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# Precise Asymptotic Spreading Behavior for an Epidemic Model with Nonlocal Dispersal

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Abstract. This paper is to derive the precise asymptotic spreading behavior for an epidemic model with nonlocal dispersal. The proof is based on a Liouville type theorem on the positive bounded entire solutions. This Liouville theorem holds for a general class of reaction-diffusion systems with nonlocal dispersal which can be useful for reaction-diffusion systems arising in ecology and epidemiology.

## 1. Introduction

In this paper, we consider the following SIR (susceptible-infective-removed) epidemic model with nonlocal dispersal

$$S_t(x,t) = d_1\mathcal{N}_1[S(\,\cdot\,,t)](x) + \mu - \mu S(x,t) - \frac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)}, \quad x \in \mathbb{R},\ t > 0, \tag{1.1}$$

$$I_t(x,t) = d_2\mathcal{N}_2[I(\,\cdot\,,t)](x) + \frac{\beta S(x,t)I(x,t)}{1+\alpha I(x,t)} - (\mu+\sigma)I(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{1.2}$$

$$R_t(x,t) = d_3\mathcal{N}_3[R(\,\cdot\,,t)](x) + \sigma I(x,t) - \mu R(x,t), \quad x \in \mathbb{R},\ t > 0, \tag{1.3}$$

where $S(x,t)$, $I(x,t)$, $R(x,t)$ represent the population densities of the susceptible, infective, removed individuals at position $x$ and time $t$, respectively. The parameters $d_1$, $d_2$, $d_3$, $\mu$, $\beta$, $\sigma$ are all positive constants in which $d_i$ is the diffusion coefficient, $i = 1,2,3$, and $\mu$ denotes the same death rates of susceptible, infective and removed populations. Also, after a suitable rescaling (cf. [21]), the inflow of newborns into the susceptible population is taken to be the same constant $\mu$. The parameter $\sigma$ is the removed/recovery rate and $\beta$ is the infective transmission rate. While the *nonnegative* constant $\alpha$ measures the saturation level (see [5,23]) in the Holling type II incidence function $\beta SI/(1+\alpha I)$.

Moreover, the nonlocal dispersal $\mathcal{N}_i$ is an operator acting on a function $\varphi$ defined by

$$\mathcal{N}_i[\varphi](x) := (J_i * \varphi)(x) - \varphi(x) = \int_{\mathbb{R}} J_i(x-y)\varphi(y)\,dy - \varphi(x), \quad x \in \mathbb{R},$$

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where the kernel $J_i$ is a probability density function, $i = 1, 2, 3$. Throughout this paper, we adopt the following class of kernels. For a given $\widehat{\lambda} \in (0, \infty]$, a function $J \colon \mathbb{R} \to [0, \infty)$ is said to be in the class $\mathcal{P}(\widehat{\lambda})$ if the following conditions hold:

(J1) the kernel $J$ is nonnegative and continuous;

(J2) it holds that

$$\int_{\mathbb{R}} J(y)\, dy = 1, \quad J(y) = J(-y) \quad \text{for all } y \in \mathbb{R};$$

(J3) it holds that $\int_{\mathbb{R}} J(y) e^{\lambda |y|}\, dy < \infty$ for any $\lambda \in (0, \widehat{\lambda})$ and

$$\int_{\mathbb{R}} J(y) e^{\lambda |y|}\, dy \to \infty \quad \text{as } \lambda \uparrow \widehat{\lambda}.$$

Unlike the classical diffusion modelling the random movements, the mechanism of nonlocal dispersal describes the individuals moving freely to have a long-range diffusion effect [22]. This nonlocal interaction nature is often presented in many diffusive systems in ecology, biology, neuroscience etc. Therefore, the study of nonlocal evolution equations has attracted a lot of attention in past years, we refer the reader to, e.g., [1–4, 6–11, 14–18, 20, 24–27] and the references cited therein.

We are concerned with the precise asymptotic spreading behaviors of solutions to system (1.1)–(1.3). Since (1.3) is decoupled from the other two equations in our SIR model, in the sequel we shall only consider the system (1.1)–(1.2). In particular, we are interested in the initial value problem for (1.1)–(1.2) with the initial condition

$$S(x, 0) = 1, \quad I(x, 0) = I_0(x), \quad x \in \mathbb{R}, \tag{1.4}$$

where $I_0$ is a nonnegative continuous function defined in $\mathbb{R}$ with a nonempty compact support.

