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# Non-Gorenstein Locus and Almost Gorenstein Property of the Ehrhart Ring of the Stable Set Polytope of a Cycle Graph

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Abstract. Let $R$ be the Ehrhart ring of the stable set polytope of a cycle graph which is not Gorenstein. We describe the non-Gorenstein locus of $\operatorname{Spec} R$. Further, we show that $R$ is almost Gorenstein. Moreover, we show that the conjecture of Hibi and Tsuchiya is true.

## 1. Introduction

In this paper, we call a simple graph consisting of exactly one cycle a cycle graph. An even cycle graph, i.e., a cycle graph with even vertices is bipartite and therefore is perfect. By the result of Ohsugi and Hibi [14, Theorem 2.1(b)], the Ehrhart ring of the stable set polytope of a perfect graph is Gorenstein if and only if sizes of maximal cliques are constant. In particular, the Ehrhart ring of the stable set polytope of an even cycle graph is Gorenstein.

On the other hand, in the course of studying the $h$-vector of graded Cohen–Macaulay rings, Hibi and Tsuchiya [8, Theorem 1] showed that the Ehrhart ring of the stable set polytope of an odd cycle graph is Gorenstein if and only if the size of the cycle is less than or equal to 5. They used the fact that cycle graphs are $t$-perfect. Later the present author vastly generalized this result to general $t$-perfect graphs and characterized when the Ehrhart ring of the stable set polytope is Gorenstein completely [12]: the Ehrhart ring of the stable set polytope of a $t$-perfect graph $G = (V, E)$ is Gorenstein if and only if (i) $E = \emptyset$, (ii) $G$ has no isolated vertex nor triangle and there is no odd cycle without chord and length at least 7 or (iii) every maximal clique of $G$ has size at least 3 and there is no odd cycle without chord and length at least 5.

In this paper, we study the Ehrhart ring of the stable set polytope of a cycle graph which is not Gorenstein. Our main tool of research is the trace ideal of the canonical module. Herzog, Hibi and Stamate [6, Lemma 2.1] showed that if $R$ is a Cohen–Macaulay local or graded ring over a field with canonical module $\omega_R$, then $R_{\mathfrak{p}}$ is Gorenstein if and

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only if $\mathfrak{p} \not\supset \operatorname{tr}(\omega_R)$ for $\mathfrak{p} \in \operatorname{Spec} R$, where $\operatorname{tr}(\omega_R)$ is the trace of $\omega_R$. In particular, $\operatorname{tr}(\omega_R)$ is a defining ideal of the non-Gorenstein locus of $R$. Therefore, one can study the non-Gorenstein locus of $\operatorname{Spec} R$ of a Cohen–Macaulay ring $R$ by examining the trace of its canonical module.

In this paper, we first describe the non-Gorenstein locus of the $\operatorname{Spec} R$ of the Ehrhart ring $R$ of the stable set polytope of an odd cycle graph with length at least 7. We describe minimal prime ideals of $\operatorname{tr}(\omega_R)$ explicitly and show that all of them have exactly the half dimension of the dimension of $R$.

Next we analyze the structure of $\omega_R$ more precisely. We show that there is a unique monomial in $\omega_R$ with minimal degree. Moreover, we show that there is a unique system of generators of $\omega_R$ consisting of monomials. By analyzing the margin of monomials in $\omega_R$ with respect to the odd cycle condition of $t$-perfect graphs, we classify the monomials in $\omega_R$ and express the structure of $\omega_R$ by using this classification. Using this expression of the structure of $\omega_R$, we show that $R$ is an almost Gorenstein graded ring (for the definition of almost Gorenstein property, see [5]).

Finally, we study the Ulrich module appeared in the investigation of almost Gorenstein property and show that the conjecture of Hibi and Tsuchiya [8, Conjecture 1] is true.

## 2. Preliminaries

In this section, we establish notation and terminology. For unexplained terminology of commutative algebra, we consult [1] and of graph theory, we consult [3].

In this paper, all rings and algebras are assumed to be commutative with an identity element. Further, all graphs are assumed to be finite, simple and without loop. We denote the set of nonnegative integers, the set of integers, the set of rational numbers and the set of real numbers by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ respectively.

For a set $X$, we denote by $\#X$ the cardinality of $X$. For sets $X$ and $Y$, we define $X \setminus Y := \{x \in X \mid x \notin Y\}$. For nonempty sets $X$ and $Y$, we denote the set of maps from $X$ to $Y$ by $Y^X$. If $X$ is a finite set, we identify $\mathbb{R}^X$ with the Euclidean space $\mathbb{R}^{\#X}$. For $f, f_1, f_2 \in \mathbb{R}^X$ and $a \in \mathbb{R}$, we define maps $f_1 \pm f_2$ and $af$ by $(f_1 \pm f_2)(x) = f_1(x) \pm f_2(x)$ and $(af)(x) = a(f(x))$ for $x \in X$. Let $A$ be a subset of $X$. We define the characteristic function $\chi_A \in \mathbb{R}^X$ of $A$ by $\chi_A(x) = 1$ for $x \in A$ and $\chi_A(x) = 0$ for $x \in X \setminus A$. For a nonempty subset $\mathscr{X}$ of $\mathbb{R}^X$, we denote by $\operatorname{conv} \mathscr{X}$ (resp. $\operatorname{aff} \mathscr{X}$) the convex hull (resp. affine span) of $\mathscr{X}$.

**Definition 2.1.** Let $X$ be a finite set and $\xi \in \mathbb{R}^X$. For $B \subset X$, we set $\xi^+(B) := \sum_{b \in B} \xi(b)$.

A stable set of a graph $G = (V, E)$ is a subset $S$ of $V$ with no two elements of $S$ are adjacent. We treat the empty set as a stable set.

**Definition 2.2.** The stable set polytope $\mathrm{STAB}(G)$ of a graph $G = (V, E)$ is

$$\mathrm{conv}\{\chi_S \in \mathbb{R}^V \mid S \text{ is a stable set of } G\}.$$

Note that $\chi_{\{v\}} \in \mathrm{STAB}(G)$ for any $v \in V$ and $\chi_\emptyset \in \mathrm{STAB}(G)$. In particular, $\dim \mathrm{STAB}(G) = \#V$.

