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A Minimum Principle for Stochastic Optimal Control Problem with Interval Cost Function

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Abstract. In this paper, we study an optimal control problem in which their cost function is interval-valued. Also, a stochastic differential equation governs their state space. Moreover, we introduce a generalized version of Bellman's optimality principle for the stochastic system with an interval-valued cost function. Also, we obtain the Hamilton–Jacobi–Bellman equations and their control decisions. Two numerical examples happen in finance in which their cost function are interval-valued functions, illustrating the efficiency of the discussed results. The obtained results provide significantly reliable decisions compared to the case where the conventional cost function is applied.

1. Introduction

In recent decades due to the growing human tendency to use theories of optimal control in real life, finding a solution to overcome the uncertainties in a problem has become a common topic in some research in this field. In optimal control problems, from a practical point of view, since the coefficients of the cost function may be uncertain or state variables may not be predictable, considering the state variables and the coefficients as deterministic may be unrealistic. Many efforts have been made to overcome this challenge which lies in the area's robust control problems. To read more about the developments and results in this field, study the review paper [20] and references therein. Regarding the importance and necessity of uncertainty modeling, we can refer to investment and stock portfolio control. These are to maximize profits or minimize risk as the goal. In such cases, one kind of uncertainty may arise because of the floating of financial market information and the lack of the investor's prediction from the result of human selections [21, 22]. Also, in the health and treatment areas, especially epidemic diseases, it is very important how to consider the uncertainties [1, 8]. In most research on modeling uncertainties in optimal control problems, the interval and fuzzy sets are conventional to show uncertainty.

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See [1, 3, 8, 10, 11, 21, 26]. Zhao and Zhu [26] studied an optimal control problem with a quadratic cost function based on fuzzy concepts. Also, Leal et al. obtained the necessary and sufficient optimization conditions for an uncertain optimal control problem with an interval cost function [11] and [10]. Hence, an essential part of the study of optimal control problems is related to how we model uncertainty in cost function such that it reflects as much as possible the problem's facts. Therefore, a reliable model can obtain through a lower and upper bound for the cost function. Moreover, the state space usually faces a type of uncertainty challenge that originate from the unpredictability of the system's behavior in the future. Commonly, in the literature related to stochastic optimal control, the state variables were considered stochastic the system's behavior was modeled by a stochastic differential equation [7, 14]. Also, maximum principle of Pontryagin uses to find necessary optimality conditions in the optimal control problems. The backbone of Bellman's principle of optimality is Hamiltonian–Jacobi–Bellman (HJB) equation. It can obtain an optimal feedback control [2]. Then, Yong and Zhou [25] and [24] obtained HJB equations for these systems. Also, a generalization of Bellman's principle is presented to control of stochastic systems with more detail by Crespo [4]. Moreover, A Stochastic maximum principle was introduced by Ji [9]. For study of other applications and developments in this area, we refer the readers to [12, 13, 15–18]. As mentioned earlier, some researchers have tried to model some uncertainties in cost function or unpredictability in state space, while considering both of them together, is a less studied subject in the optimal control research yet. Therefore, in this paper, we try to present a more realistic model of uncertainty in the cost function and the state space simultaneously. Then, we introduce and solve the Hamilton–Jacobi–Bellman equations for stochastic optimal control problem with interval cost function by using the generalization of mentioned methods.

The organization of the rest of the paper is as follows. In Section 2, we develop an optimal control problem with an interval cost function and stochastic state space. Section 3 gives a Hamiltonian for the interval stochastic optimal control. Then, we introduce the Hamilton–Jacobi–Bellman equations related to the stochastic optimal control problem with an interval cost function. In the last of this section, we state and prove a lemma and some theorems. The numerical simulations validate all mentioned in the previous parts in Section 4. In this section, we present an example which can be happen in the investment portfolio control problems [5, 6, 27]. We study it where the cost function is interval-valued function. Finally, we discuss about conclusion of the paper in Section 5. Also, a list of the notations used in this paper provides as follows:

$$\begin{aligned} U &: \text{ Metric space,} \\ (\Omega, \mathcal{F}, \mathbb{P}) &: \text{ Probability space,} \\ L^0_{F_\theta}(\Omega, \mathbb{R}) &: \text{ Set of all (F-measurable) random variables,} \end{aligned}$$

$L^p_{F_\theta}(\Omega, \mathbb{R}^n)$: Banach space,

$\mathbb{E}$: Mathematical expectation,

$\mathbb{E}_\theta[\,\cdot\,]$: Mathematical expectation on filtration F_θ,

$\mathrm{tr}(A)$: Trace of matrix A,

A' : Transpose of matrix A,

$\mathbb{S}^n$: Set of all $(n \times n)$ symmetric matrices.