Under the assumption

$$\beta > \mu + \sigma, \tag{1.5}$$

there is a unique stable positive endemic equilibrium $(S^*, I^*)$, where

$$S^* = \frac{\mu + \sigma + \alpha\mu}{\alpha\mu + \beta}, \quad I^* = \frac{\mu(\beta - \mu - \sigma)}{(\mu + \sigma)(\alpha\mu + \beta)},$$

which corresponds to the coexistence state of $(S, I)$. Hereafter we set $\gamma := \mu + \sigma$ and define

$$c^* := \inf_{0 < \lambda < \widehat{\lambda}_2} \frac{d_2 \left[ \int_{\mathbb{R}} J_2(y) e^{\lambda y}\, dy - 1 \right] + \beta - \gamma}{\lambda}. \tag{1.6}$$

Note that the constant $c^*$ is well-defined and $c^* > 0$, since $\beta - \gamma > 0$ due to (1.5).

We now state the main theorem of this paper as follows.

**Theorem 1.1.** *Let $\alpha \geq 0$ and $J_i \in \mathcal{P}(\widehat{\lambda}_i)$ for some $\widehat{\lambda}_i \in (0,\infty]$, $i = 1,2$. Assume* (1.5)*. In the case $\alpha = 0$, we further assume that $d_1 = d_2$ and $J_1 = J_2$. Let $(S, I)$ be a solution of* (1.1), (1.2) *and* (1.4) *with a nonnegative nontrivial compactly supported continuous initial data $I_0$. Then*

$$\lim_{t\to\infty} \sup_{|x|\leq ct} \{|S(x,t) - S^*| + |I(x,t) - I^*|\} = 0, \quad \forall\, c \in (0, c^*),$$

*where $c^*$ is defined by* (1.6).

To prove Theorem 1.1, we present in this paper a Liouville type theorem (see Theorem 2.1 below) to characterize entire solutions for more general reaction-diffusion systems including system (1.1)–(1.2) as a special case. Hereafter, a solution is called an entire solution if it is defined for all $t \in \mathbb{R}$. For the characterization of entire solutions in the study of the asymptotic behavior of the associated reaction-diffusion systems, we refer the reader to, e.g., the references cited in [12] for the case of classical diffusion and [13] for the fractional diffusion. In fact, the proof of Theorem 2.1 is quite similar to the one given in [13]. However, extending a Lyapunov function for ODE to a Lyapunov functional for PDE in an unbounded spatial domain relies on a suitable choice of the weight function. We are able to find such a weight function to overcome this difficulty. Consequently, Theorem 2.1 can be applied to a large class of systems in ecology and epidemiology such as those reaction-diffusion systems studied in [12, 13] with diffusions replaced by nonlocal dispersals.

The rest of this paper is organized as follows. We present a Liouville type theorem along with its proof in Section 2. Then we give the detailed proof of Theorem 1.1 for the precise asymptotic spreading behavior of system (1.1)–(1.2) in Section 3.

## 2. A Liouville type theorem

In this section, we consider the following general reaction-diffusion system

$$\frac{\partial u_i}{\partial t} = d_i \mathcal{N}_i[u_i(\,\cdot\,,t)](x) + f_i(u_1,\ldots,u_m), \quad x \in \mathbb{R}^n,\ t \in \mathbb{R},\ i = 1,\ldots,m, \tag{2.1}$$

where $m$, $n$ are positive integers, $d_i > 0$ and $f_i \colon \mathbb{R}^m \to \mathbb{R}$ is a $C^1$ function for each $i = 1,\ldots,m$. Moreover, $J_i \in \mathcal{P}(\widehat{\lambda}_i)$ for some constant $\widehat{\lambda}_i \in (0,\infty]$ for $i = 1,\ldots,m$. Note that conditions (J1)–(J3) are well-defined in $\mathbb{R}^n$ for $n \geq 1$.

We assume that (2.1) has a unique positive constant equilibrium $u^* := (u_1^*,\ldots,u_m^*)$ such that $u_i^* \in (0,\infty)$ for each $i$. Set $g(\theta) := \theta - 1 - \ln\theta$, $\theta > 0$. Note that $g$ is a strictly convex smooth function on $(0,\infty)$ such that $g(1) = 0$ and $g(\theta) > 0$ for all $\theta \neq 1$. Then we have the following Liouville type theorem for (2.1).

