TAIWANESE JOURNAL OF MATHEMATICS
Vol. 27, No. 6, pp. 1185–1210, December 2023
DOI: 10.11650/tjm/230801

A Fractional-order Quasi-reversibility Method to a Backward Problem for the Multi-term Time-fractional Diffusion Equation

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Abstract. In the present paper, we devote our efforts to a backward problem for an anomalous diffusion model with multi-term time fractional derivatives. Such a problem is ill-posed. For this purpose, we introduce a fractional-order quasi-reversibility regularization method that is a new perturbation related to the time fractional derivative into the original equation. Based on some properties of the multinomial Mittag-Leffler function as well as the Fourier method, we theoretically give some regularity results of the regularized solution, and prove the corresponding convergence rate under the a-priori regularization parameter choice rule in the general dimensional case. Finally, several numerical examples are given to demonstrate the effectiveness of the proposed method. The numerical results are well in line with our expectations.

1. Introduction

Fractional diffusion equations have a wide range of applications and great influence in mathematical physics and other fields. Fractional diffusion equations can more accurately describe some anomalous diffusion phenomena, such as underground sewage survey and oil pollution survey [1,4,6,25], so the study of fractional diffusion equations have attracted the attention of many scholars.

The multi-term time fractional diffusion equation (MTFDE in short) is a special fractional diffusion equation, which can improve the efficiency of studying anomalous diffusion problems. Because this kind of model could describe the diffusion phenomenon of a solute in a multi-scale medium. Such processes are believed to provide useful models for a crowd of non-homogeneousand non-stationary processes, for example, it is shown to be efficient models for describing some anomalous diffusion processes in the highly heterogeneous media by [6] in which the authors indicated that diffusion equation with time-fractional derivative was well performed in describing the long-tailed profile of a particle diffuses in a highly heterogeneous medium. Therefore, the study of this multi-term fractional diffusion equation is of great significance in physics and engineering.

Received April 3, 2023; Accepted August 3, 2023.
Communicated by Jenn-Nan Wang.
2020 *Mathematics Subject Classification.* 35R30, 35R25, 65M30.
Key words and phrases. multi-term time-fractional diffusion equation, fractional-order quasi-reversibility regularization method, backward problem, regularized solution, convergence rate.
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Let T be a fixed positive constant and Ω be a bounded domain in $\mathbb{R}^d$ with sufficient smooth (at least Lipschitz) boundary $\partial\Omega$. In this paper, we focus on the following initial boundary value problem (IBVP) for a multi-term time-fractional diffusion model

$$\begin{cases} \sum_{j=1}^{s} q_j \partial_t^{\alpha_j} \omega(x,t) = L\omega, & (x,t) \in \Omega \times (0,T], \\ \omega(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \\ \omega(x,0) = \omega_0(x), & x \in \Omega, \end{cases} \tag{1.1}$$

where ∂_t^α represents the Caputo fractional time derivative defined by

$$\partial_t^\alpha \omega := \frac{1}{\Gamma(1-\alpha)} \int_0^t \frac{\partial \omega(x,s)}{\partial s} \frac{ds}{(t-s)^\alpha}, \quad 0 < \alpha < 1, \ 0 < t < T,$$

where $\Gamma(\cdot)$ denotes the Gamma function. For a fixed positive integer s, the orders $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_s)$ and the coefficients $\boldsymbol{q} = (q_1, \ldots, q_s)$ are restricted in the admissible sets

$$\mathcal{A} := \big\{ (\alpha_1, \ldots, \alpha_s) \in \mathbb{R}^s \mid \overline{\alpha} \geq \alpha_1 > \alpha_2 > \cdots > \alpha_s \geq \underline{\alpha} \big\},$$
$$\mathcal{Q} := \big\{ (q_1, \ldots, q_s) \in \mathbb{R}^s \mid q_1 = 1, q_j \in [\underline{q}, \overline{q}], j = 2, \ldots, s \big\}$$

with fixed $0 < \underline{\alpha} < \overline{\alpha} < 1$ and $0 < \underline{q} < \overline{q}$.

The differential operator L is defined by

$$L\omega(x,t) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial \omega}{\partial x_j}(x,t) \right) + b(x)\omega(x,t), \tag{1.2}$$

where $a_{ij} \in C^1(\overline{\Omega})$ and $a_{ij} = a_{ji}$, $1 \leq i,j \leq d$, and $b \in C(\overline{\Omega})$, $b(x) \leq 0$, $x \in \overline{\Omega}$. Moreover, we assume that the operator L is uniformly elliptic on $\overline{\Omega}$, i.e., there exists a constant $\mu > 0$ such that

$$\sum_{i,j=1}^{d} a_{ij}(x)\xi_i\xi_j \geq \mu|\xi|^2, \quad x \in \overline{\Omega}, \ \xi = (\xi_1, \ldots, \xi_d) \in \mathbb{R}^d.$$

The initial state $\omega_0(x)$ satisfies the following compatibility condition:

$$\omega_0(x) = 0, \quad x \in \partial\Omega.$$

To study various properties and results of fractional calculus, we can refer to [11, 26, 28]. However, the multi-term time fractional diffusion equations whose mean-squared displacement behaving like $\langle \Delta x^2 \rangle \sim Ct^{\min\{\alpha_j\}}$ as $t \to \infty$ (see, e.g., [23]) grows in time slower than linear case, which are very different from the traditional diffusion equations, for example, the decay in time is slow, and the smoothness of the solution is limited. Up to now, there are extensive profound literatures concerning theory and computation of the forward problems for the multi-term time-fractional diffusion equations (see [9, 13, 15, 20, 21, 40] for an incomplete list).

In practice, some parameters in the model, such as the fractional orders, some coefficients, source term, initial data or partial boundary data may not be measured directly, which needs to be recovered by additional measured data by indirect means. Thus some inverse problems in fractional diffusion equations arise. The published works were considered from different aspect on the inverse problems for multi-term counterpart, e.g., see [7, 8, 12, 14, 16, 18, 22, 30–33] etc, and review literatures on inverse problems in anomalous models [10, 17].

Here the initial function $\omega_0(x)$ is unknown, so the inverse problem considered in this paper is to identify initial state based on problem (1.1) and an additional terminal data

$$g(x) = \omega(x, T), \quad x \in \Omega. \tag{1.3}$$

This kind of backward problems in fractional diffusion equations have significant application background and have increased extensive attention. In 2010, Liu and Yamamoto firstly proposed a quasi-reversibility regularization method to solve the backward problem in time for a time-fractional diffusion equation in one-dimensional case [19]. In 2011, Sakamoto et al. also gave the uniqueness for the backward problem based on the theoretical results of the forward problem [27]. Subsequently, some effective regularization methods continue to emerge, such as the Tikhonov regularization method [3, 34, 36], modified quasi-boundary value method [37], iterative regularization method [35], variational method [38], new adjoint technique [39], fractional Landweber method [5], fractional Tikhonov regularization method [2] and so on. Recently, Shi et al. [29] proposed a fractional-order quasi-reversibility method to a backward problem for the time fractional diffusion equation. Some of the ideas in the present paper come from the above literature.

Compared with the inverse problem of classical heat equations, it is more challenging to study the inverse problem of fractional equation, especially multi-term time-fractional diffusion equation. For example, a fundamental fact is that under the time reverse $t' = T - t$, there holds $\partial_t = -\partial_{t'}$ while $\partial_t^\alpha \neq -\partial_{t'}^\alpha$ for $\alpha \in (0, 1)$, which makes some standard techniques devoted to the backward problem for classical heat equation invalid in establishing the convergence property of the regularized solutions of the backward problem for (1.1). On the other hand, the appearance of the multinomial Mittag-Leffler function makes the study of the forward and inverse problems for fractional equation more complicated in the aspects of theoretical analysis and numerical calculation.

In this paper, we focus on the well-posedness as well as convergence analysis of the regularized solution for backward problem (3.2) based on the fractional-order quasi-reversibility method. Our main contribution lies that a new regularization method based on time fractional derivative as well as some crucial properties for the multinomial Mittag-Leffler function are proposed. And the method we proposed is very different from [19]. Under a low regularity of the initial state, we establish the convergence property of the regularizing solutions of the backward problem and a uniform Hölder type error estimate with an optimal convergence order is obtained for a priori parameter choice rules with the aid of the new quasi-reversibility method. Also, we put forward a numerical algorithm for the forward problem and the backward problem with a regularization term. By this numerical scheme, the original equation can be reduced to the form of matrix iteration, and we reconstruct the initial value by numerical solution instead of exact solution. Finally, Numerical examples demonstrate the effectiveness of the algorithm and also indicate that our new regularization technique has better reconstruction effect on non-smooth functions.

The remainder of this paper is organized as follows. In Section 2, some preliminaries are presented. In Section 3, we illustrate the ill-posedness of inverse problem and also consider the existence and uniqueness of the regularized solution. In Section 4, some properties of multinomial Mittag-Leffler function are used to study the convergence analysis of the regularized solution. In Section 5, the convergence speed of the corresponding regularized solution can be accurately expressed under an a-priori parameter choice rule. In Section 6, we present the numerical algorithm. In Section 7, three numerical examples demonstrate the effectiveness of the algorithm. Finally, we make a concluding remark in Section 8.

2. Preliminaries

In this section, we give some definitions and some necessary preparations. Denote $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $(\cdot,\cdot)$ as the inner product of $L^2(\Omega)$. Henceforth C (or C_j) denote positive constants that may have different values in each place.

