TAIWANESE JOURNAL OF MATHEMATICS
Vol. 27, No. 6, pp. 1075–1104, December 2023
DOI: 10.11650/tjm/230706

$\delta^\sharp(2,2)$-Ideal Centroaffine Hypersurfaces of Dimension 4

Handan Yıldırım* and Luc Vrancken

Abstract. Ideal submanifolds have been studied from various aspects since Chen invented δ-invariants in early 1990s (see [12] for a survey). In centroaffine differential geometry, Chen's invariants denoted by $\delta^\sharp$ are used to determine an optimal bound for the squared norm of the Tchebychev vector field of a hypersurface. We point out that a hypersurface attaining this bound is said to be an ideal centroaffine hypersurface. In this paper, we deal with $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces in $\mathbb{R}^5$ and in particularly, we focus on 4-dimensional $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of type 1.

1. Introduction

In the last decade of the last century, Chen constructed general optimal inequalities including his δ-invariants and the squared mean curvature of Riemannian submanifolds of real space forms to determine an optimal lower bound for the squared mean curvature of these Riemannian submanifolds (see [12] for details).

In [13], an optimal inequality was obtained for Lagrangian submanifolds of complex space forms. Since centroaffine differential geometry is of great interest among geometers (for instance, see [2,3,10,12,20–22,27–32,35–39]), taking into account the similarity with the Lagrangian case, the following $\delta^\sharp$-invariant was introduced for a centroaffine hypersurface M^n of $\mathbb{R}^{n+1}$:

$$\delta^\sharp(n_1,\ldots,n_k)(p) = \widehat{\tau}(p) - \sup\{\widehat{\tau}(L_1) + \cdots + \widehat{\tau}(L_k)\},$$

where $\widehat{\tau}(p)$ is the scalar curvature of M^n at $p \in M^n$ and $\widehat{\tau}(L_i)$ is the scalar curvature of L_i, $i = 1,\ldots,k$ such that $L_1,\ldots,L_k$ run over all k mutually orthogonal subspaces of T_pM^n with $\dim L_i = n_i$, $2 \leq n_1 \leq \cdots \leq n_k \leq n-1$ and $n_1 + \cdots + n_k \leq n$ (cf. [3,12]). We emphasize that the norm of Tchebychev vector field $T^\sharp$ is one of the main invariants in centroaffine differential geometry and general optimal inequalities related to $\delta^\sharp$-invariants

Received March 16, 2023; Accepted July 25, 2023.
Communicated by Mu-Tao Wang.
2020 *Mathematics Subject Classification.* 53A15, 53C42.
Key words and phrases. centroaffine differential geometry, $\delta^\sharp$-invariants, $\delta^\sharp(2,2)$-ideal, centroaffine hypersurfaces of dimension 4, type 1.
*Corresponding author.

include the squared norm of Tchebychev vector field $T^\sharp$ instead of the squared mean curvature in Riemannian geometry.

Submanifolds satisfying the equality cases of the inequalities related to either δ or $\delta^\sharp$ are said to be ideal submanifolds. It is known that these inequalities and the classification problem of ideal submanifols have been studied extensively by many geometers (for example, see [1, 3–9, 11–19, 23, 25–28, 37–39]). However, there are still some submanifolds which have not been classified yet.

We point out that the bounds for the inequalities related to $\delta^\sharp$-invariants are essentially different depending on whether $n_1 + \cdots + n_k = n$ or $n_1 + \cdots + n_k < n$. We remark that there have been some results regarding the inequalities about $\delta^\sharp$-invariants for the latter one in the literature (for instance, see [3, 27, 28, 37, 39]). However, there have been no results published related to the inequalities regarding $\delta^\sharp$-invariants for the first one yet. Because of this reason, we deal with 4-dimensional $\delta^\sharp(2,2)$-ideal centoaffine hypersurfaces in this paper. For our purpose, we mainly focus on 4-dimensional $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of type 1. Because, taking into account the classification of $\delta(2, n-2)$-ideal Lagrangian submanifolds in n-dimensional complex space forms in [14], the work on $\delta^\sharp(2, n-2)$-ideal centroaffine hypersurfaces in $\mathbb{R}^{n+1}$ is in preparation, where $n > 4$ (cf. [38]). We note that the methods developed and the results obtained in [38] are applied for 4-dimensional $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of type 2, type 3, type 4, type 5 and type 6 as explained in Section 3 and an additional assumption on the differentiable extensions of the frame is used in some cases. We emphasize that different cases depending on the form of the difference tensor are crucial for these problems.

2. Preliminaries

In this section, we recall some basic notions about centroaffine hypersurfaces by means of [35].

Let M^n be an n-dimensional C^∞-manifold and $F\colon M^n \to \mathbb{R}^{n+1}$ be a non-degenerate hypersurface whose position vector is nowhere tangent to M^n. Then, F is a transversal field along itself and $\xi = -F$ is said to be the centroaffine normal. Following [35], F together with this normalization is said to be a centroaffine hypersurface.

Let $X, Y, Z \in \chi(M^n)$. The centroaffine structure equations are given by

$$D_X F_*(Y) = F_*(\nabla_X Y) + h(X,Y)\xi, \tag{2.1}$$

$$D_X \xi = -F_*(X), \tag{2.2}$$

where D denotes the canonical flat connection of $\mathbb{R}^{n+1}$, ∇ is a torsion-free connection on M^n which is called the induced centroaffine connection and h is a non-degenerate symmetric $(0,2)$-tensor field which is called the centroaffine metric. The corresponding equations of Gauss and Codazzi are given respectively by

$$R(X,Y)Z = h(Y,Z)X - h(X,Z)Y, \tag{2.3}$$

$$(\nabla_X h)(Y,Z) = (\nabla_Y h)(X,Z). \tag{2.4}$$

The totally symmetric $(0,3)$-tensor field $\mathcal{C}(X,Y,Z) = (\nabla_X h)(Y,Z)$ is said to be the cubic form.

We assume that the centroaffine hypersurface is definite, i.e., h is definite. If h is negative definite, we shall replace $\xi = -F$ by $\xi = F$ for the affine normal. In this way, the centroaffine metric h becomes positive definite. In both cases, equations (2.1) and (2.4) hold whereas equations (2.2) and (2.3) change sign. When $\xi = -F$ (respectively, $\xi = F$), M^n is said to be positive (respectively, negative) definite.

Let $\widehat{\nabla}$ be the Levi–Civita connection of h. Moreover, denote by $\widehat{R}$ (respectively, $\widehat{\tau}$) the curvature tensor (respectively, the scalar curvature) of $\widehat{\nabla}$. The difference tensor K is then defined by

$$K_X Y = K(X,Y) = \nabla_X Y - \widehat{\nabla}_X Y$$

which is a symmetric $(1,2)$-tensor field. K and $\mathcal{C}$ are related by

$$\mathcal{C}(X,Y,Z) = -2h(K_X Y, Z).$$

Thus, for each X, K_X is self-adjoint with respect to h.

The Tchebychev form T and the Tchebychev vector field $T^\sharp$ of M^n are defined by

$$T(X) = \frac{1}{n} \operatorname{trace} K_X \quad \text{and} \quad h(T^\sharp, X) = T(X).$$

If $T = 0$ and M^n is a centroaffine hypersurface of the equiaffine space, then M^n is a so-called proper equiaffine hypersphere centered at the origin, in the sense of [35]. Note that it is an elliptic (respectively, a hyperbolic) equiaffine hypersphere when it is positive (respectively, negative) definite. If K vanishes, then M^n is a hyperquadric centered at the origin. Note that it is an ellipsoid (respectively, a two-sheeted hyperboloid) if M^n is positive (respectively, negative) definite.

It is well known in centroaffine differential geometry that

$$\begin{gathered}
h(K_X Y, Z) = h(Y, K_X Z), \\
\widehat{R}(X,Y)Z = K_Y K_X Z - K_X K_Y Z + \epsilon\big(h(Y,Z)X - h(X,Z)Y\big), \\
(\widehat{\nabla} K)(X,Y,Z) = (\widehat{\nabla} K)(Y,Z,X) = (\widehat{\nabla} K)(Z,X,Y),
\end{gathered}$$

where $\epsilon = 1$ (respectively, $\epsilon = -1$) when M^n is positive (respectively, negative) definite.

3. $\delta^\sharp$-invariants, inequalities and ideal immersions

Let M^n be an n-dimensional Riemannian manifold. For a plane section $\pi \subset T_pM^n$, where $p \in M^n$, let $\widehat{\kappa}(\pi)$ be the sectional curvature of M^n associated with π. For an orthonormal basis $\{e_1, \dots, e_n\}$ of T_pM^n, the scalar curvature $\widehat{\tau}$ at p is defined by

$$\widehat{\tau}(p) = \sum_{i<j} \widehat{\kappa}(e_i \wedge e_j), \quad 1 \leq i < j \leq n.$$

Let L be a subspace of T_pM^n with dimension $r \geq 2$ and $\{e_1, \dots, e_r\}$ be an orthonormal basis of L. The scalar curvature $\widehat{\tau}(L)$ of L is defined by

$$\widehat{\tau}(L) = \sum_{\alpha<\beta} \widehat{\kappa}(e_\alpha \wedge e_\beta), \quad 1 \leq \alpha < \beta \leq r.$$

Given two integers $n \geq 3$ and $k \geq 1$, the finite set consisting of all k-tuples $(n_1, \dots, n_k)$ of integers satisfying

$$2 \leq n_1 \leq \cdots \leq n_k \leq n-1 \quad \text{and} \quad n_1 + \cdots + n_k \leq n$$

is denoted by $\mathcal{S}(n,k)$. Moreover, the union $\bigcup_{k\geq 1} \mathcal{S}(n,k)$ is denoted by $\mathcal{S}(n)$.

For each $(n_1, \dots, n_k) \in \mathcal{S}(n)$ and each $p \in M^n$, the invariant $\delta(n_1, \dots, n_k)(p)$ is defined by

$$\delta(n_1, \dots, n_k)(p) = \widehat{\tau}(p) - \inf\{\widehat{\tau}(L_1) + \cdots + \widehat{\tau}(L_k)\},$$

where $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of T_pM^n such that $\dim L_i = n_i$, $i = 1, \dots, k$ (cf. [12]).

In [7,12], Chen proved a sharp general inequality between $\delta(n_1, \dots, n_k)$ and the squared mean curvature H^2 for submanifolds in real space forms. For Lagrangian submanifolds of complex projective spaces, the final sharp inequality was obtained in [13]. Taking into account [3] (see also [12]), it is clear that this inequality can be adapted to centroaffine differential geometry by defining the following invariant for a centroaffine hypersurface M^n of $\mathbb{R}^{n+1}$:

$$\delta^\sharp(n_1, \dots, n_k)(p) = \widehat{\tau}(p) - \sup\{\widehat{\tau}(L_1) + \cdots + \widehat{\tau}(L_k)\},$$

where $(n_1, \dots, n_k) \in \mathcal{S}(n)$, $p \in M^n$ and $L_1, \dots, L_k$ run over all k mutually orthogonal subspaces of T_pM^n such that $\dim L_i = n_i$, $i = 1, \dots, k$. The difference with the Lagrangian case is due to the difference of sign in the Gauss equation.

Throughout this paper, we deal with the case $n_1 + \cdots + n_k = n$ in centroaffine differential geometry. As a result, before stating the inequality in this case, we introduce some notations. For a given $\delta^\sharp$-invariant $\delta^\sharp(n_1, \dots, n_k)$ on M^n ($n \geq 3$, $k \geq 1$ and $2 \leq n_1 \leq \cdots \leq n_k \leq n-1$) and a given point $p \in M^n$, we consider mutually orthogonal subspaces

$L_1, \dots, L_k$ of T_pM^n maximizing the quantity $\widehat{\tau}(L_1) + \cdots + \widehat{\tau}(L_k)$. We then choose an orthonormal basis $\{e_1, \dots, e_n\}$ for T_pM^n such that

$$e_1, \dots, e_{n_1} \in L_1, \quad e_{n_1+1}, \dots, e_{n_1+n_2} \in L_2, \quad \dots, \quad e_{n_1+\cdots+n_{k-1}+1}, \dots, e_{n_1+\cdots+n_k} \in L_k$$

and define

$$\Delta_1 := \{1, \dots, n_1\}, \quad \Delta_2 := \{n_1+1, \dots, n_1+n_2\}, \quad \dots, \\ \Delta_k := \{n_1 + \cdots + n_{k-1} + 1, \dots, n_1 + \cdots + n_k\}.$$

From now on, we use the following conventions for the ranges of summation indices:

$$A, B, C \in \{1, \dots, n\}, \quad i, j \in \{1, \dots, k\}, \quad \alpha_i, \beta_i \in \Delta_i.$$

The components of the difference tensor is denoted by $K_{AB}^C = h\big(K(e_A, e_B), e_C\big)$. Due to the symmetry of the cubic form, these are symmetric with respect to three indices A, B and C. Adapting the proof of [13] (cf. also [12]), we have the following theorem in the centroaffine case:

Theorem 3.1. *Let M^n be a definite centroaffine hypersurface of $\mathbb{R}^{n+1}$. Take $\epsilon = 1$ (respectively, $\epsilon = -1$) when M^n is positive (respectively, negative) definite. Then, for each k-tuple $(n_1, \dots, n_k) \in \mathcal{S}(n)$ with $n_1 + \cdots + n_k = n$, we have*

(3.1)
$$\delta^\sharp(n_1, \dots, n_k) \geq -\frac{n^2\big(k-1-2\sum_{i=2}^k \frac{1}{n_i+2}\big)}{2\big(k-2\sum_{i=2}^k \frac{1}{n_i+2}\big)}\|T^\sharp\|^2 + \frac{1}{2}\left(n(n-1) - \sum_{i=1}^k n_i(n_i-1)\right)\epsilon.$$

The equality case of inequality (3.1) *holds at a point $p \in M^n$ if and only if one has*

(i) $K_{\alpha_i\alpha_j}^A = 0$ *if $i \neq j$ and $A \neq \alpha_i, \alpha_j$,*

(ii) *if $n_j \neq \min\{n_1, \dots, n_k\}$:*

$$K_{\alpha_i\alpha_i}^{\beta_j} = 0 \quad \text{if } i \neq j \text{ and } \sum_{\alpha_j \in \Delta_j} K_{\alpha_j\alpha_j}^{\beta_j} = 0,$$

(iii) *if $n_j = \min\{n_1, \dots, n_k\}$:*

$$\sum_{\alpha_j \in \Delta_j} K_{\alpha_j\alpha_j}^{\beta_j} = (n_i+2)K_{\alpha_i\alpha_i}^{\beta_j} \quad \text{for any } i \neq j \text{ and any } \alpha_i \in \Delta_i.$$

A centroaffine immersion of M^n into $\mathbb{R}^{n+1}$ is called $\delta^\sharp(n_1, \dots, n_k)$-ideal if it satisfies the equality case of inequality (3.1) identically. A centroaffine immersion of M^n into $\mathbb{R}^{n+1}$ is called ideal if it is $\delta^\sharp(n_1, \dots, n_k)$-ideal for the corresponding $(n_1, \dots, n_k) \in \mathcal{S}(n)$.

