TAIWANESE JOURNAL OF MATHEMATICS
Vol. 27, No. 4, pp. 629–646, August 2023
DOI: 10.11650/tjm/230203

On Derangement Polynomials of Type D

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Abstract. Enumeration of derangements in the symmetric group $\mathfrak{S}_n$ is classical. Extensions of the enumerative results to the hyperoctahedral group B_n are combinatorially sound. That in the even-signed permutation group D_n remains largely unexplored. Let $d_n^D(q) = \sum_{\sigma \in \mathcal{D}_n^D} q^{\mathrm{maj}(\sigma)}$ be the generating function of derangements in D_n by their major indices. We study in this work properties of $d_n^D(q)$, including recurrence relations and factorial generating function. By proving the ratio monotonicity of $d_n^D(q)$, the unimodality, log-concavity and spiral property of $d_n^D(q)$ are also established.

1. Introduction

Enumeration of derangements in the symmetric group $\mathfrak{S}_n$ is classical. The formula of the classical derangement numbers is readily obtained by a routine application of the principle of inclusion-exclusion [13, Chapter 2]. Enumeration of n-derangements by a certain permutation statistic is also well studied. Notable developments along this direction include Brenti [1] and Wachs [14], which enumerates n-derangements by their excedances and major indices, respectively.

Extensions of the enumerative results to other families of groups are combinatorially sound. For instance, Chow [5] enumerates derangements in the hyperoctahedral group B_n by their flag major indices; Chow [7], and Chen et al. [3] independently enumerate derangements in B_n by their number of excedances; Chow [6] enumerates multiderangements by their excedances; Chow and Mansour [8] enumerate derangements in the wreath product $C_r \wr \mathfrak{S}_n$ by their number of excedances. See Section 2 for undefined terms.

Since the symmetric group $\mathfrak{S}_n$ and the hyperoctahedral group B_n are Coxeter groups of types A and B, respectively, from the Coxeter group theoretic point of view, the next direction of development is to count derangements in the even-signed permutation group D_n. However, unlike the types A and B cases, the type D case remains largely unexplored even in purely counting the number of type D derangements. This is even so in q-counting even-signed derangements because *natural* type D permutation statistics are unavailable in the literature.

Received October 17, 2022; Accepted February 17, 2023.
Communicated by Sen-Peng Eu.
2020 *Mathematics Subject Classification.* 05A05, 05A15.
Key words and phrases. derangements, even-signed permutations, generating functions, major indices, ratio monotonicity, unimodality, log-concavity, spiral property.

This paper is the first of a series on derangement polynomials of type D. The purpose of the present work is to fill in a basic piece in the enumerative theory of type D derangements. By q-counting derangements in D_n by their major indices, we obtain properties of the type D derangement polynomial $d_n^D(q) := \sum_{\sigma\in\mathcal{D}_n^D} q^{\mathrm{maj}(\sigma)}$, including recurrence relations and factorial generating function. By letting $q\to 1$, results of plain counting even-signed derangements so obtained allow us to exhibit and contrast certain features of the type D theory with the types A and B counterparts.

Studies of other properties of derangement polynomials are available in the literature. In a study of q-analogue of the principle of inclusion-exclusion, Chen and Rota [2] proved that the unsigned derangement polynomial $d_n(q) := \sum_{\sigma\in\mathcal{D}_n} q^{\mathrm{maj}(\sigma)}$, which was shown by Wachs [14] to be expressible as $[n]_q!\sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}}/[k]_q!$, is unimodal for all n and posed the following conjecture.

Conjecture 1.1. *The maximum coefficient appearing in the derangement polynomial $d_n(q)$ is that of $q^{\lceil n(n-1)/4\rceil}$, where $\lceil x\rceil$ denotes the smallest integer $\geq x$.*

By listing the coefficients of $d_n(q)$ spirally away from that of $q^{\lceil n(n-1)/4\rceil}$, and by showing the resulting list of coefficients being weakly decreasing, Zhang [15] settled Conjecture 1.1 in the affirmative. The spirally listed coefficients being weakly decreasing is termed the spiral property.

Chen and Xia [4] later introduced the notion of ratio monotonicity of positive sequences, and proved this property of the coefficients of $d_n(q)$. The ratio monotonicity then implies the spiral property and log-concavity, the latter property in turn implies unimodality. It is important to note that the orientation of the spiral in the sense of Chen and Xia is opposite to that of Zhang, and has become the standard for later publications, we shall adhere to this standard in the sequel. See Example 4.2.

The log-concavity of the coefficients of the nth type B derangement polynomial $d_n^B(q) := \sum_{\sigma\in\mathcal{D}_n^B} q^{\mathrm{fmaj}(\sigma)}$ is recently proved by Liu and Du [11], thus yielding a proved weaker type B version of Conjecture 1.1, weaker in the sense that only unimodality is deduced, the exact location of the maximum coefficient is not asserted.

As $d_n^D(q)$ resembles $d_n(q)$, by proving the ratio monotonicity of the coefficients of $d_n^D(q)$, we are able to deduce the spiral property, log-concavity and unimodality of them.

The organization of this paper is as follows. In the next section, we gather some notations and preliminary results which will be needed in the subsequent sections. In Section 3, we introduce the type D derangement polynomial $d_n^D(q)$; by utilising a result of Foata and Han [9], we q-count derangements in D_n by their major indices. By letting $q\to 1$, the q-counting results specialize to results of plain counting derangements in D_n. In the final section, we prove the strict ratio monotonicity of the coefficients of $d_n^D(q)$, and deduce the unimodality, log-concavity and spiral property enjoyed by $d_n^D(q)$.

2. Notations and preliminaries

We collect in this section notations and results that will be needed in the sequel.

Let $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote, as usual, the sets of all non-negative integers, integers, rational numbers and real numbers, respectively. Let $n \in \mathbb{N}$. Denote by $[n]$ the interval of integers $\{1, 2, \ldots, n\}$ (in particular, $[0] = \emptyset$).

If S is a finite set, then its cardinality is denoted by $\#S$.

Let n be a positive integer. Let $\mathfrak{S}_n$ denote the symmetric group on n letters, which is a Coxeter group of type A and of rank $n-1$ generated by $\{s_1, \ldots, s_{n-1}\}$, where for $i = 1, 2, \ldots, n-1$, $s_i = (i, i+1)$ is the simple transposition exchanging i and $i+1$. Any element π of $\mathfrak{S}_n$ is a permutation of length n represented by the word $\pi(1)\pi(2)\cdots\pi(n)$.

Let B_n denote the nth hyperoctahedral group, which is a Coxeter group of type B and of rank n generated by $\{s_0^B, s_1, \ldots, s_{n-1}\}$, where $s_0^B = (1, \overline{1})$ is the sign change. Any element π of B_n is a signed permutation satisfying $\pi(-i) = -\pi(i)$ for all $i \in [n]$ and represented by the signed word $\pi(1)\pi(2)\cdots\pi(n)$, where $|\pi| := |\pi(1)||\pi(2)|\cdots|\pi(n)| \in \mathfrak{S}_n$.

Let D_n denote the nth even-signed permutation group, which is a Coxeter group of type D and of rank n generated by $\{s_0^D, s_1, \ldots, s_{n-1}\}$, where $s_0^D = (\overline{1}, \overline{2})$ is the even-sign change. An element π of D_n is a signed n-permutation such that $\#\{i \in [n] : \pi(i) < 0\}$ is even. The representation of π as the signed word $\pi(1)\pi(2)\cdots\pi(n)$ applies in the present even-signed case.

Let σ be an n-permutation, signed or not. An integer $i \in [n]$ is said to be a *fixed point* of σ if $\sigma(i) = i$; σ is said to be a *derangement* if it has no fixed point. Denote by $\mathcal{D}_n$, $\mathcal{D}_n^B$ and $\mathcal{D}_n^D$ sets of derangements in $\mathfrak{S}_n$, B_n and D_n, respectively.