Let $D(-L) := H^2(\Omega)\cap H_0^1(\Omega)$ denote the domain of the operator $-L$ defined by (1.2). Noting that $-L$ is a self adjoint and positive operator. Let $\{\mu_n,\varphi_n\}_{n=1}^{\infty}$ be an eigensystem of $-L$, i.e., $-L\varphi_n = \mu_n\varphi_n$. We know that $0<\mu_1<\mu_2<\cdots$, $\lim_{n\to\infty}\mu_n=\infty$, and $\{\varphi_n\}_{n=1}^{\infty}$ forms the standard orthogonal basis of $L^2(\Omega)$. Now we can define the Hilbert scale space for later use.

Definition 2.1. For any $\gamma>0$, define

$$D((-L)^{\gamma}) = \left\{ \psi \in L^2(\Omega) \;\middle|\; \sum_{p=1}^{\infty} \mu_p^{2\gamma} |(\psi,\varphi_p)|^2 < \infty \right\},$$

where $(\cdot,\cdot)$ is the inner product in $L^2(\Omega)$, and define its norm

$$\|\psi\|_{D((-L)^{\gamma})} = \left\{ \sum_{p=1}^{\infty} \mu_p^{2\gamma} |(\psi,\varphi_p)|^2 \right\}^{1/2}.$$

Definition 2.2. The multinomial Mittag-Leffler function is defined by

$$E_{(\theta_1,\ldots,\theta_s),\theta_0}(z_1,\ldots,z_s) := \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_s=k} \frac{(k;k_1,\ldots,k_s)\prod_{j=1}^{s} z_j^{k_j}}{\Gamma(\theta_0+\sum_{j=1}^{s}\theta_j k_j)},$$

where $\theta_0, \theta_j \in \mathbb{R}$, and $z_j \in \mathbb{C}$ $(j=1,\ldots,s)$, and $(k;k_1,\ldots,k_s)$ denotes the multinomial coefficient

$$(k;k_1,\ldots,k_s) := \frac{k!}{k_1!\cdots k_s!} \quad \text{with } k = \sum_{j=1}^{s} k_j,$$

where k_j $(j=1,\ldots,s)$ are non-negative integers.

For the convenience of the later, for $\boldsymbol{\alpha} \in \mathcal{A}$ and $\boldsymbol{q} \in \mathcal{Q}$, we adopt the abbreviation

$$E_{\boldsymbol{\alpha}',\beta}^{(\mu_n)}(t) := E_{(\alpha_1,\alpha_1-\alpha_2,\ldots,\alpha_1-\alpha_s),\beta}\big(-\mu_n t^{\alpha_1}, -q_2 t^{\alpha_1-\alpha_2}, \ldots, -q_s t^{\alpha_1-\alpha_s}\big), \quad t>0, \tag{2.1}$$

where $\boldsymbol{\alpha}' = (\alpha_1, \alpha_1-\alpha_2, \ldots, \alpha_1-\alpha_s)$ and μ_n denotes the n-th eigenvalues of elliptic operator $-L$ with the homogeneous Dirichlet boundary condition.

Lemma 2.3. [15] *Let $0<\theta<2$ and $0<\alpha_s<\alpha_{s-1}<\cdots<\alpha_1<1$ be given. Assume that $\frac{\alpha_1\pi}{2}<\mu<\alpha_1\pi$, $\mu \le |\arg(z_1)| \le \pi$, and $z_j \in \mathbb{R}$ for $j=2,\ldots,s$, and there exists $K>0$ such that $-K \le z_j < 0$ $(j=2,\ldots,s)$. Then there exists a constant $C>0$ depending only on μ, K, α_j $(j=1,\ldots,s)$ and θ such that*

$$\big|E_{(\alpha_1,\alpha_1-\alpha_2,\ldots,\alpha_1-\alpha_s),\theta}(z_1,\ldots,z_s)\big| \le \frac{C}{1+|z_1|}.$$

Lemma 2.4. (see also [31, Proposition 2.6]) *For $\mu_n>0$, we have $0<1-\mu_n t^{\alpha_1} E_{\boldsymbol{\alpha}',1+\alpha_1}^{(\mu_n)}(t)$ <1 for $t>0$. Moreover, $1-\mu_n t^{\alpha_1} E_{\boldsymbol{\alpha}',1+\alpha_1}^{(\mu_n)}(t)$ is a strictly decreasing function on $t>0$.*

Lemma 2.5. [30] *Let $0<\alpha_s<\alpha_{s-1}<\cdots<\alpha_1<1$. Then*

$$\big|1-\mu_n t^{\alpha_1} E_{\boldsymbol{\alpha}',\alpha_1+1}^{(\mu_n)}(t)\big| \le \sum_{j=2}^{s} \frac{M(1+q_j t^{\alpha_1-\alpha_j})}{1+\mu_n t^{\alpha_1}}, \quad t>0,\ n=1,2,\ldots,$$

where M is a positive constant.

Proposition 2.6. *For any $\xi \in \mathbb{R}$ and $\xi>0$, we have*

$$\left|\frac{d}{d\xi} E_{\boldsymbol{\alpha}',1+\alpha_1}^{(\xi)}(t)\right| \le \frac{1}{\alpha_1 \xi}\big|E_{\boldsymbol{\alpha}',\alpha_1}^{(\xi)}(t)\big|, \quad t>0.$$

Proof. From (2.1), we know that

$$\begin{aligned} E_{\boldsymbol{\alpha}',1+\alpha_1}^{(\xi)}(t) &= E_{(\alpha_1,\alpha_1-\alpha_2,\ldots,\alpha_1-\alpha_s),1+\alpha_1}(-\xi t^{\alpha_1}, -q_2 t^{\alpha_1-\alpha_2}, \ldots, -q_s t^{\alpha_1-\alpha_s}) \\ &= \sum_{k=0}^{\infty} \sum_{k_1+\cdots+k_s=k} \frac{(k;k_1,\ldots,k_s)(-\xi t^{\alpha_1})^{k_1}(-q_2 t^{\alpha_1-\alpha_2})^{k_2}\cdots(-q_s t^{\alpha_1-\alpha_s})^{k_s}}{\Gamma\big(\alpha_1+1+\alpha_1 k_1+\sum_{j=2}^{s}(\alpha_1-\alpha_j)k_j\big)}, \end{aligned}$$

then we have

$$\begin{aligned}
&\left|\frac{d}{d\xi}E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)\right| \\
&= \left|\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_s=k}\frac{(k;k_1,\ldots,k_s)\big((-1)^{k_1}k_1\xi^{k_1-1}t^{\alpha_1k_1}\big)(-q_2t^{\alpha_1-\alpha_2})^{k_2}\cdots(-q_st^{\alpha_1-\alpha_s})^{k_s}}{\Gamma\big(\alpha_1+1+\alpha_1k_1+\sum_{j=2}^{s}(\alpha_1-\alpha_j)k_j\big)}\right| \\
&= \left|\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_s=k}\frac{(k;k_1,\ldots,k_s)\big(\frac{k_1}{\xi}(-1)^{k_1}\xi^{k_1}t^{\alpha_1k_1}\big)(-q_2t^{\alpha_1-\alpha_2})^{k_2}\cdots(-q_st^{\alpha_1-\alpha_s})^{k_s}}{\Gamma\big(\alpha_1+1+\alpha_1k_1+\sum_{j=2}^{s}(\alpha_1-\alpha_j)k_j\big)}\right| \\
&= \left|\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_s=k}\frac{k_1}{\xi}\frac{(k;k_1,\ldots,k_s)\big((-\xi)^{k_1}t^{\alpha_1k_1}\big)(-q_2t^{\alpha_1-\alpha_2})^{k_2}\cdots(-q_st^{\alpha_1-\alpha_s})^{k_s}}{\Gamma\big(\alpha_1+1+\alpha_1k_1+\sum_{j=2}^{s}(\alpha_1-\alpha_j)k_j\big)}\right| \\
&= \left|\sum_{k=0}^{\infty}\sum_{k_1+\cdots+k_s=k}\frac{k_1}{\xi}\frac{(k;k_1,\ldots,k_s)(-\xi t^{\alpha_1})^{k_1}(-q_2t^{\alpha_1-\alpha_2})^{k_2}\cdots(-q_st^{\alpha_1-\alpha_s})^{k_s}}{\{\alpha_1+\alpha_1k_1+\sum_{j=2}^{s}(\alpha_1-\alpha_j)k_j\}\Gamma\big(\alpha_1+\alpha_1k_1+\sum_{j=2}^{s}(\alpha_1-\alpha_j)k_j\big)}\right| \\
&\le \frac{1}{\alpha_1\xi}|E^{(\xi)}_{\boldsymbol{\alpha}',\alpha_1}(t)|. \qquad \square
\end{aligned}$$

3. Ill-posedness of inverse problem and fractional-order quasi-reversibility method

In this section, we will first discuss the ill-posedness of the backward problem and then propose a fractional-order quasi-reversibility method by introducing a new perturbation related to the time fractional derivative into the original equation.

By a standard Fourier method, we know that (1.1) admits a unique solution given by [15]:

$$\omega(x,t)=\sum_{n=1}^{\infty}(\omega_0,\varphi_n)\big(1-\mu_nt^{\alpha_1}E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)\big)\varphi_n(x).$$

From (1.3), then we have

$$\sum_{n=1}^{\infty}(g,\varphi_n)\varphi_n(x)=\omega(x,T)=\sum_{n=1}^{\infty}(\omega_0,\varphi_n)\big(1-\mu_nT^{\alpha_1}E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big)\varphi_n(x).$$

So we arrive at

$$(\omega_0,\varphi_n)=\frac{(g,\varphi_n)}{1-\mu_nT^{\alpha_1}E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}. \tag{3.1}$$

From Lemma 2.4, we know the denominator of (3.1) tends to zero when n is sufficiently large. Therefore, the above inverse problem is ill-posed and we need to add some regularization strategies. By Lemma 2.5, we also find that the backward problem for multi-fractional diffusion equation is mildly ill-posed because the multiplier $1/(1-\mu_nT_1^{\alpha}E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T))$ just grows linearly to μ_n, i.e., $1/(1-\mu_nT_1^{\alpha}E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T))\sim\mu_n$ as $n\to\infty$, which is very mild compared to the exponential growth e^{μ_nT} for the case $\alpha=1$.