We now restrict ourselves to the case of $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces in $\mathbb{R}^5$. Of course, we do not know whether the decomposition obtained in the previous theorem is unique. Depending on how the Tchebychev vector field $T^\sharp$ is related to a possible decomposition of the tangent space TM^4, we can consider the following types of such ideal hypersurfaces:

1. M^4 is of type 1 if, for any possible decomposition of TM^4, $T^\sharp$ has a component in the directions of both L_1 and L_2. In this case, we can pick $e_1 \in L_1$ and $e_3 \in L_2$ such that $T^\sharp$ is a combination of e_1 and e_3.

2. M^4 is of type 2 if there exists a decomposition of TM^4 such that $T^\sharp$ is contained in L_1 and the difference tensor K restricted to L_2 does not vanish identically. In this case, we pick e_1 such that $T^\sharp$ is a multiple of e_1 and we make a rotation in L_2 such that $h\big(K(e_3,e_3),e_4\big)=0$.

3. M^4 is of type 3 if there exists a decomposition of TM^4 such that $T^\sharp$ is contained in L_1 and K restricted to L_2 vanishes identically. In this case, we pick e_1 such that $T^\sharp$ is a multiple of e_1. We still retain a rotation freedom in L_2.

4. M^4 is of type 4 if, for any decomposition of TM^4, $T^\sharp$ vanishes and the restrictions of K to both L_1 and L_2 are non-vanishing. Applying a rotation in both spaces, we may assume that the basis is chosen such that $h\big(K(e_1,e_1),e_2\big)=0=h\big(K(e_3,e_3),e_4\big)$.

5. M^4 is of type 5 if there exists a decomposition of TM^4 such that $T^\sharp$ vanishes, the restriction of K to L_1 is non-vanishing and the restriction of K to L_2 vanishes. In this case, it follows easily that this happens if and only if M^4 is a $\delta^\sharp(2)$-ideal centroaffine hypersurface with vanishing $T^\sharp$. These can be classified by following the same approach in [24] (see also [16,17]).

6. M^4 is of type 6 if there exists a decomposition of TM^4 such that $T^\sharp$ vanishes and the restrictions of K to both L_1 and L_2 vanish. We point out that this is valid if and only if K vanishes identically. Therefore, by the classical Berwald theorem (cf. [35]) and the fact that the metric is definite, we get that M^4 is either a hyperboloid or an ellipsoid.

Note that since both eigenspaces of a 4-dimensional $\delta^\sharp(2,2)$-ideal centroaffine hypersurface have the same dimension, it follows from the theorem that the conditions for a 4-dimensional $\delta^\sharp(2,2)$-centroaffine hypersurface to be ideal are weaker than the ones for a $\delta^\sharp(2,n-2)$-centroaffine hypersurface of $\mathbb{R}^{n+1}$ with $n>4$. We emphasize that, in the latter case, $T^\sharp$ would automatically be in the direction of the first distribution and therefore submanifolds of type 1 would not occur. However, this also means that in order to deal with

4-dimensional $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of type 2, …, type 6, the methods developed and the results obtained in [38] are applied, where an additional assumption on the differentiable extensions of the frame is used in some cases. For this reason, we only give the classification of 4-dimensional $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of type 1 throughout this paper.

We emphasize that, in order to complete the classification of 4-dimensional $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces, it is sufficient

1. to obtain a classification of 4-dimensional $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of type 1,
2. to show that the frames in the cases of type 2, …, type 6 can always be chosen in a differentiable way. We remark that these cases can be studied in exactly the same way as $\delta^\sharp(2,n-2)$-ideal centroaffine hypersurfaces in $\mathbb{R}^{n+1}$ with $n > 4$ (see [38] for the details and see [14] for Lagrangian case).

4. $\delta^\sharp(2,2)$-ideal centroaffine hypersurfaces of type 1 in $\mathbb{R}^5$

In this section, we suppose that M^4 is a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface of type 1 in $\mathbb{R}^5$. We assume that the affine normal is chosen such that the affine metric h is positive definite. In this case, expressing the conditions of Theorem 3.1 and choosing an appropriate orthonormal basis in each Δ_i, we have the following lemma:

Lemma 4.1. *Let M^4 be a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface of type 1 in $\mathbb{R}^5$. Then, at each point p of M^4, there exists an orthonormal frame $\{e_1,\dots,e_4\}$ such that*

$$\begin{aligned}
&K(e_1,e_1) = (a_1+3b_3)e_1 + a_2e_2 + a_3e_3, && K(e_1,e_2) = a_2e_1 + (b_3-a_1)e_2,\\
&K(e_3,e_3) = b_3e_1 + (b_1+3a_3)e_3 + b_2e_4, && K(e_3,e_4) = b_2e_3 + (a_3-b_1)e_4,\\
&K(e_2,e_2) = (b_3-a_1)e_1 - a_2e_2 + a_3e_3, && K(e_4,e_4) = b_3e_1 + (a_3-b_1)e_3 - b_2e_4,\\
&K(e_1,e_3) = a_3e_1 + b_3e_3,\ \ K(e_1,e_4) = b_3e_4, && K(e_2,e_3) = a_3e_2,\ \ K(e_2,e_4) = 0,
\end{aligned}$$

where $a_1,a_2,a_3,b_1,b_2,b_3 \in \mathbb{R}$ such that $a_3 \neq 0 \neq b_3$.

Since $a_3 \neq 0$ and $b_3 \neq 0$, we deduce that

Lemma 4.2. *Let M^4 be a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface of type 1 in $\mathbb{R}^5$. Then, if necessary by restricting to an open dense set, around each point p of M^4, there exists an orthonormal frame field which can be also denoted by $\{e_1,\dots,e_4\}$ such that*

$$\begin{aligned}
&K(e_1,e_1) = (a_1+3b_3)e_1 + a_2e_2 + a_3e_3, && K(e_1,e_2) = a_2e_1 + (b_3-a_1)e_2,\\
&K(e_3,e_3) = b_3e_1 + (b_1+3a_3)e_3 + b_2e_4, && K(e_3,e_4) = b_2e_3 + (a_3-b_1)e_4,\\
&K(e_2,e_2) = (b_3-a_1)e_1 - a_2e_2 + a_3e_3, && K(e_4,e_4) = b_3e_1 + (a_3-b_1)e_3 - b_2e_4,\\
&K(e_1,e_3) = a_3e_1 + b_3e_3,\ \ K(e_1,e_4) = b_3e_4, && K(e_2,e_3) = a_3e_2,\ \ K(e_2,e_4) = 0,
\end{aligned}$$

where a_1, a_2, a_3, b_1, b_2 and b_3 are functions such that $a_3 \neq 0 \neq b_3$.

Proof. First, we remark that

$$T^\sharp = \frac{3}{2}(b_3 e_1 + a_3 e_3).$$

This implies that $b_3^2 + a_3^2$ is a differentiable function. We then introduce a vector v by

$$v = \frac{4}{9}\left(K(T^\sharp, T^\sharp) - \frac{9}{2}(a_3^2 + b_3^2)T^\sharp\right).$$

Hence, we have

$$v = a_1 b_3^2 e_1 + a_2 b_3^2 e_2 + a_3^2 b_1 e_3 + a_3^2 b_2 e_4$$

which implies that $(a_1^2 + a_2^2)b_3^4 + a_3^4(b_1^2 + b_2^2)$ and $a_1 b_3^3 + b_1 a_3^3$ are differentiable functions. It can be easily verified that

$$\begin{aligned} &K(T^\sharp, v) - \frac{3}{2}(a_3^2 + b_3^2)v \\ &= \frac{3}{2}\big((2a_3^3 b_1 b_3 + b_3^3(a_1^2 + a_2^2 + 2a_1 b_3))e_1 + (a_3^3(2a_3 b_1 + b_1^2 + b_2^2) + 2a_1 a_3 b_3^3)e_3\big). \end{aligned}$$

Now, there are two possibilities. First, we assume that this vector and $T^\sharp$ are linearly independent. In this case, we get

$$a_3^2(b_1^2 + b_2^2) - b_3^2(a_1^2 + a_2^2) \neq 0.$$

We introduce a differentiable vector w by

$$w = K(T^\sharp, v) - \frac{3}{2}(b_3^2 + a_3^2)v - 2(a_3^3 b_1 + b_3^3 a_1)T^\sharp.$$

It can be easily seen that

$$w = \frac{3}{2}\big((a_1^2 + a_2^2)b_3^3 e_1 + (b_1^2 + b_2^2)a_3^3 e_3\big).$$

Because of our assumption, w and $T^\sharp$ are linearly independent. In order to simplify notations, we write

$$a_1 = \frac{\widetilde{a_1}}{b_3}, \quad a_2 = \frac{\widetilde{a_2}}{b_3}, \quad b_1 = \frac{\widetilde{b_1}}{a_3}, \quad b_2 = \frac{\widetilde{b_2}}{a_3}.$$

It follows respectively from the inner products $\langle T^\sharp, T^\sharp\rangle$, $\langle v, v\rangle$, $\langle T^\sharp, w\rangle$, $\langle T^\sharp, v\rangle$ and $\langle v, w\rangle$ that there exist differentiable functions $d_1, \ldots, d_5$ such that

$$b_3^2 + a_3^2 = d_1, \tag{4.1}$$

$$b_3^2(\widetilde{a_1}^2 + \widetilde{a_2}^2) + a_3^2(\widetilde{b_1}^2 + \widetilde{b_2}^2) = d_2, \tag{4.2}$$

$$b_3^2(\widetilde{a_1}^2 + \widetilde{a_2}^2) + a_3^2(\widetilde{b_1}^2 + \widetilde{b_2}^2) = d_3, \tag{4.3}$$

$$b_3^2\widetilde{a_1} + a_3^2\widetilde{b_1} = d_4,$$

$$b_3^2\widetilde{a_1}(\widetilde{a_1}^2 + \widetilde{a_2}^2) + a_3^2\widetilde{b_1}(\widetilde{b_1}^2 + \widetilde{b_2}^2) = d_5.$$

We now introduce

$$\widetilde{w} = K(v,w) - \frac{3}{2}d_3 v - d_4 w - d_5 T^\sharp$$

which gives

$$\widetilde{w} = \frac{3}{2}\big((\widetilde{a_1}^2 + \widetilde{a_2}^2)^2 b_3 e_1 + (\widetilde{b_1}^2 + \widetilde{b_2}^2)^2 a_3 e_3\big).$$

Therefore, $\langle \widetilde{w}, \widetilde{w}\rangle$ yields that

$$b_3^2(\widetilde{a_1}^2 + \widetilde{a_2}^2)^2 + a_3^2(\widetilde{b_1}^2 + \widetilde{b_2}^2)^2 = d_6 \tag{4.4}$$

is a differentiable function. It is clear that we can solve (4.1) and (4.2) for a_3^2 and b_3^2. Substituting these solutions into (4.3) and (4.4), we get solutions for $\widetilde{a_1}^2 + \widetilde{a_2}^2$ and $\widetilde{b_1}^2 + \widetilde{b_2}^2$. These solutions allow us to express e_1 and e_3 in a differentiable way by means of the differentiable vector fields $T^\sharp$ and w. Additionally, from the last two equations, we can also express $\widetilde{a_1}$ and $\widetilde{b_1}$ in a differentiable way. Now, looking at either $K(e_1,e_1)$ or $K(e_3,e_3)$, for the eigenspaces of either K_{e_1} or K_{e_3} restricted to the space $\{e_2,e_4\}$, we see that e_2 and e_4 are determined uniquely unless $a_2 = b_2 = 0 = a_1 = b_1$.

Next, we consider the case

$$a_3^2(b_1^2 + b_2^2) = b_3^2(a_1^2 + a_2^2)$$

or equivalently

$$\widetilde{b_1}^2 + \widetilde{b_2}^2 = \widetilde{a_1}^2 + \widetilde{a_2}^2.$$

We define the vectors $T^\sharp$, v and w as before. In this case, $w = (\widetilde{a_1}^2 + \widetilde{a_2}^2)T^\sharp$ implying that $\widetilde{a_1}^2 + \widetilde{a_2}^2$ is differentiable. Under the above case, we define a differentiable vector

$$\widetilde{w} = K(v,v) + (\widetilde{a_1}^2 + \widetilde{a_2}^2 - 2\widetilde{a_1}b_3^2 - 2a_3^2\widetilde{b_1})v - \frac{2}{3}(\widetilde{a_1}^2 + \widetilde{a_2}^2)(a_3^2 + b_3^2)T^\sharp.$$

Thus, we get

$$\widetilde{w} = (\widetilde{a_1}^2 + \widetilde{a_2}^2)(2\widetilde{a_1}b_3 e_1 + 2\widetilde{b_1}a_3 e_3).$$

Proceeding now in the same way as in the previous case, we obtain a differentiable frame unless $\widetilde{b_1} = \widetilde{a_1}$, in which case (if necessary by replacing e_4 by $-e_4$) we may also assume that $\widetilde{b_2} = \widetilde{a_2}$. As before, we can easily get that $\widetilde{a_1}$, $\widetilde{a_2}$ and $a_3^2 + b_3^2$ are differentiable functions. The vectors $T^\sharp$ and v are linearly independent if and only if $a_2 \neq 0$. We also have that the space spanned by $T^\sharp$ and v is invariant under K. Consequently, its orthogonal complement is also invariant under $K_{T^\sharp}$. Since $\widetilde{a_2} \neq 0$, we see that this space has two distinct eigenvalues (and so two well-determined unitary eigenvectors f_1 and f_2). Computing $h\big(K(f_1,f_1),f_2\big)$ yields the differentiability of $\frac{a_3^2 - b_3^2}{a_3 b_3}$ and therefore e_1, e_2, e_3 and e_4 can be expressed by means of $T^\sharp$, v, f_1 and f_2 in a differentiable way.