As far as Section 3 is concerned, we need several permutation statistics. An n-permutation σ is said to have a (type A) descent at position $i \in [n-1]$ if $\sigma(i) > \sigma(i+1)$. The major index of σ is defined as

$$\operatorname{maj}(\sigma) := \sum_{i=1}^{n-1} i\chi(\sigma(i) > \sigma(i+1)),$$

i.e., $\operatorname{maj}(\sigma)$ is the sum of all (type A) descents of σ, where $\chi(P) = 1$ or 0 depending on whether the statement P is true or not. The number of negative letters of σ is defined as

$$N(\sigma) := \#\{i \in [n] : \sigma(i) < 0\}.$$

The flag major index of σ is defined as

$$\operatorname{fmaj}(\sigma) := 2\operatorname{maj}(\sigma) + N(\sigma).$$

Let z and q be commuting indeterminates. For $n \in \mathbb{N}$, define the q-integer and q-factorial by

$$[n]_q := 1 + q + \cdots + q^{n-1} \quad \text{and} \quad [n]_q! := [1]_q[2]_q\cdots[n]_q,$$

respectively. Define the q-exponential $E(z;q)$ by

$$E(z;q) := \sum_{n\geq 0} \frac{q^{\binom{n}{2}} z^n}{[n]_q!}.$$

A result needed in Section 3 is the following:

$$\sum_{\sigma\in\mathcal{D}_n^B} q^{\mathrm{fmaj}(\sigma)} z^{N(\sigma)} = \sum_{k=0}^{n} (-1)^k q^{2\binom{k}{2}} (1+zq)^{n-k} \frac{[n]_{q^2}!}{[k]_{q^2}!}, \tag{2.1}$$

due to Foata and Han [9].

Let $a_1, a_2, \ldots, a_n$ be a sequence of positive numbers. It is said to be *unimodal* if there exists $1 \leq j \leq n$ such that $a_i \leq a_{i+1}$ for $i = 1, \ldots, j-1$ and $a_i \geq a_{i+1}$ for $i = j, \ldots, n-1$; it is *log-concave* if $a_k^2 \geq a_{k-1}a_{k+1}$ holds for $1 < k < n$; it is *ratio monotone* if

$$\frac{a_1}{a_n} \leq \frac{a_2}{a_{n-1}} \leq \cdots \leq \frac{a_{\lfloor n/2\rfloor}}{a_{\lceil n/2\rceil+1}} \leq 1 \tag{2.2}$$

and

$$\frac{a_n}{a_2} \leq \frac{a_{n-1}}{a_3} \leq \cdots \leq \frac{a_{\lfloor n/2\rfloor+2}}{a_{\lceil n/2\rceil}} \leq 1 \tag{2.3}$$

hold, where $\lfloor\cdot\rfloor$ and $\lceil\cdot\rceil$ denote the floor function and the ceiling function, respectively. The sequence is strictly ratio monotone if all the inequalities are strict.

The chains of inequalities (2.2)–(2.3) involve approximately half of the sequence. One can bundle up to involve the whole sequence, as follows.

Proposition 2.1. *If a sequence $a_1, a_2, \ldots, a_n$ of positive numbers is ratio monotone, then*

$$\frac{a_1}{a_n} \leq \frac{a_2}{a_{n-1}} \leq \cdots \leq \frac{a_n}{a_1} \tag{2.4}$$

and

$$\frac{a_n}{a_2} \leq \frac{a_{n-1}}{a_3} \leq \cdots \leq \frac{a_2}{a_n} \tag{2.5}$$

hold.

Proof. First note that $\lceil n/2\rceil = \lfloor n/2\rfloor + \chi(n \text{ odd})$. Consider the odd n case. Taking reciprocals of (2.2), we have

$$\frac{a_{\lceil n/2\rceil}}{a_{\lfloor n/2\rfloor+1}} = 1 \leq \frac{a_{\lceil n/2\rceil+1}}{a_{\lfloor n/2\rfloor}} \leq \cdots \leq \frac{a_{n-1}}{a_2} \leq \frac{a_n}{a_1}.$$

By chaining up (2.2) and the preceding chain of inequalities, (2.4) follows. Proofs of the even n case and that of (2.5), being similar, are omitted. □

3. A class of type D derangement polynomials

In this section, we q-count even-signed n-derangements by their major indices.

Towards this end, we define the type D derangement polynomial by

$$d_n^D(q) = \sum_{\sigma\in\mathcal{D}_n^D} q^{\mathrm{maj}(\sigma)}, \quad n = 1, 2, \ldots.$$

The first few non-trivial members of $d_n^D(q)$ are as follows:

$$\begin{aligned}
d_1^D(q) &= 0,\\
d_2^D(q) &= 1 + 2q,\\
d_3^D(q) &= 2 + 5q + 5q^2 + 2q^3,\\
d_4^D(q) &= 4 + 14q + 24q^2 + 29q^3 + 25q^4 + 15q^5 + 6q^6,\\
d_5^D(q) &= 8 + 36q + 84q^2 + 142q^3 + 192q^4 + 214q^5 + 197q^6 + 149q^7 + 91q^8 + 41q^9 + 10q^{10},\\
d_6^D(q) &= 16 + 88q + 256q^2 + 540q^3 + 924q^4 + 1352q^5 + 1730q^6 + 1956q^7 + 1970q^8\\
&\quad + 1768q^9 + 1405q^{10} + 977q^{11} + 583q^{12} + 285q^{13} + 103q^{14} + 22q^{15},\\
d_7^D(q) &= 32 + 208q + 720q^2 + 1800q^3 + 3648q^4 + 6352q^5 + 9812q^6 + 13692q^7 + 17456q^8\\
&\quad + 20480q^9 + 22210q^{10} + 22316q^{11} + 20778q^{12} + 17888q^{13} + 14182q^{14}\\
&\quad + 10285q^{15} + 6749q^{16} + 3939q^{17} + 1985q^{18} + 819q^{19} + 249q^{20} + 42q^{21}.
\end{aligned}$$

It is readily observed from the above list that for each $n \geq 4$, the sequence of coefficients of $d_n^D(q)$ has a single peak.

Theorem 3.1. *We have*

(i) *for* $n \geq 1$, $d_n^D(q) = \sum_{k=0}^{n}(-1)^k q^{\binom{k}{2}} 2^{n-1-k}\frac{[n]_q!}{[k]_q!} + \frac{(-1)^n q^{\binom{n}{2}}}{2}$;

(ii) *for* $n \geq 2$, $d_n^D(q) = 2[n]_q d_{n-1}^D(q) + (-1)^n q^{\binom{n-1}{2}}\big([n]_q + q^{n-1}\big)$;

(iii) $2 + \sum_{n\geq 1}\frac{d_n^D(q^2)u^n}{[2]_q[4]_q\cdots[2n-2]_q[n]_q} = \frac{(1+q-u)E\big(\frac{-u}{1+q}; q^2\big)}{1+q-2u} + \frac{(1+q-uq)E\big(\frac{-uq}{1+q}; q^2\big)}{1+q-2uq}$;

(iv) *for* $n \geq 2$,

$$d_{n+1}^D(q) = \left(\frac{[n]_q(2[n+1]_q + q^{n-1})}{[n]_q + q^{n-1}}\right) d_n^D(q) + \left(\frac{2q^{n-1}[n]_q([n+1]_q + q^n)}{[n]_q + q^{n-1}}\right) d_{n-1}^D(q).$$

Proof. It is clear that $\mathcal{D}_n^B = \mathcal{D}_n^D \cup \overline{\mathcal{D}}_n^D$, where $\overline{\mathcal{D}}_n^D = \{\sigma \in \mathcal{D}_n^B : N(\sigma) \equiv 1 \pmod 2\}$. Since $\mathrm{fmaj}(\sigma) = 2\,\mathrm{maj}(\sigma) + N(\sigma)$, the identity (2.1) can be written as

$$\sum_{\sigma\in\mathcal{D}_n^D} q^{2\,\mathrm{maj}(\sigma)}(qz)^{N(\sigma)} + \sum_{\sigma\in\overline{\mathcal{D}}_n^D} q^{2\,\mathrm{maj}(\sigma)}(qz)^{N(\sigma)} = \sum_{k=0}^{n}(-1)^k q^{2\binom{k}{2}}(1+zq)^{n-k}\frac{[n]_{q^2}!}{[k]_{q^2}!}.$$

