n}(\Omega),\, \|\nabla u\|_n^n - \sum_{j=1}^{l} \alpha_j \|u\|_{p_j}^n \leq 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}} \, dx$$

is attained.

1. Introduction

Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$, and $H_0^{1,n}(\Omega)$ be the completion of $C_0^\infty(\Omega)$ in the norm $\|u\|_{H_0^{1,n}(\Omega)}^n = \int_\Omega (|u|^n + |\nabla u|^n) \, dx$. The study of sharp constant for Trudinger–Moser inequality traces back to 1960s and 1970s. In 1971, Moser [23] elegantly sharpened the results of Phohozaev [27] and Trudinger [31], then established the classical Trudinger–Moser inequality:

$$\sup_{u \in H_0^{1,n}(\Omega),\, \|\nabla u\|_n = 1} \int_\Omega e^{\alpha |u|^{\frac{n}{n-1}}} \, dx < +\infty \tag{1.1}$$

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for any $\alpha \le \alpha_n = n\omega_{n-1}^{1/(n-1)}$, where $\alpha_n = n\omega_{n-1}^{1/(n-1)}$, ω_{n-1} is the surface area of the unit ball in $\mathbb{R}^n$. Here and in the sequel, $\|\cdot\|_p$ denotes the L^p-norm with respect to the Lebesgue measure. There are many generalizations of (1.1) in many ways. For instance, it was generalized to higher order derivations, to compact Riemannian manifolds, to some functions without boundary condition and to unbounded domains in $\mathbb{R}^n$. We refer the interested readers to the papers [33,34] and references therein for more details in this topic. Another meritorious question concerning to Trudinger–Moser inequalities is whether extremal function exists or not. It was firstly discussed by Carleson and Chang [2]. They put forward the existence of extremal functions for (1.1) when Ω is unit ball in $\mathbb{R}^n$. Then Flucher [13] extended this result when Ω is a general bounded smooth domain in two dimension. Lin [20] generalized the existence result to a bounded smooth domain in n dimension. Li [16,17] and Li–Liu [18] obtained the existence result on compact Riemannian manifolds with or without boundary. The Trudinger–Moser inequality (1.1) has been improved in several ways. Adimurthi and Sandeep proved a singular Trudinger–Moser inequality which generalizes (1.1) to the singular weight case. Then de Souza [10] established a sharp Trudinger–Moser type inequality for a class of Schrödinger operators in $\mathbb{R}^2$. Zhou considered a sharp form of anisotropic Moser–Trudinger inequality which involves L^n norm in [40], involves the anisotropic Dirichlet norm $\big(\int_\Omega F^n(\nabla u)\,dx\big)^{\frac{1}{n}}$ in [42], and involves the first eigenvalue and several singular points in [41]. The problem on the existence of extremals for the singular Trudinger–Moser inequality was solved by Csató and Roy [9], and by Csató, Roy and the author [7] in any dimension $n \ge 3$. Nguyen [25] extended the ones of Yang and Zhu [36] to more general cases of the nonlinearity function F and the weight function h. Yuan [38] considered an improved singular Trudinger–Moser inequality in unit ball.

Adimurthi and Druet [1] established the modified inequality in dimension two as follows: Let $\lambda(\Omega) > 0$ be the first eigenvalue of the Laplace operator with respect to the Dirichlet boundary condition. Then for any $\alpha < \lambda(\Omega)$, there holds

$$\sup_{u\in H_0^1(\Omega),\ \int_\Omega |\nabla u|^2\,dx=1} \int_\Omega e^{4\pi u^2(1+\alpha\|u\|_2^2)}\,dx < +\infty. \tag{1.2}$$

Obviously, (1.2) is stronger than (1.1). The situation is quite different when the dimension $n \ge 3$. It was proved by Yang [32] that an analog of (1.2) still holds when Ω is a smooth bounded domain in high dimension. Later, Lu–Yang [22] replaced $\|u\|_2$ with $\|u\|_p$ $(p < 1 < \infty)$ in (1.2) to get the same conclusion as in the case $p = 2$. Also, similar result holds on Riemann surfaces [39]. In [5], Chen etc. considered an improved fractional Trudinger–Moser inequalities on bounded intervals and the existence of their extremals. Recently, they also investigated the optimal concentration level of anisotropic Trudinger–Moser functionals on any bounded domain in [3]. Zhu [43] consider the improved Trudinger–Moser inequality involving L^p norm in $\mathbb{R}^n$: let

$$\lambda_p(\Omega) = \inf_{u \in H_0^{1,n}(\Omega),\, u \not\equiv 0} \frac{\|\nabla u\|_n^n}{\|u\|_p^n},$$

then for any $0 \le \alpha \le \lambda_p(\Omega)$, there holds

$$\sup_{u \in H_0^{1,n}(\Omega),\, \|\nabla u\|_n = 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}} (1+\alpha \|u\|_p^n)^{\frac{1}{n-1}}} \, dx < +\infty.$$

Adapting the ideas in the above conclusion, our main results are stated as

Theorem 1.1. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \ge 3$. If $p_j > 1$ for all $1 \le j \le l$, and satisfies*

$$\max_{1 \le j \le l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1, \quad \sum_{j=1}^{l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1,$$

then we have

$$\sup_{u \in H_0^{1,n}(\Omega),\, \|\nabla u\|_n \le 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}} \left(1+\sum_{j=1}^{l} \alpha_j \|u\|_{p_j}^n\right)^{\frac{1}{n-1}}} \, dx < +\infty.$$

Theorem 1.2. *Under the same assumptions of Theorem 1.1, there exists a function $u_1 \in H_0^{1,n}(\Omega) \cap C^1(\overline{\Omega})$ and $\|\nabla u_1\|_n = 1$ such that*

$$\begin{aligned} &\int_\Omega e^{\alpha_n |u_1|^{\frac{n}{n-1}} \left(1+\sum_{j=1}^{l} \alpha_j \|u_1\|_{p_j}^n\right)^{\frac{1}{n-1}}} \, dx \\ &= \sup_{u \in H_0^{1,n}(\Omega),\, \|\nabla u\|_n \le 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}} \left(1+\sum_{j=1}^{l} \alpha_j \|u\|_{p_j}^n\right)^{\frac{1}{n-1}}} \, dx. \end{aligned}$$

If $\alpha_j = 0$, the above result is the classical Trudinger–Moser inequality (1.1); if $j = 1$, the above partial result is established in [43], which he consider the condition only about $1 < p_j \le n$. In this paper, we consider the general case $p_j > 1$, $1 \le j \le l$.

Another improvement in the classical Trudinger–Moser inequality (1.1) has been established by Tintarev in [29] in $\mathbb{R}^2$. More precisely, he shows that

$$\sup_{u \in H_0^{1,2}(\Omega),\, \|\nabla u\|_2^2 - \alpha \|u\|_2^2 \le 1} \int_\Omega e^{4\pi u^2} \, dx < \infty \tag{1.3}$$

for any $0 \le \alpha < \lambda(\Omega)$. It is easy to see that (1.3) is stronger than (1.2). Then Yang [35] proved the existence of extremal function for the (1.3). Later, Nguyen generalize the inequality (1.3) to higher dimension $n \ge 3$ in [24]. There are many other works in this topic, such as [4, 8, 14, 15, 37, 44] and the references therein. Similar to Theorems 1.1 and 1.2, we have

Theorem 1.3. *Let Ω be a bounded smooth domain in $\mathbb{R}^n$, $n \geq 3$. If $p_j > 1$ for all $1 \leq j \leq l$, and satisfies*

$$\max_{1\leq j\leq l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1, \quad \sum_{j=1}^{l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1,$$

then we have

$$\sup_{u\in H_0^{1,n}(\Omega),\ \|\nabla u\|_n^n-\sum_{j=1}^l \alpha_j\|u\|_{p_j}^n\leq 1} \int_\Omega e^{\alpha_n|u|^{\frac{n}{n-1}}}\,dx < +\infty.$$

Theorem 1.4. *Under the same assumptions of Theorem 1.3, there exists a function $u_2 \in H_0^{1,n}(\Omega) \cap C^1(\overline{\Omega})$ and $\|\nabla u_2\|_n^n - \sum_{j=1}^l \alpha_j\|u_2\|_{p_j}^n = 1$ such that*

$$\int_\Omega e^{\alpha_n|u_2|^{\frac{n}{n-1}}}\,dx = \sup_{u\in H_0^{1,n}(\Omega),\ \|\nabla u\|_n^n-\sum_{j=1}^l \alpha_j\|u\|_{p_j}^n\leq 1} \int_\Omega e^{\alpha_n|u|^{\frac{n}{n-1}}}\,dx.$$

The remaining part of this paper is organized as follows: In Section 2, we prove Theorem 1.1 by the method of blow-up analysis, which was extensively employed by [11,12,26]. Section 3 gives the proof of the existence of extremal function for the Trudinger–Moser inequality involving various L^p-norms. Since the proof of Theorem 1.3 is almost the same as that of Theorem 1.1, we omit it here. In the last section, we will prove Theorem 1.4 by Green-function like [32].

2. Proof of Theorem 1.1

In this section, we study the proof of Theorem 1.1. For simplicity, we introduce the notations

$$J_\gamma(u) = \int_\Omega e^{\gamma|u|^{\frac{n}{n-1}}\left(1+\sum_{j=1}^l \alpha_j\|u\|_{p_j}^n\right)^{\frac{1}{n-1}}}\,dx, \quad \mathcal{H} = \left\{u \in H_0^{1,n}(\Omega) : \|\nabla u\|_n \leq 1\right\}.$$

Recall that the n-Laplacian is defined by $\Delta_n u = \operatorname{div}(|\nabla u|^{n-2}\nabla u)$ for $u \in H^{1,n}(\Omega)$. In order to prove Theorem 1.1, we consider the subcritical functional $J_{\alpha_n-\epsilon}$ firstly.