Next, we consider the case that $\widetilde{a_2} = 0$ and $\widetilde{a_1} \neq 0$ on an open set. In this case, $K(T^\sharp, T^\sharp)$ is a multiple of $T^\sharp$. Hence, we obtain that $\widetilde{a_1}$ and $a_3^2 + b_3^2$ are differentiable functions. We also find that $\widetilde{w} = -a_3 e_1 + b_3 e_3$ is a vector of the length $a_3^2 + b_3^2$ spanning a 1-dimensional eigenspace of $K_{T^\sharp}$. Consequently, $\widetilde{w}$ is a differentiable vector. Computing $h\big(K(\widetilde{w}, \widetilde{w}), \widetilde{w}\big)$, we deduce that $\frac{a_3^2 - b_3^2}{a_3 b_3}$ is also differentiable. Thus, we get e_1 and e_3 in a differentiable way. As explained in the first case, this also gives e_2 and e_4.

Finally, we consider the case with an additional condition $\widetilde{a_1} = 0$. In this case, we take the differentiable frame as the vector $T^\sharp$ together with 3 arbitrary mutually orthogonal unitary vector fields. A straightforward computation shows that this differentiable frame is compatible with Chen's equality, where $a_1 = a_2 = b_1 = b_2 = b_3 = 0$. Note that in this case, M^4 is not of type 1 with respect to the new frame since $a_1 = a_2 = b_1 = b_2 = b_3 = 0$. Moreover, in this case, we have the decompositions of TM^4 in mutually orthogonal planes, each of which realising the equality. $\square$

We note that if both a_3 and b_3 vanish, then the classification results can be obtained in a similar way to that of Lagrangian submanifolds of complex space forms (see [16, 17]).

From now on, we always work on the open dense subset introduced in Lemma 4.2. We denote by $\widehat{\Gamma}_{ij}^k$ (respectively, $\widehat{\omega}_j^k(e_i)$) the Christoffel symbols (respectively, the connection forms) according to the Levi–Civita connection of the affine metric.

Since M^4 is of type 1, $a_3 \neq 0 \neq b_3$. Note that if $a_1 = a_2 = b_1 = b_2 = 0$, we can change the frame to make it of type 3 (cf. [20]). So, this case is excluded here. Hence, there are two cases to consider (if necessary) after interchanging distributions: (i) $a_1^2 + a_2^2 \neq 0 \neq b_1^2 + b_2^2$ on an open set or (ii) $a_1^2 + a_2^2 = 0$ and $b_1^2 + b_2^2 \neq 0$ on an open set. We call the first case type (1i) and it is the case that will be considered in detail in the remainer of this paper. Using the fact that $\mathcal{C}$ is totally symmetric, we have the following Levi–Civita connections for the case (i) by a tedious, long and straightforward computations:

$$
\begin{aligned}
\widehat{\nabla}_{e_1} e_1 &= \frac{a_1 d_2 - a_2 d_3}{b_3} e_2 + c_3 e_3 + c_2 e_4,\\
\widehat{\nabla}_{e_1} e_2 &= \frac{-a_1 d_2 + a_2 d_3}{b_3} e_1,\\
\widehat{\nabla}_{e_1} e_3 &= -c_3 e_1 - \frac{b_3 c_2}{a_3} e_4,\\
\widehat{\nabla}_{e_1} e_4 &= -c_2 e_1 + \frac{b_3 c_2}{a_3} e_3,\\
\widehat{\nabla}_{e_2} e_1 &= \frac{a_3(b_3 \kappa + c_3) - b_1 c_3 - b_2 c_2}{b_3} e_2,\\
\widehat{\nabla}_{e_2} e_2 &= \frac{-a_3(b_3 \kappa + c_3) + b_1 c_3 + b_2 c_2}{b_3} e_1 + c_3 e_3 + c_2 e_4,\\
\widehat{\nabla}_{e_2} e_3 &= -c_3 e_2,\\
\widehat{\nabla}_{e_2} e_4 &= -c_2 e_2,
\end{aligned}
$$

$$
\begin{aligned}
\widehat{\nabla}_{e_3} e_1 &= -\frac{a_3 d_2}{b_3} e_2 - d_3 e_3,\\
\widehat{\nabla}_{e_3} e_2 &= \frac{a_3 d_2}{b_3} e_1 - d_2 e_3,\\
\widehat{\nabla}_{e_3} e_3 &= d_3 e_1 + d_2 e_2 + \frac{b_1 c_2 - b_2 c_3}{a_3} e_4,\\
\widehat{\nabla}_{e_3} e_4 &= -\frac{b_1 c_2 - b_2 c_3}{a_3} e_3,\\
\widehat{\nabla}_{e_4} e_1 &= -d_3 e_4,\\
\widehat{\nabla}_{e_4} e_2 &= -d_2 e_4,\\
\widehat{\nabla}_{e_4} e_3 &= \frac{-a_1 d_3 - a_2 d_2 + b_3(d_3 + a_3\kappa)}{a_3} e_4,\\
\widehat{\nabla}_{e_4} e_4 &= d_3 e_1 + d_2 e_2 + \frac{a_1 d_3 + a_2 d_2 - b_3(d_3 + a_3\kappa)}{a_3} e_3,
\end{aligned}
$$

where c_2, c_3, d_2, d_3 and κ are additional unknown functions and the coefficients a_3, b_3, a_2, b_2, a_1 and b_1 of the difference tensor satisfy respectively the following system of differential equations:

- For the function a_3, we have

$$
\begin{aligned}
e_1(a_3) &= -b_3 c_3,\\
e_2(a_3) &= 0,\\
e_3(a_3) &= -2b_2 c_2 - 2b_1 c_3 - a_2 d_2 - a_1 d_3 + b_3(d_3 + a_3\kappa),\\
e_4(a_3) &= b_1 c_2 - b_2 c_3.
\end{aligned}
$$

- For the function b_3, we obtain

$$
\begin{aligned}
e_1(b_3) &= -b_2 c_2 - b_1 c_3 - 2a_2 d_2 - 2a_1 d_3 + a_3(c_3 + b_3\kappa),\\
e_2(b_3) &= a_1 d_2 - a_2 d_3,\\
e_3(b_3) &= -a_3 d_3,\\
e_4(b_3) &= 0.
\end{aligned}
$$

- For the function a_2, we deduce

$$
\begin{aligned}
e_1(a_2) &= \frac{-a_{11} b_3 + 3a_2\big(b_1 c_3 + b_2 c_2 - a_3(c_3 + b_3\kappa)\big)}{b_3},\\
e_2(a_2) &= a_{22} - b_2 c_2 - \frac{a_2(-3a_1 d_2 + 3a_2 d_3 + b_3 d_2)}{b_3},\\
e_3(a_2) &= a_2 c_3 + \frac{3a_1 a_3 d_2}{b_3},\\
e_4(a_2) &= a_2 c_2.
\end{aligned}
$$

- For the function b_2, we find

$$e_1(b_2) = \frac{3b_1b_3c_2}{a_3} + b_2d_3,$$
$$e_2(b_2) = b_2d_2,$$
$$e_3(b_2) = -b_{11} + \frac{3b_2\big(a_2d_2 + a_1d_3 - b_3(d_3 + a_3\kappa)\big)}{a_3},$$
$$e_4(b_2) = b_{22} - \frac{b_2(a_3c_2 - 3b_1c_2 + 3b_2c_3)}{a_3} - a_2d_2.$$

- For the function a_1, we get

$$e_1(a_1) = \frac{3a_1\big(b_2c_2 + (-a_3 + b_1)c_3\big)}{b_3} - (a_{22} - b_2c_2 + a_2d_2 + 2a_1d_3 + 3a_1a_3\kappa),$$
$$e_2(a_1) = \frac{-a_{11}b_3 + (3a_1 - 2b_3)(a_1d_2 - a_2d_3)}{b_3},$$
$$e_3(a_1) = a_1c_3 - \frac{3a_2a_3d_2}{b_3},$$
$$e_4(a_1) = a_1c_2.$$

- For the function b_1, we obtain

$$e_1(b_1) = -\frac{3b_2b_3c_2}{a_3} + b_1d_3,$$
$$e_2(b_1) = b_1d_2,$$
$$e_3(b_1) = \frac{3b_1\big(a_2d_2 + (a_1 - b_3)d_3\big)}{a_3} - (b_{22} + b_2c_2 + 2b_1c_3 - a_2d_2 + 3b_1b_3\kappa),$$
$$e_4(b_1) = -b_{11} + \frac{(2a_3 - 3b_1)(-b_1c_2 + b_2c_3)}{a_3}.$$

In the above, a_{11}, a_{22}, b_{11} and b_{22} are arbitrary functions defined additionally.

We now focus on the distributions $\mathcal{D}_1 = \{e_1, e_2\}$ and $\mathcal{D}_2 = \{e_3, e_4\}$. For this reason, we recall some notions about distributions (see [34] for the details). Let (M^n, h) be a Riemannian manifold and $\widehat{\nabla}$ be its Levi–Civita connection. Then, a subbundle $E \subset TM^n$ is called autoparallel if $\widehat{\nabla}_X Y \in E$ holds for all $X, Y \in E$, whereas a subbundle E is called totally umbilical if there exists a vector field $H \in E^\perp$ such that $h(\widehat{\nabla}_X Y, Z) = h(X,Y)h(H,Z)$ for all $X, Y \in E$ and $Z \in E^\perp$. Here, H is said to be the mean curvature vector of E. If, moreover, $h(\widehat{\nabla}_X H, Z) = 0$ holds, E is called spherical.

From the formulas for the Levi–Civita connection, we deduce that both of the distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ are integrable. Moreover, since one is also the orthogonal complement of the other one, their orthogonal complements are also integrable. Furthermore, we see that each subbundle with the mean curvature vector $c_3e_3 + c_2e_4$ or $d_3e_1 + d_2e_2$ is totally umbilical in that case. In sense of Definition 3 of [33], this implies that M^4 is a TP net and in view of Proposition 4 of [33], the manifold has the structure of a twisted product.

In order to obtain a classification, we now have to consider the following subcases:

$$1.\ a_1 \neq 0 \text{ and } b_1 \neq 0, \quad 2.\ a_1 \neq 0 \text{ and } b_1 = 0, \quad 3.\ a_1 = b_1 = 0.$$

Suppose that $a_1 \neq 0$ and $b_1 \neq 0$. We first try to find coordinate vector fields. We know that M^4 is a twisted product. We try to take isothermal coordinates on each of the two components, i.e., we try to find isothermal coordinates which are linear combinations of respectively e_1, e_2 and e_3, e_4. Take $U \times V$, where $U \subset \mathbb{R}^2$, $V \subset \mathbb{R}^2$, $(u_1,u_2) \in U$ and $(v_1,v_2) \in V$ in order to define the vector fields as follows:

$$g_1 = u_1e_1 + u_2e_2, \quad g_2 = -u_2e_1 + u_1e_2, \quad g_3 = v_1e_3 + v_2e_4, \quad g_4 = -v_2e_3 + v_1e_4,$$

where

$$u_1 = \frac{\rho_2 \cos\left(\frac{1}{3}\arctan\left(\frac{a_2}{a_1}\right)\right)}{\sqrt{a_1^2+a_2^2}}, \qquad u_2 = \frac{\rho_2 \sin\left(\frac{1}{3}\arctan\left(\frac{a_2}{a_1}\right)\right)}{\sqrt{a_1^2+a_2^2}},$$
$$v_1 = \frac{\rho_1 \cos\left(\frac{1}{3}\arctan\left(\frac{b_2}{b_1}\right)\right)}{\sqrt{b_1^2+b_2^2}}, \qquad v_2 = \frac{\rho_1 \sin\left(\frac{1}{3}\arctan\left(\frac{b_2}{b_1}\right)\right)}{\sqrt{b_1^2+b_2^2}}$$

such that ρ_2 is a function depending only on the first component and ρ_1 is a function depending only on the second component. It follows now by a straightforward computation that

$$[g_1,g_3] = [g_1,g_4] = [g_2,g_3] = [g_2,g_4] = 0.$$

Moreover, $[g_1,g_2]$ and $[g_3,g_4]$ also vanish if and only if the functions ρ_1 and ρ_2 satisfy the following system of differential equations:

$$\begin{aligned}
e_1(\rho_2) &= -\frac{2\rho_2}{3(a_1^2+a_2^2)b_3}\big(a_{11}a_2b_3 + a_1b_3(a_{22} - b_2c_2 + a_2d_2) \\
&\quad + 3a_1^2(-b_2c_2 + a_3c_3 - b_1c_3 + b_3d_3 + a_3b_3\kappa) \\
&\quad + a_2^2\big(-3b_2c_2 - 3b_1c_3 + b_3d_3 + 3a_3(b_3\kappa + c_3)\big)\big), \\
e_2(\rho_2) &= \frac{2\rho_2}{3(a_1^2+a_2^2)b_3}\big(3a_1^3d_2 - 3a_1^2(b_3d_2 + a_2d_3) - a_2(-a_{22}b_3 + b_2b_3c_2 + 2a_2b_3d_2 + 3a_2^2d_3) \\
&\quad + a_1\big(-a_{11}b_3 + a_2(3a_2d_2 + 2b_3d_3)\big)\big), \\
e_3(\rho_1) &= -\frac{2\rho_1}{3a_3(b_1^2+b_2^2)}\big(-3(b_1^2+b_2^2)\big(a_2d_2 + (a_1-b_3)d_3\big) \\
&\quad + a_3\big(b_{11}b_2 + b_1(b_{22} + b_2c_2 - a_2d_2) + 3b_1^2(b_3\kappa + c_3) + b_2^2(3b_3\kappa + c_3)\big)\big), \\
e_4(\rho_1) &= -\frac{2\rho_1}{3a_3(b_1^2+b_2^2)}\big(-3(b_1^2+b_2^2)(b_1c_2 - b_2c_3) \\
&\quad + a_3\big(3b_1^2c_2 + b_1(b_{11} - 2b_2c_3) + b_2(-b_{22} + 2b_2c_2 + a_2d_2)\big)\big).
\end{aligned}$$