Setting $z = -q^{-1}$ yields

$$\sum_{\sigma\in\mathcal{D}_n^D} q^{2\,\mathrm{maj}(\sigma)} - \sum_{\sigma\in\overline{\mathcal{D}}_n^D} q^{2\,\mathrm{maj}(\sigma)} = (-1)^n q^{2\binom{n}{2}}; \tag{3.1}$$

setting $z = q^{-1}$ then yields

$$\sum_{\sigma\in\mathcal{D}_n^D} q^{2\,\mathrm{maj}(\sigma)} + \sum_{\sigma\in\overline{\mathcal{D}}_n^D} q^{2\,\mathrm{maj}(\sigma)} = \sum_{k=0}^{n}(-1)^k q^{2\binom{k}{2}} 2^{n-k}\frac{[n]_{q^2}!}{[k]_{q^2}!}. \tag{3.2}$$

Summing (3.1) and (3.2), we get

$$\sum_{\sigma\in\mathcal{D}_n^D} q^{2\,\mathrm{maj}(\sigma)} = \sum_{k=0}^{n}(-1)^k q^{2\binom{k}{2}} 2^{n-1-k}\frac{[n]_{q^2}!}{[k]_{q^2}!} + \frac{(-1)^n q^{2\binom{n}{2}}}{2}. \tag{3.3}$$

Replacing q^2 by q, (i) follows.

From (i), we have

$$\begin{aligned} d_n^D(q) &= 2[n]_q\left(\sum_{k=0}^{n-1}(-1)^k q^{\binom{k}{2}} 2^{n-2-k}\frac{[n-1]_q!}{[k]_q!} + \frac{(-1)^{n-1}q^{\binom{n-1}{2}}}{2}\right)\\ &\quad + (-1)^n q^{\binom{n-1}{2}}[n]_q + (-1)^n q^{\binom{n}{2}}\\ &= 2[n]_q d_{n-1}^D(q) + (-1)^n q^{\binom{n-1}{2}}\big([n]_q + q^{n-1}\big), \end{aligned}$$

which is (ii).

(iii) Pulling out the last summand on the right side of (3.3), followed by replacing k by $n-k$, we get

$$\sum_{\sigma\in\mathcal{D}_n^D} q^{2\,\mathrm{maj}(\sigma)} = \sum_{k=1}^{n}(-1)^{n-k} q^{2\binom{n-k}{2}} 2^{k-1}\frac{[n]_{q^2}!}{[n-k]_{q^2}!} + (-1)^n q^{2\binom{n}{2}}.$$

Multiplying the above identity by $u^n/([2]_q[4]_q\cdots[2n-2]_q[n]_q)$, followed by summing over $n \ge 1$, we get

$$\begin{aligned} &\sum_{n\ge 1}\frac{u^n\sum_{\sigma\in\mathcal{D}_n^D} q^{2\,\mathrm{maj}(\sigma)}}{[2]_q[4]_q\cdots[2n-2]_q[n]_q}\\ &= \sum_{n\ge 1}\frac{u^n}{[2]_q[4]_q\cdots[2n-2]_q[n]_q}\left(\sum_{k=1}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}2^{k-1}\frac{[n]_{q^2}!}{[n-k]_{q^2}!} + (-1)^n q^{2\binom{n}{2}}\right)\\ &= I + II, \end{aligned}$$

where

$$\begin{aligned} II &= \sum_{n\ge 1}\frac{(-u)^n q^{2\binom{n}{2}}}{[2]_q[4]_q\cdots[2n-2]_q[n]_q} = \sum_{n\ge 1}\frac{(1+q^n)\left(\frac{-u}{1+q}\right)^n q^{2\binom{n}{2}}}{[n]_{q^2}!}\\ &= E\left(\frac{-u}{1+q};q^2\right) + E\left(\frac{-uq}{1+q};q^2\right) - 2, \end{aligned}$$

and

$$
\begin{aligned}
I &= \sum_{n\geq 1} \frac{u^n}{[2]_q[4]_q\cdots[2n-2]_q[n]_q} \sum_{k=1}^{n} (-1)^{n-k} q^{2\binom{n-k}{2}} 2^{k-1} \frac{[n]_{q^2}!}{[n-k]_{q^2}!} \\
&= \sum_{n\geq 1} \frac{(1+q^n)\left(\frac{u}{1+q}\right)^n}{[n]_{q^2}!} \sum_{k=1}^{n} (-1)^{n-k} q^{2\binom{n-k}{2}} 2^{k-1} \frac{[n]_{q^2}!}{[n-k]_{q^2}!} \\
&= \frac{u}{1+q} \sum_{k\geq 1} \left(\frac{2u}{1+q}\right)^{k-1} \sum_{n\geq k} \frac{\left(\frac{-u}{1+q}\right)^{n-k} q^{2\binom{n-k}{2}}}{[n-k]_{q^2}!} \\
&\quad + \frac{uq}{1+q} \sum_{k\geq 1} \left(\frac{2uq}{1+q}\right)^{k-1} \sum_{n\geq k} \frac{\left(\frac{-uq}{1+q}\right)^{n-k} q^{2\binom{n-k}{2}}}{[n-k]_{q^2}!} \\
&= \frac{uE\left(\frac{-u}{1+q}; q^2\right)}{1+q-2u} + \frac{uqE\left(\frac{-uq}{1+q}; q^2\right)}{1+q-2uq}.
\end{aligned}
$$

Consequently,

$$
\begin{aligned}
&2 + \sum_{n\geq 1} \frac{u^n \sum_{\sigma\in\mathcal{D}_n^D} q^{2\,\mathrm{maj}(\sigma)}}{[2]_q[4]_q\cdots[2n-2]_q[n]_q} \\
&= \frac{uE\left(\frac{-u}{1+q}; q^2\right)}{1+q-2u} + \frac{uqE\left(\frac{-uq}{1+q}; q^2\right)}{1+q-2uq} + E\left(\frac{-u}{1+q}; q^2\right) + E\left(\frac{-uq}{1+q}; q^2\right) \\
&= \frac{(1+q-u)E\left(\frac{-u}{1+q}; q^2\right)}{1+q-2u} + \frac{(1+q-uq)E\left(\frac{-uq}{1+q}; q^2\right)}{1+q-2uq},
\end{aligned}
$$

which is (iii).

From (ii), we have

$$
\begin{aligned}
\frac{d_{n+1}^D(q)}{q^{\binom{n}{2}}\left([n+1]_q+q^n\right)} &= \frac{2[n+1]_q d_n^D(q)}{q^{\binom{n}{2}}\left([n+1]_q+q^n\right)} + (-1)^{n+1}, \\
\frac{d_n^D(q)}{q^{\binom{n-1}{2}}\left([n]_q+q^{n-1}\right)} &= \frac{2[n]_q d_{n-1}^D(q)}{q^{\binom{n-1}{2}}\left([n]_q+q^{n-1}\right)} + (-1)^n,
\end{aligned}
$$

whose sum can be simplified to yield

$$
d_{n+1}^D(q) = \left(\frac{[n]_q(2[n+1]_q+q^{n-1})}{[n]_q+q^{n-1}}\right) d_n^D(q) + \left(\frac{2q^{n-1}[n]_q([n+1]_q+q^n)}{[n]_q+q^{n-1}}\right) d_{n-1}^D(q),
$$

which is (iv). □

In Theorem 3.1(i), for $0 \leq k \leq n$, the kth summand $(-1)^k q^{\binom{k}{2}} 2^{n-1-k} [k+1]_q [k+2]_q \cdots [n]_q$ is a polynomial in q of degree $\binom{k}{2} + \sum_{i=k}^{n-1} i = \binom{n}{2}$. Amongst the elements of $\mathcal{D}_n^D$, the maximal major index $\binom{n}{2}$ is attained by $\overline{1}\overline{2}\cdots\overline{n} \in \mathcal{D}_n^D$ when n is even, and by $2\overline{1}\overline{3}\cdots\overline{n} \in \mathcal{D}_n^D$ when n is odd.