Lemma 2.1. *For any small ϵ, if $p_j > 1$ for all $1 \leq j \leq l$, satisfies $\max_{1\leq j\leq l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1$ and $\sum_{j=1}^l \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1$, then there exists an extremal function $u_\epsilon \in \mathcal{H} \cap C^1(\overline{\Omega})$ such that*

$$J_{\alpha_n-\epsilon}(u_\epsilon) = \sup_{u\in\mathcal{H}} J_{\alpha_n-\epsilon}(u). \tag{2.1}$$

Proof. For any $\epsilon > 0$, we choose a maximizing sequence $u_i \subset H_0^{1,n}(\Omega)$ such that $\|\nabla u_i\|_n \leq 1$ and

$$\lim_{i\to+\infty} J_{\alpha_n-\epsilon}(u_i) = \sup_{u\in\mathcal{H}} J_{\alpha_n-\epsilon}(u).$$

Since u_i is bounded in $H_0^{1,n}(\Omega)$, we can assume $u_i \rightharpoonup u_\epsilon$ weakly in $H_0^{1,n}(\Omega)$, $u_i \to u_\epsilon$ strongly in $L^n(\Omega)$ and $u_i \to u_\epsilon$ a.e. in Ω. Obviously, we have

$$\begin{aligned} f_i &= e^{(\alpha_n-\epsilon)|u_i|^{\frac{n}{n-1}}\left(1+\sum_{j=1}^l \alpha_j\|u_i\|_{p_j}^n\right)^{\frac{1}{n-1}}} \\ \to f_\epsilon &= e^{(\alpha_n-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}\left(1+\sum_{j=1}^l \alpha_j\|u_\epsilon\|_{p_j}^n\right)^{\frac{1}{n-1}}} \quad \text{a.e. in } \Omega. \end{aligned}$$

We claim that $u_\epsilon \not\equiv 0$. Suppose not, $1+\sum_{j=1}^l \alpha_j\|u_i\|_{p_j}^n \to 1$, from which one can see that f_i is bounded in $L^p(\Omega)$ for some $p>1$ and $f_i \to 1$ in $L^1(\Omega)$. Hence $|\Omega| = \sup_{u\in\mathcal{H}} J_{\alpha_n-\epsilon}(u)$, which is impossible. Therefore $u_\epsilon \not\equiv 0$. Since $\sum_{j=1}^l \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1$, then we have

$$\sum_{j=1}^l \alpha_j \frac{\|u\|_{p_j}^n}{\|\nabla u\|_n^n} = \frac{\sum_{j=1}^l \alpha_j\|u\|_{p_j}^n}{\|\nabla u\|_n^n} < 1.$$

Therefore, one can get

$$1+\sum_{j=1}^l \alpha_j\|u_i\|_{p_j}^n \to 1+\sum_{j=1}^l \alpha_j\|u_\epsilon\|_{p_j}^n < 1+\|\nabla u_\epsilon\|_n^n \le \frac{1}{1-\|\nabla u_\epsilon\|_n^n}.$$

By Lion's theorem in [21], one can see that f_i is bounded in $L^p(\Omega)$ for some $p>1$. Since $f_i \to f_\epsilon$ a.e. in Ω, then $f_i \to f_\epsilon$ strongly in $L^1(\Omega)$. Therefore $\int_\Omega f_\epsilon\,dx = \sup_{u\in\mathcal{H}} J_{\alpha_n-\epsilon}(u)$ and $\|\nabla u_\epsilon\|_n = 1$. Moreover, it is not difficult to check that the corresponding Euler–Lagrange equation of u_ϵ is

$$-\Delta_n u_\epsilon = \frac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon |u_\epsilon|^{\frac{2-n}{n-1}} e^{\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}}} + \sum_{j=1}^l \frac{\alpha_j \|u_\epsilon\|_{p_j}^{n-p_j} u_\epsilon^{p_j-1}}{1+2\sum_{j=1}^l \alpha_j\|u_\epsilon\|_{p_j}^n} \tag{2.2}$$

with

$$\begin{aligned} \alpha_\epsilon &= (\alpha_n-\epsilon)\left(1+\sum_{j=1}^l \alpha_j\|u_\epsilon\|_{p_j}^n\right)^{\frac{1}{n-1}}, \\ \beta_\epsilon &= \left(1+\sum_{j=1}^l \alpha_j\|u_\epsilon\|_{p_j}^n\right) \Big/ \left(1+2\sum_{j=1}^l \alpha_j\|u_\epsilon\|_{p_j}^n\right), \\ \lambda_\epsilon &= \int_\Omega |u_\epsilon|^{\frac{n}{n-1}} e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}}\,dx, \end{aligned}$$

where $-\Delta_n u_\epsilon = -\operatorname{div}(|\nabla u_\epsilon|^{n-2}\nabla u_\epsilon)$. By using the regularity theory for degenerate elliptic equations, see [28, p. 269, Theorem 8], [30, p. 127, Theorem 1] and [19, p. 1203, Theorem 1], we can easily get $u_\epsilon \in C^1(\overline{\Omega})$. □

The following analogy of Lemma 3.2 in [32] is important.

Lemma 2.2. *If $p_j > 1$ for all $1 \le j \le l$, and satisfies* $\max_{1\le j\le l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1$ *and* $\sum_{j=1}^{l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1$, *then we get* $\lim_{\epsilon\to 0} J_{\alpha_n-\epsilon}(u_\epsilon) = \sup_{u\in\mathcal{H}} J_{\alpha_n}(u)$.

Proof. We know that $J_{\alpha_n-\epsilon}(u_\epsilon) = \sup_{u\in\mathcal{H}} J_{\alpha_n-\epsilon}(u)$. Obviously $J_{\alpha_n-\epsilon}(u_\epsilon) \le J_{\alpha_n}(u_\epsilon) \le \sup_{u\in\mathcal{H}} J_{\alpha_n}(u)$, then we have $\lim_{\epsilon\to 0} J_{\alpha_n-\epsilon}(u_\epsilon) \le \sup_{u\in\mathcal{H}} J_{\alpha_n}(u)$. On the other hand, we get by (2.1),

$$\int_\Omega e^{\alpha_n|u|^{\frac{n}{n-1}}\left(1+\sum_{j=1}^{l}\alpha_j\|u\|_{p_j}^{n}\right)^{\frac{1}{n-1}}} dx \le \liminf_{\epsilon\to 0}\int_\Omega e^{(\alpha_n-\epsilon)|u|^{\frac{n}{n-1}}\left(1+\sum_{j=1}^{l}\alpha_j\|u\|_{p_j}^{n}\right)^{\frac{1}{n-1}}} dx$$
$$\le \liminf_{\epsilon\to 0}\int_\Omega e^{(\alpha_n-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}\left(1+\sum_{j=1}^{l}\alpha_j\|u_\epsilon\|_{p_j}^{n}\right)^{\frac{1}{n-1}}} dx,$$

which implies that $\sup_{u\in\mathcal{H}} J_{\alpha_n}(u) \le \liminf_{\epsilon\to 0} J_{\alpha_n-\epsilon}(u_\epsilon)$. Hence $\lim_{\epsilon\to 0} J_{\alpha_n-\epsilon}(u_\epsilon) = \sup_{u\in\mathcal{H}} J_{\alpha_n}(u)$. □

Next, we will use the method of blow-up analysis to study the behavior of u_ϵ in Lemma 2.1. The following several lemmas are useful.

Lemma 2.3. *Let λ_ϵ be defined in* (2.2), *there holds* $\liminf_{\epsilon\to 0}\lambda_\epsilon > 0$.

Proof. Apparently $\liminf_{\epsilon\to 0}\lambda_\epsilon \ge 0$. Using the inequality $e^t \le 1 + te^t$ for $t \le 0$, one has

$$\int_\Omega e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}} dx \le |\Omega| + \alpha_\epsilon\lambda_\epsilon. \tag{2.3}$$

By Lemma 2.2, we can know that

$$\lim_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}} dx = \sup_{u\in\mathcal{H}} J_{\alpha_n}(u) > |\Omega|. \tag{2.4}$$

Combining (2.3) and (2.4), one gets the result directly. □

Denote $c_\epsilon = |u_\epsilon|(x_\epsilon) = \max_{x\in\Omega}|u_\epsilon|(x)$. If c_ϵ is bounded, then $-\Delta_n u_\epsilon$ is bounded in $L^\infty(\Omega)$ since $\liminf_{\epsilon\to 0}\lambda_\epsilon > 0$ by Lemma 2.3. Then we can assume that u_ϵ converges to u_0 in $H_0^{1,n}(\Omega)$, strongly in $L^s(\Omega)$ for any $s > 1$ and almost everywhere in Ω. From Lemma 2.2 and the Lebesgue dominated convergence theorem, we can know u_0 is the desired extremal function for the supremum J_{α_n}. Without loss of generality, we may assume $c_\epsilon = u_\epsilon(x_\epsilon) \to +\infty$, for otherwise we consider $-u_\epsilon$ instead of u_ϵ. We assume $x_\epsilon \to p \in \overline{\Omega}$. There are two cases which contain the concentration point p lies in the interior of Ω or on $\partial\Omega$. Using the same ideas as in [24,32,43], we can exclude the boundary blow-up. From now on, we assume that p lies in the interior of Ω. The following concentration phenomenon is indispensable in our subsequent blow-up analysis.