2. Stochastic optimal control and dynamic programming method

In this section, we present a stochastic optimal problem with an interval cost function. The contents related to dynamic programming are given from [19]. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space and $f, \sigma\colon [0,T] \times \mathbb{R}^n \times U \times \Omega \to \mathbb{R}^n$ be given maps and U is a metric space. Also, $\mathcal{U}[0,t_f] = \{u\colon [0,t_f] \to U \mid u(\cdot) \text{ is } F\text{-progressively measurable}\}$, then we investigate and present a class of optimal control problems which governed by a Stochastic Differential Equation (SDE) as below [14]:

$$dX(t) = f(t, X(t), u(t))\,dt + \sigma(t, X(t), u(t))\,dw(t) \quad \text{and} \quad X(0) = x, \tag{2.1}$$

where $u(\cdot)$ is called a control process, $u(t) \in \mathcal{U}[0,t_f]$ and $X(\cdot)$ is called a state process. Also, $f(t,x,u)$ is a drift and $\sigma(t,x,u)$ is a diffusion (see [2,7]).

For any given vector $X(t)$ in $\mathbb{R}^n$ and $u(t) \in \mathcal{U}[0;t_f]$, the state equation admits a unique solution $X(\cdot) \equiv X(\,\cdot\,, x, u(\cdot))$. We present a tool as performance measuring of the control process $u(\cdot)$ by following the cost functional:

$$J(u(\cdot)) = \mathbb{E}\left[\int_0^T g(t, X(t), u(t))\,dt + h(X(T))\right].$$

We find $u^\star(t) \in \mathcal{U}[0,t_f]$ for given $x \in \mathbb{R}^n$ such that

$$J(u^\star(\cdot)) = \inf_{u(\cdot)\in\mathcal{U}[\theta,t_f]} J(u(\cdot)). \tag{2.2}$$

In this case, we call $u^\star(t)$ and its corresponding state process $X^\star(\cdot)$ are an open-loop optimal control and an open-loop optimal state process, respectively, if any $u^\star(t) \in \mathcal{U}[\theta,t_f]$ satisfying (2.2). Also, $(X^\star(\cdot); u^\star(\cdot))$ is nominated an open-loop optimal pair.

Here, let $\mathbb{J} \subseteq L^0_{F_\theta}(\Omega;\mathbb{R})$, then we define $\operatorname{ess\,inf} \mathbb{J}$ as bellow.

Definition 2.1. [24] Let $\mathcal{X}_\theta = L^p_{F_\theta}(\Omega;\mathbb{R}^n)$ for any $\theta \in \mathcal{T}[0,t_f]$ where $\mathcal{T}[0,t_f]$ is the set of all $\mathbb{F}$-stopping times valued in $[0,t_f]$. Let $\mathcal{A}$ be the set of all admissible initial pairs as follows:

$$\mathcal{A} = \{(\theta,\xi) \mid \theta \in \mathcal{T}[\theta,\xi], \xi \in \mathcal{X}_\theta\}.$$

Also, let $(\theta, \xi) \in \mathcal{A}$ be fixed and let $\mathbb{J} \subseteq L^0_{F_\theta}(\Omega; \mathbb{R})$. A random variable $\overline{J} \subseteq L^0_{F_\theta}(\Omega; \mathbb{R})$ is called the essential infimum of $\mathbb{J}$ and denote $\overline{J} = \operatorname{ess\,inf} \mathbb{J}$, if the following hold:

$$\overline{J}(\omega) \leq J(\omega), \quad \omega \in \Omega, \; \forall \, J \in \mathbb{J},$$

and if $\widehat{J} \subseteq L^0_{F_\theta}(\Omega, \mathbb{R})$ with the property that

$$\widehat{J}(\omega) \leq J(\omega), \quad \omega \in \Omega, \; \forall \, J \in \mathbb{J},$$

then

$$\widehat{J}(\omega) \leq \overline{J}(\omega).$$

Here, we consider an optimal control problem as follows.