Comparing Theorem 3.1(iii), namely

$$2+\sum_{n\geq 1} d_n^D(q^2)\frac{u^n}{[2]_q[4]_q\cdots[2n-2]_q[n]_q}=\frac{(1+q-u)E\big(\frac{-u}{1+q};q^2\big)}{1+q-2u}+\frac{(1+q-uq)E\big(\frac{-uq}{1+q};q^2\big)}{1+q-2uq}, \tag{3.4}$$

to the corresponding type B result [5, Theorem 5(ii)]:

$$\sum_{n\geq 0} d_n^B(q)\frac{u^n}{[2]_q[4]_q\cdots[2n]_q}=\frac{E\big(\frac{-u}{1+q};q^2\big)}{1-u},$$

where $d_n^B(q)=\sum_{\sigma\in\mathcal{D}_n^B} q^{\mathrm{fmaj}(\sigma)}$, the right side of (3.4) bifurcates into a sum of two terms. An explanation of this bifurcation is as follows.

With the help of Mathematica, we see that the expansion of the first term reads

$$\begin{aligned}\frac{(1+q-u)E\big(\frac{-u}{1+q};q^2\big)}{1+q-2u}&=1+(1+2q^2)\frac{\big(\frac{u}{1+q}\big)^2}{[2]_{q^2}!}+(2+5q^2+5q^4+2q^6)\frac{\big(\frac{u}{1+q}\big)^3}{[3]_{q^2}!}\\&\quad+(4+14q^2+24q^4+29q^6+25q^8+15q^{10}+6q^{12})\frac{\big(\frac{u}{1+q}\big)^4}{[4]_{q^2}!}+\cdots\\&=1+d_2^D(q^2)\frac{\big(\frac{u}{1+q}\big)^2}{[2]_{q^2}!}+d_3^D(q^2)\frac{\big(\frac{u}{1+q}\big)^3}{[3]_{q^2}!}+d_4^D(q^2)\frac{\big(\frac{u}{1+q}\big)^4}{[4]_{q^2}!}+\cdots;\end{aligned}$$

the expansion of the second term is the same as that of the first term with uq in place of u. Since $[2k]_q=(1+q)[k]_{q^2}$ and $(1+q^n)[n]_q=(1+q)[n]_{q^2}$, summing those two terms

$$\frac{(1+q-u)E\big(\frac{-u}{1+q};q^2\big)}{1+q-2u}+\frac{(1+q-uq)E\big(\frac{-uq}{1+q};q^2\big)}{1+q-2uq}=2+\sum_{n\geq 2} d_n^D(q^2)\frac{\big(\frac{u}{1+q}\big)^n(1+q^n)}{[n]_{q^2}!}$$

results in cancellation of “2” in the factor $[2n]_q=(1+q)[n]_{q^2}$ in the denominator of the summands.

Theorem 3.1(i) can also be written as

$$d_n^D(q)=2^{n-1}[n]_q!\sum_{k=0}^{n-1}\frac{(-1)^k q^{\binom{k}{2}}}{2^k[k]_q!}+(-1)^n q^{\binom{n}{2}}. \tag{Theorem 3.1(i)$'$}$$

For $n\geq 1$, let $d_n^D:=\#\mathcal{D}_n^D$ be the nth type D derangement number. Note that D_1 consists of the identity permutation of length 1 only so that $\mathcal{D}_1^D=\emptyset$ and $d_1^D=0$. It is immediate to see that as $q\to 1$, Theorem 3.1 specializes to the counterparts of Theorem 3.2.

Theorem 3.2. *The following results hold:*

(i) *for* $n \geq 1$, $d_n^D = 2^{n-1} n! \sum_{k=0}^{n-1} \frac{(-1)^k}{2^k k!} + (-1)^n$*;*

(ii) *for* $n \geq 1$, $d_n^D = 2nd_{n-1}^D + (-1)^n(n+1)$*;*

(iii) $d^D(u) := 2 + \sum_{n\geq 1} d_n^D \frac{u^n}{2^{n-1} n!} = \frac{(2-u)e^{-u/2}}{1-u}$*;*

(iv) *for* $n \geq 1$, $d_{n+1}^D = \left(\frac{n(2n+3)}{n+1}\right) d_n^D + \left(\frac{2n(n+2)}{n+1}\right) d_{n-1}^D$*;*

(v) *for* $n \geq 1$, $2^{n-1} n! = \sum_{k=0}^{n} \binom{n}{k} d_k^D$*;*

(vi) $d^D(u)$ *satisfies the following linear first order ordinary differential equation*

$$2(u-1)(u-2)(d^D)'(u) + u(u-3)d^D(u) = 0$$

in $\mathbb{Q}[[u]]$*;*

(vii) *for* $n \geq 2$, $d_{n+1}^D = 3nd_n^D + (5-2n)nd_{n-1}^D - 2(n-1)nd_{n-2}^D$.

Proof. As $q \to 1$, Theorem 3.1(i)$'$ specializes to (i), and Theorem 3.1(ii)–(iv) specialize to (ii)–(iv), respectively.

Denote by $D_{n,k}$ the set of all elements of D_n having k fixed points. Then $D_n = \bigcup_{k=0}^{n} D_{n,k}$, where the union is disjoint. Let $\sigma \in D_{n,k}$. The subword of fixed points of σ can be chosen in $\binom{n}{k}$ ways, and the non-fixed point subword is an even-signed derangement of length $n-k$. Hence, $2^{n-1}n! = \sum_{k=0}^{n} \binom{n}{k} d_{n-k}^D$ and (v) follows.

By logarithmic differentiation of $d^D(u)$, we have

$$\frac{(d^D)'(u)}{d^D(u)} = \frac{d}{du} \ln\left(\frac{(2-u)e^{-u/2}}{1-u}\right) = -\frac{u(u-3)}{2(u-1)(u-2)}$$

from which (vi) follows.

Substituting $d^D(u) = 2 + \sum_{n\geq 1} d_n^D u^n / 2^{n-1} n!$ and

$$(d^D)'(u) = \sum_{n\geq 1} d_n^D \frac{u^{n-1}}{2^{n-1}(n-1)!} = \sum_{n\geq 0} d_{n+1}^D \frac{u^n}{2^n n!}$$

into (vi), followed by extracting the coefficient of u^n, (vii) follows. $\square$

Some comments on Theorem 3.2 are in order. The formula of d_n^D in Theorem 3.2(i) resembles its type B counterpart, namely

$$d_n^B = 2^n n! \sum_{k=0}^{n} \frac{(-1)^k}{2^k k!}.$$

Theorem 3.2(iv) and (vii) are new in the sense that d_n^D satisfies a four-term recurrence relation with integral coefficients, of which some are negative, and that d_n^D also satisfies a three-term recurrence relation but with rational coefficients! These two properties are absent in the types A and B derangement theories.

Theorem 3.2(iv) is the type D analogue of a recurrence relation for the classical derangement number $d_n := n! \sum_{k=0}^{n} (-1)^k/k!$, namely

$$d_{n+1} = n(d_n + d_{n-1}),$$

which follows by eliminating $(-1)^{n+1}$ from the other recurrences

$$d_{n+1} = (n+1)d_n + (-1)^{n+1} \quad \text{and} \quad d_n = nd_{n-1} + (-1)^n$$

satisfied by d_n.

Values of the first ten members of d_n^D are

$$\begin{gathered} d_1^D = 0, \quad d_2^D = 3, \quad d_3^D = 14, \quad d_4^D = 117, \quad d_5^D = 1164, \quad d_6^D = 13975, \\ d_7^D = 195642, \quad d_8^D = 3130281, \quad d_9^D = 56345048, \quad d_{10}^D = 1126900971, \end{gathered}$$

which happen to be the first 10 numbers of direct isometries that are derangements of the $(n-1)$-dimensional facets of an n-cube [10], registered as sequence A161936 at OEIS [12]. This suggests that geometric interpretation of d_n^D is a possible direction for development.