Next we fix notation about Ehrhart rings. Let $\mathbb{K}$ be a field, $X$ a finite set and $\mathscr{P}$ a rational convex polytope in $\mathbb{R}^X$, i.e., a convex polytope whose vertices are contained in $\mathbb{Q}^X$. Let $-\infty$ be a new element with $-\infty \notin X$ and set $X^- := X \cup \{-\infty\}$. Also let $\{T_x\}_{x \in X^-}$ be a family of indeterminates indexed by $X^-$. For $f \in \mathbb{Z}^{X^-}$, we denote the Laurent monomial $\prod_{x \in X^-} T_x^{f(x)}$ by $T^f$. We set $\deg T_x = 0$ for $x \in X$ and $\deg T_{-\infty} = 1$. Then the Ehrhart ring of $\mathscr{P}$ over a field $\mathbb{K}$ is the $\mathbb{N}$-graded subring

$$\mathbb{K}\left[T^f \;\middle|\; f \in \mathbb{Z}^{X^-}, f(-\infty) > 0, \frac{1}{f(-\infty)} f\Big|_X \in \mathscr{P}\right]$$

of the Laurent polynomial ring $\mathbb{K}[T_x^{\pm 1} \mid x \in X^-]$, where $f|_X$ is the restriction of $f$ to $X$. We denote the Ehrhart ring of $\mathscr{P}$ over $\mathbb{K}$ by $E_{\mathbb{K}}[\mathscr{P}]$.

It is known that $E_{\mathbb{K}}[\mathscr{P}]$ is Noetherian and $\dim E_{\mathbb{K}}[\mathscr{P}] = \dim \mathscr{P} + 1$. It is also known that $E_{\mathbb{K}}[\mathscr{P}]$ is normal and Cohen–Macaulay by the result of Hochster [9]. Moreover, by the description of the canonical module of a normal affine semigroup ring by Stanley [15, p. 82], we see the following

**Lemma 2.3.** *The ideal*

$$\bigoplus_{f \in \mathbb{Z}^{X^-}, f(-\infty) > 0, \frac{1}{f(-\infty)} f\big|_X \in \mathrm{relint}\,\mathscr{P}} \mathbb{K} T^f$$

*of* $E_{\mathbb{K}}[\mathscr{P}]$ *is the canonical module of* $E_{\mathbb{K}}[\mathscr{P}]$*, where* relint $\mathscr{P}$ *denotes the interior of* $\mathscr{P}$ *in the topological space* aff $\mathscr{P}$.

We denote the ideal of the above lemma by $\omega_{E_{\mathbb{K}}[\mathscr{P}]}$ and call the canonical ideal of $E_{\mathbb{K}}[\mathscr{P}]$.

Let $G = (V, E)$ be a graph. $G$ is, by definition, $t$-perfect if

$$\mathrm{STAB}(G) = \left\{ f \in \mathbb{R}^V \;\middle|\; \begin{array}{l} 0 \le f(x) \le 1 \text{ for any } x \in V,\ f^+(e) \le 1 \text{ for any } e \in E \\ \text{and } f^+(C) \le \dfrac{\#C - 1}{2} \text{ for any odd cycle } C \text{ in } G \end{array} \right\}.$$

Let $G = (V, E)$ be a cycle graph with length $n$, i.e., $V = \{v_0, v_1, \ldots, v_{n-1}\}$, $E = \{\{v_i, v_j\} \mid i - j \equiv 1 \pmod{n}\}$. It is shown by Mahjoub [10] that $G$ is $t$-perfect. If $n$ is even, then $G$ is a bipartite graph and therefore is perfect. Thus, by the result of Ohsugi and Hibi [14, Theorem 2.1(b)], $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is Gorenstein.

In this paper, we study $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ the case where it is not Gorenstein. Therefore, we assume that the length of $G$ is $2\ell+1$, where $\ell$ is a positive integer. Hibi and Tsuchiya [8, Theorem 1] (see also [12, Corollary 3.9]) showed that if $G$ is a cycle graph of length $2\ell+1$, then $E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is Gorenstein if and only if $\ell \leq 2$. Therefore, we mainly consider the case where $\ell \geq 3$.

Our main tool is the result of Herzog, Hibi and Stamate [6] about the trace of the canonical module.

**Definition 2.4.** Let $R$ be a ring and $M$ an $R$-module. We set

$$\mathrm{tr}(M) := \sum_{\varphi \in \mathrm{Hom}(M,R)} \varphi(M)$$

and call $\mathrm{tr}(M)$ the trace of $M$.

We recall the following

**Fact 2.5.** [6, Lemma 1.1] Let $R$ be a ring and $I$ an ideal of $R$ containing an $R$-regular element. Also let $Q(R)$ be the total quotient ring of fractions of $R$ and set $I^{-1} := \{x \in Q(R) \mid xI \subset R\}$. Then

$$\mathrm{tr}(I) = I^{-1}I.$$

Note that if $R$ is a Noetherian normal domain and $I$ is a divisorial ideal, then $I^{-1} = I^{(-1)}$, the inverse element of $I$ in $\mathrm{Div}(R)$. Moreover, we recall the following

**Fact 2.6.** [6, Lemma 2.1] Let $R$ be a Cohen–Macaulay local or graded ring over a field with canonical module $\omega_R$. Then for $\mathfrak{p} \in \mathrm{Spec}(R)$,

$$R_{\mathfrak{p}} \text{ is Gorenstein} \iff \mathfrak{p} \not\supset \mathrm{tr}(\omega_R).$$

In particular, the non-Gorenstein locus of $\mathrm{Spec}\, R$ is $V(\mathrm{tr}(\omega_R))$.

We also recall the following our previous results.

**Definition 2.7.** Let $G' = (V', E')$ be a graph. We set

$$\begin{aligned}\mathscr{K} &= \mathscr{K}(G')\\ &:= \{K \subset V' \mid K \text{ is a clique of } G' \text{ and size of } K \text{ is less than or equal to } 3\}.\end{aligned}$$

Further, for $n \in \mathbb{Z}$, we define $t\mathcal{U}^{(n)}(G')$ to be the set of $\mu \in \mathbb{Z}^{V'^{-}}$ satisfying the following conditions:

(1) $\mu(z) \geq n$ for any $z \in V'$,

(2) $\mu^{+}(K) \leq \mu(-\infty) - n$ for any maximal element $K$ of $\mathscr{K}$ and

(3) $\mu^+(C) \le \mu(-\infty)\frac{\#C-1}{2} - n$ for any odd cycle $C$ without chord and length at least 5.

By this notation, the following holds.

**Fact 2.8.** [12, Remark 3.10] If $G'$ is a $t$-perfect graph, then

$$\omega^{(n)}_{E_{\mathbb{K}}[\mathrm{STAB}(G')]} = \bigoplus_{\mu \in t\mathcal{U}^{(n)}(G')} \mathbb{K}T^{\mu}$$

for any $n \in \mathbb{Z}$, where $\omega^{(n)}_{E_{\mathbb{K}}[\mathrm{STAB}(G')]}$ is the $n$-th power of $\omega_{E_{\mathbb{K}}[\mathrm{STAB}(G')]}$ in $\mathrm{Div}(E_{\mathbb{K}}[\mathrm{STAB}(G')])$.