Lemma 2.4. *Under the assumption that $c_\epsilon \to +\infty$, we have $u_\epsilon \rightharpoonup 0$ weakly in $H_0^{1,n}(\Omega)$, $u_\epsilon \to 0$ strongly in $L^n(\Omega)$. $|\nabla u_\epsilon|^n\,dx \rightharpoonup \delta_p$ in sense of measure, where δ_p is the Dirac measure at p. Furthermore, we have $\alpha_\epsilon \to \alpha_n$, $\beta_\epsilon \to 1$.*

Proof. Since $\|\nabla u_\epsilon\|_n = 1$ and $u_\epsilon \in H_0^{1,n}(\Omega)$, we may assume $u_\epsilon \rightharpoonup u_0$ weakly in $H_0^{1,n}(\Omega)$, $u_\epsilon \to u_0$ strongly in $L^n(\Omega)$. Suppose $u_0 \not\equiv 0$, then we have if $\sum_{j=1}^{l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1$,

$$1 + \sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n \to 1 + \sum_{j=1}^{l} \alpha_j \|u_0\|_{p_j}^n < 1 + \|\nabla u_0\|_n^n \le \frac{1}{1 - \|\nabla u_0\|_n^n}.$$

Hence by a theorem of Lions in [21] we conclude that $e^{\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}}}$ is bounded in $L^q(\Omega)$ for some $q > 1$ provided that ϵ is sufficiently small. Applying the elliptic estimates to equation (2.2), one gets c_ϵ is bounded, which contradicts $c_\epsilon \to +\infty$. Therefore, $u_0 \equiv 0$ and consequently $\alpha_\epsilon \to \alpha_n$, $\beta_\epsilon \to 1$. Assume $|\nabla u_\epsilon|^n\, dx \rightharpoonup \mu$ in sense of measure. We can choose a cut-off function $\phi \in C_0^1(\Omega)$, which is supported in $\mathbb{B}_{r_0}(p) \subset \Omega$ and equal to 1 in $\mathbb{B}_{r_0/2}(p)$ for some small $r_0 > 0$ such that

$$\int_{\mathbb{B}_{r_0}(p)} |\nabla(\phi u_\epsilon)|^n\, dx \le 1 - \eta$$

for some $\eta > 0$ provided that ϵ is sufficiently small. By the classical Trudinger–Moser inequality (1.1), we can know $e^{\alpha_\epsilon (\phi u_\epsilon)^{\frac{n}{n-1}}}$ is bounded in $L^s(\Omega)$ for some $s > 1$. Then the elliptic estimate to equation (2.2) implies that u_ϵ is bounded in $L^\infty(\mathbb{B}_{r_0/2}(p))$, which contradicts the assumption that $c_\epsilon \to +\infty$. Therefore, $|\nabla u_\epsilon|^n\, dx \rightharpoonup \delta_p$. □

Let

(2.5)
$$r_\epsilon = \lambda_\epsilon^{\frac{1}{n}} \beta_\epsilon^{-\frac{1}{n}} c_\epsilon^{-\frac{1}{n-1}} e^{-\frac{\alpha_\epsilon}{n} c_\epsilon^{n/(n-1)}}, \quad \psi_\epsilon(x) = \frac{1}{c_\epsilon} u_\epsilon(x_\epsilon + r_\epsilon x), \quad \varphi_\epsilon(x) = c_\epsilon^{\frac{1}{n-1}}(u_\epsilon(x_\epsilon + r_\epsilon x) - c_\epsilon),$$

where ψ_ϵ and φ_ϵ are defined on $\Omega_\epsilon = \{x \in \mathbb{R}^n : x_\epsilon + r_\epsilon x \in \Omega\}$. A direct computation gives

$$-\Delta_n \psi_\epsilon(x) = c_\epsilon^{-n} \psi_\epsilon^{\frac{1}{n-1}} e^{\alpha_\epsilon\left(|u_\epsilon|^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - c_\epsilon^{\frac{n}{n-1}}\right)} + \sum_{j=1}^{l} \frac{\alpha_j c_\epsilon^{p_j - n} r_\epsilon^n \|u_\epsilon\|_{p_j}^{n-p_j} \psi_\epsilon^{p_j-1}}{1 + 2\sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n}, \tag{2.6}$$

$$-\Delta_n \varphi_\epsilon(x) = \psi_\epsilon^{\frac{1}{n-1}} e^{\alpha_\epsilon\left(|u_\epsilon|^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - c_\epsilon^{\frac{n}{n-1}}\right)} + \sum_{j=1}^{l} \frac{\alpha_j c_\epsilon^{p_j} r_\epsilon^n \|u_\epsilon\|_{p_j}^{n-p_j} \psi_\epsilon^{p_j-1}}{1 + 2\sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n}. \tag{2.7}$$

For the purpose of studying the convergence of ψ_ϵ and φ_ϵ, we need the following

Lemma 2.5. *Fixed any* $0 < \delta < \alpha_n$, *we have* $r_\epsilon^n e^{\delta c_\epsilon^{n/(n-1)}} \to 0$ *as* $\epsilon \to 0$.

Proof. By the expression of r_ϵ in (2.5) and λ_ϵ in (2.2), we have

$$\begin{aligned} r_\epsilon^n e^{\delta c_\epsilon^{n/(n-1)}} &= \lambda_\epsilon \beta_\epsilon^{-1} c_\epsilon^{-\frac{n}{n-1}} e^{-\alpha_\epsilon c_\epsilon^{n/(n-1)}} e^{\delta c_\epsilon^{n/(n-1)}} \\ &= \beta_\epsilon^{-1} c_\epsilon^{-\frac{n}{n-1}} e^{(\delta - \alpha_\epsilon) c_\epsilon^{n/(n-1)}} \int_\Omega |u_\epsilon|^{\frac{n}{n-1}} e^{\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}}}\, dx \\ &\le \beta_\epsilon^{-1} c_\epsilon^{-\frac{n}{n-1}} \int_\Omega |u_\epsilon|^{\frac{n}{n-1}} e^{\delta |u_\epsilon|^{\frac{n}{n-1}}}\, dx \end{aligned} \tag{2.8}$$

for sufficiently small ϵ. Clearly, $|u_\epsilon|^{\frac{n}{n-1}} e^{\delta|u_\epsilon|^{\frac{n}{n-1}}}$ is bounded in $L^q(\Omega)$ for some $q > 1$. From (2.8) and recall that $\beta_\epsilon \to 1$, $c_\epsilon \to +\infty$, we get the result. $\square$

By the preceding lemma and elliptic estimates for quasi-linear equations [28, 30], we can get the asymptotic behavior of ψ_ϵ and φ_ϵ as $\epsilon \to 0$.

Lemma 2.6. *$\psi_\epsilon \to 1$ and $\varphi_\epsilon \to \varphi$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

Proof. Without loss of generality, we assume $p_1 \le p_2 \le \cdots \le p_l$, and consider the following three cases:

Case 1*: $1 < p_j \le n$ $(1 \le j \le l)$.* Note that $|\psi_\epsilon| \le 1$, applying elliptic estimates (see [30, Theorem 1]) to equation (2.6), we have $\|\psi_\epsilon\|_{C^{1,\alpha}(\mathbb{B}_{R/2})} \le C$, then we can apply Arzelá–Ascoli theorem to know there exists $\psi \in C^1(\mathbb{B}_{R/4})$ such that $\psi_\epsilon \to \psi$. Let $R \to \infty$, we get $\psi_\epsilon \to 1$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. It is easy to know $\Delta_n \psi = 0$ in $\mathbb{R}^n$. Liouville type theorem implies that $\psi = 1$ in $\mathbb{R}^n$. On the other hand, we have in any ball $\mathbb{B}_R(0)$,

$$\begin{aligned}|u_\epsilon|^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - c_\epsilon^{\frac{n}{n-1}} &= c_\epsilon^{\frac{n}{n-1}}\big(|\psi_\epsilon|^{\frac{n}{n-1}}(x) - 1\big)\\ &= \frac{n}{n-1}\varphi_\epsilon(x)(1 + O((\psi_\epsilon(x) - 1)^2)).\end{aligned}$$

Applying Harnack inequality for n-Laplace equation [28] and Lemma 2.5 to equation (2.7), one can see that $-\Delta_n \varphi_\epsilon(x)$ is bounded. Then elliptic estimates (see [30, Theorem 1]) implies that φ_ϵ is bounded in $C^{1,\alpha}(\mathbb{B}_{R/4})$ for some $0 < \alpha < 1$, and whence $\varphi_\epsilon \to \varphi$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$.

Case 2*: $p_j > n$ $(1 \le j \le m)$ and $1 < p_j \le n$ $(m < j \le l)$.* In this case, we should begin exploring the boundedness of $-\Delta_n \psi_\epsilon(x)$ and $-\Delta_n \varphi_\epsilon(x)$. It is obviously known that

$$\sum_{j=m+1}^{l} \frac{\alpha_j c_\epsilon^{p_j - n} r_\epsilon^n \|u_\epsilon\|_{L^{p_j}(\Omega)}^{n-p_j} \psi_\epsilon^{p_j - 1}}{1 + 2\sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n} = o_\epsilon(1).$$

For $p_j > n$ $(1 \le j \le m)$, we have the following inequality

$$\begin{aligned}\sum_{j=1}^{m} \frac{\alpha_j c_\epsilon^{p_j - n} r_\epsilon^n \|u_\epsilon\|_{L^{p_j}(\Omega)}^{n-p_j} \psi_\epsilon^{p_j - 1}}{1 + 2\sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n} &\le \sum_{j=1}^{m} \frac{\alpha_j c_\epsilon^{p_j - n} r_\epsilon^n \|u_\epsilon\|_{L^{p_j}(B_{Rr_\epsilon}(x_\epsilon))}^{n-p_j} \psi_\epsilon^{p_j - 1}}{1 + 2\sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n}\\ &= \sum_{j=1}^{m} \frac{\alpha_j c_\epsilon^{p_j - n} r_\epsilon^{\frac{n^2}{p_j}} \psi_\epsilon^{p_j - 1}}{1 + 2\sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n}\big(\|u_\epsilon\|_{L^{p_j}(B_R(0))}^{n-p_j} + o_\epsilon(1)\big).\end{aligned}$$

Due to the expression of r_ϵ, we can get $c_\epsilon^{p_j - n} r_\epsilon^{\frac{n^2}{p_j}} \to 0$. Hence, $\sum_{j=1}^{m} \frac{\alpha_j c_\epsilon^{p_j - n} r_\epsilon^n \|u_\epsilon\|_{L^{p_j}(\Omega)}^{n-p_j} \psi_\epsilon^{p_j - 1}}{1 + 2\sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n}$ is bounded. In the same way we can get $-\Delta_n \varphi_\epsilon(x)$ is bounded as well. The subsequent discussion is similar to Case 1.