Motivated by the quasi-reversibility method in [29], we consider the following problem as our regularizing scheme

$$
(3.2) \qquad \begin{cases} \sum_{j=1}^{s} q_j \partial_t^{\alpha_j} u(x,t) = Lu + \beta \sum_{j=1}^{s} q_j \partial_t^{\alpha_j}(Lu), & (x,t) \in \Omega \times (0,T], \\ u(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T], \\ u(x,T) = g(x), & x \in \Omega, \end{cases}
$$

where $\beta = \beta(\delta)$ acts as regularizing parameter depending on the known noisy level δ. After adding the regularization term, we can see that the ill-posedness of the backward problem can be effectively avoided from the following theorem. Now we need consider the existence and uniqueness of the solution to (3.2).

Remark 3.1. In the regularized problem (3.2), we have added a perturbation term of fractional derivative to overcome the ill-posedness of inverse problem. The main idea is to change the eigenvalue of the original elliptic operator. As we know from (3.1) that $1/(1 - \mu_n T_1^{\alpha} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T))$ is an amplification factor when n is sufficiently large because $\mu_n \to +\infty$ as $n \to \infty$. On the other hand, from (3.5) we have $\lambda_n \to 1/\beta$ is bounded when n is sufficiently large. So the backward problem becomes well-posed from (3.6) by taking $t = 0$.

Theorem 3.2. *Let $g \in D(-L)$ and $\beta \in (0, +\infty)$. Then problem* (3.2) *has a unique solution $u^{\beta}[g] \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$.*

Proof. Our proof process can be seen as two steps.

Step 1*: The existence and regularity of the solution.* We need to solve the above problem by the separation variable method. Let $u(x,t) = X(x)T(t)$ and substituting it into (3.2) gives

$$
\frac{\sum_{j=1}^{s} q_j \partial_t^{\alpha_j} T(t)}{T(t)} = \frac{LX(x)}{X - \beta LX(x)} = -\lambda,
$$

where $-\lambda$ is called the separation constant and not yet known. Therefore, we deduce an eigenvalue problem

$$
(3.3) \qquad \begin{cases} LX(x) - \beta\lambda LX(x) = -\lambda X(x), & x \in \Omega, \\ X(x) = 0, & x \in \partial\Omega, \end{cases}
$$

and an ODE with fractional derivative

$$
(3.4) \qquad \sum_{j=1}^{s} q_j \partial_t^{\alpha_j} T(t) + \lambda T(t) = 0, \quad t > 0.
$$

We know that the eigensystem $\{\mu_n, \varphi_n\}$ satisfies $L\mu_n = -\mu_n \varphi_n$ and $\varphi_n|_{\partial\Omega} = 0$. So eigenvalue problem (3.3) yields

$$
(3.5) \qquad \lambda_n = \frac{\mu_n}{\beta\mu_n + 1}, \quad X_n(x) = \varphi_n(x), \quad n = 1, 2, \ldots.
$$

Especially, we can easily verify the one-dimensional case. we know that $\mu_n = n^2$, $\lambda_n = n^2/(\beta n^2+1)$ and $\varphi_n = \sin nx$ for $L = \Delta$ and $\Omega = (0,\pi)$. For fixed $\lambda = \lambda_n$, we can solve (3.4) (e.g., [26]) by using the Laplace transform to obtain

$$T_n(t) = 1 - \lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t).$$

Noting $u(x,T) = g(x)$ from (1.3), and combining the above results, we have

$$u^\beta[g](x,t) = \sum_{n=1}^{\infty} \frac{1 - \lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1 - \lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} g_n \varphi_n(x), \tag{3.6}$$

where $g_n = (g, \varphi_n)$. Now we verify the regularity of $u^\beta(x,t)$. From Lemmas 2.4 and 2.5, we obtain

$$\begin{aligned} \sum_{n=1}^{\infty} \left| \mu_n g_n \frac{1 - \lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1 - \lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right|^2 &= \sum_{n=1}^{\infty} \mu_n^2 |g_n|^2 \left| \frac{1 - \lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1 - \lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right|^2 \\ &\le C^2 \sum_{n=1}^{\infty} \mu_n^2 |g_n|^2 \quad \text{for any } t > 0, \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} \sum_{n=1}^{\infty} \left| g_n \frac{1 - \lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1 - \lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right|^2 &\le \sum_{n=1}^{\infty} |g_n|^2 \left| \frac{1}{1 - \lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right|^2 \\ &\le C^2 \sum_{n=1}^{\infty} \mu_n^2 |g_n|^2 \quad \text{for any } t \ge 0, \end{aligned} \tag{3.8}$$

where C is a constant. Since $g \in D(-L)$, the series $\sum_{n=1}^{\infty} \mu_n^2 |g_n|^2$ is convergent, (3.7) and (3.8) are also uniform convergent. So we have from the Lebesgue theorem that $u^\beta \in C([0,T]; L^2(\Omega)) \cap C((0,T]; H^2(\Omega) \cap H_0^1(\Omega))$.

Step 2: The uniqueness of the solution. Let u be the solution to (3.2) for $g = 0$. Then

$$u(x,t) = \sum_{n=1}^{\infty} u_n(t) \varphi_n(x), \quad n = 1, 2, \ldots,$$

where $u_n(t) = (u(\,\cdot\,,t), \varphi_n)$, and

$$\sum_{j=1}^{s} q_j \partial_t^{\alpha} u_n(t) = -\lambda_n u_n(t).$$

The condition $u(x,T) = 0$ yields $u_n(T) = 0$. With the help of the existence and uniqueness of the ordinary fractional differential equation, we have $u_n(t) \equiv 0$, $n = 1, 2, \ldots$ in $t \in [0,T]$. Because $\varphi_n(x)$ is a complete orthogonal system in $L^2(\Omega)$, we have $u \equiv 0$. The proof is complete. $\square$

4. Convergence analysis and error estimate

In this section, our goal is to give the main theoretical result: convergence analysis of the regularized solution.

Theorem 4.1. *Let $\omega(x,T) = g(x)$ be the final value of (1.1) and $\omega_0 \in L^2(\Omega)$ be the initial value. Then*

$$u^\beta[g] \to \omega$$

in $C((0,T];L^2(\Omega))$ as $\beta \to 0$. Moreover,

$$\|u^\beta[g](\,\cdot\,,0) - \omega(\,\cdot\,,0)\| \to 0$$

as $\beta \to 0$.

Proof. After calculation we have

$$\begin{aligned}
&u^\beta[g](x,t) - \omega(x,t) \\
&= \sum_{n=1}^{\infty} \frac{1-\lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} g_n\varphi_n(x) - \sum_{n=1}^{\infty} \frac{1-\mu_n t^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} g_n\varphi_n(x) \\
&=: \sum_{n=1}^{\infty} h_{n,\boldsymbol{\alpha},T}(t) g_n\varphi_n(x),
\end{aligned}$$

where $g_n = (g,\varphi_n)$, and

$$\begin{aligned}
&h_{n,\boldsymbol{\alpha},T}(t) \\
&= \frac{1-\lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - \frac{1-\mu_n t^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \\
&= \frac{1-\mu_n t^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \left(\frac{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}{1-\mu_n t^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)} \cdot \frac{1-\lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - 1 \right).
\end{aligned}$$

By Lemmas 2.4 and 2.5, we have

(4.1)

$$\begin{aligned}
&\sum_{n=1}^{\infty} h^2_{n,\boldsymbol{\alpha},T}(t) g_n^2 \\
&\le \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty} \right) \left(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right)^2 \\
&\quad \times \left(\frac{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}{1-\mu_n t^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)} \cdot \frac{1-\lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - 1 \right)^2
\end{aligned}$$

$$\leq \sum_{n=1}^{N} \left(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right)^2 \left(\frac{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}{1-\mu_n t^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)} \cdot \frac{1-\lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - 1 \right)^2$$
$$+ 2 \sum_{n=N+1}^{\infty} \left(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right)^2$$
$$\times \left(\left(\frac{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}{1-\mu_n t^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)} \cdot \frac{1-\lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right)^2 + 1 \right)^2$$
$$\leq \sum_{n=1}^{N} \left(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right)^2 \left(\frac{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}{1-\mu_n t^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)} \cdot \frac{1-\lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - 1 \right)^2$$
$$+ 2C \sum_{n=N+1}^{\infty} \left(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right)^2 \left(\left(1 + \left(\frac{T}{t} \right)^{\alpha_1} \right)^2 + 1 \right)^2 \quad \text{for all } t > 0.$$

Since $\omega(\,\cdot\,,0) \in L^2(\Omega)$, we have from (3.1) that

$$\sum_{n=1}^{\infty} \frac{1}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\alpha',1+\alpha_1}(T)} g_n \varphi_n \in L^2(\Omega), \tag{4.2}$$

which shows that $\sum_{n=1}^{\infty} \big(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \big)^2$ converges. Hence, for arbitrary $\varepsilon > 0$, we can find $N = N(\varepsilon)$ such that

$$\sum_{n=N(\varepsilon)+1}^{\infty} \left(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right)^2 < \frac{\varepsilon}{2}. \tag{4.3}$$

By means of the continuity of λ_n in β and $1 - \xi t^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)$ in ξ, we can choose an appropriate $\beta = \beta(\varepsilon)$ such that

$$\left(\frac{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}{1-\mu_n t^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)} \cdot \frac{1-\lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - 1 \right)^2 < \frac{\varepsilon}{2} \tag{4.4}$$

for all $n = 1, 2, \ldots, N(\varepsilon)$. Combining (4.1)–(4.4) we have

$$\sum_{n=1}^{\infty} h^2_{n,\boldsymbol{\alpha},T}(t) g_n^2 \leq \frac{\varepsilon}{2} \sum_{n=1}^{N} \left(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\alpha',1+\alpha_1}(T)} \right)^2 + C(t)\varepsilon \leq C(t)(1 + \|\omega_0\|^2)\varepsilon. \tag{4.5}$$

Since ε has been arbitrary chosen, it follows that $\sum_{n=1}^{\infty} h^2_{n,\boldsymbol{\alpha},T}(t) g_n^2 \to 0$ as $\varepsilon \to 0$. We can get the following result

$$\|u^\beta[g](\,\cdot\,,t) - \omega(\,\cdot\,,t)\| \to 0, \quad \beta \to 0$$

holds for all $0 < t \leq T$.