We abbreviate $t\mathcal{U}^{(n)}(G)$ as $t\mathcal{U}^{(n)}$ in the rest of this paper. The following lemma is very easily proved but very useful.

**Lemma 2.9.** *Suppose that $\eta \in t\mathcal{U}^{(1)}$ and $\zeta \in t\mathcal{U}^{(-1)}$. If $x \in V$ and $(\eta + \zeta)(x) = 0$, then $\eta(x) = 1$ and $\zeta(x) = -1$.*

## 3. Non-Gorenstein loci of the Ehrhart rings of the stable set polytopes of cycle graphs

In this section, we state the non-Gorenstein loci of the Ehrhart rings of the stable set polytopes of cycle graphs. Since it is known that the Ehrhart rings of the stable set polytopes of even graphs and odd cycle graphs with length at most 5 are Gorenstein, we focus our attention to odd cycles with length at least 7.

Let $G = (V, E)$ be a cycle graph with length $2\ell+1$, where $\ell$ is an integer with $\ell \ge 3$. We set $R = E_{\mathbb{K}}[\mathrm{STAB}(G)]$, $V = \{v_0, v_1, \ldots, v_{2\ell}\}$ and $E = \{\{v_i, v_j\} \mid i - j \equiv 1 \pmod{2\ell+1}\}$. Further, we set $e_j = \{v_j, v_{j+1}\}$ for $0 \le j \le 2\ell - 1$ and $e_{2\ell} = \{v_{2\ell}, v_0\}$.

Let $\mathfrak{p}_i$ be the ideal of $R$ generated by $\{T^{\mu} \mid \mu \in t\mathcal{U}^{(0)}, \mu(v_i) > 0 \text{ or } \mu^+(V) < \ell\mu(-\infty)\}$, i.e.,

$$\mathfrak{p}_i = \bigoplus_{\substack{\mu \in t\mathcal{U}^{(0)} \\ \mu(v_i) > 0 \text{ or } \mu^+(V) < \ell\mu(-\infty)}} \mathbb{K}T^{\mu}$$

for $0 \le i \le 2\ell$. Then $\mathfrak{p}_i$ is the prime ideal corresponding to the face $\mathscr{P}_i = \{f \in \mathrm{STAB}(G) \mid f(v_i) = 0, f^+(V) = \ell\}$ of $\mathrm{STAB}(G)$, i.e., $E_{\mathbb{K}}[\mathscr{P}_i] = R/\mathfrak{p}_i$. With this notation, we see the following

**Theorem 3.1.** *In the above setting, it holds that*

$$\sqrt{\mathrm{tr}(\omega_R)} = \bigcap_{i=0}^{2\ell} \mathfrak{p}_i.$$

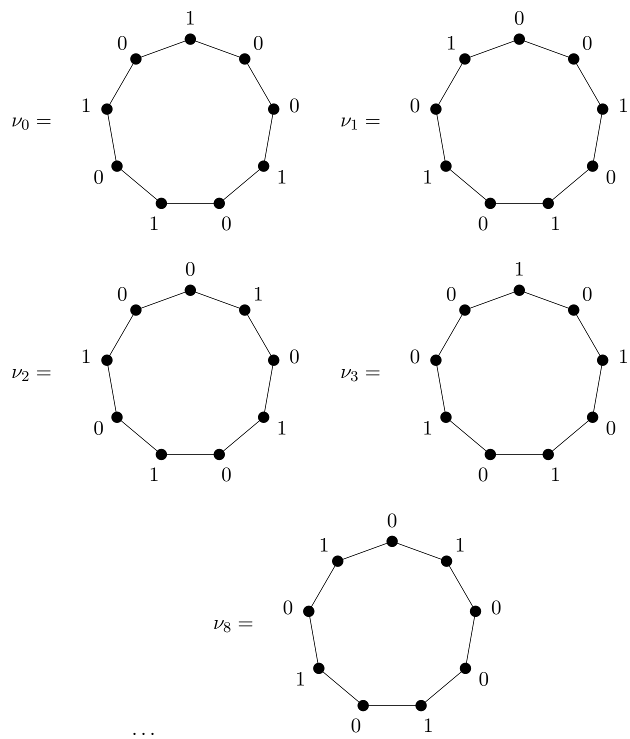
*In particular, non-Gorenstein locus of the Ehrhart ring $R$ of the stable set polytope of $G$ is a closed subset of $\operatorname{Spec} R$ of dimension $\ell + 1$.*

This theorem follows from Lemmas 3.2, 3.3, 3.4 and 3.5 below.

Before stating the lemmas, we need some preparation. For $i$ with $0 \le i \le 2\ell$, we define $\mu_i \in \mathbb{Z}^{V^-}$ by

$$\mu_i(v_j) = \begin{cases} 1 & \text{if } j - i \equiv 0, 2, 4, \dots, 2\ell - 2 \pmod{2\ell+1}, \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \mu_i(-\infty) = 1$$

and set $\nu_i = \mu_i|_V$. For example, the case where $\ell = 4$, $\nu_i$ are as follows, where the top vertex is $v_0$ and $v_1, v_2, \dots, v_8$ are aligned anti-clockwise.



Next we analyze the dimension of $\mathscr{P}_i$. By symmetry, we see that $\dim \mathscr{P}_i = \dim \mathscr{P}_0$ for any $i$.

If $f \in \mathscr{P}_0$, then $f(v_0) = 0$ and $f^+(V) = \ell$. Since $f^+(V) = f(v_0) + \sum_{j=1}^{\ell} f^+(e_{2j-1})$ and $f^+(e_j) \le 1$ for any $j$, we see that

$$f^+(e_{2j-1}) = 1 \quad \text{for } 1 \le j \le \ell.$$

These linear equations and $f(v_0) = 0$ are independent. Therefore we see that $\dim \mathscr{P}_0 \le 2\ell + 1 - (\ell + 1) = \ell$.

Next we prove the reverse inequality. It is easily verified that $\nu_2, \nu_4, \dots, \nu_{2\ell}$ and $\nu_1$ are elements of $\mathscr{P}_0$. The matrix whose columns correspond to $v_0, v_1, \dots, v_{2\ell}$ and rows correspond to $\nu_4 - \nu_2, \nu_6 - \nu_4, \dots, \nu_{2\ell} - \nu_{2\ell-2}, \nu_1 - \nu_{2\ell}$ respectively is

$$\begin{bmatrix} 0 & 1 & -1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 & \cdots & \cdots & 0 & 0 & 0 & 0 \\ & & & & \ddots & \ddots & & & & & \\ & & & & & \ddots & \ddots & & & & \\ & & & & & & \ddots & \ddots & & & \\ 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 0 & 1 & -1 \end{bmatrix}.$$

This is a rank $\ell$ matrix. Therefore, we see the following

**Lemma 3.2.** *It holds that* $\dim \mathscr{P}_i = \ell$ *for any* $i$ *with* $0 \le i \le 2\ell$. *In particular,* $\dim R/\mathfrak{p}_i = \ell + 1$ *for any* $i$ *with* $0 \le i \le 2\ell$.