Of course, we have to check if such functions ρ_1 and ρ_2 exist or not. In order to do so, we have to verify that the so called integrability conditions stated as

$$e_i\big(e_j(\rho_k)\big) - e_j\big(e_i(\rho_k)\big) = \big(\widehat{\nabla}_{e_i} e_j - \widehat{\nabla}_{e_j} e_i\big)(\rho_k)$$

are satisfied. It can be easily verified by using the Gauss equations and by checking the integrability conditions for the other functions. We remark that the Gauss equation yields the following:

$$\begin{aligned}
e_2(c_2) &= e_2(c_3) = 0,\\
e_4(d_2) &= e_4(d_3) = 0,\\
e_1(c_3) &= -\frac{b_3(a_3^2 + c_2^2)}{a_3},\\
e_1(c_2) &= \frac{b_3 c_2 c_3}{a_3},\\
e_3(d_3) &= -\frac{a_3(b_3^2 + d_2^2)}{b_3},\\
e_3(d_2) &= \frac{a_3 d_2 d_3}{b_3},\\
e_1(d_2) &= \frac{d_3(a_2 d_3 - a_1 d_2)}{b_3} - a_2 b_3 + d_2 d_3,\\
e_3(c_2) &= -\frac{c_3(b_1 c_2 - b_2 c_3)}{a_3} - a_3 b_2 + c_2 c_3,\\
e_4(c_3) &= -a_3 b_2 + c_2 c_3 - \frac{c_2\big(a_2 d_2 + a_1 d_3 - b_3(d_3 + a_3\kappa)\big)}{a_3},\\
e_2(d_3) &= -a_2 b_3 + d_2 d_3 - \frac{d_2\big(b_2 c_2 + b_1 c_3 - a_3(c_3 + b_3\kappa)\big)}{b_3},\\
e_3(c_3) &= -2a_3^2 - a_3 b_1 - a_1 b_3 - 2b_3^2 + c_3^2 + d_3^2 + q_1 + \epsilon,\\
e_1(d_3) &= \frac{b_1 b_3 c_2^2 - b_2 b_3 c_2 c_3 + a_1 a_3 d_2^2 - a_2 a_3 d_2 d_3 - a_3 b_3 q_1}{a_3 b_3},\\
e_2(d_2) &= 2a_1 b_3 + \frac{b_1 c_2^2 - b_2 c_2 c_3 + a_3 d_2^2 - a_3 q_1}{a_3}\\
&\quad + \frac{b_3^3 + (b_2 c_2 - a_3 c_3 + b_1 c_3) d_3 - b_3 d_3(d_3 + a_3\kappa)}{b_3},\\
e_4(c_2) &= \frac{1}{a_3}\big(- a_3^3 + a_3^2 b_1 - a_1 a_3 b_3 - 2a_3 b_3^2 + a_3 c_2^2 - b_1 c_2^2 + b_2 c_2 c_3 + a_2 c_3 d_2\\
&\qquad + a_1 c_3 d_3 - b_3 c_3 d_3 + a_3 d_3^2 + a_3 q_1 + a_3\epsilon - a_3 b_3 c_3 \kappa\big),
\end{aligned}$$

where q_1 is an additional unknown function. We now denote $e_i(\kappa)$ by $\widetilde{\kappa}_i$. Then, it follows from the integrability conditions for c_3 and d_3 that

$$\widetilde{\kappa}_1 = \frac{1}{a_3^2 b_3^2}\big(b_3(a_1 + 2b_3)c_2(b_1 c_2 - b_2 c_3) + a_2^2 a_3(-b_3^2 + d_3^2)$$

$$
\begin{aligned}
&+ a_2a_3d_2(3b_2c_2 - 3a_3c_3 + 3b_1c_3 - 2a_1d_3 + 2b_3d_3 - a_3b_3\kappa)\\
&+ a_3^2\big(b_2b_3c_2\kappa - a_1d_3(3c_3 + b_3\kappa) + b_1b_3(-b_3 + c_3\kappa)\big)\\
&+ a_3\big(a_1^2d_2^2 + a_1\big(b_3^3 + 3(b_2c_2 + b_1c_3)d_3 - b_3(d_2^2 + d_3^2 + q_1)\big)\\
&+ b_3(2b_3^3 - a_{11}d_2 - a_{22}d_3 + 2b_2c_2d_3 + b_1c_3d_3 - b_3\epsilon + b_3^2c_3\kappa)\big)\\
&- a_3^3b_3\big(c_3\kappa + b_3(-2 + \kappa^2)\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\kappa}_2 = \frac{1}{a_3^2b_3^2}\big(&3a_1^2a_3d_2d_3 - 3a_2^2a_3d_2d_3 + a_3b_3(a_{22}d_2 + b_1c_3d_2 - a_{11}d_3)\\
&+ a_1a_3\big(-d_2(b_2c_2 - a_3c_3 + b_1c_3 + 3b_3d_3) + a_2(b_3^2 + 3d_2^2 - 3d_3^2)\big)\\
&+ a_2\big(b_3c_2(b_1c_2 - b_2c_3) - a_3^2c_3d_3 + a_3\big(2b_3^3 + (b_2c_2 + b_1c_3)d_3 - b_3(d_2^2 - 2d_3^2 + q_1)\big)\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\kappa}_3 = \frac{1}{a_3^2b_3^2}\big(&2a_3^4b_3 + b_3\big(-b_1b_2c_2c_3 + b_2^2c_3^2 + 3a_2b_2c_2d_2 + 3a_2b_1c_3d_2\\
&+ 3(a_1 - b_3)(b_2c_2 + b_1c_3)d_3\big) + a_3^3b_3(-b_1 + d_3\kappa)\\
&- a_3^2(b_1^2b_3 + b_2^2b_3 + a_1b_3^2 - 2b_3^3 - 2a_1d_2^2 + 2a_2d_2d_3 + b_3\epsilon + b_3^3\kappa^2)\\
&+ a_3b_3\big(-b_{11}c_2 - b_{22}c_3 + 2b_2c_2c_3 + 2a_2c_3d_2 - b_3(b_2c_2 - a_2d_2 + b_3d_3)\kappa\\
&+ a_1\big(-b_1b_3 + d_3(c_3 + b_3\kappa)\big) + b_1\big(-c_2^2 + d_3^2 + q_1 + \epsilon - b_3(2b_3 + c_3\kappa)\big)\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
\widetilde{\kappa}_4 = \frac{1}{a_3^2b_3}\big(&(b_1c_2 - b_2c_3)(2b_2c_2 + 3b_1c_3 - a_2d_2 + b_3d_3)\\
&+ a_1(-a_3b_2b_3 + a_3c_2d_3 - b_1c_2d_3 + b_2c_3d_3)\\
&+ a_3\big(b_{22}c_2 - (b_{11} + 3b_1c_2)c_3 + b_2(-2b_3^2 - c_2^2 + 3c_3^2 + d_3^2 + q_1 + \epsilon)\big)\big).
\end{aligned}
$$

A long but straightforward computation shows that all the Gauss equations as well as the integrability conditions for the functions a_3 and b_3 are satisfied. The integrability conditions for the functions a_1, b_1, a_2 and b_2 give that

$$
\begin{aligned}
e_1(a_{11}) = \frac{1}{a_3b_3^2}\big(&-a_2^2a_3b_3d_2d_3 + w_1 + a_2b_3\big(3b_3c_2(b_1c_2 - b_2c_3)\\
&+ a_3(b_3^3 + 2b_3d_2^2 + 4a_{22}d_3 - 5b_2c_2d_3 - b_1c_3d_3 + 3b_3d_3^2 - 3b_3q_1)\\
&+ a_3^2d_3(c_3 + b_3\kappa)\big)\\
&+ a_3b_3\big(b_3(3a_{22} - 4b_2c_2)d_2 + 4a_{11}\big(b_2c_2 + b_1c_3 - b_3d_3 - a_3(c_3 + b_3\kappa)\big)\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
e_2(a_{11}) = \frac{1}{a_3b_3^2}\big(&-t_1 + a_2a_3\big(-12a_1^2d_2d_3\\
&+ a_1d_2\big(-12b_2c_2 - 12b_1c_3 + 11b_3d_3 + 12a_3(c_3 + b_3\kappa)\big)\\
&+ b_3\big(-4a_{11}d_3 + d_2(-b_2c_2 + a_3c_3 - b_1c_3 + b_3d_3 + a_3b_3\kappa)\big)\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
e_3(a_{11}) = \frac{1}{b_3^2}\big(&2b_3\big(a_{11}b_3c_3 + 2a_3(a_{22} - b_2c_2)d_2\big) - 3a_1^2a_3d_2d_3 + 3a_2^2a_3d_2d_3\\
&+ a_2\big(b_3(3b_1c_2^2 - 3b_2c_2c_3 - 2b_3c_3d_3)\\
&+ a_3\big(3(b_2c_2 + b_1c_3)d_3 + b_3\big(b_3^2 + 2d_2^2 - 3(d_3^2 + q_1)\big)\big) - 3a_3^2d_3(c_3 + b_3\kappa)\big)
\end{aligned}
$$

$$
\begin{aligned}
&\quad + 3a_1\big(a_2a_3(3b_3^2 - 3d_2^2 - d_3^2) + d_2\big(b_3^2c_3 - a_3(b_2c_2 + b_1c_3 - 2b_3d_3)\\
&\quad + a_3^2(c_3 + b_3\kappa)\big)\big)\big),\\
e_4(a_{11}) &= c_2(2a_{11} + 3a_1d_2 - 2a_2d_3),\\
e_1(a_{22}) &= \frac{1}{a_3b_3^2}\big(12a_2^3a_3d_2d_3 + t_1 + a_2^2a_3\big(- 7b_3^3 + 12\big((b_2c_2 - a_3c_3 + b_1c_3)d_3\\
&\quad + a_1(-d_2^2 + d_3^2)\big) + b_3\big(6d_2^2 - d_3(5d_3 + 12a_3\kappa)\big)\big)\\
&\quad + b_3\big(b_3^2c_2(3b_1c_2 + b_2c_3) + a_3\big(a_{11}(4a_1 - b_3)d_2 + b_2c_2(-4b_2c_2 - 4b_1c_3 + b_3d_3)\big)\\
&\quad + 4a_3^2b_2c_2(c_3 + b_3\kappa) + 4a_{22}a_3\big(b_2c_2 + b_1c_3 - a_3(c_3 + b_3\kappa)\big)\big)\big),\\
e_2(a_{22}) &= \frac{1}{a_3b_3^2}\big(w_1 + a_1a_3\big(12a_2^2d_2d_3\\
&\quad + 4d_2\big(- 3a_1(b_2c_2 - a_3c_3 + b_1c_3 + a_1d_3) + b_3\big(a_{22} - b_2c_2 + 3a_1(d_3 + a_3\kappa)\big)\big)\\
&\quad + a_2\big(- 8b_3^3 + 12(-a_1d_2^2 + b_2c_2d_3 - a_3c_3d_3 + b_1c_3d_3 + a_1d_3^2)\\
&\quad + b_3\big(5d_2^2 - 6d_3(d_3 + 2a_3\kappa)\big)\big)\big)\big),\\
e_3(a_{22}) &= \frac{1}{a_3b_3^2}\big(3a_1^2a_3^2(2b_3^2 - d_2^2) + 3a_2^2a_3^2(-b_3^2 + 2d_2^2 + d_3^2)\\
&\quad + a_2d_2\big(3b_2b_3^2c_2 - 2a_3b_3^2c_3 + a_3^2(3b_2c_2 + 3b_1c_3 + 7b_3d_3) - 3a_3^3(c_3 + b_3\kappa)\big)\\
&\quad - 3a_1\big(a_3b_3c_2(-b_1c_2 + b_2c_3) - b_2b_3^2c_2d_3\\
&\quad - a_3^2\big(b_3^3 + (b_2c_2 + b_1c_3 + 2a_2d_2)d_3 - b_3(d_2^2 + d_3^2 + q_1)\big) + a_3^3d_3(c_3 + b_3\kappa)\big)\\
&\quad - b_3\big(a_3^2(b_2^2b_3 + 4a_{11}d_2) + b_2b_3(b_1c_2c_3 - b_2c_3^2 + 3b_3c_2d_3)\\
&\quad + a_3b_3\big(- 2a_{22}c_3 + c_2(b_{11} + b_2c_3 + 3b_2b_3\kappa)\big)\big)\big),\\
e_4(a_{22}) &= \frac{1}{a_3}\big(- a_3^3b_2 + a_3^2b_1b_2 + b_2\big(2b_1c_2^2 + c_3(-2b_2c_2 + a_2d_2 + a_1d_3 - b_3d_3)\big)\\
&\quad + a_3\big(- a_1b_2b_3 + c_2(2a_{22} + b_{22} - 3a_2d_2)\\
&\quad + b_2\big(- 2c_2^2 + d_3^2 + q_1 + \epsilon - b_3(2b_3 + c_3\kappa)\big)\big)\big),\\
e_1(b_{11}) &= \frac{1}{a_3^2}\big(- 5a_3^3b_2b_3 + a_3^2(6b_1b_2b_3 + 2b_{11}d_3 + 3b_1c_2d_3 - 2b_2c_3d_3)\\
&\quad + 3b_3\big(- b_1^2c_2c_3 + b_2c_3(2b_2c_2 + a_2d_2 + a_1d_3 - b_3d_3)\\
&\quad - b_1\big(4b_2c_2^2 + b_2c_3^2 + a_2c_2d_2 + (a_1 - b_3)c_2d_3\big)\big)\\
&\quad + a_3b_3\big(- 3a_1b_2b_3 + c_2(4b_{22} + 6b_1c_3 - 4a_2d_2 + 3b_1b_3\kappa)\\
&\quad + b_2\big(2c_2^2 + 3(d_3^2 + q_1 + \epsilon) - 3b_3(2b_3 + c_3\kappa)\big)\big)\big),\\
e_2(b_{11}) &= (2b_{11} + 3b_1c_2 - 2b_2c_3)d_2,\\
e_3(b_{11}) &= \frac{1}{a_3^2}\big(z_1 + a_3\big(4b_{11}\big(a_2d_2 + (a_1 - b_3)d_3\big) + b_2c_3(4b_{22} + 2b_2c_2 - 5a_2d_2 - a_1d_3 + b_3d_3)\\
&\quad - 12b_1^2c_2(c_3 + b_3\kappa) + b_1\big(- 4b_{22}c_2 + 4a_2c_2d_2 + b_2\big(- 8c_2^2 + 6c_3(c_3 + 2b_3\kappa)\big)\big)\big)\big),
\end{aligned}
$$