4. Ratio monotonicity of $d_n^D(q)$ and consequences

The goal of this section is to prove the (strict) ratio monotonicity of the coefficients (see Theorem 4.8), hence deducing the unimodality, log-concavity and spiral property, of $d_n^D(q)$. A few little lemmas are needed for the proof of the main theorem. The next lemma is similar to [4, Lemma 2.3]. We provide below a slick proof of it.

Lemma 4.1. *For positive numbers $c_0, c_1, \ldots, c_{k+1}, d_0, d_1, \ldots, d_{k+1}$ satisfying*

$$\frac{d_0}{c_0} < \frac{d_1}{c_1} < \cdots < \frac{d_k}{c_k} < \frac{d_{k+1}}{c_{k+1}},$$

there hold

$$\text{(a)}\ \frac{\sum_{i=1}^{k} d_i}{\sum_{i=1}^{k} c_i} < \frac{\sum_{i=1}^{k+1} d_i}{\sum_{i=1}^{k+1} c_i}, \qquad \text{(b)}\ \frac{\sum_{i=1}^{k} d_i}{\sum_{i=1}^{k} c_i} < \frac{\sum_{i=2}^{k+1} d_i}{\sum_{i=2}^{k+1} c_i}, \qquad \text{(c)}\ \frac{\sum_{i=1}^{k} d_i}{\sum_{i=0}^{k} c_i} < \frac{\sum_{i=1}^{k+1} d_i}{\sum_{i=0}^{k+1} c_i}.$$

Proof. (a) follows from

$$\begin{aligned} \frac{\sum_{i=1}^{k+1} d_i}{\sum_{i=1}^{k+1} c_i} - \frac{\sum_{i=1}^{k} d_i}{\sum_{i=1}^{k} c_i} &= \frac{c_{k+1}\big(\frac{d_{k+1}}{c_{k+1}} \sum_{i=1}^{k} c_i - \sum_{i=1}^{k} d_i\big)}{\sum_{i=1}^{k+1} c_i \sum_{i=1}^{k} c_i} > \frac{c_{k+1}\big(\frac{d_k}{c_k} \sum_{i=1}^{k-1} c_i - \sum_{i=1}^{k-1} d_i\big)}{\sum_{i=1}^{k+1} c_i \sum_{i=1}^{k} c_i} \\ &> \cdots > \frac{c_{k+1}\big(\frac{d_2}{c_2} c_1 - d_1\big)}{\sum_{i=1}^{k+1} c_i \sum_{i=1}^{k} c_i} > 0, \end{aligned}$$

(b) from

$$\begin{aligned}&\frac{\sum_{i=2}^{k+1} d_i}{\sum_{i=2}^{k+1} c_i} - \frac{\sum_{i=1}^{k} d_i}{\sum_{i=1}^{k} c_i}\\ &= \frac{(d_{k+1}c_1 - c_{k+1}d_1) + \left(d_{k+1}\sum_{i=2}^{k} c_i - c_{k+1}\sum_{i=2}^{k} d_i\right) + \left(c_1\sum_{i=2}^{k} d_i - d_1\sum_{i=2}^{k} c_i\right)}{\sum_{i=2}^{k+1} c_i \sum_{i=1}^{k} c_i} > 0,\end{aligned}$$

where the positivity of successive terms of the numerator follows from the assumption, and from the same approach in (a), and (c) from

$$\frac{\sum_{i=1}^{k+1} d_i}{\sum_{i=0}^{k+1} c_i} - \frac{\sum_{i=1}^{k} d_i}{\sum_{i=0}^{k} c_i} = \frac{\left(d_{k+1}\sum_{i=1}^{k} c_i - c_{k+1}\sum_{i=1}^{k} d_i\right) + c_0 d_{k+1}}{\sum_{i=0}^{k+1} c_i \sum_{i=0}^{k} c_i} > 0. \qquad \square$$

Example 4.2. By listing the coefficients of $d_4^D(q) = 4+14q+24q^2+29q^3+25q^4+15q^5+6q^6$ spirally, we get the chain of (strict) inequalities

$$4 < 6 < 14 < 15 < 24 < 25 < 29,$$

and this defines the (strict) spiral property of the coefficients of $d_4^D(q)$. The chains of inequalities (2.2)–(2.3) in this case are $\frac{4}{6} < \frac{14}{15} < \frac{24}{25} < 1$ and $\frac{6}{14} < \frac{15}{24} < \frac{25}{29} < 1$, thus showing the coefficients of $d_4^D(q)$ being strictly ratio monotone. The chains of inequalities (2.4)–(2.5) in this case are

$$\frac{4}{6} < \frac{14}{15} < \frac{24}{25} < \frac{29}{29} < \frac{25}{24} < \frac{15}{14} < \frac{6}{4} \quad \text{and} \quad \frac{6}{14} < \frac{15}{24} < \frac{25}{29} < \frac{29}{25} < \frac{24}{15} < \frac{14}{6},$$

which also hold.

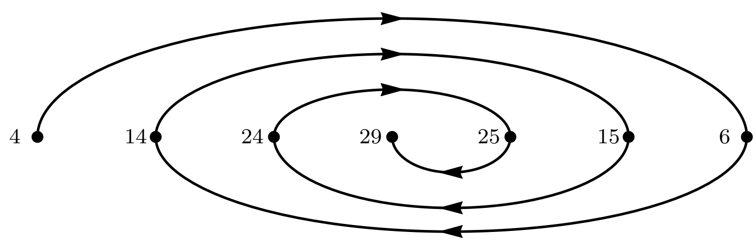

Figure 4.1

Example 4.3. Listing the coefficients of $d_5^D(q)$ spirally, we have

$$8 < 10 < 36 < 41 < 84 < 91 < 142 < 149 < 192 < 197 < 214,$$

i.e., the coefficients of $d_5^D(q)$ enjoy the strict spiral property. The chains of inequalities (2.2)–(2.3) in this case are

$$\frac{8}{10} < \frac{36}{41} < \frac{84}{91} < \frac{142}{149} < \frac{192}{197} < 1 \quad \text{and} \quad \frac{10}{36} < \frac{41}{84} < \frac{91}{142} < \frac{149}{192} < \frac{197}{214} < 1,$$

i.e., the coefficients of $d_5^D(q)$ are strictly ratio monotone. The chains of inequalities (2.4)–(2.5) in this case are

$$\frac{8}{10} < \frac{36}{41} < \frac{84}{91} < \frac{142}{149} < \frac{192}{197} < \frac{214}{214} < \frac{197}{192} < \frac{149}{142} < \frac{91}{84} < \frac{41}{36} < \frac{10}{8},$$
$$\frac{10}{36} < \frac{41}{84} < \frac{91}{142} < \frac{149}{192} < \frac{197}{214} < \frac{214}{197} < \frac{192}{149} < \frac{142}{91} < \frac{84}{41} < \frac{36}{10},$$

respectively, which also hold.

Interested readers can verify that $d_6^D(q)$ and $d_7^D(q)$ listed at the beginning of Section 3 also enjoy the strict spiral property. Write $d_n^D(q) = \sum_{k=0}^{\delta_n} c_{n,k} q^k$, where $\delta_n = \binom{n}{2}$. Explicit formulas of certain $c_{n,k}$'s are obtainable by analyzing Theorem 3.1(i).