**Theorem 2.1.** *Let* $u = (u_1, \dots, u_m)$ *be an entire solution of* (2.1) *such that* $0 < a_i \le u_i \le A_i < \infty$ *for* $i = 1, \dots, m$ *for some positive constants* $\{a_i, A_i\}$ *with* $a_i \le u_i^* \le A_i$. *Suppose that the corresponding diffusion-free system of* (2.1) *admits a nonnegative bounded Lyapunov function in the form*

$$F(u) = \sum_{i=1}^{m} F_i(u_i), \quad u = (u_1, \dots, u_m) \in \mathbb{R}_+^m,$$

*where* $F_i(u_i) = c_i g(u_i/u_i^*)$ *for some positive constant* $c_i$ *for* $i = 1, \dots, m$ *such that*

$$\sum_{i=1}^{m} F_i'(u_i) f_i(u) \le -\kappa F(u) \quad \text{for } u_i \in [a_i, A_i],\ 1 \le i \le m \tag{2.2}$$

*for some positive constant* $\kappa$. *Then* $u = u^*$.

To prove Theorem 2.1, we first prepare the following lemma.

**Lemma 2.2.** *Let* $J \in \mathcal{P}(\widehat{\lambda})$ *for some* $\widehat{\lambda} \in (0, \infty]$. *Then for any* $\varepsilon > 0$ *there exists* $R > 0$ *sufficiently large such that*

$$\mathcal{N}[\rho_R](x) := (J * \rho_R)(x) - \rho_R(x) \le \varepsilon \rho_R(x), \quad \forall\, x \in \mathbb{R}^n,$$

*where*

$$\rho_R(x) := e^{-|x|/R}.$$

*Proof.* First, $J * e^{-|x|/R}$ is well-defined because of (J2).

Next, writing

$$\mathcal{N}[\rho_R](x) = \int_{\mathbb{R}^n} J(y) \big[ e^{-|x-y|/R} - e^{-|x|/R} \big]\, dy = \rho_R(x) \int_{\mathbb{R}^n} J(y) \big[ e^{|x|/R - |x-y|/R} - 1 \big]\, dy$$

and using $|x| - |x-y| \le |y|$ for any $x, y \in \mathbb{R}^n$, we obtain

$$\mathcal{N}[\rho_R](x) \le \rho_R(x) \int_{\mathbb{R}^n} J(y) \big\{ e^{|y|/R} - 1 \big\}\, dy. \tag{2.3}$$

Now, let $\varepsilon > 0$ be given and let $R_0 > 1$ be sufficiently large such that $1/R_0 < \widehat{\lambda}$. Then, by (J2) and (J3), there exists $r > 0$ sufficiently large such that

$$0 < \int_{|y| \ge r} J(y) \big\{ e^{|y|/R} - 1 \big\}\, dy \le \int_{|y| \ge r} J(y) \big\{ e^{|y|/R_0} - 1 \big\}\, dy < \varepsilon/2, \quad \forall\, R \ge R_0. \tag{2.4}$$

Moreover, since the sequence $J(y)\big\{ e^{|y|/R} - 1 \big\}$ converges to 0 as $R \to \infty$ uniformly over $\{|y| \le r\}$, we may choose a large enough $R \ge R_0$ such that

$$0 < \int_{|y| \le r} J(y) \big\{ e^{|y|/R} - 1 \big\}\, dy < \varepsilon/2.$$

Then the lemma follows from this estimate together with (2.3) and (2.4). □

With the weight $\rho_R$, we introduce the functional

$$\mathcal{F}_R(t) := \int_{\mathbb{R}^n} F(u(x,t))\rho_R(x)\,dx$$

for an entire solution $u$ of (2.1) satisfying

$$0 < a_i \le u_i \le A_i < \infty, \quad i = 1, \dots, m.$$

Note that $F(u(x,t))$ is uniformly bounded over $\mathbb{R}^n \times \mathbb{R}$ and $\rho_R$ is integrable over $\mathbb{R}^n$. Hence $\mathcal{F}_R(t)$ is well-defined and uniformly bounded for $t \in \mathbb{R}$. Then Theorem 2.1 can be proved in the same manner as that for [13, Theorem 1.1]. To be self-contained and for the reader's convenience, we provide some details here.