Next we state the following

**Lemma 3.3.** *Let* $i$ *be an integer with* $0 \le i \le 2\ell$. *Then* $\mathfrak{p}_i \supset \operatorname{tr}(\omega_R)$.

*Proof.* We may assume that $i = 0$. Let $T^\mu$ be an arbitrary monomial in $\operatorname{tr}(\omega_R)$. We deduce a contradiction by assuming $T^\mu \notin \mathfrak{p}_0$.

Since $T^\mu \in \operatorname{tr}(\omega_R)$, there are $\eta \in t\mathcal{U}^{(1)}$ and $\zeta \in t\mathcal{U}^{(-1)}$ with $\mu = \eta + \zeta$. Since $T^\mu \notin \mathfrak{p}_0$, it holds that $\mu(v_0) = 0$ and therefore $\eta(v_0) = 1$ by Lemma 2.9. Thus,

$$\eta^+(V) = \sum_{j=1}^{\ell} \eta^+(e_{2j-1}) + 1.$$

Since $\eta^+(e_{2j-1}) + 1 \le \eta(-\infty)$ for $1 \le j \le \ell$, we see that

$$\eta^+(V) \le \sum_{j=1}^{\ell} (\eta(-\infty) - 1) + 1 = \ell\eta(-\infty) - \ell + 1.$$

On the other hand, since $T^\mu \notin \mathfrak{p}_0$,

$$\mu^+(V) = \ell\mu(-\infty).$$

Moreover, since

$$\eta^+(V) + 1 \le \ell\eta(-\infty), \quad \zeta^+(V) - 1 \le \ell\zeta(-\infty),$$
$$\eta(-\infty) + \zeta(-\infty) = \mu(-\infty) \quad \text{and} \quad \eta^+(V) + \zeta^+(V) = \mu^+(V) = \ell\mu(-\infty),$$

we see that

$$\eta^+(V)+1=\ell\eta(-\infty).$$

Therefore,

$$\ell\eta(-\infty)-1=\eta^+(V)\leq\ell\eta(-\infty)-\ell+1$$

and we see that $\ell\leq 2$. This contradicts to the assumption. □

By Lemma 3.3, we see that

$$\operatorname{tr}(\omega_R)\subset\bigcap_{i=0}^{2\ell}\mathfrak{p}_i.$$

Since the right hand side is a radical ideal, we see that

$$\sqrt{\operatorname{tr}(\omega_R)}\subset\bigcap_{i=0}^{2\ell}\mathfrak{p}_i.$$

In order to show the reverse inclusion, we first state the following

**Lemma 3.4.** *If $\mu\in t\mathcal{U}^{(0)}$ and $\mu(v_i)>0$ for any $i$, then $T^\mu\in\sqrt{\operatorname{tr}(\omega_R)}$.*

*Proof.* Define $\eta,\zeta\in\mathbb{Z}^{V^-}$ by

$$\eta(x)=\begin{cases}\ell-1 & \text{if } x\in V,\\ 2\ell-1 & \text{if } x=-\infty\end{cases}\quad\text{and}\quad\zeta(x)=\begin{cases}(\ell-2)\mu(x)-\ell+1 & \text{if } x\in V,\\ (\ell-2)\mu(x)-2\ell+1 & \text{if } x=-\infty.\end{cases}$$

Then $\eta+\zeta=(\ell-2)\mu$.

For $x\in V$,

$$\eta(x)=\ell-1\geq 1\quad\text{and}\quad\zeta(x)=(\ell-2)\mu(x)-\ell+1\geq\ell-2-\ell+1=-1$$

since $\ell\geq 3$ and $\mu(x)\geq 1$. Further, for $e\in E$,

$$\eta^+(e)+1=2(\ell-1)+1=2\ell-1=\eta(-\infty)$$

and

$$\zeta^+(e)-1=(\ell-2)\mu^+(e)-2(\ell-1)-1\leq(\ell-2)\mu(-\infty)-2\ell+1=\zeta(-\infty).$$

Finally,

$$\eta^+(V)+1=(2\ell+1)(\ell-1)+1=2\ell^2-\ell=\ell(2\ell-1)=\ell\eta(-\infty)$$

and

$$\begin{aligned}\zeta^+(V)-1&=(\ell-2)\mu^+(V)-(2\ell+1)(\ell-1)-1\leq\ell(\ell-2)\mu(-\infty)-2\ell^2+\ell\\&=\ell((\ell-2)\mu(-\infty)-2\ell+1)=\ell\zeta(-\infty).\end{aligned}$$

Therefore, $\eta \in t\mathcal{U}^{(1)}$ and $\zeta \in t\mathcal{U}^{(-1)}$. Thus, we see that

$$(T^\mu)^{\ell-2} = T^{(\ell-2)\mu} = T^\eta T^\zeta \in \omega_R \omega_R^{(-1)} = \mathrm{tr}(\omega_R)$$

and therefore

$$T^\mu \in \sqrt{\mathrm{tr}(\omega_R)}. \qquad \square$$

Finally, we state the following

**Lemma 3.5.** *Let $\mu \in t\mathcal{U}^{(0)}$ and $\mu^+(V) < \ell\mu(-\infty)$. Then $T^\mu \in \sqrt{\mathrm{tr}(\omega_R)}$.*

*Proof.* Define $\eta, \zeta \in \mathbb{Z}^{V^-}$ by

$$\eta(x) = \begin{cases} 1 & \text{if } x \in V, \\ 3 & \text{if } x = -\infty \end{cases} \quad \text{and} \quad \zeta(x) = \begin{cases} (\ell-2)\mu(x) - 1 & \text{if } x \in V, \\ (\ell-2)\mu(-\infty) - 3 & \text{if } x = -\infty. \end{cases}$$

Then $\eta + \zeta = (\ell-2)\mu$.