$$
\begin{aligned}
e_4(b_{11}) = {} & \frac{1}{a_3^2 b_3}\big(- x_1 + a_3 b_3\big(4(b_{22} - a_2 d_2)(a_2 d_2 + a_1 d_3 - b_3 d_3) \\
& - b_2\big(4b_{11}c_3 + a_2 c_2 d_2 + (a_1 - b_3)c_2 d_3\big) \\
& + b_1 c_2(4b_{11} + 11 b_2 c_3 + 12 b_2 b_3 \kappa) + b_2^2(6c_2^2 - 5c_3^2 - 12 b_3 c_3 \kappa)\big)\big), \\
e_1(b_{22}) = {} & \frac{1}{a_3^2 b_3}\big(a_2^2 a_3^2(-b_3^2 + d_3^2) - a_2 d_2\big(- 3b_3^2(b_2 c_2 + b_1 c_3) \\
& + a_3^2\big(- 3b_2 c_2 - 3b_1 c_3 + (a_1 + b_3)d_3\big) + 3a_3^3(c_3 + b_3\kappa)\big) \\
& - b_3\big(3a_3^3 b_1 b_3 + a_3^2(-3b_1^2 b_3 + 3b_2^2 b_3 + a_{11}d_2 - 2b_{22}d_3 + 2b_2 c_2 d_3) \\
& + 3b_3\big(2b_1^2 c_2^2 - 2b_2^2 c_2^2 - 3b_1 b_2 c_2 c_3 - b_2^2 c_3^2 + (-a_1 + b_3)(b_2 c_2 + b_1 c_3)d_3\big) \\
& + a_3 b_3\big(3a_1 b_1 b_3 + c_2(4b_{11} - 7b_2 c_3 + 3b_2 b_3 \kappa) \\
& + 3b_1(2b_3^2 + c_2^2 - d_3^2 - q_1 - \epsilon + b_3 c_3 \kappa)\big)\big)\big), \\
e_2(b_{22}) = {} & \frac{1}{a_3 b_3}\big(a_3 b_3(a_{22} + 2b_{22} - 3b_2 c_2)d_2 + a_1 a_2 a_3(2b_3^2 + 3d_2^2) - 3a_2^2 a_3 d_2 d_3 \\
& - a_2\big(b_3 c_2(-b_1 c_2 + b_2 c_3) - a_3\big(b_3^3 + (b_2 c_2 + b_1 c_3)d_3 - b_3(2d_2^2 + d_3^2 + q_1)\big) \\
& + a_3^2 d_3(c_3 + b_3\kappa)\big)\big), \\
e_3(b_{22}) = {} & \frac{1}{a_3^2 b_3}\big(12 b_2 b_3(-b_1 c_2 + b_2 c_3)(b_2 c_2 + b_1 c_3 + a_2 d_2 + a_1 d_3 - b_3 d_3) \\
& + a_3^3\big(- 7b_2^2 b_3 + d_2(3a_1 d_2 + a_2 d_3)\big) + x_1 \\
& + a_3^2 b_3\big(- b_{11}c_2 + b_2 c_2 c_3 + a_2 c_3 d_2 + b_3(-4b_{22} + b_2 c_2 + 4a_2 d_2)\kappa\big)\big), \\
e_4(b_{22}) = {} & \frac{1}{a_3^2}\big(5a_3^4 b_2 - 5a_3^3 b_1 b_2 - 12 b_1(b_1 c_2 - b_2 c_3)(b_2 c_2 + b_1 c_3 + a_2 d_2 + a_1 d_3 - b_3 d_3) \\
& + z_1 + a_3^2\big(3a_1 b_2 b_3 - 3b_{22}c_2 + 4a_2 c_2 d_2 + 4b_{11}(c_3 + b_3\kappa) \\
& + b_2\big(- 2c_2^2 - 3(2c_3^2 + d_3^2 + q_1 + \epsilon) + b_3(6b_3 - c_3\kappa)\big)\big)\big),
\end{aligned}
$$

where t_1, x_1, w_1 and z_1 are additional unknown functions. It now follows that the integrability conditions for the functions ρ_1 and ρ_2 are also satisfied. Thus, g_1, g_2, g_3 and g_4 can be interpreted as coordinate vector fields. Moreover, using Proposition 1 of [33], we see that these vector fields are parallel vector fields for the underlying structure on the twisted product related to both components. Furthermore, using the formulas for the derivatives of the functions, it follows that

$$
\begin{aligned}
g_1\left(g_3\left(\ln\frac{u_1^2 + u_2^2}{v_1^2 + v_2^2}\right)\right) &= 0, & g_1\left(g_4\left(\ln\frac{u_1^2 + u_2^2}{v_1^2 + v_2^2}\right)\right) &= 0, \\
g_2\left(g_3\left(\ln\frac{u_1^2 + u_2^2}{v_1^2 + v_2^2}\right)\right) &= 0, & g_2\left(g_4\left(\ln\frac{u_1^2 + u_2^2}{v_1^2 + v_2^2}\right)\right) &= 0.
\end{aligned}
$$

This implies that we can write $\sqrt{u_1^2 + u_2^2} = \frac{e^{\kappa_1}}{\mu}$ and $\sqrt{v_1^2 + v_2^2} = \frac{e^{\kappa_2}}{\mu}$, where κ_1 is a function depending only on the first component and κ_2 is a function depending only on the second component. We also get that there exist functions $k_1, \dots, k_6$ such that

$$
\begin{aligned}
K(g_1, g_1) &= (4k_1 + k_3)g_1 + k_2 g_2 + k_5 \frac{u_1^2 + u_2^2}{v_1^2 + v_2^2} g_3 + k_4 \frac{u_1^2 + u_2^2}{v_1^2 + v_2^2} g_4,\\
K(g_1, g_2) &= k_2 g_1 - k_3 g_2,\\
K(g_1, g_3) &= k_5 g_1 + k_1 g_3,\\
K(g_1, g_4) &= k_4 g_1 + k_1 g_4,\\
K(g_2, g_2) &= -k_3 g_1 + 3k_2 g_2 + k_5 \frac{u_1^2 + u_2^2}{v_1^2 + v_2^2} g_3 + k_4 \frac{u_1^2 + u_2^2}{v_1^2 + v_2^2} g_4,\\
K(g_2, g_3) &= k_5 g_2 + k_2 g_3,\\
K(g_2, g_4) &= k_4 g_2 + k_2 g_4,\\
K(g_3, g_3) &= k_1 \frac{v_1^2 + v_2^2}{u_1^2 + u_2^2} g_1 + k_2 \frac{v_1^2 + v_2^2}{u_1^2 + u_2^2} g_2 + (4k_5 + k_6) g_3 + k_4 g_4,\\
K(g_3, g_4) &= k_4 g_3 - k_6 g_4,\\
K(g_4, g_4) &= k_1 \frac{v_1^2 + v_2^2}{u_1^2 + u_2^2} g_1 + k_2 \frac{v_1^2 + v_2^2}{u_1^2 + u_2^2} g_2 - k_6 g_3 + 3k_4 g_4.
\end{aligned}
$$

Consequently, in order to complete the classification, it is now sufficient to check which twister spaces can be immersed as centroaffine hypersurfaces with cubic form as described above. In order to do so, we need to express everything in terms of the vector fields g_1, g_2, g_3, g_4 and the twistor metric. We start with $U \times V$, where $U \subset \mathbb{R}^2$ and $V \subset \mathbb{R}^2$ are equipped with its canonical metric. We can consider g_1 and g_2 (respectively, g_3 and g_4) as the standard basis on the first (respectively, second) component. The twistor metric is determined by the functions κ_1, κ_2 and μ, where κ_1 is a function depending only on the first component and κ_2 is a function depending only on the second component. We point out that since each surface is conformally flat, this can be also considered as the twistor product of two surfaces with the same twistor function $\frac{1}{\mu}$. We emphasize that these surfaces correspond to equiaffine spheres (cf. [32]).

On the other hand, we note that the case (2) $a_1 \neq 0$ and $b_1 = 0$ and the case (3) $a_1 = 0$ and $b_1 = 0$, using similar computations, lead to the same result.

Summarizing the previous computations, we have shown the following:

Theorem 4.3. *Let $F\colon M^4 \to \mathbb{R}^5$ be a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface of type* (1i)*. Then, we have locally*

$$M^4 = \frac{1}{\mu}(M_1^2 \times M_2^2)$$

such that $M_1^2 \subset \mathbb{R}^3$ is a surface with metric $\langle \cdot, \cdot \rangle = e^{2\kappa_1}(dx_1^2 + dx_2^2)$ and $M_2^2 \subset \mathbb{R}^3$ is a surface with metric $\langle \cdot, \cdot \rangle = e^{2\kappa_2}(dy_1^2 + dy_2^2)$. Moreover, in terms of the standard basis

$\left\{\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right\}$, *we have*

$$\begin{aligned}
K\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_1}\right) &= (4k_1+k_3)\frac{\partial}{\partial x_1} + k_2\frac{\partial}{\partial x_2} + k_5\frac{\lambda_1^2}{\lambda_2^2}\frac{\partial}{\partial y_1} + k_4\frac{\lambda_1^2}{\lambda_2^2}\frac{\partial}{\partial y_2},\\
K\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}\right) &= k_2\frac{\partial}{\partial x_1} - k_3\frac{\partial}{\partial x_2},\\
K\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}\right) &= k_5\frac{\partial}{\partial x_1} + k_1\frac{\partial}{\partial y_1},\\
K\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_2}\right) &= k_4\frac{\partial}{\partial x_1} + k_1\frac{\partial}{\partial y_2},\\
K\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_2}\right) &= -k_3\frac{\partial}{\partial x_1} + 3k_2\frac{\partial}{\partial x_2} + k_5\frac{\lambda_1^2}{\lambda_2^2}\frac{\partial}{\partial y_1} + k_4\frac{\lambda_1^2}{\lambda_2^2}\frac{\partial}{\partial y_2},\\
K\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_1}\right) &= k_5\frac{\partial}{\partial x_2} + k_2\frac{\partial}{\partial y_1},\\
K\left(\frac{\partial}{\partial x_2}, \frac{\partial}{\partial y_2}\right) &= k_4\frac{\partial}{\partial x_2} + k_2\frac{\partial}{\partial y_2},\\
K\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_1}\right) &= k_1\frac{\lambda_2^2}{\lambda_1^2}\frac{\partial}{\partial x_1} + k_2\frac{\lambda_2^2}{\lambda_1^2}\frac{\partial}{\partial x_2} + (4k_5+k_6)\frac{\partial}{\partial y_1} + k_4\frac{\partial}{\partial y_2},\\
K\left(\frac{\partial}{\partial y_1}, \frac{\partial}{\partial y_2}\right) &= k_4\frac{\partial}{\partial y_1} - k_6\frac{\partial}{\partial y_2},\\
K\left(\frac{\partial}{\partial y_2}, \frac{\partial}{\partial y_2}\right) &= k_1\frac{\lambda_2^2}{\lambda_1^2}\frac{\partial}{\partial x_1} + k_2\frac{\lambda_2^2}{\lambda_1^2}\frac{\partial}{\partial x_2} - k_6\frac{\partial}{\partial y_1} + 3k_4\frac{\partial}{\partial y_2},
\end{aligned}$$

where $\lambda_1 = \frac{e^{\kappa_1}}{\mu}$ *and* $\lambda_2 = \frac{e^{\kappa_2}}{\mu}$.

We emphasize that any M^4 constructed in this theorem is a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface of type (1i) in $\mathbb{R}^5$.

In the following, "," used after the index in the expressions k_i, κ_j, n_{11} and r_{11} shows that partial derivative(s) is (are) taken, where $i = 1, \ldots, 6$ and $j = 1, 2$.

Lemma 4.4. *If M^4 is as described in Theorem* 4.3*, then there exist constants C_1, C_2 and C such that*

$$(k_1+k_3)e^{2\kappa_1} = C_1, \tag{4.5}$$

$$(k_5+k_6)e^{2\kappa_2} = C_2, \tag{4.6}$$

$$2C_1^2 e^{-6\kappa_1} + (\kappa_{1,x_1x_1} + \kappa_{1,x_2x_2})e^{-2\kappa_1} = C, \tag{4.7}$$

$$2C_2^2 e^{-6\kappa_2} + (\kappa_{2,y_1y_1} + \kappa_{2,y_2y_2})e^{-2\kappa_2} = -C. \tag{4.8}$$