*Proof of Theorem* 2.1. First, we compute

$$\frac{d}{dt}\mathcal{F}_R(t) = \sum_{i=1}^m \int_{\mathbb{R}^n} F_i'(u_i) f_i(u)\rho_R\,dx + \sum_{i=1}^m d_i \int_{\mathbb{R}^n} F_i'(u_i)\mathcal{N}_i[u_i]\rho_R\,dx.$$

It follows from (2.2) and $g'(\theta) = 1 - 1/\theta$ that

$$\frac{d}{dt}\mathcal{F}_R(t) \le -\kappa\mathcal{F}_R(t) + \sum_{i=1}^m d_i c_i \frac{1}{u_i^*}\int_{\mathbb{R}^n}\left(1 - \frac{u_i^*}{u_i}\right)\mathcal{N}_i[u_i]\rho_R\,dx. \tag{2.5}$$

Next, for a fixed $i$ set

$$I_i(t) := \int_{\mathbb{R}^n}\left(1 - \frac{u_i^*}{u_i(x,t)}\right)\mathcal{N}_i[u_i](x,t)\rho_R(x)\,dx.$$

Then we obtain from

$$\mathcal{N}_i[u_i](x,t) = \int_{\mathbb{R}^n} J_i(x-y)\{u_i(y,t) - u_i(x,t)\}\,dy,$$

that

$$I_i(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)u_i^*\left[1 - \frac{u_i(x,t)}{u_i^*} + \frac{u_i(y,t)}{u_i^*} - \frac{u_i(y,t)}{u_i(x,t)}\right]\rho_R(x)\,dydx. \tag{2.6}$$

By changing the order of integration, we also have

$$I_i(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)u_i^*\left[1 - \frac{u_i(x,t)}{u_i^*} + \frac{u_i(y,t)}{u_i^*} - \frac{u_i(y,t)}{u_i(x,t)}\right]\rho_R(x)\,dxdy. \tag{2.7}$$

On the other hand, by exchanging the roles of $x$ and $y$ and using $J(x-y) = J(y-x)$, we get from (2.6) that

$$I_i(t) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)u_i^*\left[1 - \frac{u_i(y,t)}{u_i^*} + \frac{u_i(x,t)}{u_i^*} - \frac{u_i(x,t)}{u_i(y,t)}\right]\rho_R(y)\,dxdy. \tag{2.8}$$

Summing over (2.7) and (2.8), we obtain $2I_i(t) = I_{i1}(t) + I_{i2}(t)$, where

$$I_{i1}(t) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)u_i^* \left[\rho_R(x) + \rho_R(y) - \frac{u_i(x,t)\rho_R(y)}{u_i(y,t)} - \frac{u_i(y,t)\rho_R(x)}{u_i(x,t)}\right] dxdy,$$

$$I_{i2}(t) := \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)\big\{[u_i(y,t) - u_i(x,t)]\rho_R(x) + [u_i(x,t) - u_i(y,t)]\rho_R(y)\big\}\, dxdy.$$

Then, by exchanging $x$ and $y$ and using $J(x-y) = J(y-x)$, we obtain

$$I_{i1}(t) = 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)u_i^* \left[1 - \frac{u_i(y,t)}{u_i(x,t)}\right] \rho_R(x)\, dxdy,$$

$$I_{i2}(t) = 2\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)\big[u_i(y,t) - u_i(x,t)\big]\rho_R(x)\, dxdy.$$

It follows that

$$\begin{aligned} I_i(t) &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)\left[u_i(y,t) + u_i^* - u_i(x,t) - u_i^*\frac{u_i(y,t)}{u_i(x,t)}\right]\rho_R(x)\, dxdy \\ &\le \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)\left[u_i(y,t) - u_i(x,t) + u_i^* \ln\left(\frac{u_i(x,t)}{u_i(y,t)}\right)\right]\rho_R(x)\, dxdy, \end{aligned}$$

using $1 - X \le \ln(1/X)$ for all $X > 0$. Thus we get

$$I_i(t) \le \frac{u_i^*}{c_i}\int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)\big[F_i(u_i(y,t)) - F_i(u_i(x,t))\big]\rho_R(x)\, dxdy. \tag{2.9}$$

Moreover, using

$$\begin{aligned} \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)F_i(u_i(y,t))\rho_R(x)\, dxdy &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(y-x)F_i(u_i(x,t))\rho_R(y)\, dydx \\ &= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} J_i(x-y)F_i(u_i(x,t))\rho_R(y)\, dxdy, \end{aligned}$$

it follows from (2.9) that

$$I_i(t) \le \frac{u_i^*}{c_i}\int_{\mathbb{R}^n} F_i(u_i(x,t))\mathcal{N}_i[\rho_R](x)\, dx.$$