It is obvious that $\eta(x) \geq 1$ and $\zeta(x) \geq -1$ for any $x \in V$. Let $e$ be an arbitrary edge in $G$. Then

$$\eta^+(e) + 1 = 2 + 1 = \eta(-\infty)$$

and

$$\zeta^+(e) - 1 = (\ell-2)\mu^+(e) - 2 - 1 \leq (\ell-2)\mu(-\infty) - 3 = \zeta(-\infty).$$

Further,

$$\eta^+(V) + 1 = (2\ell+1) + 1 \leq 3\ell = \ell\eta(-\infty)$$

and, since $\mu^+(V) + 1 \leq \ell\mu(-\infty)$ by assumption, we see that

$$\begin{aligned} \zeta^+(V) - 1 &= (\ell-2)\mu^+(V) - (2\ell+1) - 1 = (\ell-2)(\mu^+(V)+1) - 3\ell \\ &\leq (\ell-2)\ell\mu(-\infty) - 3\ell = \ell\zeta(-\infty). \end{aligned}$$

Therefore, $\eta \in t\mathcal{U}^{(1)}$ and $\zeta \in t\mathcal{U}^{(-1)}$. Thus, we see that

$$(T^\mu)^{\ell-2} = T^{(\ell-2)\mu} = T^\eta T^\zeta \in \omega_R \omega_R^{(-1)} = \mathrm{tr}(\omega_R)$$

and therefore

$$T^\mu \in \sqrt{\mathrm{tr}(\omega_R)}. \qquad \square$$

## 4. Almost Gorenstein property

In this section, we show that the Ehrhart rings of the stable set polytopes of cycle graphs are almost Gorenstein graded rings. For the definition of almost Gorenstein property, see [5]. In this paper, we only treat almost Gorenstein graded property and we say almost Gorenstein graded as almost Gorenstein for short. We focus our attention to odd cycle graphs of length at least 7 by the same reason as the previous section.

Our main purpose of this section is to show the following

**Theorem 4.1.** *Let $G$ be a cycle graph with length $2\ell+1$, where $\ell$ is an integer with $\ell \geq 3$. Then $R = E_{\mathbb{K}}[\mathrm{STAB}(G)]$ is an almost Gorenstein ring.*

In order to prove this theorem, we need some preparation. Let $\ell$, $G$, $R$, $v_i$, $e_i$, $\mu_i$ and $\nu_i$ for $0 \leq i \leq 2\ell$ be as in Section 3. Further, for integer $k$ with $1 \leq k \leq \ell - 1$, we define $\eta_k \in \mathbb{Z}^{V^-}$ by

$$\eta_k(x) = \begin{cases} k & \text{if } x \in V, \\ 2k+1 & \text{if } x = -\infty. \end{cases}$$

Then it is easily verified that $\eta_k \in t\mathcal{U}^{(1)}$ for any $k$. Note that $\ell\eta_k(-\infty) - \eta_k^+(V) = \ell - k$ and $\eta_k^+(e) + 1 = \eta_k(-\infty)$ for any $e \in E$ and $1 \leq k \leq \ell - 1$.

Since for any $\eta \in t\mathcal{U}^{(1)}$ and for any $i$ with $0 \leq i \leq 2\ell$, it holds that

$$\eta(-\infty) \geq \eta^+(e_i) + 1 \geq 3,$$

we see that $T^{\eta_1}$ is the unique monomial in $\omega_R$ with minimum degree 3. In particular, $a(R) = -\deg T^{\eta_1} = -3$. Therefore, we consider the morphism $\varphi\colon R \to \omega_R(3)$ of graded $R$-modules with $\varphi(1) = T^{\eta_1}$. $R$ is, by definition, almost Gorenstein if and only if $\operatorname{Cok}\varphi$ is an Ulrich module.

**Lemma 4.2.** *It holds that*

$$(\operatorname{Im}\varphi)(-3) = \bigoplus_{\substack{\eta \in t\mathcal{U}^{(1)} \\ \ell\eta(-\infty) - \eta^+(V) \geq \ell - 1}} \mathbb{K}T^{\eta}.$$

*Proof.* First note that $(\operatorname{Im}\varphi)(-3) = T^{\eta_1}R$.

Let $\eta$ be an arbitrary element of $t\mathcal{U}^{(1)}$ with $\ell\eta(-\infty) - \eta^+(V) \geq \ell - 1$. Set $\mu = \eta - \eta_1$. Then

$$\mu(x) = \eta(x) - \eta_1(x) = \eta(x) - 1 \geq 0$$

for any $x \in V$,

$$\mu^+(e) = \eta^+(e) - \eta_1^+(e) = \eta^+(e) - 2 = (\eta^+(e) + 1) - 3 \leq \eta(-\infty) - \eta_1(-\infty) = \mu(-\infty)$$

for any $e \in E$ and

$$\begin{aligned}\ell\mu(-\infty) - \mu^+(V) &= \ell\eta(-\infty) - \eta^+(V) - (\ell\eta_1(-\infty) - \eta_1^+(V))\\ &\geq (\ell-1) - (\ell-1)\\ &= 0.\end{aligned}$$

Therefore, we see that $\mu \in t\mathcal{U}^{(0)}$. Thus, we see that

$$T^\eta = T^{\eta_1}T^\mu \in T^{\eta_1}R.$$

On the other hand, if $T^\eta$ is a monomial in $T^{\eta_1}R$, then there is $\mu \in t\mathcal{U}^{(0)}$ with $\eta = \mu + \eta_1$. Since $T^\eta \in T^{\eta_1}R \subset \omega_R$, we see that $\eta \in t\mathcal{U}^{(1)}$ by Fact 2.8. Further,

$$\begin{aligned}\ell\eta(-\infty) - \eta^+(V) &= \ell\mu(-\infty) - \mu^+(V) + \ell\eta_1(-\infty) - \eta_1^+(V)\\ &\geq \ell\eta_1(-\infty) - \eta_1^+(V)\\ &= \ell - 1.\end{aligned}$$

Thus we see that

$$(\operatorname{Im}\varphi)(-3) = T^{\eta_1}R = \bigoplus_{\substack{\eta \in t\mathcal{U}^{(1)}\\ \ell\eta(-\infty)-\eta^+(V)\geq \ell-1}} \mathbb{K}T^\eta. \qquad \square$$

Next, we set

$$\mathscr{P} := \{f \in \mathrm{STAB}(G) \mid f^+(V) = \ell\}.$$

Then $\mathscr{P}$ is a face of $\mathrm{STAB}(G)$. Further, we set

$$t\mathcal{U}_0^{(0)} := \{\mu \in t\mathcal{U}^{(0)} \mid \mu^+(V) = \ell\mu(-\infty)\} \quad \text{and} \quad R^{(0)} := \bigoplus_{\mu \in t\mathcal{U}_0^{(0)}} \mathbb{K}T^\mu.$$

Then $R^{(0)}$ is a subalgebra of $R$ and the Ehrhart ring $E_{\mathbb{K}}[\mathscr{P}]$ of $\mathscr{P}$. Note that $\mu_i \in t\mathcal{U}_0^{(0)}$ for $0 \leq i \leq 2\ell$.