Case 3*: $p_j > n$ $(1 \le j \le l)$.* Under the circumstance, it is not difficult to know that

$$\sum_{j=1}^{l} \frac{\alpha_j c_\epsilon^{p_j-n} r_\epsilon^n \|u_\epsilon\|_{L^{p_j}(\Omega)}^{n-p_j} \psi_\epsilon^{p_j-1}}{1+2\sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n} = o_\epsilon(1).$$

Therefore, $-\Delta_n \psi_\epsilon(x)$ and $-\Delta_n \varphi_\epsilon(x)$ are all bounded. In conclusion, for any $p_j > 1$, we have $\psi_\epsilon \to 1$ and $\varphi_\epsilon \to \varphi$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$. Moreover,

$$\begin{aligned}
\int_{\mathbb{B}_{R/8}(0)} e^{\frac{n}{n-1}\alpha_n \varphi}\, dx &\le \liminf_{\epsilon\to 0} \int_{\mathbb{B}_{R/8}(0)} e^{\alpha_\epsilon \left(|u_\epsilon|^{\frac{n}{n-1}}(x_\epsilon + r_\epsilon x) - c_\epsilon^{\frac{n}{n-1}}\right)}\, dx \\
&= \liminf_{\epsilon\to 0} \int_{\mathbb{B}_{Rr_\epsilon/8}(x_\epsilon)} e^{\alpha_\epsilon \left(|u_\epsilon^{\frac{n}{n-1}}| - c_\epsilon^{\frac{n}{n-1}}\right)} r_\epsilon^{-n}\, dx \\
&\le \liminf_{\epsilon\to 0} (1+o(1)) \frac{1}{\lambda_\epsilon} \int_{\mathbb{B}_{Rr_\epsilon/8}(x_\epsilon)} |u_\epsilon|^{\frac{n}{n-1}} e^{\alpha_\epsilon \left|u_\epsilon^{\frac{n}{n-1}}\right|}\, dx \\
&\le 1.
\end{aligned}$$

Hence φ satisfies the following equation

$$-\Delta_n \varphi = e^{\frac{n}{n-1}\alpha_n \varphi} \text{ in } \mathbb{R}^n, \quad \varphi(0) = 0 = \sup_{\mathbb{R}^n} \varphi, \quad \int_{\mathbb{R}^n} e^{\frac{n}{n-1}\alpha_n \varphi}\, dx \le 1.$$

From Lemma 4.2 in [17] and Lemma 2.1 in [6], we obtain the solution of the above equation is

$$\varphi(x) = -\frac{n-1}{\alpha_n} \ln \left(1 + \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n-1}} |x|^{\frac{n}{n-1}}\right). \qquad \square$$

Define $u_\epsilon^A = \min\left\{u_\epsilon, \frac{c_\epsilon}{A}\right\}$. Similar to [17, 32], we have the following

Lemma 2.7. $\lim_{\epsilon\to 0} \int_\Omega |\nabla u_\epsilon^A|^n\, dx = \frac{1}{A}$ *for any $A > 1$.*

Proof. We have by the equation (2.2) and the divergence theorem,

$$\begin{aligned}
\int_\Omega \left|\nabla \left(u_\epsilon - \frac{c_\epsilon}{A}\right)^+\right|^n dx &= -\int_\Omega \left(u_\epsilon - \frac{c_\epsilon}{A}\right)^+ \Delta_n u_\epsilon\, dx \\
&= \int_\Omega \left(u_\epsilon - \frac{c_\epsilon}{A}\right)^+ \left(\frac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon^{\frac{1}{n-1}} e^{\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}}} + \sum_{j=1}^{l} \frac{\alpha_j \|u_\epsilon\|_{p_j}^{n-p_j} u_\epsilon^{p_j-1}}{1+2\sum_{j=1}^{l}\alpha_j \|u_\epsilon\|_{p_j}^n}\right) dx \\
&\ge \int_{B_{Rr_\epsilon}(x_\epsilon)} \left(1 - \frac{1}{A}\right) c_\epsilon (1+o_\epsilon(1)) \frac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon^{\frac{1}{n-1}} e^{\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}}}\, dx + o_\epsilon(1) \\
&\ge \int_{B_R} \left(1 - \frac{1}{A}\right) c_\epsilon (1+o_\epsilon(1)) \frac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon^{\frac{1}{n-1}} e^{\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}}} r_\epsilon^n\, dx + o_\epsilon(1) \\
&\ge \left(1 - \frac{1}{A}\right) \int_{B_R} e^{\frac{n}{n-1}\alpha_n \varphi}\, dx + o_\epsilon(1).
\end{aligned}$$

Letting $\epsilon \to 0$, we obtain

$$\liminf_{\epsilon\to 0} \int_\Omega \left|\nabla \left(u_\epsilon - \frac{c_\epsilon}{A}\right)^+\right|^n dx \ge 1 - \frac{1}{A}.$$

By the same argument, we establish that

$$\liminf_{\epsilon\to 0}\int_\Omega |\nabla u_\epsilon^A|^n\,dx \geq \frac{1}{A}.$$

Since

$$\int_\Omega |\nabla u_\epsilon^A|^n\,dx + \int_\Omega \left|\nabla\left(u_\epsilon - \frac{c_\epsilon}{A}\right)^+\right|^n\,dx = 1,$$

we get the result. □

The following lemma is used in proving the existence of extremal functions of the Trudinger–Moser inequality. Because it provides the asymptotic behavior of u_ϵ, we include it here.

Lemma 2.8. $\limsup_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^{\frac{n}{n-1}}}\,dx \leq |\Omega| + \lim_{R\to+\infty}\limsup_{\epsilon\to 0}\int_{\mathbb{B}_{Rr_\epsilon(x_\epsilon)}} e^{\alpha_\epsilon u_\epsilon^{\frac{n}{n-1}}}\,dx.$

Proof. For any $A>1$, from the expression of λ_ϵ in (2.2), we have

$$\begin{aligned}\int_\Omega e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}}\,dx &= \int_{u_\epsilon<\frac{c_\epsilon}{A}} e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}}\,dx + \int_{u_\epsilon\geq\frac{c_\epsilon}{A}} e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}}\,dx\\ &\leq \int_\Omega e^{\alpha_\epsilon|u_\epsilon^A|^{\frac{n}{n-1}}}\,dx + \frac{A^{\frac{n}{n-1}}\lambda_\epsilon}{c_\epsilon^{\frac{n}{n-1}}}.\end{aligned}$$

By Lemma 2.7, one has $\int_\Omega e^{\alpha_\epsilon|u_\epsilon^A|^{\frac{n}{n-1}}}\,dx \to |\Omega|$ as $\epsilon\to 0$. Let $\epsilon\to 0$ first, then $A\to 1$, one has

$$\limsup_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon u_\epsilon^{\frac{n}{n-1}}}\,dx \leq |\Omega| + \limsup_{\epsilon\to 0}\frac{\lambda_\epsilon}{c_\epsilon^{\frac{n}{n-1}}}. \tag{2.9}$$

On the other hand, from the definition of r_ϵ in (2.5),

$$\int_{\mathbb{B}_{Rr_\epsilon(x_\epsilon)}} e^{\alpha_\epsilon u_\epsilon^{\frac{n}{n-1}}}\,dx = \frac{\lambda_\epsilon}{\beta_\epsilon c_\epsilon^{\frac{n}{n-1}}}\left(\int_{\mathbb{B}_R(0)} e^{\frac{n}{n-1}\alpha_n\varphi}\,dx + o_\epsilon(1)\right),$$

which gives

$$\lim_{R\to+\infty}\limsup_{\epsilon\to 0}\int_{\mathbb{B}_{Rr_\epsilon(x_\epsilon)}} e^{\alpha_\epsilon u_\epsilon^{\frac{n}{n-1}}}\,dx = \limsup_{\epsilon\to 0}\frac{\lambda_\epsilon}{c_\epsilon^{\frac{n}{n-1}}}.$$

Together with (2.9), the lemma is completed. □

Using the same method of Lemma 4.9 in [32], one can prove without any difficulty that

$$\lim_{\epsilon\to 0}\int_\Omega \phi c_\epsilon\frac{\beta_\epsilon}{\lambda_\epsilon}u_\epsilon^{\frac{1}{n-1}}e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}}\,dx = \phi(p), \quad \forall\,\phi\in C_0^\infty(\Omega).$$

The following is similar to Lemma 3.10 in [43].

Lemma 2.9. *Let $f \in L^1(\Omega)$, and $u \in C^1(\overline{\Omega}) \cap H_0^{1,n}(\Omega)$ satisfy the following equation*

$$-\Delta_n u = f + \sum_{j=1}^{l} \alpha_j \|u\|_{p_j}^{n-p_j} u^{p_j-1},$$

where $\sum_{j=1}^{l} \frac{\alpha_j}{\lambda_{p_j}(\Omega)} < 1$. Then for any $1 < s < n$, we have $\|\nabla u\|_s \le C\|f\|_1$ for some constant C depending only on p_j, s, α_j, n, $\lambda_{p_j}(\Omega)$.