We find that constant in (4.5) depends on the variable $t > 0$, and hence, the convergence at $t = 0$ also need us to consider. For the particular case $t = 0$, there holds

$$h_{n,\boldsymbol{\alpha},T}(0) = \frac{1}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \left(\frac{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - 1 \right).$$

We can easily get the results below

$$u^\beta[g](x,0) - \omega(x,0) = \sum_{n=1}^{\infty} h_{n,\boldsymbol{\alpha},T}(0) g_n \varphi_n(x).$$

On the other hand, we have from Lemma 2.5 that

$$\begin{aligned}
&\sum_{n=1}^{\infty} h^2_{n,\boldsymbol{\alpha},T}(0) g_n^2 \\
&= \left(\sum_{n=1}^{N} + \sum_{n=N+1}^{\infty}\right) \left(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right)^2 \left(\frac{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - 1 \right)^2 \\
&\le \sum_{n=1}^{N} \left(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right)^2 \left(\frac{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - 1 \right)^2 \\
&\quad + 4 \sum_{n=N+1}^{\infty} \left(\frac{g_n}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} \right)^2.
\end{aligned}$$

Similar to the case at $t > 0$, we obtain

$$\|u^\beta[g](\,\cdot\,,0) - \omega(\,\cdot\,,0)\| \to 0$$

as $\beta \to 0$. The proof is complete. □

However, problems encountered in real life, we only know additional noisy data. Here we consider $g^\delta(x)$, the noisy input data of $g(x)$ with a weak regularity, as our input data which satisfies

$$\|g^\delta - g\| \le \delta. \tag{4.6}$$

Theorem 4.2. *Let $\boldsymbol{\alpha} \in \mathcal{A}$, $\boldsymbol{q} \in \mathcal{Q}$ and $g^\delta(x)$ denote the noisy data which satisfies (4.6). Then*

$$\|u^{\beta(\delta)}[g^\delta](\,\cdot\,,t) - \omega(\,\cdot\,,t)\| \to 0$$

as $\delta \to 0$, where $\beta = \beta(\delta)$ denotes the regularization parameter satisfying

$$\beta(\delta) \to 0, \quad \frac{\delta}{\beta(\delta)} \to 0 \tag{4.7}$$

as $\delta \to 0$.

Proof. Note that for $t \in (0,T)$,

$$\|u^\beta[g^\delta](\,\cdot\,,t) - \omega(\,\cdot\,,t)\| \le \|u^\beta[g^\delta](\,\cdot\,,t) - u^\beta[g](\,\cdot\,,t)\| + \|u^\beta[g](\,\cdot\,,t) - \omega(\,\cdot\,,T)\|.$$

By virtue of Theorem 4.1, the second item of the above equation tends to zero as $\delta \to 0$ because of (4.7), and hence, we only estimate the first item. Our estimation process is as follows:

$$\begin{aligned}
&\|u^\beta[g^\delta](\,\cdot\,,t) - u^\beta[g](\,\cdot\,,t)\|^2 \\
&= \sum_{n=1}^\infty \left(\frac{1 - \lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\alpha',1+\alpha_1}(t)}{1 - \lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\alpha',1+\alpha_1}(T)} \right)^2 (g_n^\delta - g_n)^2 \\
&\le \sum_{n=1}^\infty \left(\frac{1}{1 - \lambda_n T^{\alpha_1} \cdot \frac{C_6}{1+\lambda_n T^{\alpha_1}}} \right)^2 (g_n^\delta - g_n)^2 \\
&\le \sum_{n=1}^\infty C \left(\frac{1 + \lambda_n T^{\alpha_1}}{1 + \lambda_n T^{\alpha_1} - \lambda_n T^{\alpha_1}} \right)^2 (g_n^\delta - g_n)^2 \\
&\le C \sum_{n=1}^\infty (1 + \lambda_n T^{\alpha_1})^2 (g_n^\delta - g_n)^2 \le 2C \cdot \left(1 + \left(\frac{T^{\alpha_1}}{\beta} \right)^2 \right) \cdot \delta^2.
\end{aligned}$$

Here $t \in (0,T)$. From the above discussion, we get the corresponding results. □

5. The convergence rate of the corresponding solution

In addition, if some additional conditions can be given to the initial data, the convergence rate of the corresponding solution can be accurately expressed, and we can prove the following conclusion.

Theorem 5.1. *Assume the assumption $\omega_0 \in H_0^1(\Omega)$ (i.e., $\|\omega_0\|_{H_0^1(\Omega)} \le M^2$) holds, then*

$$\|u^\beta[g](\,\cdot\,,t) - \omega(\,\cdot\,,t)\| \le C_8 M \beta^{1/2},$$

where C_8 is a constant depending on $\underline{\alpha}$, $\overline{\alpha}$ and T. Moreover, if $\|g^\delta - g\| < \delta$ and $\beta(\delta) = C\delta^{2/3}$, we have

$$\|u^{\beta(\delta)}[g^\delta](\,\cdot\,,t) - \omega(\,\cdot\,,t)\| \le C_9 M \delta^{1/3},$$

where C_9 is a constant depending on $\underline{\alpha}$, $\overline{\alpha}$ and T.

Proof. Taking advantage of the previous discussion, we have

$$\|u^\beta[g](\,\cdot\,,t) - \omega(\,\cdot\,,t)\|^2 = \sum_{n=1}^\infty g_n^2 h_{n,\boldsymbol{\alpha},T}^2(t)$$

and

$$h_{n,\boldsymbol{\alpha},T}(t) = \frac{1-\lambda_n t^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\lambda_n T^{\alpha_1} E^{(\lambda_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - \frac{1-\mu_n t^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}$$
$$= Q'(\xi)(\lambda_n - \mu_n) = Q'(\xi) \cdot \frac{-\beta\mu_n^2}{\beta\mu_n + 1},$$

where $Q(\xi) = \frac{1-\xi t^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)}$ and $\xi \in (\lambda_n, \mu_n)$. For $\xi \neq 0$, it follows from Lemmas 2.3, 2.4 and Proposition 2.6 that

$$\begin{aligned}
&|Q'(\xi)| \\
&= \frac{\big|\big(t^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(t) + \xi t^{\alpha_1} \frac{d}{d\xi} E^{(\xi)}_{\boldsymbol{\alpha},1+\alpha_1}(t)\big)\big(1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big)}{\big|1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big|^2} \\
&\quad - \frac{\big(1-\xi t^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)\big)\big(T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T) + \xi T^{\alpha_1} \frac{d}{d\xi} E^{(\xi)'}_{\boldsymbol{\alpha},1+\alpha_1}(T)\big)\big|}{\big|1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big|^2} \\
&\leq \frac{\big|t^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)\big| + \big|\xi t^{\alpha_1} \frac{d}{d\xi} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)\big|}{\big|1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big|} + \frac{\big|T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big| + \big|\xi T^{\alpha_1} \frac{d}{d\xi} E^{(\xi)}_{\boldsymbol{\alpha},1+\alpha_1}(T)\big|}{\big|1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big|^2} \\
&\leq \frac{C\big|1+\frac{1}{\alpha_1}\big| \cdot \big|\frac{t^{\alpha_1}}{1+|\xi t^{\alpha_1}|}\big|}{\big|1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big|} + \frac{C\big|1+\frac{1}{\alpha_1}\big| \cdot \big|\frac{T^{\alpha_1}}{1+|\xi T^{\alpha_1}|}\big|}{\big|1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big|^2} \\
&\leq C\left|1+\frac{1}{\alpha_1}\right| T^{\alpha_1} \left|\frac{1}{C_- \xi T^{\alpha_1}} + \frac{1}{C_-}\right| + \frac{C\big|1+\frac{1}{\alpha_1}\big|T^{\alpha_1}\big|1+|\xi T^{\alpha_1}|\big|}{\big(1+|\xi T^{\alpha_1}| - C_- \xi T^{\alpha_1}\big)^2} \\
&\to C\left(1+\frac{1}{\alpha_1}\right) T^{\alpha_1}, \quad \xi \to +\infty,
\end{aligned}$$

where C_- is a constant greater than 0 and less than 1. We can obtain there must exist a constant $\widetilde{C} = \widetilde{C}(\boldsymbol{\alpha})$ such that

$$|Q'(\xi)| < \widetilde{C} \cdot \left(1+\frac{1}{\alpha_1}\right) \cdot T^{\alpha_1}.$$

For $\xi = 0$, we obtain

$$\begin{aligned}
|Q'(0)| &= \lim_{\xi\to 0} \frac{\Big|\frac{1-\xi t^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)}{1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)} - 1\Big|}{|\xi|} = \lim_{\xi\to 0} \frac{\big|\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T) - \xi t^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)\big|}{\big|\xi\big(1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big)\big|} \\
&= \lim_{\xi\to 0} \frac{\big|T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T) - t^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(t)\big|}{\big|1-\xi T^{\alpha_1} E^{(\xi)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\big|} \leq \left|\frac{t^{\alpha_1} + T^{\alpha_1}}{\Gamma(\alpha_1+1)}\right|.
\end{aligned}$$

From the above two inequalities and trigonometric inequalities, we draw a conclusion

$$|Q'(t)| \leq C'(\alpha_1) \max\left\{\left(1+\frac{1}{\alpha_1}\right)\cdot T^{\alpha_1}, 1, 2T^{\alpha_1}\right\}, \quad t \in [0,T],$$

where $C'(\alpha_1) = \max\left\{\widetilde{C}, \frac{1}{\Gamma(1+\alpha_1)}\right\}$.