**Lemma 4.3.** *It holds that*

$$R^{(0)} = \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \dots, T^{\mu_{2\ell}}].$$

*Proof.* Since $\mu_i \in t\mathcal{U}_0^{(0)}$ for $0 \leq i \leq 2\ell$, it is clear that $R^{(0)} \supset \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \dots, T^{\mu_{2\ell}}]$. Let $\mu$ be an arbitrary element of $t\mathcal{U}_0^{(0)}$. We prove by induction on $\mu(-\infty)$ that $T^\mu \in \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \dots, T^{\mu_{2\ell}}]$.

If $\mu(-\infty) = 0$, then $\mu = 0$ and $T^\mu = 1 \in \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \dots, T^{\mu_{2\ell}}]$. Suppose that $\mu(-\infty) > 0$. We first consider the case where $\mu(x) > 0$ for any $x \in V$. Since

$$(2\ell+1)\mu(-\infty) > 2\ell\mu(-\infty) = 2\mu^+(V) = \sum_{i=0}^{2\ell} \mu^+(e_i),$$

we see that there is $i$ with $\mu^+(e_i) < \mu(-\infty)$. By symmetry, we may assume that

$$\mu^+(e_0) < \mu(-\infty).$$

Set $\mu' = \mu - \mu_2$. Then

$$\mu'(x) = \mu(x) - \mu_2(x) \geq \mu(x) - 1 \geq 0$$

for any $x \in V$, since $\mu(x) > 0$. If $e \in E$ and $e \neq e_0$, then

$$(\mu')^+(e) = \mu^+(e) - \mu_2^+(e) = \mu^+(e) - 1 \leq \mu(-\infty) - 1 = \mu'(-\infty).$$

Further,

$$(\mu')^+(e_0) = \mu^+(e_0) - \mu_2^+(e_0) = \mu^+(e_0) < \mu(-\infty)$$

by assumption. Therefore,

$$(\mu')^+(e_0) \leq \mu(-\infty) - 1 = \mu'(-\infty).$$

Moreover,

$$(\mu')^+(V) = \mu^+(V) - \mu_2^+(V) = \ell\mu(-\infty) - \ell\mu_2(-\infty) = \ell\mu'(-\infty).$$

Therefore, $\mu' \in t\mathcal{U}_0^{(0)}$ and by induction hypothesis, we see that $T^{\mu'} \in \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \ldots, T^{\mu_{2\ell}}]$. Thus,

$$T^\mu = T^{\mu'} T^{\mu_2} \in \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \ldots, T^{\mu_{2\ell}}].$$

Next consider the case where $\mu(x) = 0$ for some $x \in V$. By symmetry, we may assume that $\mu(v_0) = 0$. Since $\sum_{i=0}^{\ell} \mu^+(e_{2i-1}) = \mu(v_0) + \sum_{i=0}^{\ell} \mu^+(e_{2i-1}) = \mu^+(V) = \ell\mu(-\infty)$ and $\mu^+(e_j) \leq \mu(-\infty)$ for any $j$, we see that

$$\mu(e_{2i-1}) = \mu(-\infty) \quad \text{for any } 1 \leq i \leq \ell.$$

First consider the case where $\mu(v_{2i}) = 0$ for any $1 \leq i \leq \ell$. In this case, $\mu(v_{2i-1}) = \mu^+(e_{2i-1}) = \mu(-\infty) > 0$ for any $1 \leq i \leq \ell$. Set $\mu' = \mu - \mu_1$. Then

$$\mu'(v_{2i-1}) = \mu(v_{2i-1}) - \mu_1(v_{2i-1}) = \mu(v_{2i-1}) - 1 \geq 0$$

for $1 \leq i \leq \ell$ and

$$\mu'(x) = \mu(x) - \mu_1(x) = \mu(x) \geq 0$$

for any $x \in V \setminus \{v_1, v_3, \ldots, v_{2\ell-1}\}$. Further,

$$(\mu')^+(e_{2i-1}) = \mu^+(e_{2i-1}) - \mu_1^+(e_{2i-1}) = \mu(-\infty) - 1 = \mu'(-\infty)$$

for any $1 \le i \le \ell$,

$$(\mu')^+(e_{2i}) = \mu^+(e_{2i}) - \mu_1^+(e_{2i}) = \mu(v_{2i+1}) - 1 = \mu(-\infty) - 1 = \mu'(-\infty)$$

for $0 \le i \le \ell - 1$ and

$$(\mu')^+(e_{2\ell}) = \mu^+(e_{2\ell}) - \mu_1^+(e_{2\ell}) = \mu(v_{2\ell}) + \mu(v_0) = 0 \le \mu(-\infty) - 1 = \mu'(-\infty).$$

Moreover,

$$(\mu')^+(V) = \mu^+(V) - \mu_1^+(V) = \ell\mu(-\infty) - \ell\mu_1(-\infty) = \ell\mu'(-\infty).$$

Therefore, $\mu' \in t\mathcal{U}_0^{(0)}$ and by induction hypothesis, we see that $T^{\mu'} \in \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \ldots, T^{\mu_{2\ell}}]$. Thus,

$$T^{\mu} = T^{\mu'}T^{\mu_1} \in \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \ldots, T^{\mu_{2\ell}}].$$

If $\mu(v_{2i}) \neq 0$ for some $i$ with $1 \le i \le \ell$, set $j = \min\{i \mid 1 \le i \le \ell, \mu(v_{2i}) \neq 0\}$. For $i$ with $1 \le i \le j-1$,

$$\mu(v_{2i-1}) = \mu(v_{2i-1}) + \mu(v_{2i}) = \mu^+(e_{2i-1}) = \mu(-\infty) > 0$$

since $\mu^+(e_{2i-1}) = \mu(-\infty)$ and $\mu(v_{2i}) = 0$. Further, if $j < \ell$, then $\mu(v_{2j}) > 0$ and $\mu(v_{2j}) + \mu(v_{2j+1}) = \mu^+(e_{2j}) \le \mu(-\infty)$, we see that $\mu(v_{2j+1}) < \mu(-\infty)$. Moreover, since $\mu(v_{2j+1}) + \mu(v_{2j+2}) = \mu^+(e_{2j+1}) = \mu(-\infty)$, we see that $\mu(v_{2j+2}) > 0$. By the same argument and induction, we see that

$$\mu(v_{2i}) > 0 \quad \text{for } j \le i \le \ell.$$

Set $\mu' = \mu - \mu_{2j}$. Since

$$\mu_{2j}(v_k) = \begin{cases} 1 & \text{if } k \in \{1, 3, \ldots, 2j-3, 2j, 2j+2, \ldots, 2\ell\}, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\mu(v_k) > 0 \quad \text{if } k \in \{1, 3, \ldots, 2j-3, 2j, 2j+2, \ldots, 2\ell\},$$

we see that

$$\mu(x) \ge 0 \quad \text{for any } x \in V.$$

Further, if $i \neq 2j-2$, then, since $\mu_{2j}^+(e_i) = \mu_{2j}(-\infty) = 1$, we see that

$$(\mu')^+(e_i) = \mu^+(e_i) - \mu_{2j}^+(e_i) \le \mu(-\infty) - \mu_{2j}(-\infty) = \mu'(-\infty).$$