Problem 2.2. [25] For any given $(\theta; \xi) \in \mathcal{A}$, find a $\overline{u}(\cdot) \in \mathcal{U}[\theta; t_f]$ such that for any $u_k(\cdot) \in \mathcal{U}[\theta, t_f]$,

$$J(\theta, \xi; \overline{u}(\cdot)) = \operatorname{ess\,inf} J(\theta, \xi, u_k(\cdot)) \equiv \mathbf{V}(\theta, \xi); \quad \text{a.s.}$$

We call $\mathbf{V}$ the value function of Problem 2.2.

Now, for any $(\theta, \xi) \in \mathcal{A}$, we define the value function, called the Bellman's optimality principle, as bellow:

$$\mathbf{V}(\theta, \xi) = \inf_{u(\cdot) \in \mathcal{U}[\theta, t_f]} \mathbb{E} \left\{ \int_\theta^{t_f} g(s; X(s), u(s)) \, ds + \mathbf{V}(t_f; X(t_f)) \right\}, \tag{2.3}$$

where $X(\cdot)$ is the state process corresponding to the control $u(\cdot)$ with an initial pair (θ, ξ). Let $V(\,\cdot\,, \cdot\,)$ be the restriction of $\mathbf{V}$ on $[0, t_f] \times \mathbb{R}^n$ defined by

$$V(t, X) = \mathbf{V}(t, X), \quad \forall \, (t, X) \in [0, t_f] \times \mathbb{R}^n.$$

We call $V(\,\cdot\,, \cdot\,)$ the restricted value function.

Lemma 2.3. *For any $(\theta, \xi_1), (\theta, \xi_2) \in \mathcal{A}$, there exists a constant positive number $K > 0$ such that*

$$|\mathbf{V}(\theta, \xi_1) - \mathbf{V}(\theta, \xi_2)| \leq K |\xi_1 - \xi_2|.$$

Proof. See [24]. □

2.1. Interval stochastic optimal control

The current section describes the basic concepts of an interval as a function. We use the stochastic optimal control with an interval cost function.

Definition 2.4. An interval $[\underline{x}, \overline{x}]$ is the real single-valued function $X^I(\lambda_x)$, where

$$X^I(\lambda_x) = (1 - \lambda_x)\underline{x} + \lambda_x \overline{x}.$$

This means that $X^I(\lambda_x)$ is a single-valued real linear function with two coefficients.

Theorem 2.5. *Let $[\underline{x}, \overline{x}]$ be an interval, then*

$$\mathbb{E}[\underline{x}, \overline{x}] = [\mathbb{E}(\underline{x}), \mathbb{E}(\overline{x})].$$

Proof. The proof is straightforward. Due to Definition 2.4, we can consider interval $[\underline{x}, \overline{x}]$ as a $X^I(\lambda_x)$ which is a single-valued real linear function. Also this is onto and $[\underline{x}, \overline{x}]$ is a convex set. So,

$$\begin{aligned}\mathbb{E}[\underline{x}, \overline{x}] &= \mathbb{E}(X^I(\lambda_x)) = \{\mathbb{E}((1 - \lambda_x)\underline{x} + \lambda_x \overline{x}) \mid 0 \le \lambda_x \le 1\} \\ &= \{(1 - \lambda_x)\mathbb{E}(\underline{x}) + \lambda_x \mathbb{E}(\overline{x}) \mid 0 \le \lambda_x \le 1\} = [\mathbb{E}(\underline{x}), \mathbb{E}(\overline{x})].\end{aligned}$$ □