Moreover, we have the following system of differential equations:

$$k_{1,x_1} = \frac{e^{2\kappa_1}n_{11} + k_1\kappa_{1,x_1}\mu - k_2\kappa_{1,x_2}\mu - (2k_1 + C_1e^{-2\kappa_1})\mu_{x_1}}{\mu},$$

$$
\begin{aligned}
k_{1,x_2} &= \frac{k_2\kappa_{1,x_1}\mu + k_1\kappa_{1,x_2}\mu - k_2\mu_{x_1} + (-k_1 + C_1e^{-2\kappa_1})\mu_{x_2}}{\mu},\\
k_{1,y_1} &= -\frac{(k_5\mu_{x_1} + k_1\mu_{y_1})}{\mu},\\
k_{1,y_2} &= -\frac{(k_4\mu_{x_1} + k_1\mu_{y_2})}{\mu},\\
k_{2,x_1} &= \frac{k_2\kappa_{1,x_1}\mu + k_1\kappa_{1,x_2}\mu - k_2\mu_{x_1} + (-k_1 + C_1e^{-2\kappa_1})\mu_{x_2}}{\mu},\\
k_{2,x_2} &= \frac{e^{2\kappa_1}n_{11} - k_1\kappa_{1,x_1}\mu + k_2\kappa_{1,x_2}\mu + C_1e^{-2\kappa_1}\mu_{x_1} - 2k_2\mu_{x_2}}{\mu},\\
k_{2,y_1} &= -\frac{(k_5\mu_{x_2} + k_2\mu_{y_1})}{\mu},\\
k_{2,y_2} &= -\frac{(k_4\mu_{x_2} + k_2\mu_{y_2})}{\mu},\\
k_{3,x_1} &= \frac{-k_1\kappa_{1,x_1}\mu - 2C_1e^{-2\kappa_1}\kappa_{1,x_1}\mu + k_2\kappa_{1,x_2}\mu - e^{2\kappa_1}n_{11} + (2k_1 + C_1e^{-2\kappa_1})\mu_{x_1}}{\mu},\\
k_{3,x_2} &= -\frac{k_2\kappa_{1,x_1}\mu + k_1\kappa_{1,x_2}\mu - k_2\mu_{x_1} - k_1\mu_{x_2} + C_1e^{-2\kappa_1}(2\kappa_{1,x_2}\mu + \mu_{x_2})}{\mu},\\
k_{3,y_1} &= \frac{k_5\mu_{x_1} + k_1\mu_{y_1}}{\mu},\\
k_{3,y_2} &= \frac{k_4\mu_{x_1} + k_1\mu_{y_2}}{\mu},\\
k_{4,x_1} &= -\frac{(k_4\mu_{x_1} + k_1\mu_{y_2})}{\mu},\\
k_{4,x_2} &= -\frac{(k_4\mu_{x_2} + k_2\mu_{y_2})}{\mu},\\
k_{4,y_1} &= \frac{k_4\kappa_{2,y_1}\mu + k_5\kappa_{2,y_2}\mu - k_4\mu_{y_1} + (-k_5 + C_2e^{-2\kappa_2})\mu_{y_2}}{\mu},\\
k_{4,y_2} &= \frac{e^{2\kappa_2}n_{11} - k_5\kappa_{2,y_1}\mu + k_4\kappa_{2,y_2}\mu + C_2e^{-2\kappa_2}\mu_{y_1} - 2k_4\mu_{y_2}}{\mu},\\
k_{5,x_1} &= -\frac{(k_5\mu_{x_1} + k_1\mu_{y_1})}{\mu},\\
k_{5,x_2} &= -\frac{(k_5\mu_{x_2} + k_2\mu_{y_1})}{\mu},\\
k_{5,y_1} &= \frac{e^{2\kappa_2}n_{11} + k_5\kappa_{2,y_1}\mu - k_4\kappa_{2,y_2}\mu - (2k_5 + C_2e^{-2\kappa_2})\mu_{y_1}}{\mu},\\
k_{5,y_2} &= \frac{k_4\kappa_{2,y_1}\mu + k_5\kappa_{2,y_2}\mu - k_4\mu_{y_1} + (-k_5 + C_2e^{-2\kappa_2})\mu_{y_2}}{\mu},\\
k_{6,x_1} &= \frac{k_5\mu_{x_1} + k_1\mu_{y_1}}{\mu},\\
k_{6,x_2} &= \frac{k_5\mu_{x_2} + k_2\mu_{y_1}}{\mu},
\end{aligned}
$$

$$k_{6,y_1} = \frac{-k_5\kappa_{2,y_1}\mu - 2C_2e^{-2\kappa_2}\kappa_{2,y_1}\mu + k_4\kappa_{2,y_2}\mu - e^{2\kappa_2}n_{11} + (2k_5 + C_2e^{-2\kappa_2})\mu_{y_1}}{\mu},$$
$$k_{6,y_2} = -\frac{k_4\kappa_{2,y_1}\mu + k_5\kappa_{2,y_2}\mu - k_4\mu_{y_1} - k_5\mu_{y_2} + C_2e^{-2\kappa_2}(2\kappa_{2,y_2}\mu + \mu_{y_2})}{\mu},$$

where n_{11} is expressed by

$$n_{11} = -e^{-2\kappa_1}d_{11} + e^{-2\kappa_1}k_1\kappa_{1,x_1}\mu - e^{-2\kappa_1}k_2\kappa_{1,x_2}\mu + e^{-2\kappa_1}(3k_1 + k_3)\mu_{x_1} + 2e^{-2\kappa_1}k_2\mu_{x_2}$$

or

$$n_{11} = -e^{-2\kappa_2}m_{11} + e^{-2\kappa_2}k_5\kappa_{2,y_1}\mu - e^{-2\kappa_2}k_4\kappa_{2,y_2}\mu + e^{-2\kappa_2}(3k_5 + k_6)\mu_{y_1} + 2e^{-2\kappa_2}k_4\mu_{y_2}$$

such that d_{11} and m_{11} are functions.

Remark 4.5. Because of equations (4.5)–(4.8) and the paper [32], M_1^2 and M_2^2 correspond to equiaffine spheres.

It can be easily verified that

$$\begin{aligned}
n_{11,x_1} &= \frac{e^{-2(\kappa_1+\kappa_2)}}{2\mu}\big(k_1e^{2\kappa_2}\big((k_1^2+k_2^2)\mu^2 + e^{2\kappa_1}(-\epsilon + C\mu^2) - \mu_{x_1}^2 - \mu_{x_2}^2\big)\\
&\quad + k_1e^{2\kappa_1}\big((k_4^2+k_5^2)\mu^2 - \mu_{y_1}^2 - \mu_{y_2}^2\big)\big),\\
n_{11,x_2} &= \frac{e^{-2(\kappa_1+\kappa_2)}}{2\mu}\big(k_2e^{2\kappa_2}\big((k_1^2+k_2^2)\mu^2 + e^{2\kappa_1}(-\epsilon + C\mu^2) - \mu_{x_1}^2 - \mu_{x_2}^2\big)\\
&\quad + k_2e^{2\kappa_1}\big((k_4^2+k_5^2)\mu^2 - \mu_{y_1}^2 - \mu_{y_2}^2\big)\big),\\
n_{11,y_1} &= \frac{e^{-2(\kappa_1+\kappa_2)}}{2\mu}\big(-k_5e^{2\kappa_2}\big(-(k_1^2+k_2^2)\mu^2 + e^{2\kappa_1}(\epsilon + C\mu^2) + \mu_{x_1}^2 + \mu_{x_2}^2\big)\\
&\quad + k_5e^{2\kappa_1}\big((k_4^2+k_5^2)\mu^2 - \mu_{y_1}^2 - \mu_{y_2}^2\big)\big),\\
n_{11,y_2} &= \frac{e^{-2(\kappa_1+\kappa_2)}}{2\mu}\big(-k_4e^{2\kappa_2}\big(-(k_1^2+k_2^2)\mu^2 + e^{2\kappa_1}(\epsilon + C\mu^2) + \mu_{x_1}^2 + \mu_{x_2}^2\big)\\
&\quad + k_4e^{2\kappa_1}\big((k_4^2+k_5^2)\mu^2 - \mu_{y_1}^2 - \mu_{y_2}^2\big)\big).
\end{aligned}$$

We can write

$$\begin{aligned}
n_{11,x_1} &= k_1\left(\frac{C\mu}{2} + r_{11}\right), & n_{11,x_2} &= k_2\left(\frac{C\mu}{2} + r_{11}\right),\\
n_{11,y_1} &= k_5\left(-\frac{C\mu}{2} + r_{11}\right), & n_{11,y_2} &= k_4\left(-\frac{C\mu}{2} + r_{11}\right),
\end{aligned}$$

where

$$\begin{aligned}
r_{11} = \frac{e^{-2(\kappa_1+\kappa_2)}}{2\mu}&\big(e^{2\kappa_2}\big((k_1^2+k_2^2)\mu^2 + e^{2\kappa_1}(-\epsilon) - \mu_{x_1}^2 - \mu_{x_2}^2\big)\\
&+ e^{2\kappa_1}\big((k_4^2+k_5^2)\mu^2 - \mu_{y_1}^2 - \mu_{y_2}^2\big)\big).
\end{aligned}$$

Moreover, it can be easily seen that

$$r_{11,x_1} = k_1 n_{11} - \frac{1}{2} C\mu_{x_1}, \qquad r_{11,x_2} = k_2 n_{11} - \frac{1}{2} C\mu_{x_2},$$
$$r_{11,y_1} = k_5 n_{11} + \frac{1}{2} C\mu_{y_1}, \qquad r_{11,y_2} = k_4 n_{11} + \frac{1}{2} C\mu_{y_2}.$$

On the other hand, a straightforward computation yields

$$\begin{aligned}
\mu_{x_1x_1} &= \frac{1}{2\mu} e^{-2(\kappa_1+\kappa_2)} \big(- 2k_1 C_1 e^{2\kappa_2} \mu^2 \\
&\quad + e^{2(\kappa_1+\kappa_2)} \big(- (3k_1^2 + k_2^2)\mu^2 + 2\kappa_{1,x_1}\mu\mu_{x_1} + \mu_{x_1}^2 - 2\kappa_{1,x_2}\mu\mu_{x_2} + \mu_{x_2}^2 \big) \\
&\quad + e^{4\kappa_1} \big(- (k_4^2 + k_5^2)\mu^2 + e^{2\kappa_2}(\epsilon + C\mu^2) + \mu_{y_1}^2 + \mu_{y_2}^2 \big) \big), \\
\mu_{x_1x_2} &= k_2(-k_1 + C_1 e^{-2\kappa_1})\mu + \kappa_{1,x_2}\mu_{x_1} + \kappa_{1,x_1}\mu_{x_2}, \\
\mu_{x_1y_1} &= -k_1 k_5 \mu, \\
\mu_{x_1y_2} &= -k_1 k_4 \mu, \\
\mu_{x_2x_2} &= \frac{1}{2\mu} e^{-2(\kappa_1+\kappa_2)} \big(2k_1 C_1 e^{2\kappa_2} \mu^2 \\
&\quad + e^{2(\kappa_1+\kappa_2)} \big(- (k_1^2 + 3k_2^2)\mu^2 - 2\kappa_{1,x_1}\mu\mu_{x_1} + \mu_{x_1}^2 + 2\kappa_{1,x_2}\mu\mu_{x_2} + \mu_{x_2}^2 \big) \\
&\quad + e^{4\kappa_1} \big(- (k_4^2 + k_5^2)\mu^2 + e^{2\kappa_2}(\epsilon + C\mu^2) + \mu_{y_1}^2 + \mu_{y_2}^2 \big) \big), \\
\mu_{x_2y_1} &= -k_2 k_5 \mu, \\
\mu_{x_2y_2} &= -k_2 k_4 \mu, \\
\mu_{y_1y_1} &= \frac{1}{2\mu} e^{-2(\kappa_1+\kappa_2)} \big(- 2k_5 C_2 e^{2\kappa_1} \mu^2 \\
&\quad + e^{4\kappa_2} \big(- (k_1^2 + k_2^2)\mu^2 + e^{2\kappa_1}(\epsilon - C\mu^2) + \mu_{x_1}^2 + \mu_{x_2}^2 \big) \\
&\quad + e^{2(\kappa_1+\kappa_2)} \big(- (k_4^2 + 3k_5^2)\mu^2 + 2\kappa_{2,y_1}\mu\mu_{y_1} + \mu_{y_1}^2 - 2\kappa_{2,y_2}\mu\mu_{y_2} + \mu_{y_2}^2 \big) \big), \\
\mu_{y_1y_2} &= k_4 \big(- k_5 + C_2 e^{-2\kappa_2} \big)\mu + \kappa_{2,y_2}\mu_{y_1} + \kappa_{2,y_1}\mu_{y_2}, \\
\mu_{y_2y_2} &= \frac{1}{2\mu} e^{-2(\kappa_1+\kappa_2)} \big(2k_5 C_2 e^{2\kappa_1} \mu^2 + e^{4\kappa_2} \big(- (k_1^2 + k_2^2)\mu^2 + e^{2\kappa_1}(\epsilon - C\mu^2) + \mu_{x_1}^2 + \mu_{x_2}^2 \big) \\
&\quad + e^{2(\kappa_1+\kappa_2)} \big(- (3k_4^2 + k_5^2)\mu^2 - 2\kappa_{2,y_1}\mu\mu_{y_1} + \mu_{y_1}^2 + 2\kappa_{2,y_2}\mu\mu_{y_2} + \mu_{y_2}^2 \big) \big).
\end{aligned}$$

We now define

$$\begin{aligned}
\mathfrak{f}_1 &= e^{-2\kappa_1}(k_1\mu - \mu_{x_1})(k_1\mu + \mu_{x_1}), \\
\mathfrak{f}_2 &= e^{-2\kappa_1}(k_2\mu - \mu_{x_2})(k_2\mu + \mu_{x_2}), \\
\mathfrak{f}_3 &= \left(n_{11} - r_{11} + \frac{1}{2} C\mu \right)(k_1\mu - \mu_{x_1}), \\
\mathfrak{f}_4 &= e^{-2\kappa_1}(k_1\mu - \mu_{x_1})(k_2\mu + \mu_{x_2}), \\
\mathfrak{f}_5 &= e^{-2\kappa_1}(k_2\mu - \mu_{x_2})(k_1\mu + \mu_{x_1}),
\end{aligned}$$

$$
\begin{aligned}
\mathfrak{f}_6 &= \left(n_{11} - r_{11} + \frac{1}{2}C\mu\right)(k_2\mu - \mu_{x_2}),\\
\mathfrak{f}_7 &= e^{2\kappa_1}\left(n_{11} - r_{11} + \frac{1}{2}C\mu\right)\left(n_{11} + r_{11} - \frac{1}{2}C\mu\right),\\
\mathfrak{f}_8 &= \left(n_{11} + r_{11} - \frac{1}{2}C\mu\right)(k_1\mu + \mu_{x_1}),\\
\mathfrak{f}_9 &= \left(n_{11} + r_{11} - \frac{1}{2}C\mu\right)(k_2\mu + \mu_{x_2}),\\
\mathfrak{g}_1 &= e^{-2\kappa_2}(k_5\mu - \mu_{y_1})(k_5\mu + \mu_{y_1}),\\
\mathfrak{g}_2 &= e^{-2\kappa_2}(k_4\mu - \mu_{y_2})(k_4\mu + \mu_{y_2}),\\
\mathfrak{g}_3 &= \left(n_{11} - r_{11} - \frac{1}{2}C\mu\right)(k_5\mu - \mu_{y_1}),\\
\mathfrak{g}_4 &= e^{-2\kappa_2}(k_5\mu + \mu_{y_1})(k_4\mu - \mu_{y_2}),\\
\mathfrak{g}_5 &= e^{-2\kappa_2}(k_5\mu - \mu_{y_1})(k_4\mu + \mu_{y_2}),\\
\mathfrak{g}_6 &= \left(n_{11} - r_{11} - \frac{1}{2}C\mu\right)(k_4\mu - \mu_{y_2}),\\
\mathfrak{g}_7 &= e^{2\kappa_2}\left(n_{11} - r_{11} - \frac{1}{2}C\mu\right)\left(n_{11} + r_{11} + \frac{1}{2}C\mu\right),\\
\mathfrak{g}_8 &= \left(n_{11} + r_{11} + \frac{1}{2}C\mu\right)(k_5\mu + \mu_{y_1}),\\
\mathfrak{g}_9 &= \left(n_{11} + r_{11} + \frac{1}{2}C\mu\right)(k_4\mu + \mu_{y_2})
\end{aligned}
$$