We omit the proof here. The interested readers can refer [32] and its corrigendum in [32] to get the detailed process of argumentation. Using Lemma 2.9, we can prove the following

Lemma 2.10. *For any $1 < s < n$, $c_\epsilon^{\frac{1}{n-1}} u_\epsilon$ is bounded in $H_0^{1,s}(\Omega)$. $c_\epsilon^{\frac{1}{n-1}} u_\epsilon \rightharpoonup G$ weakly in $H^{1,s}(\Omega)$ for any $1 < s < n$, where G is a Green function satisfying*

$$\begin{cases} -\Delta_n G = \delta_p + \sum_{j=1}^{l} \alpha_j \|G\|_{p_j}^{n-p_j} G^{p_j-1} & \text{in } \Omega, \\ G = 0 & \text{on } \partial\Omega. \end{cases}$$

Furthermore, $c_\epsilon^{\frac{1}{n-1}} u_\epsilon \to G$ in $C^1(\overline{\Omega'})$ for any domain $\Omega' \subset\subset \overline{\Omega} \setminus \{p\}$.

Proof. By (2.2) we have

$$-\Delta_n\big(c_\epsilon^{\frac{1}{n-1}} u_\epsilon\big) = c_\epsilon \frac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon^{\frac{1}{n-1}} e^{\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}}} + \sum_{j=1}^{l} \frac{\alpha_j \big\|c_\epsilon^{\frac{1}{n-1}} u_\epsilon\big\|_{p_j}^{n-p_j} \big(c_\epsilon^{\frac{1}{n-1}} u_\epsilon\big)^{p_j-1}}{1 + 2\sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^{n}}. \tag{2.10}$$

From Lemmas 2.8 and 2.9, we can get $c_\epsilon^{\frac{1}{n-1}} u_\epsilon$ is bounded in $H_0^{1,s}(\Omega)$. Assume $c_\epsilon^{\frac{1}{n-1}} u_\epsilon \rightharpoonup G$ weakly in $H^{1,s}(\Omega)$. Testing equation (2.10) with $\phi \in C_0^\infty(\Omega)$, we have

$$\begin{aligned} \int_\Omega -\phi \Delta_n\big(c_\epsilon^{\frac{1}{n-1}} u_\epsilon\big)\, dx &= \int_\Omega \phi c_\epsilon \frac{\beta_\epsilon}{\lambda_\epsilon} u_\epsilon^{\frac{1}{n-1}} e^{\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}}}\, dx \\ &\quad + \sum_{j=1}^{l} \int_\Omega \phi \frac{\alpha_j}{1 + 2\sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^{n}} \big\|c_\epsilon^{\frac{1}{n-1}} u_\epsilon\big\|_{p_j}^{n-p_j} \big(c_\epsilon^{\frac{1}{n-1}} u_\epsilon\big)^{p_j-1}\, dx \\ &\to \phi(p) + \sum_{j=1}^{l} \int_\Omega \alpha_j \|G\|_{p_j}^{n-p_j} \phi G^{p_j-1}\, dx. \end{aligned}$$

Hence

$$\int_\Omega \nabla\phi |\nabla G|^{n-2} \nabla G\, dx = \phi(p) + \sum_{j=1}^{l} \int_\Omega \alpha_j \|G\|_{p_j}^{n-p_j} \phi G^{p_j-1}\, dx.$$

Therefore,

$$-\Delta_n G = \delta_p + \sum_{j=1}^{l} \alpha_j \|G\|_{p_j}^{n-p_j} G^{p_j-1}$$

in a distributional sense. The usual elliptic estimates give the second assertion, which completes the proof of Lemma 2.10. □

Proof of Theorem 1.1. When $c_\epsilon \to +\infty$, a straightforward calculation gives

$$\begin{aligned} J_{\alpha_n-\epsilon}(u_\epsilon) &= \int_\Omega e^{(\alpha_n-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}\left(\left(1+\sum_{j=1}^l \alpha_j\|u_\epsilon\|_{p_j}^n\right)^{\frac{1}{n-1}}-1\right)} e^{(\alpha_n-\epsilon)|u_\epsilon|^{\frac{n}{n-1}}}\,dx \\ &\le e^{\alpha_n c_\epsilon^{\frac{n}{n-1}}\left(\left(1+\sum_{j=1}^l \alpha_j\|u_\epsilon\|_{p_j}^n\right)^{\frac{1}{n-1}}-1\right)} \int_\Omega e^{\alpha_n|u_\epsilon|^{\frac{n}{n-1}}}\,dx \\ &= e^{\frac{\alpha_n}{n-1}\sum_{j=1}^n \alpha_j\|c_\epsilon^{\frac{1}{n-1}}u_\epsilon\|_{p_j}^n + c_\epsilon^{-\frac{n}{n-1}} O\left(\sum_{j=1}^l \left\|c_\epsilon^{\frac{1}{n-1}}u_\epsilon\right\|_{p_j}^{2n}\right)} \int_\Omega e^{\alpha_n|u_\epsilon|^{\frac{n}{n-1}}}\,dx. \end{aligned}$$

By using Lemma 2.10 and the classical Trudinger–Moser inequality (1.1) completes the proof of Theorem 1.1. □

3. Proof of Theorem 1.2

In this section, we give the proof of Theorem 1.2 by dividing it into two steps:

Step 1*: Upper bound of* J_{α_n}. Under the assumption that $c_\epsilon \to +\infty$ and $u_\epsilon \to p \in \Omega$, the following holds:

$$\sup_{u\in H_0^{1,n}(\Omega),\ \|\nabla u\|_n\le 1} \int_\Omega e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}}\,dx \le |\Omega| + \frac{\omega_{n-1}}{n} e^{\alpha_n A_p + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}, \tag{3.1}$$

where A_p is defined in (3.2).

Inspired by [18, 32], we need the following result due to Carleson and Chang [2]: Let $\mathbb{B}$ be the unit ball in R^n. Given a function sequence $(u_\epsilon)_{\epsilon>0} \subset H_0^{1,n}(\mathbb{B})$ with $\|\nabla u_\epsilon\|_n = 1$. If $|\nabla u_\epsilon|^n\,dx \rightharpoonup \delta_0$ weakly in sense of measure, then

$$\limsup_{\epsilon\to 0} \int_{\mathbb{B}} e^{\alpha_n|u_\epsilon|^{\frac{n}{n-1}}}\,dx \le |\mathbb{B}|\left(1 + e^{1+\frac{1}{2}+\cdots+\frac{1}{n-1}}\right).$$

Let G be as in Lemma 2.10, then G takes the form

$$G = -\frac{n}{\alpha_n}\ln|x-p| + A_p + g(x), \tag{3.2}$$

where A_p is a constant depending only on α, β, p, $g(p) = 0$, $g(x)$ is continuous at p, and $g(x) \in C^1(\overline{\Omega}\backslash\{p\})$. Readers can refer [17] to get more information about the representation of $G(x)$.

Denote $\mathbb{B}_\delta(p) = \{x \in \mathbb{R}^n : |x-p| \le \delta\}$ by $\mathbb{B}_\delta$ and $\partial\mathbb{B}_\delta(p)$ by $\partial\mathbb{B}_\delta$ for simplicity. By Lemma 2.10 and (3.2), we have

$$\int_{\Omega\backslash\mathbb{B}_\delta} |\nabla u_\epsilon|^n\,dx = c_\epsilon^{-\frac{n}{n-1}}\left(\int_{\Omega\backslash\mathbb{B}_\delta} |\nabla G|^n\,dx + o_\epsilon(1)\right)$$

$$= c_\epsilon^{-\frac{n}{n-1}} \left(\int_{\partial \mathbb{B}_\delta} G|\nabla G|^{n-2} \frac{\partial G}{\partial n} \, ds + \sum_{j=1}^{l} \alpha_j \|G\|_{p_j}^n + o_\epsilon(1) \right)$$
$$= c_\epsilon^{-\frac{n}{n-1}} \left(\frac{1}{\alpha_n} \ln \frac{1}{\delta^n} + A_p + \sum_{j=1}^{l} \alpha_j \|G\|_{p_j}^n + o_\delta(1) \right),$$

where $o_\epsilon(1) \to 0$ as $\epsilon \to 0$, $o_\delta(1) \to 0$ as $\delta \to 0$. Let $b_\epsilon = \sup_{\partial \mathbb{B}_\delta} u_\epsilon$ and $\overline{u}_\epsilon = (u_\epsilon - b_\epsilon)^+$. Then $\overline{u}_\epsilon \in H_0^{1,n}(\mathbb{B}_\delta)$ and

$$\int_{\mathbb{B}_\delta} |\nabla \overline{u}_\epsilon|^n \, dx \le \tau_\epsilon = 1 - c_\epsilon^{-\frac{n}{n-1}} \left(\frac{1}{\alpha_n} \ln \frac{1}{\delta^n} + A_p + \sum_{j=1}^{l} \alpha_j \|G\|_{p_j}^n + o_\epsilon(1) + o_\delta(1) \right).$$

By the result of Carleson and Chang at the beginning of the section,

$$\limsup_{\epsilon \to 0} \int_{\mathbb{B}_\delta} e^{\alpha_n \left|\overline{u}_\epsilon / \tau_\epsilon^{1/n}\right|^{\frac{n}{n-1}}} \, dx \le \delta^n \frac{\omega_{n-1}}{n} \left(1 + e^{1 + \frac{1}{2} + \cdots + \frac{1}{n-1}} \right). \tag{3.3}$$