It suffices to estimate the term $\sum_{n=1}^{\infty} \mu_n^3 g_n^2$, which can be achieved by the fact $\omega_0 \in H_0^1(\Omega)$ and Lemma 2.5:

$$\begin{aligned} M^2 = \|\omega_0\|^2_{H_0^1(\Omega)} &= \sum_{n=1}^{\infty} \frac{\mu_n g_n^2}{\left(1-\mu_n T^{\alpha_1} E^{(\mu_n)}_{\boldsymbol{\alpha}',1+\alpha_1}(T)\right)^2} \\ &\geq \sum_{n=1}^{\infty} \frac{\mu_n g_n^2}{\left(\frac{C_7 \sum_{j=2}^{s} T^{\alpha_1-\alpha_j}}{1+\mu_n T^{\alpha_1}}\right)^2} \geq C_9 \sum_{n=1}^{\infty} \frac{\mu_n g_n^2 \cdot (1+\mu_n T^{\alpha_1})^2}{\left(\sum_{j=2}^{s} T^{\alpha_1-\alpha_j}\right)^2} \\ &= C_9 T^{2\alpha_1} \sum_{n=1}^{\infty} \frac{\left(\mu_n + \frac{1}{T^{\alpha_1}}\right)^2}{\left(\sum_{j=2}^{s} T^{\alpha_1-\alpha_j}\right)^2} \mu_n g_n^2 \geq C_9 T^{2\alpha_1} \sum_{n=1}^{\infty} \frac{\mu_n^3 g_n^2}{\left(\sum_{j=2}^{s} T^{\alpha_1-\alpha_j}\right)^2}, \end{aligned}$$

that is,

$$\sum_{n=1}^{\infty} \mu_n^3 g_n^2 \leq \frac{C_7 M^2 \cdot \left(\sum_{j=2}^{s} T^{\alpha_1-\alpha_j}\right)^2}{T^{2\alpha_1}}.$$

From the above discussion, we come to a conclusion

$$\begin{aligned} &\|u^\beta[g](\,\cdot\,,t) - \omega(\,\cdot\,,t)\|^2 \\ &= \sum_{n=1}^{\infty} |Q'(\xi)|^2 \left(\frac{-\beta\mu_n^2}{\beta\mu_n+1}\right)^2 g_n^2 \leq \sum_{n=1}^{\infty} \frac{\beta}{4} |Q'(\xi)|^2 \mu_n^3 g_n^2 \\ &\leq \beta \cdot C'(\alpha_1) \max\left\{\left(1+\frac{1}{\alpha_1}\right)\cdot T^{\alpha_1}, 1, 2T^{\alpha_1}\right\} \cdot \frac{C_9 M^2 \cdot \left(\sum_{j=2}^{s} T^{\alpha_1-\alpha_j}\right)^2}{T^{2\alpha}} \\ &= \beta \cdot M^2 \cdot C_{10}^2, \end{aligned}$$

where $C_{10} = C'(\alpha_1) \max\left\{\left(1+\frac{1}{\alpha_1}\right)\cdot T^{\alpha_1}, 1, 2T^{\alpha_1}\right\} \frac{C_9 M^2 \cdot \left(\sum_{j=2}^{s} T^{\alpha_1-\alpha_j}\right)^2}{T^{2\alpha_1}}$. Combining the result in Theorem 4.2, we have

$$\|u^{\beta(\delta)}[g^\delta](\,\cdot\,,t) - \omega(\,\cdot\,,t)\| \leq 2C_7\delta + 2C_7 T^{\alpha_1} \cdot \frac{\delta}{\beta} + C_{10} M \beta^{1/2}.$$

Assume the assumption $\omega_0 \in H_0^1(\Omega)$ holds, further we choose $\beta(\delta) = \left[\left(\frac{C_7}{C_9 C'(\alpha_j)}\right)^{1/2} \frac{1}{M} \min\left\{\left(1+\frac{1}{\alpha_1}\right) T^{2\alpha_1}, 1\right\}\right]^{2/3} \delta^{2/3}$, then we can get the convergence rate of the regularized solution $u^{\beta(\delta)}$ with noise data $g^\delta(x)$, as shown below

$$\begin{aligned}
&\|u^{\beta(\delta)}[g^\delta](\,\cdot\,,t)-\omega(\,\cdot\,,t)\| \\
&= 2C_7\delta + 2C_7T^{\alpha_1}\delta\biggl(\biggl[\biggl(\frac{C_7}{C_9C'(\alpha_j)}\biggr)^{1/2}\frac{1}{M}\min\biggl\{\biggl(1+\frac{1}{\alpha_1}\biggr)T^{\alpha_1},1\biggr\}\biggr]^{2/3}\delta^{2/3}\biggr)^{-1} \\
&\quad + MC_{10}\biggl[\biggl(\frac{C_7}{C_9C'(\alpha_j)}\biggr)^{1/2}\frac{1}{M}\min\biggl\{\biggl(1+\frac{1}{\alpha_1}\biggr)T^{\alpha_1},1\biggr\}\biggr]^{1/3}\delta^{1/3} \\
&\le 2\biggl(C_7+\biggl(C_7M\bigl(C'(\alpha_1)C_9\bigr)^{1/2}\min\biggl\{\biggl(1+\frac{1}{\alpha_1}\biggr)T^{\alpha_1},1\biggr\}^{-1}\biggr)^{2/3} \\
&\qquad + \biggl(MC'(\alpha_1)C_9\max\biggl\{\biggl(1+\frac{1}{\alpha_1}\biggr)T^{2\alpha_1},1\biggr\}\biggr)^{2/3}\biggr)\delta^{1/3}
\end{aligned}$$

for all $t\in[0,T)$ and arbitrarily fixed $\alpha_1\in(\overline{\alpha},\underline{\alpha})$, where g^δ satisfying $\|g^\delta-g\|\le\delta$ for all $\delta\in[0,1)$. $\square$

6. Numerical algorithm

This section considers a finite difference method for solving a class of multi-term time-fractional diffusion equation. For simplicity, we mainly consider the following IBVP in one-dimensional case:

$$\begin{aligned}
&\sum_{j=1}^{m} q_j\partial_t^{\alpha_j}\omega(x,t)=\omega_{xx}, \quad (x,t)\in(0,\pi)\times(0,T), \\
&\omega(0,t)=\omega(\pi,t)=0, \quad t\in(0,T], \\
&\omega(x,0)=\varphi(x), \quad x\in[0,\pi],
\end{aligned} \tag{6.1}$$

where $0<\alpha_j<1$, φ is a well-known function, and $\varphi(0)=\varphi(\pi)=0$.

Taking the positive integers M and T, let $\Delta x=\frac{\pi}{M}$, $\Delta t=\frac{1}{T}$. We remember $x_i=i\Delta x$ $(0\le i\le M)$, $t_n=n\Delta t$ $(0\le n\le N)$, $\Omega_x=\{x_i \mid 0\le i\le M\}$, $\Omega_t=\{t_n \mid 0\le n\le N\}$. Define the following grid function $u_i^n\approx u(x_i,t_n)$.

Using Murio's scheme [24] for time-fractional derivative and the central difference scheme for Δ, The following result can be obtained from (6.1):

$$\begin{aligned}
&\sum_{j=1}^{s}\frac{q_j(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\sum_{k=0}^{n}(u_i^{n+1-k}-u_i^{n-k})\bigl((k+1)^{1-\alpha_j}-k^{1-\alpha_j}\bigr) \\
&=\frac{1}{(\Delta x)^2}(u_{i+1}^{n+1}-2u_i^{n+1}+u_{i-1}^{n+1}).
\end{aligned}$$

Denote $U^{n+1}=(u_1^{n+1},u_2^{n+1},\dots,u_{M-1}^{n+1})^T$, after processing, we can get the following iterative format

$$BU^1=U^0, \quad BU^{n+1}=c_1U^n+c_2U^{n-1}+\cdots+\frac{d_n}{d_0}U^0,$$

where

$$d_n = \sum_{j=1}^{m} \frac{q_j(\Delta t)^{-\alpha_j}}{\Gamma(2-\alpha_j)}\big((n+1)^{1-\alpha_j} - n^{1-\alpha_j}\big), \quad n = 0, 1, \ldots, N-1,$$

$$c_n = \frac{d_{n-1} - d_n}{d_0}, \quad n = 1, \ldots, N-1,$$

and B is a tridiagonal matrix, which is given by $B_{ii} = 1 + \frac{2}{d_0(\Delta x)^2}$ for $i = 1, 2, \ldots, M-1$ and $B_{i,i-1} = -\frac{1}{d_0(\Delta x)^2}$ for $i = 2, 3, \ldots, M-1$ and $B_{i,i+1} = -\frac{1}{d_0(\Delta x)^2}$ for $i = 1, 2, \ldots, M-2$.