According to Lemma 2.2, for each $J_i$, for any $\varepsilon \in (0, \kappa/2)$, there exists $R_i > 0$ such that

$$\mathcal{N}_i[\rho_{R_i}] \le \frac{\varepsilon}{md_i}\rho_{R_i}.$$

If we choose $R \ge \max_{1\le i\le m} R_i$, then

$$I_i(t) \le \frac{\varepsilon u_i^*}{md_i c_i}\int_{\mathbb{R}^n} F_i(u_i(x,t))\rho_R(x)\, dx. \tag{2.10}$$

Finally, from (2.5) and (2.10) it follows that

$$\frac{d}{dt}\mathcal{F}_R(t) \le -(\kappa - \varepsilon)\mathcal{F}_R(t) \le -\frac{\kappa}{2}\mathcal{F}_R(t), \quad \forall\, t \in \mathbb{R}.$$

By integrating in time from $-\infty$ to $t$, we deduce that $\mathcal{F}_R(t) = 0$ for all $t \in \mathbb{R}$. Hence $F(u(x,t)) \equiv 0$ and so $u(x,t) \equiv u^*$ for all $(x,t) \in \mathbb{R}^n \times \mathbb{R}$. Theorem 2.1 is thereby proved. □

## 3. Proof of Theorem 1.1

First, we recall the following proposition from [11].

**Proposition 3.1.** *Let $\alpha > 0$ and let $(S, I)$ be a solution of* (1.1), (1.2) *and* (1.4) *with a nonnegative nontrivial compactly supported continuous initial data $I_0$. Assume the condition* (1.5) *is enforced. Then the constant $c^*$ defined in* (1.6) *is the (asymptotic) spreading speed of $I$ in the sense*

$$(3.1)\qquad \lim_{t\to\infty} \sup_{|x|>ct} I(x,t) = 0, \ \ \forall\, c > c^*; \quad \liminf_{t\to\infty} \inf_{|x|<ct} I(x,t) > 0, \ \ \forall\, c \in (0, c^*).$$

Let $\alpha \geq 0$. Since 0 is a sub-solution of (1.2) with $I(x,0) \geq 0$ for any $S(x,t) \in \mathbb{R}$, by comparison we obtain $I \geq 0$ in $\mathbb{R} \times [0,\infty)$. Similarly, 0 is a sub-solution and 1 is a super-solution of (1.1) with $S(\,\cdot\,,0) \equiv 1$. Hence, by comparison, we have $0 \leq S \leq 1$ in $\mathbb{R}\times[0,\infty)$. With this information, one can check that the proof of [11, Theorem 1.1] works well for $\alpha = 0$. Hence we obtain

**Corollary 3.2.** *Let $\alpha = 0$ and let $(S, I)$ be a solution of* (1.1), (1.2) *and* (1.4) *with a nontrivial nonnegative compactly supported continuous initial data $I_0$. Assume the condition* (1.5) *is enforced. Then* (3.1) *holds with the constant $c^*$ defined in* (1.6)*.*

Note that, for a given $\alpha > 0$, the uniform persistence of $S$ follows from that $S \geq \alpha\mu/(\beta + \alpha\mu)$ in $\mathbb{R} \times [0,\infty)$, since $\alpha\mu/(\beta + \alpha\mu)$ is a sub-solution of (1.1) with $S(\,\cdot\,,0) \equiv 1$. Recall also from [11, (2.2)] that $I \leq \max\{\|I_0\|_\infty, (\beta - \gamma)/(\alpha\gamma)\}$.

The case for $\alpha = 0$ is more delicate. We only consider the case when $d_1 = d_2 := d$ and $J_1 = J_2 := J$. Then equations (1.1)–(1.2) are reduced to

$$(3.2)\qquad \begin{aligned} S_t(x,t) &= d\mathcal{N}[S(\,\cdot\,,t)](x) + \mu - \mu S(x,t) - \beta S(x,t)I(x,t), && t > 0,\ x \in \mathbb{R},\\ I_t(x,t) &= d\mathcal{N}[I(\,\cdot\,,t)](x) + \beta S(x,t)I(x,t) - (\mu+\sigma)I(x,t), && t > 0,\ x \in \mathbb{R}, \end{aligned}$$

where

$$\mathcal{N}[\varphi](x) := \int_{\mathbb{R}} J(x-y)\varphi(y)\,dy - \varphi(x), \quad x \in \mathbb{R}.$$