Moreover, since $\mu(v_{2j-1}) < \mu(v_{2j-1}) + \mu(v_{2j}) = \mu^+(e_{2j-1}) = \mu(-\infty)$ and $\mu(v_{2j-2}) = 0$ by the definition of $j$, we see that

$$(\mu')^+(e_{2j-2}) = \mu^+(e_{2j-2}) - \mu_{2j}^+(e_{2j-2}) = \mu(v_{2j-2}) + \mu(v_{2j-1}) \le \mu(-\infty) - 1 = \mu'(-\infty).$$

Finally,

$$(\mu')^+(V) = \mu^+(V) - \mu_{2j}^+(V) = \ell\mu(-\infty) - \ell\mu_{2j}(-\infty) = \ell\mu'(-\infty).$$

Thus, we see that $\mu' \in t\mathcal{U}_0^{(0)}$. Therefore, by induction hypothesis, we see that $T^{\mu'} \in \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \ldots, T^{\mu_{2\ell}}]$ and

$$T^{\mu} = T^{\mu'}T^{\mu_{2j}} \in \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \ldots, T^{\mu_{2\ell}}]. \qquad \square$$

Next we consider the dimension of $\mathscr{P}$. The matrix whose columns correspond to $v_0, v_1, \ldots, v_{2\ell}$ and rows correspond to $\nu_3-\nu_1, \nu_4-\nu_2, \ldots, \nu_{2\ell}-\nu_{2\ell-2}, \nu_0-\nu_{2\ell-1}$ and $\nu_1-\nu_{2\ell}$ respectively is

$$\begin{bmatrix} 1 & -1 & 0 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & \cdots & \cdots & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & \cdots & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & & \ddots & \ddots & & \vdots & \vdots & \vdots \\ \vdots & \vdots & & & \ddots & \ddots & \vdots & \vdots & \vdots \\ \vdots & \vdots & & & & \ddots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & \cdots & \cdots & 0 & 1 & -1 \end{bmatrix}.$$

This is a matrix of rank $2\ell$. Since $\nu_i \in \mathscr{P}$ for $0 \leq i \leq 2\ell$, we see that $\dim \mathscr{P} \geq 2\ell$. On the other hand, $E_{\mathbb{K}}[\mathscr{P}] = R^{(0)} = \mathbb{K}[T^{\mu_0}, T^{\mu_1}, \ldots, T^{\mu_{2\ell}}]$ by Lemma 4.3. Since $\dim E_{\mathbb{K}}[\mathscr{P}] = \dim \mathscr{P} + 1$, we see that $\dim \mathscr{P} = 2\ell$. Moreover, we see that $T^{\mu_0}, T^{\mu_1}, \ldots, T^{\mu_{2\ell}}$ are algebraically independent over $\mathbb{K}$. Since $\deg T^{\mu_i} = \mu(-\infty) = 1$ for $0 \leq i \leq 2\ell$, we see the following

**Lemma 4.4.** *$R^{(0)}$ is isomorphic to the polynomial ring with $2\ell+1$ variables equipped with the standard grading.*

For $k$ with $2 \leq k \leq \ell - 1$, we set

$$C_k := \bigoplus_{\substack{\eta \in t\mathcal{U}^{(1)} \\ \ell\eta(-\infty)-\eta^+(V)=\ell-k}} \mathbb{K}T^{\eta}.$$

Then we see the following

**Lemma 4.5.** *$C_k$ is a rank 1 free $R^{(0)}$-module with basis $T^{\eta_k}$ for $2 \leq k \leq \ell - 1$.*

*Proof.* First, if $\mu \in t\mathcal{U}_0^{(0)}$, then it is easily verified that $\mu + \eta_k \in t\mathcal{U}^{(1)}$ and

$$\ell(\mu + \eta_k)(-\infty) - (\mu + \eta_k)^+(V) = \ell\eta_k(-\infty) - \eta_k^+(V) = \ell - k.$$

Therefore $T^\mu T^{\eta_k} \in C_k$.

Conversely, assume that $\eta \in t\mathcal{U}^{(1)}$ and $\ell\eta(-\infty) - \eta^+(V) = \ell - k$. Set $\mu = \eta - \eta_k$. Then

$$\begin{aligned}\mu^+(V) &= \eta^+(V) - \eta_k^+(V) = \ell\eta(-\infty) - (\ell - k) - (\ell\eta_k(-\infty) - (\ell - k)) \\ &= \ell(\eta(-\infty) - \eta_k(-\infty)) = \ell\mu(-\infty)\end{aligned}$$

and, since $\eta_k^+(e) + 1 = \eta_k(-\infty)$,

$$\mu^+(e) = (\eta^+(e) + 1) - (\eta_k^+(e) + 1) \le \eta(-\infty) - \eta_k(-\infty) = \mu(-\infty).$$

Finally, we show that $\mu(x) \ge 0$ for any $x \in V$. Assume the contrary. Then, by symmetry, we may assume that $\mu(v_0) < 0$. Then

$$\mu^+(V) = \mu(v_0) + \sum_{i=1}^{\ell} \mu^+(e_{2i-1}) < \ell\mu(-\infty),$$

contradicting the fact shown above. Therefore, $\mu(x) \ge 0$ for any $x \in V$ and we see that $\mu \in t\mathcal{U}_0^{(0)}$. Thus,

$$C_k = \Bigl(\bigoplus_{\mu \in t\mathcal{U}_0^{(0)}} \mathbb{K}T^\mu\Bigr) T^{\eta_k} = T^{\eta_k} R^{(0)}.$$

Since $C_k$ and $R^{(0)}$ are contained in a domain $R$, we see that $C_k$ is a rank 1 free $R^{(0)}$-module with basis $T^{\eta_k}$. $\square$