For the regularization problem (3.2) corresponding to the quasi-reversibility method, we can get the following format

$$\begin{aligned} BU_i^{i+1} - \beta DU^{n+1} = U_i^n &- \sum_{k=1}^{n}(u_i^{n+1-k} - u_i^{n-k})\frac{d_k}{d_0} \\ &- \beta DU^n + \beta \sum_{k=1}^{n}(DU^{n+1-k} - DU^{n-k})\frac{d_k}{d_0}. \end{aligned}$$

After a simple calculation, the above formula can be reduced to the following formula

$$\begin{aligned} (B-\beta D)U^{n+1} &= AU^n - \sum_{k=1}^{n}(u_i^{n+1-k} - u_i^{n-k})\frac{d_k}{d_0} + \beta \sum_{k=1}^{n}(DU^{n+1-k} - DU^{n-k})\frac{d_k}{d_0} \\ &= AU^n - \sum_{k=1}^{n}(u_i^{n+1-k} - u_i^{n-k})\frac{d_k}{d_0} + \beta D \sum_{k=1}^{n}(U^{n+1-k} - U^{n-k})\frac{d_k}{d_0}. \end{aligned}$$

Further we can get the following iterative form

$$(B-\beta D)U^1 = AU^0, \quad (B-\beta D)U^{n+1} = c_1 AU^n + c_2 AU^{n-1} + \cdots + \frac{d_n}{d_0}AU^0,$$

where D is a tridiagonal matrix, which is given by $D_{ii} = -\frac{2}{(\Delta x)^2}$ for $i = 1, 2, \ldots, M-1$ and $D_{i,i-1} = \frac{1}{(\Delta x)^2}$ for $i = 2, 3, \ldots, M-1$ and $D_{i,i+1} = \frac{1}{(\Delta x)^2}$ for $i = 1, 2, \ldots, M-2$. Moreover, A is a tridiagonal matrix, which is given by $A_{ii} = 1 + \frac{2\beta}{(\Delta x)^2}$ for $i = 1, 2, \ldots, M-1$ and $A_{i,i-1} = -\frac{\beta}{(\Delta x)^2}$ for $i = 2, 3, \ldots, M-1$ and $A_{i,i+1} = -\frac{\beta}{(\Delta x)^2}$ for $i = 1, 2, \ldots, M-2$. With the above algorithm we can effectively perform numerical examples.

7. Numerical experiments

In this section, we provide the numerical results for three examples in one-dimensional case to verify the validity of the algorithm.

The noisy data is generated by adding a random-perturbation, i.e.,

$$u^\delta(x,T) = u(x,T) + \epsilon u(x,T) \cdot (2 \cdot \mathrm{rand}(\mathrm{size}(u(x,T))) - 1), \quad x \in [0,\pi],$$

and δ is noise level which can be calculated by

$$\delta = \|u^{\delta}(x,T) - u(x,T)\|_{L^2(0,\pi)}.$$

We compute the approximate L^2 error to show the accuracy of numerical solution, and error denoted by

$$e(u,\epsilon) = \frac{\|u(x,0) - u^{\beta,\delta}(x,0)\|}{\|u(x,0)\|},$$

where $u^{\beta,\delta}(x,0)$ is the regularization solution corresponding to the fractional-order quasi-reversibility method.

In the following examples, we reconstruct the initial data from the information of the solution at $T = 1$. Here we set fractional order terms $s = 3$, $\Omega = (0,\pi)$ and $q_j = 1$ for $j = 1,2,3$. The grid points on $[0,\pi]$ and $[0,T]$ are 101 and 201, respectively.

Example 7.1.

$$\begin{cases} \partial_t^{\alpha} u + \partial_t^{\alpha_1} u + \partial_t^{\alpha_2} u = \frac{\partial^2 u}{\partial x^2}, & (x,t) \in (0,\pi) \times (0,T), \\ u(0,t) = u(\pi,t) = 0, & t \in (0,T], \\ u(x,0) = f(x), & x \in [0,\pi]. \end{cases}$$

Let the fractional orders $\alpha = 0.9$, $\alpha_1 = 0.7$ and $\alpha_2 = 0.5$. Assume that $f(x) = \sin(x)$, the additional data $u(x,T)$ is obtained by solving the forward problem using the finite difference method.

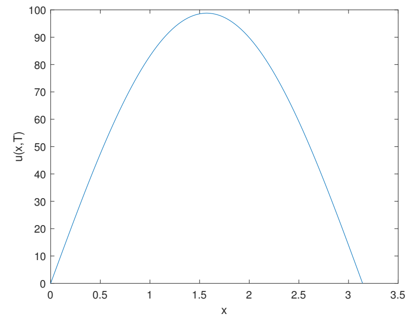
Table 7.1: Relative error of the regularization methods of Example 7.1 for $\alpha = (0.9, 0.7, 0.5)$ with $\epsilon = 1\%$, 0.1%, 0.01% and $\beta = (0.01 * \delta)^{2/3}$.

$\beta \backslash \epsilon$	$\epsilon = 1\%$	$\epsilon = 1\%$	$\epsilon = 0.01\%$
$\beta = (0.01 * \delta)^{2/3}$	0.0396	0.0130	0.0049

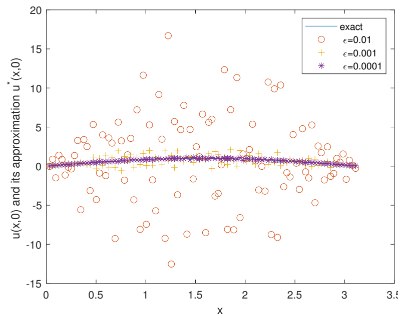
The results for Example 7.1 are shown in Figure 7.1. The input data $u(x,T)$ is shown in Figure 7.1(a). In order to illustrate the ill-posedness of inverse problem, we present numerical result without regularization method shown in Figure 7.1(b). We can see that the inverse problem of the non-regularization method is not fixed, and the effect of reconstructing the initial value is not ideal.

Choosing a reasonable regularization parameter $\beta = (0.01 * \delta)^{2/3}$, moreover, in order to verify the effectiveness of the fractional-order quasi-reversibility regularization method, the reconstruction result are presented in Figures 7.1(c), (d) and (e) with different noisy levels $\epsilon = 0.01$, 0.001, 0.0001. We can see that the numerical results of the initial state for

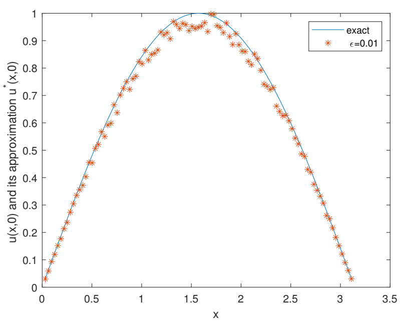
Example 7.1 match the exact ones quite well even up to 1% noise added in the "exact" Terminal data $u(x, T)$, and the relative error also decreases as the noise data decreases, and the fitting in the image is more optimal.



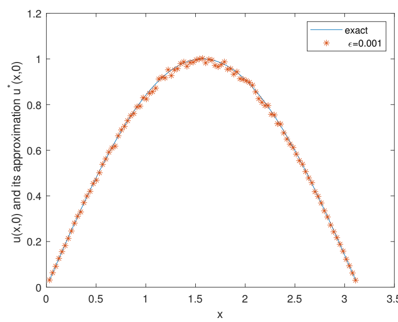
(a) The direct problem



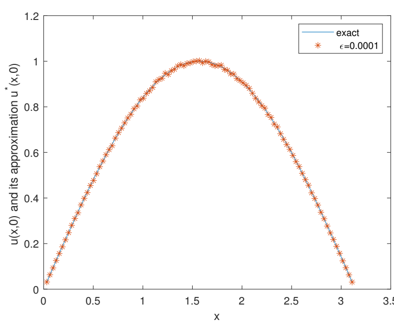
(b) No regularization method



(c) $\epsilon = 0.01$ and $\beta = (0.01 * \delta)^{2/3}$



(d) $\epsilon = 0.001$ and $\beta = (0.01 * \delta)^{2/3}$



(e) $\epsilon = 0.0001$ and $\beta = (0.01 * \delta)^{2/3}$

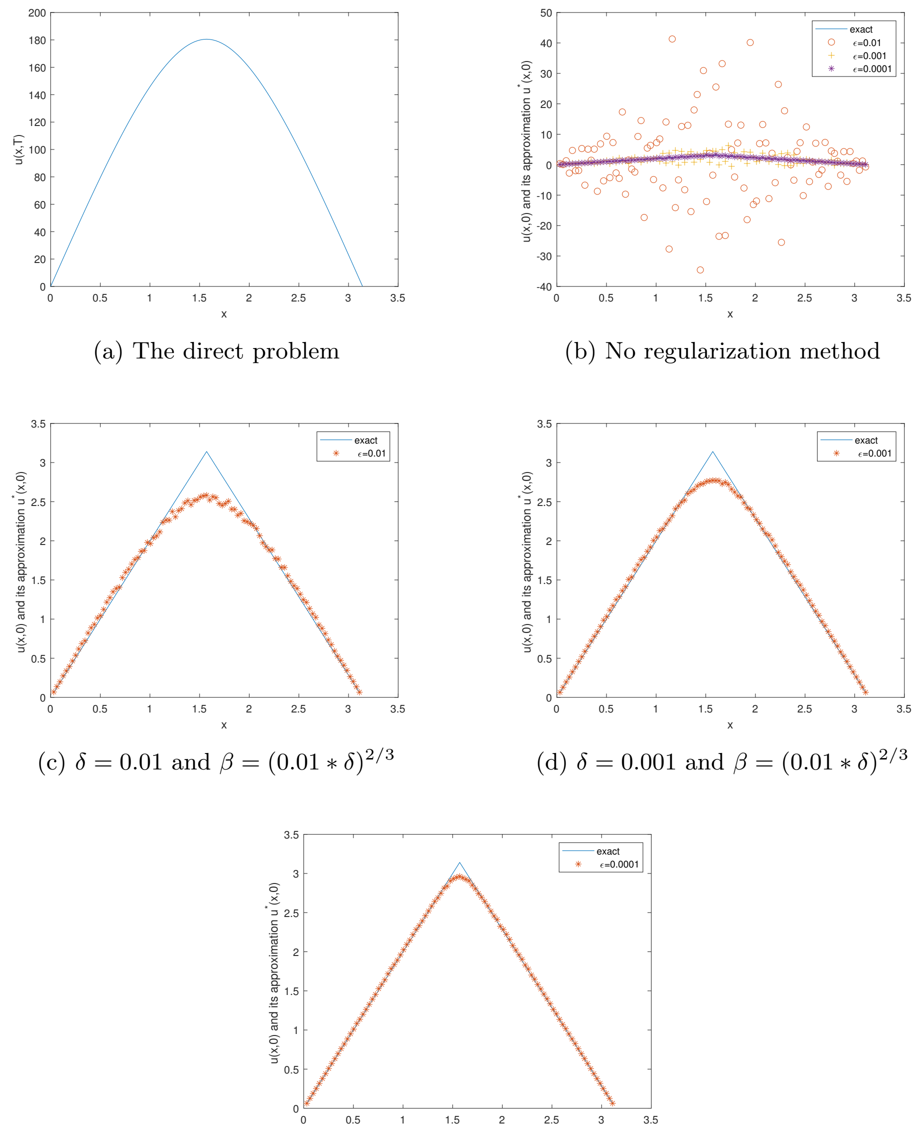
Figure 7.1: The numerical results for Example 7.1.