Set $W := 1 - (S + I)$. Then $W$ satisfies

$$W_t = d\mathcal{N}[W] - \mu W + \sigma I \geq d\mathcal{N}[W] - \mu W, \quad x \in \mathbb{R},\ t > 0.$$

It follows that

$$(e^{\mu t}W)_t(x,t) \geq d\mathcal{N}\big[e^{\mu t}W(\,\cdot\,,t)\big](x), \quad x \in \mathbb{R},\ t > 0.$$

Since $W(x,0) = -I_0(x) \geq -\|I_0\|_\infty$, by comparison, we obtain the estimate

$$S(x,t) + I(x,t) \leq 1 + e^{-\mu t}\|I_0\|_\infty \leq 1 + \|I_0\|_\infty := \theta, \quad x \in \mathbb{R},\ t > 0.$$

Using $S \geq 0$ and $I \geq 0$, we conclude that

$$I \text{ is uniformly bounded in } \mathbb{R} \times [0, \infty).$$

Moreover, since the constant $\mu/(\mu + \beta\theta)$ is a sub-solution of $S$-equation in (3.2), we obtain

$$S \geq \mu/(\mu + \beta\theta) > 0 \text{ in } \mathbb{R} \times [0, \infty),$$

by comparison.

Next, with the help of Theorem 2.1, Proposition 3.1, Corollary 3.2, and a uniform persistent result on $(S, I)$ in the zone $\{(x, t) \mid |x| \leq ct, t \gg 1\}$ for $c \in (0, c^*)$, the proof of Theorem 1.1 can be done by a similar argument as that of [12, Theorem 1.4] with some modifications due to the regularity of solutions. We provide a proof as follows.

*Proof of Theorem* 1.1. First, we recall from [18, 20] that both $S(\cdot, t)$ and $I(\cdot, t)$ are uniformly continuous on $\mathbb{R}$ for each $t \geq 0$. Moreover, the uniform boundedness of $(S, I)$ and (1.1)–(1.2) implies that both $S_t$ and $I_t$ are uniformly bounded. This implies that both $S$ and $I$ are uniformly continuous in $\mathbb{R} \times [0, \infty)$. Furthermore, it follows from (1.1)–(1.2) that both $S_t$ and $I_t$ are uniformly continuous in $\mathbb{R} \times [0, \infty)$.

Next, we let

$$k_0 \leq S(x, t) \leq 1, \quad 0 \leq I(x, t) \leq k_1 < \infty, \quad x \in \mathbb{R},\ t \geq 0 \tag{3.3}$$

for some positive constants $k_0$, $k_1$. Following [12], we assume for contradiction that there is a positive constant $\delta$ such that

$$|S(x_j, t_j) - S^*| + |I(x_j, t_j) - I^*| \geq \delta, \quad \forall\, j \geq 1 \tag{3.4}$$

for some sequence $\{(x_j, t_j)\}$ with $t_j \to \infty$ as $j \to \infty$ and $|x_j| \leq c_0 t_j$ for all $j \geq 1$ for some constant $c_0 \in (0, c^*)$. Set

$$(S_j, I_j)(x, t) := (S, I)(x + x_j, t + t_j), \quad (x, t) \in \mathbb{R}^2,\ j \geq 1.$$

It follows from the above regularity result that $\{(S_j, I_j)\}$ and $\{((S_j)_t, (I_j)_t)\}$ are uniformly bounded and equi-continuous sequences on $\mathbb{R}^2$. Hence, by Arzelá–Ascoli theorem with the help of a diagonal process, the limit

$$(S_\infty, I_\infty)(x, t) := \lim_{j \to \infty} (S_j, I_j)(x, t), \quad (x, t) \in \mathbb{R}^2,$$

exists (up to a subsequence) such that $(S_\infty, I_\infty)$ is an entire solution of system (1.1)–(1.2).

Finally, note that (3.3) holds for $(S_\infty, I_\infty)$ in $\mathbb{R}^2$. Also, by (3.1) with $c \in (c_0, c^*)$, there is a positive constant $k_3$ such that $I_\infty \geq k_3$ in $\mathbb{R}^2$. Hence $(S_\infty, I_\infty) = (S^*, I^*)$ by Theorem 2.1 and a Lyapunov function given in [12,19]. However, $|S_\infty(0,0) - S^*| + |I_\infty(0,0) - I^*| \geq \delta > 0$ by (3.4), a contradiction. This completes the proof of Theorem 1.1. □

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