Now, let $\widetilde{g}(s, X(s), u(s))$ be an interval function as bellow:

$$\widetilde{g}(s, X(s), u(s)) = \overbrace{\big[\underbrace{g^L(s, X(s), u(s))}_{\text{Lower bound}}, \underbrace{g^U(s, X(s), u(s))}_{\text{Upper bound}}\big]}^{\text{Interval function}}$$

and the state space is still as (2.1), then we are facing a stochastic optimal control problem with an interval cost function as below:

$$\begin{aligned}I^L(s, X(s), u(s)) &= \int_s^{t_f} g^L(s, X(s), u(s))\, ds + h(X(t_f)), \\ I^U(s, X(s), u(s)) &= \int_s^{t_f} g^U(s, X(s), u(s))\, ds + h(X(t_f)).\end{aligned}$$

In summary, we nominate the following as Problem (A):

$$\mathcal{J}(s, z, u(\cdot)) = \inf_{u \in \mathcal{U}[s, t_f]} \big[\mathbb{E}(I^L(s, X(s), u(s))), \mathbb{E}(I^U(s, X(s), u(s)))\big],$$

$$dX(t) = f(t, X(t), u(t))\, dt + \sigma(t, X(t), u(t))\, dw(t) \quad \text{and} \quad X(s) = z.$$

To complete our research objectives, we will present more concepts about interval stochastic optimal control and its solution in the next section.

3. Generalized Hamiltonian's function and interval stochastic optimal control

As in [2,15] have defined a Hamiltonian for a stochastic optimal problem, we should define a Hamiltonian for Problem (A), Therefore, we define an interval-valued function called the generalized Hamiltonian function. Here, $\widetilde{H}$ is the generalized Hamiltonian function and has been introduced as follows:

Definition 3.1. Let $\widetilde{H}$ be defined by

$$\widetilde{H}(t,x,p',q) = \min_{u\in U} \widetilde{\mathcal{H}}(t,x,u,p',q),$$

where

$$\widetilde{\mathcal{H}}(t,x,u,p',q) = \lambda_1 g^L(t,x,u,p,q) + \lambda_2 g^U(t,x,u,p,q) + p' f(t,x,u) + \frac{1}{2}\operatorname{tr}(q(\sigma\cdot\sigma'))$$

for all $(t,x,u,p,q) \in [0,T]\times\mathbb{R}^n\times U\times\mathbb{R}^n\times\mathbb{S}^n$, where $0\le\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_1+\lambda_2=1$ which is called a generalized Hamiltonian.

Theorem 3.2 (Interval stochastic HJB equation)**.** *Let the restriction $\widetilde{\mathcal{V}}(\,\cdot\,,\cdot\,) = \lambda_1 V^L(\,\cdot\,,\cdot\,) + \lambda_2 V^U(\,\cdot\,,\cdot\,)$ of $\widetilde{\mathbf{V}}(\,\cdot\,,\cdot\,)$ is a deterministic function where $0\le\lambda_1,\lambda_2\in\mathbb{R}$, $\lambda_1+\lambda_2=1$. Moreover, $V_t^U(\,\cdot\,,\cdot\,)$, $V_t^L(\,\cdot\,,\cdot\,)$, $V_x^U(\,\cdot\,,\cdot\,)$, $V_x^L(\,\cdot\,,\cdot\,)$ and $V_{xx}^U(\,\cdot\,,\cdot\,)$, $V_{xx}^L(\,\cdot\,,\cdot\,)$ are continuous. Then $\mathcal{V}^M(\,\cdot\,,\cdot\,)$ is a solution to the following Interval Stochastic Hamilton–Jacobi–Bellman (ISHJB, for short) equation:*

$$\mathcal{V}_t^M + \widetilde{H}(t,x,\mathcal{V}_x^M,\mathcal{V}_{xx}^M) = 0 \quad \textit{and} \quad \mathcal{V}^M(T,X) = h(x), \tag{3.1}$$