Now we focus on $\mathbb{B}_{Rr_\epsilon}(x_\epsilon)$. By Lemma 2.6, $\varphi_\epsilon \to \varphi$ in $C^1_{\mathrm{loc}}(\mathbb{R}^n)$, and whence $u_\epsilon = c_\epsilon + o_\epsilon(1)$. We have

$$\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}} = (\alpha_n - \epsilon) \left(1 + \sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n \right)^{\frac{1}{n-1}} |u_\epsilon|^{\frac{n}{n-1}}$$
$$\le \alpha_n \left(1 + \sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^n \right)^{\frac{1}{n-1}} |\overline{u}_\epsilon + b_\epsilon|^{\frac{n}{n-1}}$$
$$\le \alpha_n \overline{u}_\epsilon^{\frac{n}{n-1}} + \alpha_n \sum_{j=1}^{l} \frac{\alpha_j}{n-1} \|G\|_{p_j}^n + \frac{n}{n-1} \alpha_n \overline{u}_\epsilon^{\frac{n}{n-1}} b_\epsilon + o_\epsilon(1).$$

Using Lemma 2.10 again, we have by the definition of τ_ϵ that

$$\alpha_n \overline{u}_\epsilon^{\frac{n}{n-1}} = \alpha_n \overline{u}_\epsilon^{\frac{n}{n-1}} \left(\frac{1}{\tau_\epsilon} - c_\epsilon^{-\frac{n}{n-1}} \left(\frac{1}{\alpha_n} \ln \frac{1}{\delta^n} + A_p + \sum_{j=1}^{l} \alpha_j \|G\|_{p_j}^n + o_\epsilon(1) + o_\delta(1) \right) \right)^{\frac{1}{n-1}}$$
$$\le \alpha_n \frac{|\overline{u}_\epsilon|^{\frac{n}{n-1}}}{\tau_\epsilon^{\frac{1}{n-1}}} - \frac{1}{n-1} \left(\ln \frac{1}{\delta^n} + \alpha_n A_p + \alpha_n \sum_{j=1}^{l} \alpha_j \|G\|_{p_j}^n + o_\epsilon(1) + o_\delta(1) \right).$$

Similarly we have

$$\overline{u}_\epsilon^{\frac{n}{n-1}} b_\epsilon = \overline{u}_\epsilon^{\frac{n}{n-1}} \left(c_\epsilon^{-\frac{1}{n-1}} G + o_\epsilon(1) \right) = \frac{1}{\alpha_n} \ln \frac{1}{\delta^n} + A_p + o_\epsilon(1) + o_\delta(1).$$

Combining the above inequalities, we obtain on $\mathbb{B}_{Rr_\epsilon}(x_\epsilon)$,

$$\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}} \le \alpha_n \left|\overline{u}_\epsilon / \tau_\epsilon^{1/n}\right|^{\frac{n}{n-1}} + \ln \frac{1}{\delta^n} + \alpha_n A_p + o_\epsilon(1) + o_\delta(1),$$

which together with (3.3) and Lemma 2.8 gives

$$\lim_{\epsilon\to 0}\int_\Omega e^{\alpha_\epsilon|u_\epsilon|^{\frac{n}{n-1}}}\,dx \le |\Omega| + \frac{\omega_{n-1}}{n}e^{\alpha_n A_p+1+\frac{1}{2}+\cdots+\frac{1}{n-1}}.$$

Then according to Lemma 2.2, we conclude

$$\sup_{u\in\mathcal{H}} J_{\alpha_n}(u) \le |\Omega| + \frac{\omega_{n-1}}{n}e^{\alpha_n A_p+1+\frac{1}{2}+\cdots+\frac{1}{n-1}}.$$

Step 2: Existence of extremal functions. We will construct a blow-up sequence $\phi_\epsilon \in H_0^{1,n}(\Omega)$ such that $\|\nabla\phi_\epsilon\|_n = 1$ and

$$\int_\Omega e^{\alpha_n|\phi_\epsilon|^{\frac{n}{n-1}}\left(1+\sum_{j=1}^{l}\right)^{\frac{1}{n-1}}}\,dx > |\Omega| + \frac{\omega_{n-1}}{n}e^{\alpha_n A_p+1+\frac{1}{2}+\cdots+\frac{1}{n-1}}.$$

Let $r=|x-p|$, where p is a concentration point. Set $\widetilde{G} = G + \frac{n}{\alpha_n}\ln r - A_p$. Define

$$\phi_\epsilon = \begin{cases} \dfrac{c+c^{-\frac{1}{n-1}}\left(-\frac{n-1}{\alpha_n}\ln\left(1+c_n\left(\frac{r}{\epsilon}\right)^{\frac{n}{n-1}}\right)+B\right)}{\left(1+c^{-\frac{n}{n-1}}\sum_{j=1}^{l}\alpha_j\|G\|_{p_j}^{n}\right)^{\frac{1}{n}}} & \text{for } r\le R\epsilon,\\ \dfrac{G-\eta\widetilde{G}}{\left(c^{\frac{n}{n-1}}+\sum_{j=1}^{l}\alpha_j\|G\|_{p_j}^{n}\right)^{\frac{1}{n}}} & \text{for } R\epsilon<r<2R\epsilon,\\ \dfrac{G}{\left(c^{\frac{n}{n-1}}+\sum_{j=1}^{l}\alpha_j\|G\|_{p_j}^{n}\right)^{\frac{1}{n}}} & \text{for } r\ge 2R\epsilon,\end{cases}$$

where $c_n = \left(\frac{\omega_{n-1}}{n}\right)^{\frac{1}{n-1}}$, $\eta\in C_0^\infty(\mathbb{B}_{2R\epsilon}(p))$ is a cutoff function, $\eta=1$ on $\mathbb{B}_{R\epsilon}(p)$, $\|\nabla\eta\|_{L^\infty} = O\left(\frac{1}{R\epsilon}\right)$, B is a constant to be determined later, and R, c depending on ϵ will also be chosen later such that $R\epsilon\to 0$ and $R\to+\infty$. In order to assure that $\phi_\epsilon\in H_0^{1,n}(\Omega)$, we set

$$c+c^{-\frac{1}{n-1}}\left(-\frac{n-1}{\alpha_n}\ln\left(1+c_n\left(\frac{R\epsilon}{\epsilon}\right)^{\frac{n}{n-1}}\right)+B\right) = c^{-\frac{1}{n-1}}\left(-\frac{n}{\alpha_n}\ln R\epsilon + A_p\right).$$

Multiplying both sides of the above equation by $c^{\frac{1}{n-1}}$, we get

$$c^{\frac{n}{n-1}} - \frac{n-1}{\alpha_n}\ln\left(1+c_nR^{\frac{n}{n-1}}+B\right) = -\frac{n}{\alpha_n}\ln R\epsilon + A_p,$$

which gives that

$$c^{\frac{n}{n-1}} = -\frac{n}{\alpha_n}\ln\epsilon + \frac{n-1}{\alpha_n}\ln c_n - B + A_p + O\left(R^{-\frac{n}{n-1}}\right). \tag{3.4}$$

A straightforward calculation shows

$$\begin{aligned}&\int_{r\le R\epsilon}|\nabla\phi_\epsilon|^n\,dx\\ &= \frac{n-1}{\alpha_n\left(c^{\frac{n}{n-1}}+\sum_{j=1}^{l}\alpha_j\|G\|_{p_j}^{n}\right)}\int_0^{c_nR^{\frac{n}{n-1}}}\frac{z^{n-1}}{(1+z)^n}\,dz\end{aligned}$$

$$
\begin{aligned}
&= \frac{n-1}{\alpha_n\big(c^{\frac{n}{n-1}} + \sum_{j=1}^l \alpha_j \|G\|_{p_j}^n\big)} \int_0^{c_n R^{\frac{n}{n-1}}} \frac{((1+z)-1)^{n-1}}{(1+z)^n}\, dz \\
&= \frac{n-1}{\alpha_n\big(c^{\frac{n}{n-1}} + \sum_{j=1}^l \alpha_j \|G\|_{p_j}^n\big)} \left(\sum_{k=0}^{n-2} \frac{C_{n-1}^k (-1)^{n-1-k}}{n-k-1} + \ln\big(1 + c_n R^{\frac{n}{n-1}}\big) + O\big(R^{-\frac{n}{n-1}}\big) \right) \\
&= \frac{n-1}{\alpha_n\big(c^{\frac{n}{n-1}} + \sum_{j=1}^l \alpha_j \|G\|_{p_j}^n\big)} \left(\ln\big(1 + c_n R^{\frac{n}{n-1}}\big) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) + O\big(R^{-\frac{n}{n-1}}\big) \right),
\end{aligned}
$$

where $C_{n-2}^k = \frac{(n-2)!}{(n-2-k)!k!}$, and we have used

$$
\sum_{k=0}^{n-2} \frac{(-1)^{n-1-k} C_{n-1}^k}{n-k-1} = 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}
$$

by induction. Taking into account the expression of ϕ_ϵ, then

$$
\begin{aligned}
\int_\Omega |\nabla \phi_\epsilon|^n\, dx = &\frac{n-1}{\alpha_n\big(c^{\frac{n}{n-1}} + \sum_{j=1}^l \alpha_j \|G\|_{p_j}^n\big)} \\
&\times \Bigg(-\frac{n}{n-1} \ln \epsilon + \ln c_n + \frac{\alpha_n A_p}{n-1} - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) \\
&\quad + \frac{\alpha_n}{n-1} \sum_{j=1}^l \alpha_j \|G\|_{p_j}^n + O\big(R^{-\frac{n}{n-1}}\big) + O\bigg((R\epsilon)^n \log^n \frac{1}{R\epsilon} \log R\bigg)\Bigg).
\end{aligned}
$$

Since $\int_\Omega |\nabla \phi_\epsilon|^n\, dx = 1$, we obtain

$$
\begin{aligned}
c^{\frac{n}{n-1}} = &-\frac{n}{\alpha_n} \ln \epsilon + \frac{n-1}{\alpha_n} \ln c_n + A_p - \frac{n-1}{\alpha_n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) \\
&+ O\big(R^{-\frac{n}{n-1}}\big) + O\left((R\epsilon)^n \ln^n \frac{1}{R\epsilon} \ln R\right).
\end{aligned} \tag{3.5}
$$