From Figure 7.1 we can find that the numerical results of the inverse problem based on FDM with regularization method are obviously better than those without regularization method. As our inverse problem is mildly ill-posed, so the reconstruction result doesn't work very well without regularization terms.

Example 7.2. We also consider a numerical result for the same backward problem in case of a non-smooth unknown initial state. Assume the fractional orders $\alpha = 0.8$, $\alpha_1 = 0.6$ and $\alpha_2 = 0.4$. Let the exact initial function be

$$f(x) = \begin{cases} 2x & \text{if } 0 \le x \le \pi/2, \\ -2x + 2\pi & \text{if } \pi/2 \le x < \pi. \end{cases}$$

The additional data $u(x,T)$ is obtained by solving the forward problem using the finite difference method, and the result is shown in Figure 7.2(a).



(a) The direct problem

(b) No regularization method

(c) $\delta = 0.01$ and $\beta = (0.01 * \delta)^{2/3}$

(d) $\delta = 0.001$ and $\beta = (0.01 * \delta)^{2/3}$

(e) $\delta = 0.0001$ and $\beta = (0.01 * \delta)^{2/3}$

Figure 7.2: The numerical results for Example 7.2.

Table 7.2: Relative error of the regularization methods of Example 7.2 for $\alpha = (0.8, 0.6, 0.4)$ with $\epsilon = 1\%$, 0.1%, 0.01% and $\beta = (0.01 * \delta)^{2/3}$.

$\beta \backslash \epsilon$	$\epsilon = 1\%$	$\epsilon = 1\%$	$\epsilon = 0.01\%$
$\beta = (0.01 * \delta)^{2/3}$	0.0947	0.0484	0.0177

Moreover, to verify the effectiveness of the fractional-order quasi-reversibility regularization method, we firstly present numerical result without regularization method shown in Figure 7.2(b). We can see that effect of reconstructing the initial value is not ideal. This is easy to predict because our inverse problem is mildly ill-posed. On the other hand, the numerical results under regularization method are presented in Figures 7.2(c)–(e) in case of the noisy levels $\epsilon = 0.01$, 0.001, 0.0001 with reasonable choice strategy $\beta = (0.01 * \delta)^{2/3}$. We can also see that the relative error decreases as the noise data decreases, and the fitting in the image is more optimal.

Example 7.3. We continue to consider the backward problem for a three-term time fractional diffusion equation in one-dimensional case. But here we consider a initial state that is a piecewise function with two cusps. All other definite conditions are the same as in the previous examples.

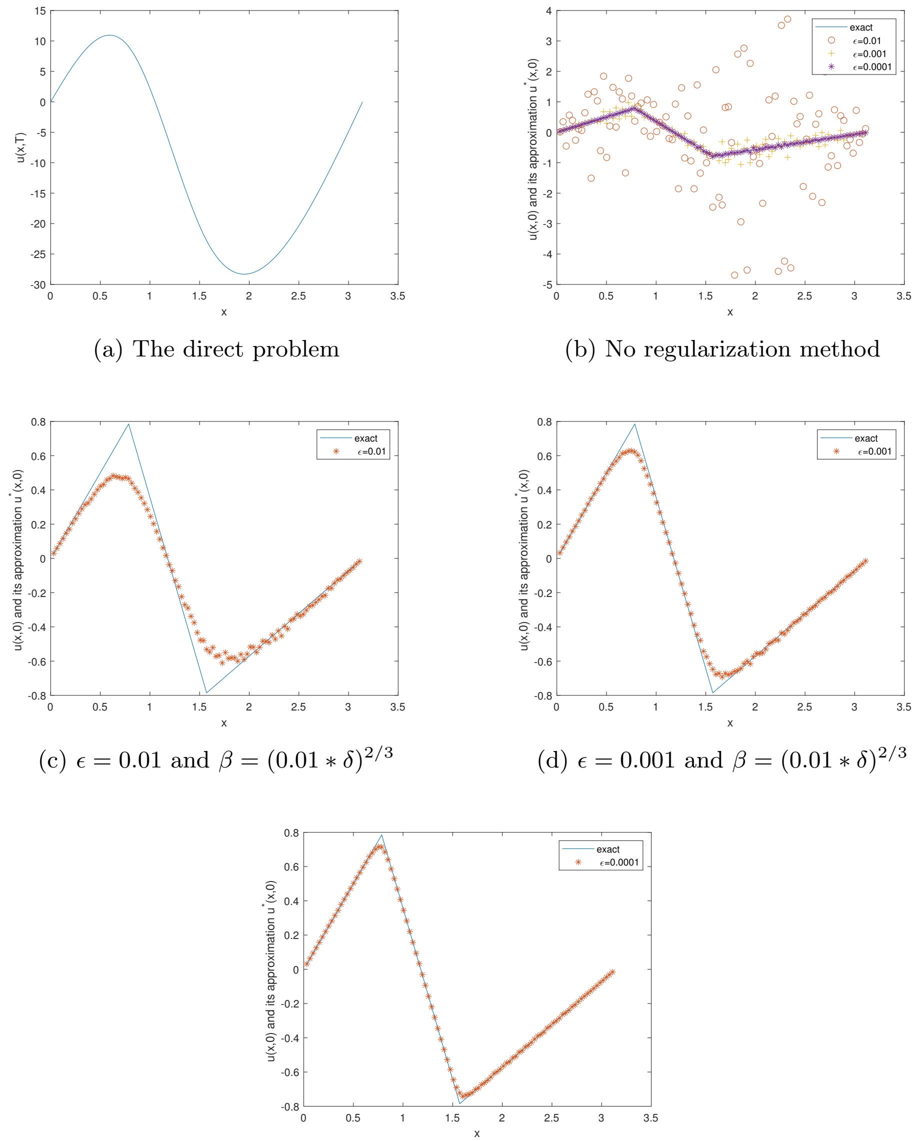
Let the exact initial function be

$$f(x) = \begin{cases} x & \text{if } 0 \leq x < \frac{\pi}{4}, \\ -2x + \frac{3}{4}\pi & \text{if } \frac{\pi}{4} \leq x < \frac{\pi}{2}, \\ \frac{1}{2} - \frac{1}{2}\pi & \text{if } \frac{\pi}{2} \leq x \leq \pi. \end{cases}$$

Table 7.3: Relative error of the regularization methods of Example 7.3 for $\alpha = (0.9, 0.7, 0.5)$ with $\epsilon = 1\%$, 0.1%, 0.01% and $\beta = (0.01 * \delta)^{2/3}$.

$\beta \backslash \epsilon$	$\epsilon = 1\%$	$\epsilon = 0.1\%$	$\epsilon = 0.01\%$
$\beta = (0.01 * \delta)^{2/3}$	0.2384	0.0870	0.0280

The input data $u(x, T)$ is also obtained by solving the forward problem using the finite difference method, and the result is displayed in Figure 7.3(a). The numerical result without regularization method is shown in Figure 7.3(b). And in case of noisy levels $\epsilon = 0.01$, 0.001, 0.0001 with a fine choice strategy $\beta = (0.01 * \delta)^{2/3}$, the reconstruction results are presented in Figures 7.3(c), 7.3(d) and 7.3(e). We come to a conclusion similar to the previous example. In Tables 7.1–7.3, we show the relative error $e(u, \epsilon)$ with different noise levels ϵ under the prior selection rule for regularization parameter β.



(a) The direct problem

(b) No regularization method

(c) $\epsilon = 0.01$ and $\beta = (0.01 * \delta)^{2/3}$

(d) $\epsilon = 0.001$ and $\beta = (0.01 * \delta)^{2/3}$

(e) $\epsilon = 0.0001$ and $\beta = (0.01 * \delta)^{2/3}$

Figure 7.3: The numerical results for Example 7.3.

Now we can draw the following conclusions from above numerical test. First, the numerical result of the inverse problem is very dependent on the regularization parameter although the backward problem is a mildly ill-posed. Secondly, as with the other inverse problem calculation results, the reconstruction effect for backward problem becomes unsatisfactory with the increase of noise level. Finally, the reconstruction effect of the fractional-order quasi-reversibility regularization method proposed in this paper is not better than that of the classical quasi-reversibility method for smooth unknown initial values, but the inversion results for non-smooth functions are still quite well. This can be seen from the numerical results in Examples 7.2 and 7.3.