where

$$\begin{aligned}
\widetilde{H}(t,x,\mathcal{V}_x^M,\mathcal{V}_{xx}^M) &= \inf_{u\in U}\widetilde{\mathcal{H}}(t,x,u,\mathcal{V}_x^M,\mathcal{V}_{xx}^M),\\
\widetilde{\mathcal{H}}(t,x,u,\mathcal{V}_x^M,\mathcal{V}_{xx}^M) &= \mathcal{V}_x^{M^t} f(t,x,u) + \frac{1}{2}\operatorname{tr}(\mathcal{V}_{xx}^{M^t}\sigma(t,x,u)\sigma^t(t,x,u)) + \mathcal{G}^M(t,x,u),\\
\mathcal{G}^M(t,x,u) &= \lambda_1 g^L(t,x,u) + \lambda_2 g^U(t,x,u),\\
\mathcal{V}^M(t,x,u) &= \lambda_1 V^L(t,x,u) + \lambda_2 V^U(t,x,u),\\
\mathcal{V}_x^M(t,x,u) &= \lambda_1 V_x^L(t,x,u) + \lambda_2 V_x^U(t,x,u),\\
\mathcal{V}_{xx}^M(t,x,u) &= \lambda_1 V_{xx}^L(t,x,u) + \lambda_2 V_{xx}^U(t,x,u).
\end{aligned}$$

Proof. Let (t,s) be such that $t\in[0,t_f]$, $s\in\mathbb{R}^n$ and for any $u\in U$, we consider the constant control $u(\cdot)=u$. Let $\widetilde{g}(s,X(s),u(s)) = [g^L(s,X(s),u(s)), g^U(s,X(s),u(s))]$ be an interval function. The following is a local form of Bellman's principle of optimality noting (2.3) for g^L and g^U:

$$\begin{aligned}
V^L(t,\xi) &= \inf_{u(\cdot)\in\mathcal{U}[t,t_f]} \mathbb{E}\big\{I^L(s,X(s),u(s)) + V^L(t_f,X(t_f))\big\},\\
V^U(t,\xi) &= \inf_{u(\cdot)\in\mathcal{U}[t,t_f]} \mathbb{E}\big\{I^U(s,X(s),u(s)) + V^U(t_f,X(t_f))\big\},
\end{aligned}$$

therefore

$$\begin{aligned}
V^L(t,\xi) &= \inf_{u(\cdot)\in\mathcal{U}[t,t_f]} \mathbb{E}\left\{\int_t^{t_f} g^L(s,X(s),u(s))\,ds + V^L(t_f,X(t_f))\right\},\\
V^U(t,\xi) &= \inf_{u(\cdot)\in\mathcal{U}[t,t_f]} \mathbb{E}\left\{\int_t^{t_f} g^U(s,X(s),u(s))\,ds + V^U(t_f,X(t_f))\right\}.
\end{aligned}$$

Let $0 \leq \lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 + \lambda_2 = 1$, therefore

$$\begin{aligned}0 &\leq \lambda_1 \mathbb{E}_t\Big[\int_t^{t_f} g^L(s, X(s), u(s))\, ds + V^L(t+\epsilon, X(t+\epsilon)) - V^L(t,x)\Big]\\ &= \mathbb{E}_t\Big[\int_t^{t_f}\Big(\lambda_1(g^L(s;X(s);u)) + V^L(s;X(s)) + V^L_x(s,X(s))f(s,X(s),u(s))\\ &\qquad + \frac{1}{2}\big(\operatorname{tr} V^L_{xx}(s,X(s))\sigma(s,X(s),u)\sigma(s,X(s),u)\big)\Big)\, ds\Big].\end{aligned}$$

Similarly,

$$\begin{aligned}0 &\leq \lambda_2 \mathbb{E}_t\Big[\int_t^{t_f} g^U(s, X(s), u(s))\, ds + V^U(t+\epsilon, X(t+\epsilon)) - V^U(t,x)\Big]\\ &= \mathbb{E}_t\Big[\int_t^{t_f}\Big(\lambda_2(g^U(s;X(s);u)) + V^L(s;X(s)) + V^U_x(s,X(s))f(s,X(s),u(s))\\ &\qquad + \frac{1}{2}\big(\operatorname{tr} V^U_{xx}(s,X(s))\sigma(s,X(s),u)\sigma(s,X(s),u)\big)\Big)\, ds\Big].\end{aligned}$$

We take the expected value of the above relations and let $\epsilon \to 0$, then

$$\begin{aligned}0 &\leq \lambda_1\left[g^L(t,x,u) + V^L_t(t,x) + V^L_x(t,x)f(t,x,u) + \frac{1}{2}\operatorname{tr}\left(V^L_{xx}(t,x)\sigma^t(t,x,u)\sigma(t,x,u)\right)\right],\\ 0 &\leq \lambda_2\left[g^U(t,x,u) + V^U_t(t,x) + V^U_x(t,x)f(t,x,u) + \frac{1}{2}\operatorname{tr}\left(V^U_{xx}(t,x)\sigma^t(t,x,u)\sigma(t,x,u)\right)\right].\end{aligned}$$