Combining (3.4) and (3.5), we have

$$
B = \frac{n-1}{\alpha_n}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) + O\big(R^{-\frac{n}{n-1}}\big) + O\left((R\epsilon)^n \ln^n \frac{1}{R\epsilon} \ln R\right).
$$

Set $R = -\ln \epsilon$, which satisfies $R\epsilon \to 0$ as $\epsilon \to 0$. Since

$$
\|\phi_\epsilon\|_{p_j}^n = \frac{\|G\|_{p_j}^n + O\big(c^{\frac{n^2}{n-1}} R^{\frac{n^2}{p_j}} \epsilon^{\frac{n^2}{p_j}}\big) + O\big((R\epsilon)^{\frac{n^2}{p_j}} (-\ln(R\epsilon))^n\big)}{c^{\frac{n}{n-1}} + \sum_{j=1}^l \alpha_j \|G\|_{p_j}^n},
$$

then it is easy to see that

$$
\alpha_n |\phi_\epsilon|^{\frac{n}{n-1}} \left(1 + \sum_{j=1}^l \alpha_j \|\phi_\epsilon\|_{p_j}^n\right)^{\frac{1}{n-1}}
$$

$$
\begin{aligned}
&\geq \alpha_n c^{\frac{n}{n-1}} - n\ln\left(1+c_n\left(\frac{r}{\epsilon}\right)^{\frac{n}{n-1}}\right) + \frac{n\alpha_n}{n-1}B + O\big(c^{-\frac{2n}{n-1}}\big)\\
&\quad - \frac{\alpha_n}{(n-1)c^{\frac{n}{n-1}}}\sum_{j=1}^{l}\alpha_j^2\|G\|_{p_j}^{2n} + O\left(c^{\frac{n^2}{n-1}}\sum_{j=1}^{n} R^{\frac{n^2}{p}}\epsilon^{\frac{n^2}{p}}\right) + O\left(\sum_{j=1}^{l}(R\epsilon)^{\frac{n^2}{p_j}}(-\ln(R\epsilon))^n\right)\\
&\geq -n\ln\epsilon + (n-1)\ln c_n + \alpha_n A_p + \left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)\\
&\quad - n\ln\left(1+c_n\left(\frac{r}{\epsilon}\right)^{\frac{n}{n-1}}\right) - \frac{\alpha_n\sum_{j=1}^{l}\alpha_j^2\|G\|_{p_j}^{2n}}{(n-1)c^{\frac{n}{n-1}}} + L,
\end{aligned}
$$

where

$$
L = O\left(c^{\frac{n^2}{n-1}}\sum_{j=1}^{n} R^{\frac{n^2}{p}}\epsilon^{\frac{n^2}{p}}\right) + O\left(\sum_{j=1}^{l}(R\epsilon)^{\frac{n^2}{p_j}}(-\ln(R\epsilon))^n\right) \quad \text{and} \quad D = \frac{\alpha_n\sum_{j=1}^{l}\alpha_j^2\|G\|_{p_j}^{2n}}{(n-1)c^{\frac{n}{n-1}}}.
$$

With the above estimates, we get

$$
\begin{aligned}
&\int_{\mathbb{B}_{R\epsilon}} e^{\alpha_n|\phi_\epsilon|^{\frac{n}{n-1}}\left(1+\sum_{j=1}^{l}\alpha\|\phi_\epsilon\|_{p_j}^{n}\right)^{\frac{1}{n-1}}}\,dx\\
&\geq e^{-n\ln\epsilon+(n-1)\ln c_n+\alpha_n A_p+\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)-D+L}\int_{\mathbb{B}_{R\epsilon}} e^{-n\ln\left(1+c_n\left(\frac{r}{\epsilon}\right)^{\frac{n}{n-1}}\right)}\,dx\\
&\geq c_n^{n-1}e^{\alpha_n A_p+\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)-D+L}\int_0^{c_n R^{\frac{n}{n-1}}}\frac{z^{n-2}}{(1+z)^n}\,dz\\
&\geq \frac{(n-1)\omega_{n-1}}{n}e^{\alpha_n A_p+\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right)-D+L}\left(\frac{1}{n-1}+O\big(R^{-\frac{n}{n-1}}\big)\right)\\
&\geq \frac{\omega_{n-1}}{n}e^{\alpha_n A_p+1+\frac{1}{2}+\cdots+\frac{1}{n-1}} - \frac{\omega_{n-1}}{n}De^{\alpha_n A_p+1+\frac{1}{2}+\cdots+\frac{1}{n-1}} + L,
\end{aligned}
$$

where we have used

$$
\sum_{k=0}^{n-2}\frac{(-1)^{n-1-k}C_{n-1}^{k}}{n-k-1} = \frac{1}{n-1}.
$$

On the other hand,

$$
\begin{aligned}
\int_{\Omega\setminus\mathbb{B}_{R\epsilon}} e^{\alpha_n|\phi_\epsilon|^{\frac{n}{n-1}}\left(1+\sum_{j=1}^{l}\alpha\|\phi_\epsilon\|_{p_j}^{n}\right)^{\frac{1}{n-1}}}\,dx &\geq \int_{\Omega\setminus\mathbb{B}_{2R\epsilon}}\left(1+\alpha_n|\phi_\epsilon|^{\frac{n}{n-1}}\right)dx\\
&\geq |\Omega| + \frac{\alpha_n}{c^{\frac{n}{(n-1)^2}}}\sum_{j=1}^{l}\alpha_j\|G\|_{p_j}^{\frac{n}{n-1}} + O\big(R^{-\frac{2n}{(n-1)^2}}\big).
\end{aligned}
$$

So we conclude

$$
\int_{\Omega} e^{\alpha_n|\phi_\epsilon|^{\frac{n}{n-1}}\left(1+\sum_{j=1}^{l}\alpha\|\phi_\epsilon\|_{p_j}^{n}\right)^{\frac{1}{n-1}}}\,dx > |\Omega| + \frac{\omega_{n-1}}{n}e^{\alpha_n A_p+1+\frac{1}{2}+\cdots+\frac{1}{n-1}}. \tag{3.6}
$$

The contradiction between (3.1) and (3.6) implies that c_ϵ is bounded. Then elliptic estimate implies that Theorem 1.2 holds.

4. Proof of Theorem 1.4

In this section, we will prove Theorem 1.4. Similarly, we first consider the subcritical function, it is not difficult to check that the Euler–Lagrange equation of u_ϵ is

$$-\Delta_n u_\epsilon = \frac{1}{\lambda_\epsilon} u_\epsilon |u_\epsilon|^{\frac{2-n}{n-1}} e^{\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}}} + \sum_{j=1}^{l} \alpha_j \|u_\epsilon\|_{p_j}^{n-p_j} u_\epsilon^{p_j-1} \tag{4.1}$$

with

$$\alpha_\epsilon = (\alpha_n - \epsilon), \quad \lambda_\epsilon = \int_\Omega |u_\epsilon|^{\frac{n}{n-1}} e^{\alpha_\epsilon |u_\epsilon|^{\frac{n}{n-1}}} \, dx.$$

Comparing (4.1) and (2.2), we conclude that the above discussion of the proof of Theorem 1.1 is nearly same as that of Theorem 1.3, so we omit here. Unlike the proof of Theorem 1.2, we use a different approach to prove Theorem 1.4. First we have the following upper bound inequality

$$\sup_{u \in H_0^{1,n}(\Omega),\ \|\nabla u\|_n^n - \sum_{j=1}^{l} \alpha_j \|u\|_{p_j}^n \le 1} \int_\Omega e^{\alpha_n |u|^{\frac{n}{n-1}}} \, dx \le |\Omega| + \frac{\omega_{n-1}}{n} e^{\alpha_n A_p + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}. \tag{4.2}$$

Then we need to construct a test function contradicting (4.2). The following lemma is crucial in the subsequent computation process.

Lemma 4.1. *Let G be the n-Green function in the above* (3.2)*.*

(a) *The sets $\{G > t\}$ form a sequence of approximately small balls of radii $\rho_t = e^{\omega_{n-1}^{\frac{1}{n-1}}(A_p - t)}$. In other words, $B_{\rho_t - r_t}(p) \subset \{G > t\} \subset B_{\rho_t + r_t}(p)$ with $r_t/\rho_t \to 0$ as $t \to +\infty$. In particular, $\lim_{t\to+\infty} e^{\alpha_n t} |\{G > t\}| = \frac{\omega_{n-1}}{n} e^{\alpha_n A_p}$.*

(b) *$\int_{G<t} |\nabla G|^n \, dx = t + \sum_{j=1}^{l} \alpha_j \|G\|_{p_j}^n + O(t^n e^{-\alpha_n t})$ as $t \to +\infty$.*

(c) *$\int_{G=t} |\nabla G|^{n-1} \, dx = 1 + O(t^n e^{-\alpha_n t})$ as $t \to +\infty$.*

(d) *$\int_{G=t} \frac{1}{|\nabla G|} \, ds \ge \omega_{n-1}^{\frac{n}{n-1}} e^{\alpha_n (A_p - t)} \big(1 + O(t^n e^{-\alpha_n t})\big)$ as $t \to +\infty$.*