Now we state the proof of Theorem 4.1. Since

$$\begin{aligned}\omega_R &= \bigoplus_{\eta \in t\mathcal{U}^{(1)}} \mathbb{K}T^\eta \\ &= \Bigl(\bigoplus_{\substack{\eta \in t\mathcal{U}^{(1)} \\ \ell\eta(-\infty) - \eta^+(V) \ge \ell - 1}} \mathbb{K}T^\mu\Bigr) \oplus \bigoplus_{k=2}^{\ell-1} \Bigl(\bigoplus_{\substack{\eta \in t\mathcal{U}^{(1)} \\ \ell\eta(-\infty) - \eta^+(V) = \ell - k}} \mathbb{K}T^\eta\Bigr) \\ &= (\operatorname{Im}\varphi)(-3) \oplus \bigoplus_{k=2}^{\ell-1} C_k,\end{aligned}$$

we see that

$$\operatorname{Cok}\varphi \cong \left(\bigoplus_{k=2}^{\ell-1} C_k\right)(3)$$

as graded $R^{(0)}$-modules. Since each $C_k$ is a free $R^{(0)}$-module, $R^{(0)}$ is isomorphic to a polynomial ring with $2\ell+1$ variables over $\mathbb{K}$ and multiplicity of a module can be computed by its Hilbert series, we see that

$$e(\operatorname{Cok}\varphi) = \ell - 2.$$

Next, we show that $T^{\eta_1}, T^{\eta_2}, \ldots, T^{\eta_{\ell-1}}$ is a minimal system of generators of $\omega_R$. Assume the contrary. Then there are $i$ and $j$ with $i \neq j$ and $\mu \in t\mathcal{U}^{(0)}$ such that

$$T^{\eta_i} = T^{\mu} T^{\eta_j}.$$

Since $i = \eta_i(v_0) = \mu(v_0) + \eta_j(v_0) \geq j$, we see that $i > j$ and $\mu(x) = \eta_i(x) - \eta_j(x) = i - j$ for any $x \in V$. On the other hand, it holds that

$$\mu(-\infty) = \eta_i(-\infty) - \eta_j(-\infty) = (2i+1) - (2j+1) = 2(i-j).$$

Therefore,

$$\ell\mu(-\infty) - \mu^+(V) = 2\ell(i-j) - (2\ell+1)(i-j) < 0,$$

contradicting the fact that $\mu \in t\mathcal{U}^{(0)}$.

Thus, we see that $T^{\eta_1}, T^{\eta_2}, \ldots, T^{\eta_{\ell-1}}$ is a minimal system of generators of $\omega_R$ and

$$\mu(\operatorname{Cok}\varphi) = \ell - 2.$$

Therefore, we see that $\operatorname{Cok}\varphi$ is an Ulrich module and $R$ is an almost Gorenstein ring.

## 5. Hibi–Tsuchiya's conjecture

In [7], Hibi made several conjectures on the $h$-vectors of Cohen–Macaulay standard graded algebras. In particular, he conjectured that an $h$-vector of a standard graded Cohen–Macaulay domain is flawless [7, Conjecture 1.4]. The $h$-vector $(h_0, h_1, \ldots, h_s)$, $h_s \neq 0$ of a Cohen–Macaulay standard graded algebra is flawless if $h_i \leq h_{s-i}$ for $0 \leq i \leq \lfloor s/2 \rfloor$ and $h_{i-1} \leq h_i$ for $1 \leq i \leq \lfloor s/2 \rfloor$. Niesi and Robbiano [13] disproved Hibi's conjecture by constructing a Cohen–Macaulay standard graded domain whose $h$-vector is $(1,3,5,4,4,1)$.

Recently, Hibi and Tsuchiya [8] computed by Normaliz [2] if $\ell = 3$, 4 or 5 then the $h$-vector of $R$ is

$$(1,21,84,85,21,1), \quad (1,66,744,2305,2304,745,66,1)$$
$$\text{or} \quad (1,187,5049,37247,96448,96449,37246,5050,187,1)$$

respectively and made the following

**Conjecture 5.1.** [8, Conjecture 1] *If $\ell \geq 3$, then the $h$-vector of $R$ is the following form:*

$$\bigl(1, h_1, h_2, \ldots, h_{\ell-1}, h_{\ell-1} + (-1)^{\ell-1}, h_{\ell-2} + (-1)^{\ell-2}, \ldots, h_3 - 1, h_2 + 1, h_1, 1\bigr).$$

Note that since $\dim R = 2\ell + 2$ and $a(R) = -3$, it holds that $s = \dim R + a(R) = 2\ell - 1$. Now we prove the following

**Theorem 5.2.** *Conjecture* 5.1 *is true.*

*Proof.* We use the notation of the previous section. Also for a finitely generated graded $R$-module $M$, we denote by $H(M,\lambda)$ the Hilbert series of $M$, i.e.,

$$H(M,\lambda) = \sum_{n\in\mathbb{Z}} (\dim_{\mathbb{K}} M_n)\lambda^n.$$

Then

$$H(R,\lambda) = \frac{h_0 + h_1\lambda + \cdots + h_s\lambda^s}{(1-\lambda)^{2\ell+2}}.$$

Further, by the second proof of [15, Theorem 4.1], we see that

$$H(\omega_R(3),\lambda) = \frac{h_s + h_{s-1}\lambda + \cdots + h_0\lambda^s}{(1-\lambda)^{2\ell+2}}.$$

On the other hand, since $C_k$ is a rank 1 free $R^{(0)}$-module with basis $T^{\eta_k}$ and $\deg T^{\eta_k} = 2k+1$, we see that

$$H(C_k,\lambda) = \frac{\lambda^{2k+1}}{(1-\lambda)^{2\ell+1}}$$

as $R^{(0)}$-modules for $2 \le k \le \ell-1$. Further, since $\operatorname{Cok}\varphi = \left(\bigoplus_{k=2}^{\ell-1} C_k\right)(3)$, we see that

$$\dim_{\mathbb{K}}(\operatorname{Cok}\varphi)_n = \sum_{k=2}^{\ell-1} \dim_{\mathbb{K}}(C_k)_{n+3}$$

for any $n \in \mathbb{Z}$. Thus

$$\begin{aligned} H(\operatorname{Cok}\varphi,\lambda) &= \frac{\lambda^2 + \lambda^4 + \cdots + \lambda^{2\ell-4}}{(1-\lambda)^{2\ell+1}} \\ &= \frac{\lambda^2 - \lambda^3 + \lambda^4 - \lambda^5 + \cdots + \lambda^{2\ell-4} - \lambda^{2\ell-3}}{(1-\lambda)^{2\ell+2}}. \end{aligned}$$

Since

$$H(\omega_R(3),\lambda) = H(R,\lambda) + H(\operatorname{Cok}\varphi,\lambda),$$

we see that

$$h_s = h_0, \quad h_{s-1} = h_1 \quad \text{and} \quad h_{s-i} = h_i + (-1)^i \quad \text{for } 2 \le i \le 2\ell-3.$$

The assertion follows from these equations, since $h_0 = 1$. □

By [4, Theorem 8.1] and [11, Theorem 2.4], we see that $R$ is standard graded. Therefore, we see the following result, since the $h$-vectors of the Ehrhart rings of odd cycle graphs with length at least 9 are non-flawless.

**Corollary 5.3.** *There is an infinite sequence of standard graded Cohen–Macaulay domains whose $h$-vectors are non-flawless.*

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