Therefore,

$$\begin{aligned}0 &\leq \lambda_1 g^L(t,x,u) + \lambda_2 g^U(t,x,u) + \lambda_1 V^L(t,x) + \lambda_2 V^U(t,x)\lambda_1 V^L_t(t,x) + \lambda_2 V^U_t(t,x),\\ &\quad + \lambda_1 V^L_x(t,x) + \lambda_2 V^U_x(t,x) + \frac{1}{2}\operatorname{tr}\left(\{\lambda_1 V^L_{xx}(t,x) + \lambda_2 V^U_{xx}(t,x)\}\sigma^t(t,x,u)\sigma(t,x,u)\right),\end{aligned}$$

and thus

$$\mathcal{V}^M_t + \inf_{u\in U}\widetilde{\mathcal{H}}(t,x,u,\mathcal{V}^M_x, V^M_{xx}) \geq 0.$$

Next, there exists a $u^{\delta,\epsilon} \in \mathcal{U}[t,t_f]$ (for any arbitrary $\epsilon, \delta > 0$), such that

(3.2)

$$\begin{aligned}\lambda_1\delta\epsilon &\geq \mathbb{E}_t\Big[\int_t^{t+\epsilon}\lambda_1\big(g^L(s;X^{\delta,\epsilon}(s);u^{\delta,\epsilon}(s))\big)\, ds + V^L(t+\epsilon, X^{\delta,\epsilon}(t+\epsilon)) - V^L(t,x)\Big]\\ &= \mathbb{E}_t\Big[\int_t^{t+\epsilon}\Big(\lambda_1(g^L(s,X^{\delta,\epsilon}(s),u)) + V^L_s(s;X^{\delta,\epsilon}(s)) + V^L_x(s,X^{\delta,\epsilon}(s))f(s,X^{\delta,\epsilon}(s),u)\\ &\qquad + \frac{1}{2}\operatorname{tr} V^L_{xx}(s;X^{\delta,\epsilon}(s))\sigma^t(s;X^{\delta,\epsilon}(s),u)\sigma(s;X^{\delta,\epsilon}(s);u^{\delta,\epsilon})\Big)\, ds\Big]\end{aligned}$$

and

(3.3)

$$\begin{aligned}\lambda_2\delta\epsilon \geq \mathbb{E}_t\Big[\int_t^{t+\epsilon} \lambda_2\big(g^U(s, X^{\delta,\epsilon}(s); u^{\delta,\epsilon}(s))\big)\,ds + V^U(t+\epsilon, X^{\delta,\epsilon}(t+\epsilon)) - V^U(t,x)\Big]\\ = \mathbb{E}_t\Big[\int_t^{t+\epsilon}\Big(\lambda_2(g^U(s, X^{\delta,\epsilon}(s); u)) + V_s^U(s, X^{\delta,\epsilon}(s)) + V_x^U(s, X^{\delta,\epsilon}(s))f(s; X^{\delta,\epsilon}(s); u)\\ + \frac{1}{2}\operatorname{tr} V_{xx}^U(s; X^{\delta,\epsilon}(s))\sigma^t(s; X^{\delta,\epsilon}(s), u)\sigma(s; X^{\delta,\epsilon}(s); u^{\delta,\epsilon})\Big)\,ds\Big].\end{aligned}$$