The proof is similar to that in [32] so we omit the process of proof here. Then we take

$$f_\epsilon(t) = \begin{cases} c - c^{-\frac{1}{n-1}} \left(\frac{n-1}{\alpha_n} \ln \left(1 + c_n \epsilon^{-\frac{n}{n-1}} e^{-\frac{\alpha_n}{n-1} t}\right) + B \right) & \text{for } t \ge t_\epsilon, \\ c^{-\frac{1}{n-1}} t & \text{for } t < t_\epsilon \end{cases}$$

with $c_n = (\omega_{n-1}/n)^{1/(n-1)}$, $t_\epsilon = \frac{n}{\alpha_n} \ln \frac{1}{R\epsilon}$, R, B and C are constants to be chosen later such that $R \to +\infty$ and $R\epsilon \to 0$ as $\epsilon \to 0$. Let G be as above. Set

$$\phi_\epsilon(x) = f_\epsilon(G(x)).$$

To ensure $\phi_\epsilon \in H_0^{1,n}(\Omega)$, we assume

$$c - c^{-\frac{1}{n-1}} \left(\frac{n-1}{\alpha_n} \ln\left(1 + c_n R^{\frac{n}{n-1}}\right) + B \right) = \frac{n}{\alpha_n} c^{-\frac{1}{n-1}} \ln \frac{1}{R\epsilon}. \tag{4.3}$$

We have by Lemma 4.1(b) that

$$\int_{G<t_\epsilon} |\nabla \phi_\epsilon|^n = c^{-\frac{n}{n-1}} \left(\frac{n}{\alpha_n} \ln \frac{1}{R\epsilon} + \sum_{j=1}^{l} \alpha_j \|G\|_{p_j}^n + O\left((R\epsilon)^n \ln^n \frac{1}{R\epsilon} \right) \right).$$

An elementary calculation shows

$$\begin{aligned}
\int_{t_\epsilon}^{+\infty} |f_\epsilon'(t)|^n \, dt &= c^{-\frac{n}{n-1}} \int_{t_\epsilon}^{+\infty} \left(\frac{c_n \epsilon^{-\frac{n}{n-1}} e^{-\frac{\alpha_n}{n-1} t}}{1 + c_n \epsilon^{-\frac{n}{n-1}} e^{-\frac{\alpha_n}{n-1} t}} \right)^n dt \\
&= \frac{n-1}{\alpha_n} c^{-\frac{n}{n-1}} \int_0^{c_n R^{\frac{n}{n-1}}} \frac{s^{n-1}}{(1+s)^n} \, ds \\
&= \frac{n-1}{\alpha_n} c^{-\frac{n}{n-1}} \left(\ln\left(1 + c_n R^{\frac{n}{n-1}}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) \right) \\
&\quad + c^{-\frac{n}{n-1}} O\big(R^{-\frac{n}{n-1}}\big).
\end{aligned}$$

Hence we have by Lemma 4.1(c) that

$$\begin{aligned}
\int_{G>t_\epsilon} |\nabla \phi_\epsilon|^n \, dx &= \int_{t_\epsilon}^{+\infty} |f_\epsilon'(t)|^n \left(\int_{G=t} |\nabla G|^n \frac{1}{|\nabla G|} \, ds \right) dt \\
&= \int_{t_\epsilon}^{+\infty} |f_\epsilon'(t)|^n \big(1 + O(t^n e^{-\alpha_n t})\big) \, dt \\
&= \frac{n-1}{\alpha_n} c^{-\frac{n}{n-1}} \left(\ln\left(1 + c_n R^{\frac{n}{n-1}}\right) - \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) \right) \\
&\quad + c^{-\frac{n}{n-1}} \left(O\big(R^{-\frac{n}{n-1}}\big) + O\left((R\epsilon)^n \log^n \frac{1}{R\epsilon} \log R \right) \right).
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_\Omega |\nabla \phi_\epsilon|^n \, dx &= -\frac{n-1}{\alpha_n} c^{-\frac{n}{n-1}} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) + c^{-\frac{n}{n-1}} \sum_{j=1}^{l} \alpha_j \|G\|_{p_j}^n \\
&\quad + \frac{n-1}{\alpha_n} c^{-\frac{n}{n-1}} \ln\left(1 + c_n R^{\frac{n}{n-1}}\right) + \frac{n}{\alpha_n} c^{-\frac{n}{n-1}} \ln \frac{1}{R\epsilon} \\
&\quad + c^{-\frac{n}{n-1}} \left(O\big(R^{-\frac{n}{n-1}}\big) + O\left((R\epsilon)^n \ln^n \frac{1}{R\epsilon} \ln R \right) \right).
\end{aligned}$$

Since $\|\nabla \phi_\epsilon\|_n^n - \sum_{j=1}^{l} \alpha_j \|\phi_\epsilon\|_{p_j}^n = 1$, then we have

$$\begin{aligned}
c^{\frac{n}{n-1}} &= -\frac{n-1}{\alpha_n} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n-1}\right) + \frac{n-1}{\alpha_n} \ln\left(1 + c_n R^{\frac{n}{n-1}}\right) + \frac{n}{\alpha_n} \ln \frac{1}{R\epsilon} \\
&\quad + O\big(R^{-\frac{n}{n-1}}\big) + O\left((R\epsilon)^n \ln^n \frac{1}{R\epsilon} \ln R \right).
\end{aligned}$$

Combining (4.2) and (4.3), which gives

$$B = -\frac{n-1}{\alpha_n}\left(1+\frac{1}{2}+\cdots+\frac{1}{n-1}\right) + O\big(R^{-\frac{n}{n-1}}\big) + O\left((R\epsilon)^n \ln^n \frac{1}{R\epsilon}\ln R\right).$$

For $t > t_\epsilon$, one can check that

$$\begin{aligned}\alpha_n f_\epsilon(t)^{\frac{n}{n-1}} &= \alpha_n c^{\frac{n}{n-1}} - n\ln\big(1+c_n\epsilon^{-\frac{n}{n-1}}e^{-\frac{\alpha_n}{n-1}t}\big) - \frac{n}{n-1}\alpha_n B \\ &\quad + O\big(R^{-\frac{n}{n-1}}\big) + O\left((R\epsilon)^n \ln^n \frac{1}{R\epsilon}\ln R\right) \\ &= 1+\frac{1}{2}+\cdots+\frac{1}{n-1} + (n-1)\ln c_n - n\ln\epsilon - n\ln\big(1+c_n\epsilon^{-\frac{n}{n-1}}e^{-\frac{\alpha_n}{n-1}t}\big) \\ &\quad + O\big(R^{-\frac{n}{n-1}}\big) + O\left((R\epsilon)^n \ln^n \frac{1}{R\epsilon}\ln R\right) + O\big(c^{-\frac{n}{n-1}}\ln^2 R\big).\end{aligned}$$

Then by Lemma 4.1, it is easy to see that

$$\begin{aligned}\int_{G\geq t_\epsilon} e^{\alpha_n|\phi_\epsilon|^{\frac{n}{n-1}}}\,dx &= \int_{t_\epsilon}^{+\infty} e^{\alpha_n|f_\epsilon(t)|^{\frac{n}{n-1}}}\left(\int_{G=t}\frac{1}{|\nabla G|}\,ds\right)dt \\ &\geq \frac{\omega_{n-1}}{n}\epsilon^{-n}e^{\alpha_n A_p+1+\frac{1}{2}+\cdots+\frac{1}{n-1}} \\ &\quad \times\left(1+O\big(R^{-\frac{n}{n-1}}\big)+O\left((R\epsilon)^n\ln^n\frac{1}{R\epsilon}\ln R\right)+O\big(c^{-\frac{n}{n-1}}\ln^2 R\big)\right) \\ &\quad \times \omega_{n-1}^{\frac{n}{n-1}}\int_{t_\epsilon}^{+\infty}\frac{e^{-\alpha_n t}}{\big(1+c_n\epsilon^{-\frac{n}{n-1}}e^{-\frac{\alpha_n}{n-1}t}\big)^n}\big(1+O(t^n e^{-\alpha_n t})\big)\,dt.\end{aligned}$$

Then we obtain

$$\begin{aligned}\int_{G\geq t_\epsilon} e^{\alpha_n|\phi_\epsilon|^{\frac{n}{n-1}}}\,dx &\geq \frac{\omega_{n-1}}{n}e^{\alpha_n A_p+1+\frac{1}{2}+\cdots+\frac{1}{n-1}} \\ &\quad \times\left(1+O\big(R^{-\frac{n}{n-1}}\big)+O\left((R\epsilon)^n\ln^n\frac{1}{R\epsilon}\ln R\right)+O\big(c^{-\frac{n}{n-1}}\ln^2 R\big)\right).\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_{G<t_\epsilon} e^{\alpha_n|\phi_\epsilon|^{\frac{n}{n-1}}}\,dx &\geq \int_{G<t_\epsilon}\big(1+\alpha_n|\phi_\epsilon|^{\frac{n}{n-1}}\big)\,dx \\ &\geq |\Omega| + O((R\epsilon)^n) + O\left(c^{-\frac{n}{(n-1)^2}}(R\epsilon)^n\ln^{\frac{n}{n-1}}\frac{1}{R\epsilon}\right).\end{aligned}$$

Combining the above estimates, we get

$$\begin{aligned}\int_\Omega e^{\alpha_n|\phi_\epsilon|^{\frac{n}{n-1}}}\,dx &\geq |\Omega| + \frac{\omega_{n-1}}{n}e^{\alpha_n A_p+1+\frac{1}{2}+\cdots+\frac{1}{n-1}} \\ &\quad + \alpha_n c^{-\frac{n}{(n-1)^2}}\left(O\left(c^{-\frac{n}{(n-1)^2}}(R\epsilon)^n\ln^n\frac{1}{R\epsilon}\ln R\right)\right. \\ &\quad \left. + O\big(c^{-\frac{n}{(n-1)^2}}R^{-\frac{n}{n-1}}\big) + O\big(c^{-\frac{n^2-2n}{(n-1)^2}}\ln^2 R\big)\right).\end{aligned}$$

Taking $R = -\log \epsilon$, we immediately have

$$\int_\Omega e^{\alpha_n |\phi_\epsilon|^{\frac{n}{n-1}}}\, dx > |\Omega| + \frac{\omega_{n-1}}{n} e^{\alpha_n A_p + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1}}.$$

This completes the proof of Theorem 1.4.

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