such that

$$
\begin{aligned}
k_1 &= \frac{e^{2\kappa_1}\big(\mathfrak{f}_3(-2(n_{11} + r_{11}) + C\mu) - \mathfrak{f}_8(2n_{11} - 2r_{11} + C\mu)\big)}{-4\mu\mathfrak{f}_7},\\
k_2 &= \frac{e^{2\kappa_1}\big(\mathfrak{f}_6(-2(n_{11} + r_{11}) + C\mu) - \mathfrak{f}_9(2n_{11} - 2r_{11} + C\mu)\big)}{-4\mu\mathfrak{f}_7},\\
k_4 &= \frac{e^{2\kappa_2}\big(\mathfrak{g}_9(-2n_{11} + 2r_{11} + C\mu) - \mathfrak{g}_6(2(n_{11} + r_{11}) + C\mu)\big)}{-4\mu\mathfrak{g}_7},\\
k_5 &= \frac{e^{2\kappa_2}\big(\mathfrak{g}_8(-2n_{11} + 2r_{11} + C\mu) - \mathfrak{g}_3(2(n_{11} + r_{11}) + C\mu)\big)}{-4\mu\mathfrak{g}_7},\\
\mu_{x_1} &= \frac{e^{2\kappa_1}\big(\mathfrak{f}_8(2n_{11} - 2r_{11} + C\mu) - \mathfrak{f}_3(2(n_{11} + r_{11}) - C\mu)\big)}{4\mathfrak{f}_7},\\
\mu_{x_2} &= \frac{e^{2\kappa_1}\big(\mathfrak{f}_9(2n_{11} - 2r_{11} + C\mu) - \mathfrak{f}_6(2(n_{11} + r_{11}) - C\mu)\big)}{4\mathfrak{f}_7},\\
\mu_{y_1} &= \frac{e^{2\kappa_2}\big(\mathfrak{g}_3(2(n_{11} + r_{11}) + C\mu) + \mathfrak{g}_8(-2n_{11} + 2r_{11} + C\mu)\big)}{-4\mathfrak{g}_7},\\
\mu_{y_2} &= \frac{e^{2\kappa_2}\big(\mathfrak{g}_6(2(n_{11} + r_{11}) + C\mu) + \mathfrak{g}_9(-2n_{11} + 2r_{11} + C\mu)\big)}{-4\mathfrak{g}_7}.
\end{aligned}
$$

We point out that whereas $\mathfrak{f}_1$, $\mathfrak{f}_2$, $\mathfrak{f}_3$, $\mathfrak{f}_4$, $\mathfrak{f}_5$, $\mathfrak{f}_6$, $\mathfrak{f}_7$, $\mathfrak{f}_8$ and $\mathfrak{f}_9$ depend only on x_1 and x_2; $\mathfrak{g}_1$, $\mathfrak{g}_2$, $\mathfrak{g}_3$, $\mathfrak{g}_4$, $\mathfrak{g}_5$, $\mathfrak{g}_6$, $\mathfrak{g}_7$, $\mathfrak{g}_8$ and $\mathfrak{g}_9$ depend only on y_1 and y_2. Moreover, we define two maps F_1 and F_2 by

$$
\begin{aligned}
F_1\colon \quad M_1^2 &\longrightarrow N\\
(x_1,x_2) &\longmapsto \begin{pmatrix} \mathfrak{f}_1 & \mathfrak{f}_5 & \mathfrak{f}_8\\ \mathfrak{f}_4 & \mathfrak{f}_2 & \mathfrak{f}_9\\ \mathfrak{f}_3 & \mathfrak{f}_6 & \mathfrak{f}_7 \end{pmatrix}
\end{aligned}
\tag{4.9}
$$

and

$$
\begin{aligned}
F_2\colon \quad M_2^2 &\longrightarrow N\\
(y_1,y_2) &\longmapsto \begin{pmatrix} \mathfrak{g}_1 & \mathfrak{g}_4 & \mathfrak{g}_8\\ \mathfrak{g}_5 & \mathfrak{g}_2 & \mathfrak{g}_9\\ \mathfrak{g}_3 & \mathfrak{g}_6 & \mathfrak{g}_7 \end{pmatrix},
\end{aligned}
\tag{4.10}
$$

where $N \subset \mathbb{R}^{3\times 3}$ is the space of 3×3 matrices of rank 1 over $\mathbb{R}$.

Note that F_1 is determined by

$$
\begin{aligned}
\frac{\partial \mathfrak{f}_1}{\partial x_1} &= \mathfrak{f}_3 - \kappa_{1,x_2}\mathfrak{f}_4 - \kappa_{1,x_2}\mathfrak{f}_5 + \mathfrak{f}_8,\\
\frac{\partial \mathfrak{f}_1}{\partial x_2} &= (\kappa_{1,x_1} + C_1 e^{-2\kappa_1})\mathfrak{f}_4 + (\kappa_{1,x_1} - C_1 e^{-2\kappa_1})\mathfrak{f}_5,\\
\frac{\partial \mathfrak{f}_2}{\partial x_1} &= \kappa_{1,x_2}\mathfrak{f}_4 + \kappa_{1,x_2}\mathfrak{f}_5,\\
\frac{\partial \mathfrak{f}_2}{\partial x_2} &= -(\kappa_{1,x_1} + C_1 e^{-2\kappa_1})\mathfrak{f}_4 - (\kappa_{1,x_1} - C_1 e^{-2\kappa_1})\mathfrak{f}_5 + \mathfrak{f}_6 + \mathfrak{f}_9,\\
\frac{\partial \mathfrak{f}_3}{\partial x_1} &= Ce^{2\kappa_1}\mathfrak{f}_1 + (\kappa_{1,x_1} + C_1 e^{-2\kappa_1})\mathfrak{f}_3 - \kappa_{1,x_2}\mathfrak{f}_6 + \mathfrak{f}_7,\\
\frac{\partial \mathfrak{f}_3}{\partial x_2} &= \kappa_{1,x_2}\mathfrak{f}_3 + Ce^{2\kappa_1}\mathfrak{f}_4 + (\kappa_{1,x_1} - C_1 e^{-2\kappa_1})\mathfrak{f}_6,\\
\frac{\partial \mathfrak{f}_4}{\partial x_1} &= \kappa_{1,x_2}\mathfrak{f}_1 - \kappa_{1,x_2}\mathfrak{f}_2 + 2C_1 e^{-2\kappa_1}\mathfrak{f}_4 + \mathfrak{f}_9,\\
\frac{\partial \mathfrak{f}_4}{\partial x_2} &= (-\kappa_{1,x_1} + C_1 e^{-2\kappa_1})\mathfrak{f}_1 + (\kappa_{1,x_1} - C_1 e^{-2\kappa_1})\mathfrak{f}_2 + \mathfrak{f}_3,\\
\frac{\partial \mathfrak{f}_5}{\partial x_1} &= \kappa_{1,x_2}\mathfrak{f}_1 - \kappa_{1,x_2}\mathfrak{f}_2 - 2C_1 e^{-2\kappa_1}\mathfrak{f}_5 + \mathfrak{f}_6,\\
\frac{\partial \mathfrak{f}_5}{\partial x_2} &= -(\kappa_{1,x_1} + C_1 e^{-2\kappa_1})\mathfrak{f}_1 + (\kappa_{1,x_1} + C_1 e^{-2\kappa_1})\mathfrak{f}_2 + \mathfrak{f}_8,\\
\frac{\partial \mathfrak{f}_6}{\partial x_1} &= \kappa_{1,x_2}\mathfrak{f}_3 + Ce^{2\kappa_1}\mathfrak{f}_5 + (\kappa_{1,x_1} - C_1 e^{-2\kappa_1})\mathfrak{f}_6,\\
\frac{\partial \mathfrak{f}_6}{\partial x_2} &= Ce^{2\kappa_1}\mathfrak{f}_2 - (\kappa_{1,x_1} + C_1 e^{-2\kappa_1})\mathfrak{f}_3 + \kappa_{1,x_2}\mathfrak{f}_6 + \mathfrak{f}_7,
\end{aligned}
$$

$$\frac{\partial \mathfrak{f}_7}{\partial x_1} = Ce^{2\kappa_1}\mathfrak{f}_3 + 2\kappa_{1,x_1}\mathfrak{f}_7 + Ce^{2\kappa_1}\mathfrak{f}_8,$$
$$\frac{\partial \mathfrak{f}_7}{\partial x_2} = Ce^{2\kappa_1}\mathfrak{f}_6 + 2\kappa_{1,x_2}\mathfrak{f}_7 + Ce^{2\kappa_1}\mathfrak{f}_9,$$
$$\frac{\partial \mathfrak{f}_8}{\partial x_1} = Ce^{2\kappa_1}\mathfrak{f}_1 + \mathfrak{f}_7 + (\kappa_{1,x_1} - C_1e^{-2\kappa_1})\mathfrak{f}_8 - \kappa_{1,x_2}\mathfrak{f}_9,$$
$$\frac{\partial \mathfrak{f}_8}{\partial x_2} = Ce^{2\kappa_1}\mathfrak{f}_5 + \kappa_{1,x_2}\mathfrak{f}_8 + (\kappa_{1,x_1} + C_1e^{-2\kappa_1})\mathfrak{f}_9,$$
$$\frac{\partial \mathfrak{f}_9}{\partial x_1} = Ce^{2\kappa_1}\mathfrak{f}_4 + \kappa_{1,x_2}\mathfrak{f}_8 + (\kappa_{1,x_1} + C_1e^{-2\kappa_1})\mathfrak{f}_9,$$
$$\frac{\partial \mathfrak{f}_9}{\partial x_2} = Ce^{2\kappa_1}\mathfrak{f}_2 + \mathfrak{f}_7 + (-\kappa_{1,x_1} + C_1e^{-2\kappa_1})\mathfrak{f}_8 + \kappa_{1,x_2}\mathfrak{f}_9.$$

Furthermore, F_2 is determined by

$$\frac{\partial \mathfrak{g}_1}{\partial y_1} = \mathfrak{g}_3 - \kappa_{2,y_2}\mathfrak{g}_4 - \kappa_{2,y_2}\mathfrak{g}_5 + \mathfrak{g}_8,$$
$$\frac{\partial \mathfrak{g}_1}{\partial y_2} = (\kappa_{2,y_1} - C_2e^{-2\kappa_2})\mathfrak{g}_4 + (\kappa_{2,y_1} + C_2e^{-2\kappa_2})\mathfrak{g}_5,$$
$$\frac{\partial \mathfrak{g}_2}{\partial y_1} = \kappa_{2,y_2}\mathfrak{g}_4 + \kappa_{2,y_2}\mathfrak{g}_5,$$
$$\frac{\partial \mathfrak{g}_2}{\partial y_2} = (-\kappa_{2,y_1} + C_2e^{-2\kappa_2})\mathfrak{g}_4 - (\kappa_{2,y_1} + C_2e^{-2\kappa_2})\mathfrak{g}_5 + \mathfrak{g}_6 + \mathfrak{g}_9,$$
$$\frac{\partial \mathfrak{g}_3}{\partial y_1} = -Ce^{2\kappa_2}\mathfrak{g}_1 + (\kappa_{2,y_1} + C_2e^{-2\kappa_2})\mathfrak{g}_3 - \kappa_{2,y_2}\mathfrak{g}_6 + \mathfrak{g}_7,$$
$$\frac{\partial \mathfrak{g}_3}{\partial y_2} = \kappa_{2,y_2}\mathfrak{g}_3 - Ce^{2\kappa_2}\mathfrak{g}_5 + (\kappa_{2,y_1} - C_2e^{-2\kappa_2})\mathfrak{g}_6,$$
$$\frac{\partial \mathfrak{g}_4}{\partial y_1} = \kappa_{2,y_2}\mathfrak{g}_1 - \kappa_{2,y_2}\mathfrak{g}_2 - 2C_2e^{-2\kappa_2}\mathfrak{g}_4 + \mathfrak{g}_6,$$
$$\frac{\partial \mathfrak{g}_4}{\partial y_2} = -(\kappa_{2,y_1} + C_2e^{-2\kappa_2})\mathfrak{g}_1 + (\kappa_{2,y_1} + C_2e^{-2\kappa_2})\mathfrak{g}_2 + \mathfrak{g}_8,$$
$$\frac{\partial \mathfrak{g}_5}{\partial y_1} = \kappa_{2,y_2}\mathfrak{g}_1 - \kappa_{2,y_2}\mathfrak{g}_2 + 2C_2e^{-2\kappa_2}\mathfrak{g}_5 + \mathfrak{g}_9,$$
$$\frac{\partial \mathfrak{g}_5}{\partial y_2} = (-\kappa_{2,y_1} + C_2e^{-2\kappa_2})\mathfrak{g}_1 + (\kappa_{2,y_1} - C_2e^{-2\kappa_2})\mathfrak{g}_2 + \mathfrak{g}_3,$$
$$\frac{\partial \mathfrak{g}_6}{\partial y_1} = \kappa_{2,y_2}\mathfrak{g}_3 - Ce^{2\kappa_2}\mathfrak{g}_4 + (\kappa_{2,y_1} - C_2e^{-2\kappa_2})\mathfrak{g}_6,$$
$$\frac{\partial \mathfrak{g}_6}{\partial y_2} = -Ce^{2\kappa_2}\mathfrak{g}_2 - (\kappa_{2,y_1} + C_2e^{-2\kappa_2})\mathfrak{g}_3 + \kappa_{2,y_2}\mathfrak{g}_6 + \mathfrak{g}_7,$$
$$\frac{\partial \mathfrak{g}_7}{\partial y_1} = -Ce^{2\kappa_2}\mathfrak{g}_3 + 2\kappa_{2,y_1}\mathfrak{g}_7 - Ce^{2\kappa_2}\mathfrak{g}_8,$$
$$\frac{\partial \mathfrak{g}_7}{\partial y_2} = -Ce^{2\kappa_2}\mathfrak{g}_6 + 2\kappa_{2,y_2}\mathfrak{g}_7 - Ce^{2\kappa_2}\mathfrak{g}_9,$$
$$\frac{\partial \mathfrak{g}_8}{\partial y_1} = -Ce^{2\kappa_2}\mathfrak{g}_1 + \mathfrak{g}_7 + (\kappa_{2,y_1} - C_2e^{-2\kappa_2})\mathfrak{g}_8 - \kappa_{2,y_2}\mathfrak{g}_9,$$

$$\frac{\partial \mathfrak{g}_8}{\partial y_2} = -Ce^{2\kappa_2}\mathfrak{g}_4 + \kappa_{2,y_2}\mathfrak{g}_8 + (\kappa_{2,y_1} + C_2 e^{-2\kappa_2})\mathfrak{g}_9,$$
$$\frac{\partial \mathfrak{g}_9}{\partial y_1} = -Ce^{2\kappa_2}\mathfrak{g}_5 + \kappa_{2,y_2}\mathfrak{g}_8 + (\kappa_{2,y_1} + C_2 e^{-2\kappa_2})\mathfrak{g}_9,$$
$$\frac{\partial \mathfrak{g}_9}{\partial y_2} = -Ce^{2\kappa_2}\mathfrak{g}_2 + \mathfrak{g}_7 - (\kappa_{2,y_1} - C_2 e^{-2\kappa_2})\mathfrak{g}_8 + \kappa_{2,y_2}\mathfrak{g}_9.$$