Lemma 4.4. *For $n \geq 4$, we have*

$$\begin{aligned}
c_{n,0} &= 2^{n-2}, \quad c_{n,1} = 2^{n-3}(2n-1), \quad c_{n,2} = 2^{n-3}(n^2-4),\\
c_{n,3} &= \frac{2^{n-4}}{3}(2n^3+3n^2-23n+3),\\
c_{n,\delta_n-3} &= \frac{1}{81}\left[2^{n-3}(-73-192n-18n^2+36n^3)+(-1)^{n+1}\right],\\
c_{n,\delta_n-2} &= \frac{1}{27}\left[2^{n-2}(18n^2-24n-23)+(-1)^{n+1}\right],\\
c_{n,\delta_n-1} &= \frac{1}{9}\left[2^{n-1}(6n-7)+(-1)^{n+1}\right],\\
c_{n,\delta_n} &= \frac{2}{3}\left[(-1)^n+2^{n-1}\right].
\end{aligned}$$

Proof. Consider $c_{n,2}$. Since for $0 \leq k \leq n$, the kth summand $(-1)^k q^{\binom{k}{2}} 2^{n-1-k}[k+1]_q[k+2]_q \cdots [n]_q$ of $d_n^D(q)$ is a polynomial in q with lowest order term $(-1)^k q^{\binom{k}{2}} 2^{n-1-k}$, only the first three summands contribute the monomial q^2, whose coefficient is

$$\begin{aligned}
&2^{n-1}\left[\binom{n-1}{2}+\binom{n-2}{1}\right] - 2^{n-2}\left[\binom{n-1}{2}+\binom{n-2}{1}\right] + 2^{n-3}\binom{n-2}{1}\\
&= 2^{n-3}(n^2-4).
\end{aligned}$$

Formulas of other coefficients are derived in a similar manner, whose details are omitted. $\square$

Lemma 4.5. *For $n \geq 4$, we have*

$$c_{n+1,k} = \begin{cases}
2\sum_{i=0}^{k} c_{n,i} & \text{for } 0 \leq k \leq n,\\
2\sum_{i=k-n}^{k} c_{n,i} & \text{for } n+1 \leq k \leq \delta_n - 1,\\
2\sum_{i=k-n}^{\delta_n} c_{n,i} + (-1)^{n+1} & \text{for } \delta_n \leq k \leq \delta_{n+1} - 1,\\
2c_{n,\delta_n} + 2(-1)^{n+1} & \text{for } k = \delta_{n+1}.
\end{cases}$$

Proof. Theorem 3.1(ii) with $n+1$ in place of n reads

$$\begin{aligned} d_{n+1}^D(q) &= 2[n+1]_q d_n^D(q) + (-1)^{n+1} q^{\binom{n}{2}}\big([n+1]_q + q^n\big) \\ &= 2\sum_{j=0}^{n}\sum_{i=0}^{\delta_n} c_{n,i}q^{i+j} + (-1)^{n+1}q^{\delta_n}(1+q+\cdots+q^{n-1}+2q^n). \end{aligned}$$

The lemma then follows from equating the coefficients of q^k on both sides. □

Lemma 4.6. *If the coefficients $c_{n,0}, c_{n,1}, \ldots, c_{n,\delta_n}$ of $d_n^D(q)$ satisfy*

$$\frac{c_{n,0}}{c_{n,\delta_n}} < \frac{c_{n,1}}{c_{n,\delta_n-1}} < \cdots < \frac{c_{n,\delta_n}}{c_{n,0}} \quad \textit{and} \quad \frac{c_{n,\delta_n}}{c_{n,1}} < \frac{c_{n,\delta_n-1}}{c_{n,2}} < \cdots < \frac{c_{n,1}}{c_{n,\delta_n}},$$

there also hold

$$\frac{\widetilde{c}_{n,0}}{\widetilde{c}_{n,\delta_n}} < \frac{\widetilde{c}_{n,1}}{\widetilde{c}_{n,\delta_n-1}} < \cdots < \frac{\widetilde{c}_{n,\delta_n}}{\widetilde{c}_{n,0}} \quad \textit{and} \quad \frac{\widetilde{c}_{n,\delta_n}}{\widetilde{c}_{n,1}} < \frac{\widetilde{c}_{n,\delta_n-1}}{\widetilde{c}_{n,2}} < \cdots < \frac{\widetilde{c}_{n,1}}{\widetilde{c}_{n,\delta_n}},$$

where $\widetilde{c}_{n,k} := 2c_{n,k}$ for $k = 0, 1, \ldots, \delta_n - 1$ and $\widetilde{c}_{n,\delta_n} := 2c_{n,\delta_n} \pm 1$.

Proof. It is clear that

$$\frac{\widetilde{c}_{n,1}}{\widetilde{c}_{n,\delta_n-1}} < \frac{\widetilde{c}_{n,2}}{\widetilde{c}_{n,\delta_n-2}} < \cdots < \frac{\widetilde{c}_{n,\delta_n-1}}{\widetilde{c}_{n,1}} \quad \text{and} \quad \frac{\widetilde{c}_{n,\delta_n-1}}{\widetilde{c}_{n,2}} < \frac{\widetilde{c}_{n,\delta_n-2}}{\widetilde{c}_{n,3}} < \cdots < \frac{\widetilde{c}_{n,2}}{\widetilde{c}_{n,\delta_n-1}}$$

hold. It remains to check that

$$\frac{\widetilde{c}_{n,0}}{\widetilde{c}_{n,\delta_n}} < \frac{\widetilde{c}_{n,1}}{\widetilde{c}_{n,\delta_n-1}} \quad \text{and} \quad \frac{\widetilde{c}_{n,\delta_n}}{\widetilde{c}_{n,1}} < \frac{\widetilde{c}_{n,\delta_n-1}}{\widetilde{c}_{n,2}},$$

which follow from

$$\begin{aligned} \frac{\widetilde{c}_{n,1}}{\widetilde{c}_{n,\delta_n-1}} - \frac{\widetilde{c}_{n,0}}{\widetilde{c}_{n,\delta_n}} &= \frac{2^{n-2}(2n-1)}{\frac{2}{9}[2^{n-1}(6n-7)+(-1)^{n+1}]} - \frac{2^{n-1}}{\frac{4}{3}[(-1)^n+2^{n-1}]\pm 1} \\ &= \frac{2^{n-3}[3\cdot 2^{n+1}+(-1)^n(18n-6)\pm\frac{3}{4}(18n-9)]}{[2^{n-1}(6n-7)+(-1)^{n+1}][(-1)^n+2^{n-1}\pm\frac{3}{4}]} \\ &> 0, \\ \frac{\widetilde{c}_{n,\delta_n-1}}{\widetilde{c}_{n,2}} - \frac{\widetilde{c}_{n,\delta_n}}{\widetilde{c}_{n,1}} &= \frac{\frac{2}{9}[2^{n-1}(6n-7)+(-1)^{n+1}]}{2^{n-2}(n^2-4)} - \frac{\frac{4}{3}[(-1)^n+2^{n-1}\pm\frac{3}{4}]}{2^{n-2}(2n-1)} \\ &= \frac{2^{n-1}(6n^2-20n+31)+(-1)^{n+1}(6n^2+2n-25)\mp\frac{9}{2}(n^2-4)}{9\cdot 2^{n-3}(n^2-4)(2n-1)} \\ &> 0. \end{aligned}$$

□

Lemma 4.7. *For $n \geq 6$, we have $n+2 < \lfloor\frac{\delta_{n+1}+1}{2}\rfloor \leq \lceil\frac{\delta_{n+1}+1}{2}\rceil < \delta_n$.*

Proof. The second inequality is trivial. The remaining inequalities follow from

$$\delta_n - \left\lceil \frac{\delta_{n+1}+1}{2} \right\rceil = \left\lfloor \frac{n^2-3n-2}{4} \right\rfloor = \begin{cases} (k-2)(4k+5)+9 & \text{if } n=4k,\ k\geq 2, \\ (k-2)(4k+7)+13 & \text{if } n=4k+1,\ k\geq 2, \\ (k-1)(4k+5)+4 & \text{if } n=4k+2,\ k\geq 1, \\ (k-1)(4k+7)+6 & \text{if } n=4k+3,\ k\geq 1, \end{cases}$$

$$\left\lfloor \frac{\delta_{n+1}+1}{2} \right\rfloor - (n+2) = \left\lfloor \frac{n^2-3n-6}{4} \right\rfloor = \begin{cases} (k-2)(4k+5)+8 & \text{if } n=4k,\ k\geq 2, \\ (k-2)(4k+7)+12 & \text{if } n=4k+1,\ k\geq 2, \\ (k-1)(4k+5)+3 & \text{if } n=4k+2,\ k\geq 1, \\ (k-1)(4k+7)+5 & \text{if } n=4k+3,\ k\geq 1. \end{cases} \qquad \square$$