8. Concluding remark

This paper considers a backward problem for a multi-term time-fractional diffusion diffusion equation by using the fractional-order quasi-reversibility method. Adding a small perturbation related to the time fractional derivative to the original equation make the original problem become well-posed. Based on the properties of multinomial Mittag-Leffler function, we demonstrate the well-posedness and regularity of the regularized solution, and we also illustrate that the backward problem for time-fractional diffusion equation is mildly ill-posed. Under an a-priori parameter choice rule, we prove the convergence and convergence rate of the regularized solution. Also, we propose a finite difference sequence for the direct problem as well as the regularization problem. Finally, the numerical experiments for three numerical examples show that our proposed method is effective.

Acknowledgments

This work is supported by the NSF of China (grant no. 12201502), the Youth Science and Technology Fund of Gansu Province (grant no. 20JR10RA099), the Innovation Capacity Improvement Project for Colleges and Universities of Gansu Province (grant no. 2020B-088) and the Northwest Normal University Graduate Research Project (grant no. 2022KYZZ-S119).

References

[1] E. E. Adams and L. W. Gelhar, *Field study of dispersion in a heterogeneous aquifer: 2. Spatial moments analysis*, Water Resour. Res. **28** (1992), no. 12, 3293–3307.

[2] S. Djennadi, N. Shawagfeh and O. Abu Arqub, *A fractional Tikhonov regularization method for an inverse backward and source problems in the time-space fractional diffusion equations*, Chaos Solitons Fractals **150** (2021), Paper No. 111127, 9 pp.

[3] X. Feng, M. Zhao and Z. Qian, *A Tikhonov regularization method for solving a backward time-space fractional diffusion problem*, J. Comput. Appl. Math. **411** (2022), Paper No. 114236, 20 pp.

[4] M. Giona, S. Cerbelli and H. E. Roman, *Fractional diffusion equation and relaxation in complex viscoelastic materials*, Phys. A: Stat. Mech. Appl. **191** (1992), no. 1-4, 449–453.

[5] Y. Han, X. Xiong and X. Xue, *A fractional Landweber method for solving backward time-fractional diffusion problem*, Comput. Math. Appl. **78** (2019), no. 1, 81–91.

[6] Y. Hatano and N. Hatano, *Dispersive transport of ions in column experiments: An explanation of long-tailed profiles*, Water Resour. Res. **34** (1998), no. 5, 1027–1033.

[7] D. Jiang, Z. Li, Y. Liu and M. Yamamoto, *Weak unique continuation property and a related inverse source problem for time-fractional diffusion-advection equations*, Inverse Problems **33** (2017), no. 5, 055013, 22 pp.

[8] S. Jiang and Y. Wu, *An inverse space-dependent source problem for a multi-term time fractional diffusion equation*, J. Math. Phys. **61** (2020), no. 12, 121502, 16 pp.

[9] B. Jin, R. Lazarov, Y. Liu and Z. Zhou, *The Galerkin finite element method for a multi-term time-fractional diffusion equation*, J. Comput. Phys. **281** (2015), 825–843.

[10] B. Jin and W. Rundell, *A tutorial on inverse problems for anomalous diffusion processes*, Inverse Problems **31** (2015), no. 3, 035003, 40 pp.

[11] A. A. Kilbas, H. M. Srivastava and J. J. Trujillo, *Theory and Applications of Fractional Differential Equations*, North-Holland Math. Stud. **204**, Elsevier Science B.V., Amsterdam, 2006.

[12] Y. S. Li, L. L. Sun, Z. Q. Zhang and T. Wei, *Identification of the time-dependent source term in a multi-term time-fractional diffusion equation*, Numer. Algorithms **82** (2019), no. 4, 1279–1301.

[13] Z. Li, X. Huang and M. Yamamoto, *Initial-boundary value problems for multi-term time-fractional diffusion equations with x-dependent coefficients*, Evol. Equ. Control Theory **9** (2020), no. 1, 153–179.

[14] Z. Li, O. Y. Imanuvilov and M. Yamamoto, *Uniqueness in inverse boundary value problems for fractional diffusion equations*, Inverse Problems **32** (2016), no. 1, 015004, 16 pp.

[15] Z. Li, Y. Liu and M. Yamamoto, *Initial-boundary value problems for multi-term time-fractional diffusion equations with positive constant coefficients*, Appl. Math. Comput. **257** (2015), 381–397.

[16] Z. Li and M. Yamamoto, *Uniqueness for inverse problems of determining orders of multi-term time-fractional derivatives of diffusion equation*, Appl. Anal. **94** (2015), no. 3, 570–579.

[17] ______, *Inverse problems of determining coefficients of the fractional partial differential equations*, in: *Handbook of Fractional Calculus with Applications: Vol. 2*, 443–464, De Gruyter, Berlin, 2019.

[18] C.-L. Lin and G. Nakamura, *Unique continuation property for multi-terms time fractional diffusion equations*, Math. Ann. **373** (2019), no. 3-4, 929–952.

[19] J. J. Liu and M. Yamamoto, *A backward problem for the time-fractional diffusion equation*, Appl. Anal. **89** (2010), no. 11, 1769–1788.

[20] Y. Liu, H. Sun, X. Yin and L. Feng, *Fully discrete spectral method for solving a novel multi-term time-fractional mixed diffusion and diffusion-wave equation*, Z. Angew. Math. Phys. **71** (2020), no. 1, Paper No. 21, 19 pp.

[21] Y. Luchko, *Initial-boundary-value problems for the generalized multi-term time-fractional diffusion equation*, J. Math. Anal. Appl. **374** (2011), no. 2, 538–548.

[22] W. Ma and L. Sun, *Inverse potential problem for a semilinear generalized fractional diffusion equation with spatio-temporal dependent coefficients*, Inverse Problems **39** (2023), no. 1, Paper No. 015005, 29 pp.

[23] F. Mainardi, A. Mura, G. Pagnini and R. Gorenflo, *Time-fractional diffusion of distributed order*, J. Vib. Control **14** (2008), no. 9-10, 1267–1290.

[24] D. A. Murio, *Implicit finite difference approximation for time fractional diffusion equations*, Comput. Math. Appl. **56** (2008), no. 4, 1138–1145.

[25] R. R. Nigmatullin, *The realization of the generalized transfer equation in a medium with fractal geometry*, Phys. Stat. Sol. (b) **133** (1986), no. 1, 425–430.

[26] I. Podlubny, *Fractional Differential Equations*, Math. Sci. Engrg. **198**, Academic Press, San Diego, CA, 1999.

[27] K. Sakamoto and M. Yamamoto, *Initial value/boundary value problems for fractional diffusion-wave equations and applications to some inverse problems*, J. Math. Anal. Appl. **382** (2011), no. 1, 426–447.

[28] S. G. Samko, A. A. Kilbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and applications*, Gordon and Breach Science Publishers, Yverdon, 1993.

[29] W. Shi, X. Xiong and X. Xue, *A fractional-order quasi-reversibility method to a backward problem for the time fractional diffusion equation*, J. Comput. Appl. Math. **394** (2021), Paper No. 113552, 14 pp.

[30] C. Sun, G. Li and X. Jia, *Numerical inversion for the initial distribution in the multi-term time-fractional diffusion equation using final observations*, Adv. Appl. Math. Mech. **9** (2017), no. 6, 1525–1546.

[31] L. L. Sun and M. L. Chang, *Galerkin spectral method for a multi-term time-fractional diffusion equation and an application to inverse source problem*, Netw. Heterog. Media **18** (2023), no. 1, 212–243.

[32] L. L. Sun, Y. S. Li and Y. Zhang, *Simultaneous inversion of the potential term and the fractional orders in a multi-term time-fractional diffusion equation*, Inverse Problems **37** (2021), no. 5, Paper No. 055007, 26 pp.

[33] L. Sun, Y. Zhang and T. Wei, *Recovering the time-dependent potential function in a multi-term time-fractional diffusion equation*, Appl. Numer. Math. **135** (2019), 228–245.

[34] N. H. Tuan, L. D. Long and S. Tatar, *Tikhonov regularization method for a backward problem for the inhomogeneous time-fractional diffusion equation*, Appl. Anal. **97** (2018), no. 5, 842–863.

[35] J.-G. Wang and T. Wei, *An iterative method for backward time-fractional diffusion problem*, Numer. Methods Partial Differential Equations **30** (2014), no. 6, 2029–2041.

[36] J.-G. Wang, T. Wei and Y.-B. Zhou, *Tikhonov regularization method for a backward problem for the time-fractional diffusion equation*, Appl. Math. Model. **37** (2013), no. 18-19, 8518–8532.

[37] T. Wei and J.-G. Wang, *A modified quasi-boundary value method for the backward time-fractional diffusion problem*, ESAIM Math. Model. Numer. Anal. **48** (2014), no. 2, 603–621.

[38] T. Wei and J. Xian, *Variational method for a backward problem for a time-fractional diffusion equation*, ESAIM Math. Model. Numer. Anal. **53** (2019), no. 4, 1223–1244.

[39] L. Yuan, X. Cheng and K. Liang, *Solving a backward problem for a distributed-order time fractional diffusion equation by a new adjoint technique*, J. Inverse Ill-Posed Probl. **28** (2020), no. 4, 471–488.

[40] R. Zheng, F. Liu and X. Jiang, *A Legendre spectral method on graded meshes for the two-dimensional multi-term time-fractional diffusion equation with non-smooth solutions*, Appl. Math. Lett. **104** (2020), 106247, 8 pp.

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