It is easy to check that these derivatives preserve the rank one conditions as well as the conditions

$$-\mathfrak{f}_7 e^{-2\kappa_1} + C\mathfrak{f}_1 + C\mathfrak{f}_2 = D_1, \quad \mathfrak{g}_7 e^{-2\kappa_2} + C\mathfrak{g}_1 + C\mathfrak{g}_2 = D_2,$$

where $D_1 + D_2 - C\epsilon = 0$. We point out that if we take the initial conditions in N, the solutions will remain in N.

Conversely, starting with two equiaffine spheres (see [32]) and 2 maps in the space of 3×3 matrices of rank 1 over $\mathbb{R}$ with suitable initial conditions, we obtain

$$\begin{aligned}(4.11)\qquad &(k_1\mu + \mu_{x_1})^2 + (k_2\mu + \mu_{x_2})^2 + e^{4\kappa_1}\left(n_{11} - r_{11} + \frac{1}{2}C\mu\right)^2 \\ &= (k_1\mu - \mu_{x_1})^2 + (k_2\mu - \mu_{x_2})^2 + e^{4\kappa_1}\left(n_{11} + r_{11} - \frac{1}{2}C\mu\right)^2\end{aligned}$$

and

$$\begin{aligned}(4.12)\qquad &(k_5\mu + \mu_{y_1})^2 + (k_4\mu + \mu_{y_2})^2 + e^{4\kappa_2}\left(n_{11} - r_{11} - \frac{1}{2}C\mu\right)^2 \\ &= (k_5\mu - \mu_{y_1})^2 + (k_4\mu - \mu_{y_2})^2 + e^{4\kappa_2}\left(n_{11} + r_{11} + \frac{1}{2}C\mu\right)^2.\end{aligned}$$

Applying the existence and uniqueness theorem gives an ideal definite $\delta^\sharp(2,2)$-immersion of type (1i). As a result, we have the following corollary:

Corollary 4.6. *Let $F\colon M^4 \to \mathbb{R}^5$ be a definite centroaffine hypersurface. It is $\delta^\sharp(2,2)$-ideal and of type* (1i) *if and only if*

$$M^4 = \frac{1}{\mu}(M_1^2 \times M_2^2),$$

where $M_1^2 \subset \mathbb{R}^3$ is a surface with metric $\langle \cdot , \cdot \rangle = e^{2\kappa_1}(dx_1^2 + dx_2^2)$ and $M_2^2 \subset \mathbb{R}^3$ is a surface with metric $\langle \cdot , \cdot \rangle = e^{2\kappa_2}(dy_1^2 + dy_2^2)$ such that F_1 is a map given in (4.9) *and F_2 is a map given in* (4.10) *with suitable initial conditions.*

Proof. We have already seen that if M^4 is a $\delta^\sharp(2,2)$-ideal definite centroaffine hypersurface of type (1i), we have the twisted product decomposition and the corresponding maps F_1 and F_2. Conversely, given a twisted product decomposition and such maps, it follows from the definitions of $\mathfrak{f}_l$ and $\mathfrak{g}_l$, where $l = 1, \ldots, 9$, that we can determine functions k_1, k_2, k_4, k_5, μ_{x_1}, μ_{x_2}, μ_{y_1}, μ_{y_2}, n_{11} and r_{11} (by means of $\mathfrak{f}_l$ and $\mathfrak{g}_l$ and therefore in terms of F_1 and F_2). Moreover, equations (4.11) and (4.12) show that these functions satisfy the differential equations for k_{i,x_j}, k_{i,y_j}, n_{11,x_j}, n_{11,y_j}, r_{11,x_j}, r_{11,y_j}, where $i = 1, \ldots, 6$ and $j = 1, 2$, as well as the second order differential equations for the function μ. Applying then the existence and uniqueness theorem for the definite centroaffine hypersurface, we obtain the desired result. □

Acknowledgments

This work was supported by the Scientific Research Projects Coordination Unit of Istanbul University with the project numbered 33525.

References

[1] P. Alegre, B.-Y. Chen and M. I. Munteanu, *Riemannian submersions, δ-invariants, and optimal inequality*, Ann. Global Anal. Geom. **42** (2012), no. 3, 317–331.

[2] T. Binder, *MathSciNet review of "B.-Y. Chen, An optimal inequality and extremal classes of affine spheres in centroaffine geometry, Geom. Dedicate 111 (2005), 187–210"*, MR2155181 (2006d:53011).

[3] J. Bolton, F. Dillen, J. Fastenakels and L. Vrancken, *A best possible inequality for curvature-like tensor fields*, Math. Inequal. Appl. **12** (2009), no. 3, 663–681.

[4] J. Bolton and L. Vrancken, *Lagrangian submanifolds attaining equality in the improved Chen's inequality*, Bull. Belg. Math. Soc. Simon Stevin **14** (2007), no. 2, 311–315.

[5] B.-Y. Chen, *Some pinching and classification theorems for minimal submanifolds*, Arch. Math. (Basel) **60** (1993), no. 6, 568–578.

[6] ______, *A Riemannian invariant and its applications to submanifold theory*, Results Math. **27** (1995), no. 1-2, 17–26.

[7] ______, *Strings of Riemannian invariants, inequalities, ideal immersions and their applications*, in: *The Third Pacific Rim Geometry Conference (Seoul, 1996)*, 7–60, Monogr. Geom. Topology **25**, International Press, Cambridge, MA, 1998.

[8] ______, *Riemannian DNA, inequalities and their applications*, J. Appl. Sci. Eng. **3** (2000), no. 3, 123–130.

[9] ______, *Some new obstructions to minimal and Lagrangian isometric immersions*, Japan. J. Math. (N.S.) **26** (2000), no. 1, 105–127.

[10] ______, *An optimal inequality and extremal classes of affine spheres in centroaffine geometry*, Geom. Dedicate **111** (2005), 187–210.

[11] ______, *δ-invariants, inequalities of submanifolds and their applications*, in: *Topics in Differential Geometry*, 29–155, Editura Academiei Române, Bucharest, 2008.

[12] ______, *Pseudo-Riemannian Geometry, δ-Invariants and Applications*, World Scientific Publishing, Hackensack, NJ, 2011.

[13] B.-Y. Chen, F. Dillen, J. Van der Veken and L. Vrancken, *Curvature inequalities for Lagrangian submanifolds: The final solution*, Differential Geom. Appl. **31** (2013), no. 6, 808–819.

[14] ______, *Classification of $\delta(2, n-2)$-ideal Lagrangian submanifolds in n-dimensional complex space forms*, J. Math. Anal. Appl. **458** (2018), no. 2, 1456–1485.

[15] B.-Y. Chen, F. Dillen, L. Verstraelen and L. Vrancken, *Characterizations of Riemannian space forms, Einstein spaces and conformally flat spaces*, Proc. Amer. Math. Soc. **128** (2000), no. 2, 589–598.

[16] B.-Y. Chen and A. Prieto-Martín, *Classification of Lagrangian submanifolds in complex space forms satisfying a basic equality involving $\delta(2, 2)$*, Differential Geom. Appl. **30** (2012), no. 1, 107–123.

[17] B.-Y. Chen, A. Prieto-Martín and X. Wang, *Lagrangian submanifolds in complex space forms satisfying an improved equality involving $\delta(2, 2)$*, Publ. Math. Debrecen **82** (2013), no. 1, 193–217.

[18] B.-Y. Chen, L. Vrancken and X. Wang, *Lagrangian submanifolds in complex space forms satisfying equality in the optimal inequality involving $\delta(2, \dots, 2)$*, Beitr. Algebra Geom. **62** (2021), no. 1, 251–264.

[19] B.-Y. Chen and H. Yıldırım, *Classification of ideal submanifolds of real space forms with type number* ≤ 2, J. Geom. Phys. **92** (2015), 167–180.

[20] X. Cheng and Z. Hu, *An optimal inequality on locally strongly convex centroaffine hypersurfaces*, J. Geom. Anal. **28** (2018), no. 1, 643–655.

[21] X. Cheng, Z. Hu and M. Moruz, *Classification of the locally strongly convex centroaffine hypersurfaces with parallel cubic form*, Results Math. **72** (2017), no. 1-2, 419–469.

[22] V. Cortés, M. Nardmann and S. Suhr, *Completeness of hyperbolic centroaffine hypersurfaces*, Comm. Anal. Geom. **24** (2016), no. 1, 59–92.

[23] F. Dillen, M. Petrovic and L. Verstraelen, *Einstein, conformally flat and semi-symmetric submanifolds satisfying Chen's equality*, Israel J. Math. **100** (1997), 163–169.

[24] F. Dillen, C. Scharlach, K. Schoels and L. Vrancken, *Special Lagrangian* 4*-folds with* $SO(2) \rtimes S_3$*-symmetry in complex space forms*, Taiwanese J. Math. **19** (2015), no. 3, 759–792.

[25] F. Dillen and L. Vrancken, *Totally real submanifolds in* $S^6(1)$ *satisfying Chen's equality*, Trans. Amer. Math. Soc. **348** (1996), no. 4, 1633–1646.

[26] M. Djorić and L. Vrancken, *Three-dimensional minimal CR submanifolds in* S^6 *satisfying Chen's equality*, J. Geom. Phys. **56** (2006), no. 11, 2279–2288.

[27] M. Kriele, C. Scharlach and L. Vrancken, *An extremal class of* 3*-dimensional elliptic affine spheres*, Hokkaido Math. J. **30** (2001), no. 1, 1–23.

[28] M. Kriele and L. Vrancken, *An extremal class of three-dimensional hyperbolic affine spheres*, Geom. Dedicate **77** (1999), no. 3, 239–252.

[29] A.-M. Li, H. L. Liu, A. Schwenk-Schellschmidt, U. Simon and C. P. Wang, *Cubic form methods and relative Tchebychev hypersurfaces*, Geom. Dedicate **66** (1997), no. 2, 203–221.

[30] A.-M. Li and C. P. Wang, *Canonical centroaffine hypersurfaces in* $\mathbb{R}^{n+1}$, Results Math. **20** (1991), no. 3-4, 660–681.

[31] H. L. Liu and C. P. Wang, *The centroaffine Tchebychev operator*, Results Math. **27** (1995), no. 1-2, 77–92.

[32] J. Loftin and I. McIntosh, *Cubic differentials in the differential geometry of surfaces*, in: *Handbook of Teichmüller Theory, Vol. VI*, 231–274, IRMA Lect. Math. Theor. Phys. **27**, European Mathematical Society (EMS), Zürich, 2016.

[33] M. Meumertzheim, H. Reckziegel and M. Schaaf, *Decomposition of twisted and warped product nets*, Results Math. **36** (1999), no. 3-4, 297–312.

[34] S. Nölker, *Isometric immersions of warped products*, Differential Geom. Appl. **6** (1996), no. 1, 1–30.

[35] K. Nomizu and T. Sasaki, *Affine Differential Geometry: Geometry of Affine Immersions*, Cambridge Tracts in Math. **111**, Cambridge University Press, Cambridge, 1994.

[36] B. Opozda, *Some relations between Riemannian and affine geometry*, Geom. Dedicate **47** (1993), no. 2, 225–236.

[37] C. Scharlach, U. Simon, L. Verstraelen and L. Vrancken, *A new intrinsic curvature invariant for centroaffine hypersurfaces*, Beiträge Algebra Geom. **38** (1997), no. 2, 437–458.

[38] J. Van der Veken, L. Vrancken and H. Yıldırım, $\delta^\sharp(2, n-2)$*-ideal centroaffine hypersurfaces in* $\mathbb{R}^{n+1}$, in preparation.

[39] H. Yıldırım and L. Vrancken, $\delta^\sharp(2, 2)$*-ideal centroaffine hypersurfaces of dimension* 5, Taiwanese J. Math. **21** (2017), no. 2, 283–304.

Handan Yıldırım
Mathematics Department, Science Faculty, Istanbul University, Vezneciler-Fatih, Istanbul, 34134, Türkiye
E-mail address: `handanyildirim@istanbul.edu.tr`

Luc Vrancken
CERAMATHS-Laboratoire de Matériaux Céramiques et de Mathématiques, Université de Polytechnique Hauts-de-France, Valenciennes, F-59313, France
and
CERAMATHS, INSA Hauts-de-France, Valenciennes, F-59313, France
and
Departement Wiskunde, KU Leuven, Leuven, 3001, Belgium
E-mail address: `luc.vrancken@uphf.fr`