Theorem 4.8. *For $n \geq 4$, the coefficients of $d_n^D(q)$ satisfy*

$$\frac{c_{n,0}}{c_{n,\delta_n}} < \frac{c_{n,1}}{c_{n,\delta_n-1}} < \cdots < \frac{c_{n,\lfloor(\delta_n+1)/2\rfloor-1}}{c_{n,\lceil(\delta_n+1)/2\rceil}} < 1, \tag{4.1}$$

$$\frac{c_{n,\delta_n}}{c_{n,1}} < \frac{c_{n,\delta_n-1}}{c_{n,2}} < \cdots < \frac{c_{n,\lfloor(\delta_n+1)/2\rfloor+1}}{c_{n,\lceil(\delta_n+1)/2\rceil-1}} < 1. \tag{4.2}$$

Proof. We proceed by induction on n, (4.1)–(4.2) are explicitly shown to be true in Example 4.2 for $n = 4$. Assume that (4.1) and (4.2) hold for n. Proposition 2.1 insures that

$$\frac{c_{n,0}}{c_{n,\delta_n}} < \frac{c_{n,1}}{c_{n,\delta_n-1}} < \cdots < \frac{c_{n,\delta_n-1}}{c_{n,1}} < \frac{c_{n,\delta_n}}{c_{n,0}}, \tag{4.3}$$

$$\frac{c_{n,\delta_n}}{c_{n,1}} < \frac{c_{n,\delta_n-1}}{c_{n,2}} < \cdots < \frac{c_{n,2}}{c_{n,\delta_n-1}} < \frac{c_{n,1}}{c_{n,\delta_n}}. \tag{4.4}$$

For $j = 0, 1, \ldots, \delta_n$, let $d_j = c_{n,j}$ and $c_j = c_{n,\delta_n-j}$. By (4.3) and Lemma 4.1, the following inequalities hold:

$$\begin{aligned} &\frac{\sum_{i=0}^{k+1} c_{n,i}}{\sum_{i=\delta_n-(k+1)}^{\delta_n} c_{n,i}} > \frac{\sum_{i=0}^{k} c_{n,i}}{\sum_{i=\delta_n-k}^{\delta_n} c_{n,i}}, \quad && k = 1, 2, \ldots, n-1, \\ &\frac{\sum_{i=1}^{n+1} c_{n,i}}{\sum_{i=\delta_n-(n+1)}^{\delta_n-1} c_{n,i}} > \frac{\sum_{i=0}^{n} c_{n,i}}{\sum_{i=\delta_n-n}^{\delta_n} c_{n,i}}, && \\ &\frac{\sum_{i=k+1-n}^{k+1} c_{n,i}}{\sum_{i=\delta_n-(k+1)}^{\delta_{n+1}-(k+1)} c_{n,i}} > \frac{\sum_{i=k-n}^{k} c_{n,i}}{\sum_{i=\delta_n-k}^{\delta_{n+1}-k} c_{n,i}}, \quad && k = n+1, n+2, \ldots, \delta_n - 1, \\ &\frac{\sum_{i=k+1-n}^{\delta_n} c_{n,i}}{\sum_{i=0}^{\delta_{n+1}-(k+1)} c_{n,i}} > \frac{\sum_{i=k-n}^{\delta_n} c_{n,i}}{\sum_{i=0}^{\delta_{n+1}-k} c_{n,i}}, \quad && k = \delta_n, \delta_n + 1, \ldots, \delta_{n+1} - 2. \end{aligned}$$

By Lemma 4.6, the above inequalities also hold with $c_{n,i}$ replaced by $\widetilde{c}_{n,i}$. By Lemma 4.4, we have

$$\frac{c_{n+1,1}}{c_{n+1,\delta_{n+1}-1}}-\frac{c_{n+1,0}}{c_{n+1,\delta_{n+1}}}=\frac{2^{n-2}(2n+1)}{\frac{1}{9}(2^n(6n-1)+(-1)^{n+2})}-\frac{2^{n-1}}{\frac{2}{3}(2^n+(-1)^{n+1})}$$
$$=\frac{3\cdot 2^{n-1}(2^{n+1}+(-1)^{n+1}(3n+2))}{(2^n+(-1)^{n+1})(2^n(6n-1)+(-1)^{n+2})}>0,$$

for $1\le k\le n-1$,

$$\frac{c_{n+1,k+1}}{c_{n+1,\delta_{n+1}-(k+1)}}-\frac{c_{n+1,k}}{c_{n+1,\delta_{n+1}-k}}$$
$$=\frac{\sum_{i=0}^{k+1}2c_{n,i}}{\sum_{i=\delta_n-(k+1)}^{\delta_n-1}2c_{n,i}+(2c_{n,\delta_n}\pm 1)}-\frac{\sum_{i=0}^{k}2c_{n,i}}{\sum_{i=\delta_n-k}^{\delta_n-1}2c_{n,i}+(2c_{n,\delta_n}\pm 1)}$$
$$=\frac{\sum_{i=0}^{k+1}\widetilde{c}_{n,i}}{\sum_{i=\delta_n-(k+1)}^{\delta_n}\widetilde{c}_{n,i}}-\frac{\sum_{i=0}^{k}\widetilde{c}_{n,i}}{\sum_{i=\delta_n-k}^{\delta_n}\widetilde{c}_{n,i}}>0,$$

$$\frac{c_{n+1,n+1}}{c_{n+1,\delta_{n+1}-(n+1)}}-\frac{c_{n+1,n}}{c_{n+1,\delta_{n+1}-n}}=\frac{\sum_{i=1}^{n+1}2c_{n,i}}{\sum_{i=\delta_n-(n+1)}^{\delta_n-1}2c_{n,i}}-\frac{\sum_{i=0}^{n}2c_{n,i}}{\sum_{i=\delta_n-n}^{\delta_n-1}2c_{n,i}+(2c_{n,\delta_n}\pm 1)}$$
$$=\frac{\sum_{i=1}^{n+1}\widetilde{c}_{n,i}}{\sum_{i=\delta_n-(n+1)}^{\delta_n-1}\widetilde{c}_{n,i}}-\frac{\sum_{i=0}^{n}\widetilde{c}_{n,i}}{\sum_{i=\delta_n-n}^{\delta_n}\widetilde{c}_{n,i}}>0,$$

for $n+1\le k\le \delta_n-2$,

$$\frac{c_{n+1,k+1}}{c_{n+1,\delta_{n+1}-(k+1)}}-\frac{c_{n+1,k}}{c_{n+1,\delta_{n+1}-k}}=\frac{\sum_{i=k+1-n}^{k+1}2c_{n,i}}{\sum_{i=\delta_n-(k+1)}^{\delta_{n+1}-(k+1)}2c_{n,i}}-\frac{\sum_{i=k-n}^{k}2c_{n,i}}{\sum_{i=\delta_n-k}^{\delta_{n+1}-k}2c_{n,i}}>0,$$

$$\frac{c_{n+1,\delta_n}}{c_{n+1,n}}-\frac{c_{n+1,\delta_n-1}}{c_{n+1,n+1}}=\frac{\sum_{i=\delta_n-n}^{\delta_n-1}2c_{n,i}+(2c_{n,\delta_n}\pm 1)}{\sum_{i=0}^{n}2c_{n,i}}-\frac{\sum_{i=\delta_n-(n+1)}^{\delta_n-1}2c_{n,i}}{\sum_{i=1}^{n+1}2c_{n,i}}$$
$$=\frac{\sum_{i=\delta_n-n}^{\delta_n}\widetilde{c}_{n,i}}{\sum_{i=0}^{n}\widetilde{c}_{n,i}}-\frac{\sum_{i=\delta_n-(n+1)}^{\delta_n-1}\widetilde{c}_{n,i}}{\sum_{i=1}^{n+1}\widetilde{c}_{n,i}}>0,$$

for $\delta_n\le k\le \delta_{n+1}-2$,

$$\frac{c_{n+1,k+1}}{c_{n+1,\delta_{n+1}-(k+1)}}-\frac{c_{n+1,k}}{c_{n+1,\delta_{n+1}-k}}$$
$$=\frac{\sum_{i=k+1-n}^{\delta_n-1}2c_{n,i}+(2c_{n,\delta_n}\pm 1)}{\sum_{i=0}^{\delta_{n+1}-(k+1)}2c_{n,i}}-\frac{\sum_{i=k-n}^{\delta_n-1}2c_{n,i}+(2c_{n,\delta_n}\pm 1)}{\sum_{i=0}^{\delta_{n+1}-k}2c_{n,i}}$$
$$=\frac{\sum_{i=k+1-n}^{\delta_n}\widetilde{c}_{n,i}}{\sum_{i=0}^{\delta_{n+1}-(k+1)}\widetilde{c}_{n,i}}-\frac{\sum_{i=k-n}^{\delta_n}\widetilde{c}_{n,i}}{\sum_{i=0}^{\delta_{n+1}-k}\widetilde{c}_{n,i}}>0,$$

$$\frac{c_{n+1,\delta_{n+1}}}{c_{n+1,0}}-\frac{c_{n+1,\delta_{n+1}-1}}{c_{n+1,1}}=\frac{\frac{2}{3}(2^n+(-1)^{n+1})}{2^{n-1}}-\frac{\frac{1}{9}(2^n(6n-1)+(-1)^n)}{2^{n-2}(2n+1)}$$
$$=\frac{2^{n+1}+(-1)^{n+1}(3n+2)}{9(2n+1)2^{n-3}}>0,$$

(4.3) also holds with $n+1$ in place of n. This finishes the induction and the proof of (4.3). As (4.3) includes (4.1) except the rightmost inequality, a proof of this latter inequality with $n+1$ in place of n is needed. By Lemma 4.7, we have $n+2 < \lfloor \frac{\delta_{n+1}+1}{2} \rfloor \leq \lceil \frac{\delta_{n+1}+1}{2} \rceil < \delta_n$ so that

$$\begin{aligned}
&1 - \frac{c_{n+1,\lfloor(\delta_{n+1}+1)/2\rfloor-1}}{c_{n+1,\lceil(\delta_{n+1}+1)/2\rceil}} \\
&= 1 - \frac{\sum_{i=\lfloor(\delta_{n+1}+1)/2\rfloor-(n+1)}^{\lfloor(\delta_{n+1}+1)/2\rfloor-1} 2c_{n,i}}{\sum_{i=\lceil(\delta_{n+1}+1)/2\rceil-n}^{\lceil(\delta_{n+1}+1)/2\rceil} 2c_{n,i}} \\
&= \begin{cases} \frac{c_{n,(\delta_{n+1}+1)/2}-c_{n,(\delta_{n+1}+1)/2-(n+1)}}{\sum_{i=(\delta_{n+1}+1)/2-n}^{(\delta_{n+1}+1)/2} c_{n,i}} & \text{if } \delta_{n+1} \text{ is odd,} \\ \frac{c_{n,\delta_{n+1}/2}+c_{n,\delta_{n+1}/2+1}-(c_{n,\delta_{n+1}/2-(n+1)}+c_{n,\delta_{n+1}/2-n})}{\sum_{i=(\delta_{n+1}/2)-(n-1)}^{(\delta_{n+1}/2)+1} c_{n,i}} & \text{if } \delta_{n+1} \text{ is even} \end{cases} \\
&> 0
\end{aligned}$$

because $0 < \frac{\delta_{n+1}+1}{2} - \frac{\delta_n+1}{2} < \frac{\delta_n+1}{2} - \left(\frac{\delta_{n+1}+1}{2} - (n+1)\right) \Rightarrow c_{n,(\delta_{n+1}+1)/2-(n+1)} < c_{n,(\delta_{n+1}+1)/2}$, etc. This finishes the proof of (4.1) with $n+1$ in place of n. Proofs of (4.2) and (4.4), being similar, are omitted. □

Corollary 4.9. *For $n \geq 4$, the sequence of coefficients of $d_n^D(q)$ is log-concave, that is,*

$$\frac{c_{n,0}}{c_{n,1}} < \frac{c_{n,1}}{c_{n,2}} < \cdots < \frac{c_{n,\delta_n-2}}{c_{n,\delta_n-1}} < \frac{c_{n,\delta_n-1}}{c_{n,\delta_n}}. \tag{4.5}$$

Proof. First note that (4.1) and (4.2) have, respectively, $\lfloor \frac{\delta_n+1}{2} \rfloor$ and $\lceil \frac{\delta_n+1}{2} \rceil - 1$ parts, excluding the rightmost 1. Consider the case that δ_n is odd. Multiplying (4.1) to (4.2) with the rightmost 1 included, we get

$$\frac{c_{n,0}}{c_{n,1}} < \frac{c_{n,1}}{c_{n,2}} < \cdots < \frac{c_{n,(\delta_n+1)/2-1}}{c_{n,(\delta_n+1)/2}} < 1. \tag{4.6}$$

Multiplying (4.1) with $c_{n,0}/c_{n,\delta_n}$ dropped to (4.2), we get

$$\frac{c_{n,\delta_n}}{c_{n,\delta_n-1}} < \frac{c_{n,\delta_n-1}}{c_{n,\delta_n-2}} < \cdots < \frac{c_{n,(\delta_n+1)/2+1}}{c_{n,(\delta_n+1)/2}} < 1.$$

By taking reciprocals of the preceding chain of inequalities, we then have

$$1 < \frac{c_{n,(\delta_n+1)/2}}{c_{n,(\delta_n+1)/2+1}} < \cdots < \frac{c_{n,\delta_n-2}}{c_{n,\delta_n-1}} < \frac{c_{n,\delta_n-1}}{c_{n,\delta_n}}. \tag{4.7}$$

Chaining up (4.6) and (4.7), (4.5) follows. The proof of the even δ_n case, being similar, is omitted. □

It is readily checked that for $2 \leq n \leq 3$, coefficients of $d_n^D(q)$ are unimodal, log-concave and possess the spiral property. For $n \geq 4$, the strict ratio monotonicity of the coefficients of $d_n^D(q)$ insures that the maximum coefficient of $d_n^D(q)$ occurs as the last spirally listed one, which is that of q^{k_n}, where $k_n = \lfloor \frac{1}{2}(\binom{n}{2}+1) \rfloor = \lceil \frac{n(n-1)}{4} \rceil$. Summarising, we have the next theorem.

Theorem 4.10. *For $n \geq 4$, coefficients of $d_n^D(q)$ are strictly unimodal, log-concave, and possess the spiral property. Moreover, the maximal coefficient of $d_n^D(q)$ is uniquely attained as that of $q^{\lceil n(n-1)/4 \rceil}$.*

We have studied in this work the derangement polynomial $d_n^D(q) = \sum_{\sigma \in \mathcal{D}_n^D} q^{\mathrm{maj}(\sigma)}$ by the major indices of $\sigma \in \mathcal{D}_n^D$, and proved the strict ratio monotonicity of $d_n^D(q)$, which in turn implies the log-concavity, spiral property and unimodality of $d_n^D(q)$. A next extension is to study the derangement polynomial $\sum_{\sigma \in \mathcal{D}_n^D} q^{\mathrm{fmaj}_D(\sigma)}$, where fmaj_D is a suitably defined flag major index over D_n. This is a subject for further research.

Acknowledgments

The author thanks the referees for careful reading of the manuscript and valuable comments/suggestions that have led to much improved readability of this paper.

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