By Lemma 2.3 and summation of (3.2), (3.3), we have

$$\begin{aligned}\delta\epsilon \geq \mathbb{E}_t\Big[\int_t^{t+\epsilon}\Big(\lambda_1 g^L(s; X^{\delta,\epsilon}(s); u) + \lambda_2 g^U(s; X^{\delta,\epsilon}(s); u) + \lambda_1 V_s^L(s; X^{\delta,\epsilon}(s))\lambda_2 V_s^U(s; X^{\delta,\epsilon}(s))\\ + \lambda_1 V_x^L(s; X^{\delta,\epsilon}(s))\lambda_2 V_x^U(s; X^{\delta,\epsilon}(s))(s, X^{\delta,\epsilon}(s))b(s; X^{\delta,\epsilon}(s); u)\\ + \frac{1}{2}\big(\operatorname{tr}\{\lambda_1 V_x^L(s; X^{\delta,\epsilon}(s)) + \lambda_2 V_x^U(s; X^{\delta,\epsilon}(s))\}\sigma^t(s; X^{\delta,\epsilon}(s), u)\sigma(s, X^{\delta,\epsilon}(s); u^{\delta,\epsilon})\big)\Big)\,ds\Big]\\ \geq \mathbb{E}_t\Big[\int_t^{t+\epsilon}\Big(\mathcal{V}_s^M(s, X^{\delta,\epsilon}(s)) + \inf_{u\in U}\widetilde{\mathcal{H}}(t,x,u,\mathcal{V}_x^M,\mathcal{V}_{xx}^M)\Big)\Big]\\ \geq \epsilon\Big[\mathcal{V}_t^M(t,x) + \inf_{u\in U}\widetilde{\mathcal{H}}(t,x,u,\mathcal{V}_x^M,\mathcal{V}_{xx}^M)\Big] - K\mathbb{E}_t\Big[\int_t^{t+\epsilon}\big|X^{\delta,\epsilon}(s) - x\big|\,ds\Big].\end{aligned}$$

We divide $\epsilon > 0$ and let $\epsilon \to 0$, then the following relation

$$\delta \geq \mathcal{V}_t^M(t,x) + \inf_{u\in U}\widetilde{\mathcal{H}}(t,x,u,\mathcal{V}_x^M,\mathcal{V}_{xx}^M)$$

holds. Since $\delta > 0$ is arbitrary, we obtain (3.1) and the proof is complete. □

If $\mathcal{V}^M(\cdot,\cdot)$ is not a deterministic function, we will need an interval stochastic HJB equation with much more complication by It'o type formula. Since we usually construct an optimal control through the value function, this can be a basis for motivation of the dynamic programming method, which comes in the form of the following theorem.

Corollary 3.3 (Verification Theorem). *Let $\mathcal{V}^M(\cdot,\cdot)$ be a solution of interval stochastic HJB equation and let the function $\psi\colon [0;T]\times\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{S}^n \to U$ satisfy*

$$\widetilde{H}(t,x,\psi(t,x,\mathcal{V}_x^M,\mathcal{V}_{xx}^M),\mathcal{V}_x^M,\mathcal{V}_{xx}^M) = \inf_{u\in U}\widetilde{\mathcal{H}}(t,x,u,\mathcal{V}_x^M,\mathcal{V}_{xx}^M).$$

Further, under the control

$$u(t) = \psi(t, X(t), \mathcal{V}_x^M(t,X(t)), \mathcal{V}_{xx}^M(t,X(t))), \quad t\in[0,t_f],$$

the state equation admits a unique solution. Then $u(\cdot)$ defined as above is an optimal control.

Proof. First, we make $u(\cdot)$, as mentioned in the theorem, then we suppose that $(t,x) \in [0,t_f) \times \mathbb{R}^n$ is given. Let $X(\cdot) = X(\,\cdot\,, t, x, u(\cdot))$. Here, Ito's formula comes in handy, and we have

$$\begin{aligned}
\mathbb{E}_t[h(X(t_f))] - \mathcal{V}^M(t,x) &= \mathbb{E}_t[\mathcal{V}^M(T, X(t_f))] - \mathcal{V}^M(t,x) \\
&= \mathbb{E}_t\bigg(\int_t^{t_f} \Big(\mathcal{V}_s^M(s; X(s)) + \mathcal{V}_x^M(s; X(s)) f(s, X(s), u(s)) \\
&\qquad + \frac{1}{2} \operatorname{tr}\left(\mathcal{V}_s^M(s, X(s)) \sigma(s, X(s), u(s)) \sigma'(s, X(s), u(s)) \right) \Big) \, ds \\
&= -\mathbb{E}_t \int_t^{t_f} \left(\mathcal{G}^M(s, X(s), u(s)) \right) ds.
\end{aligned}$$

Hence,

$$\mathcal{V}^M(t,x) = \mathcal{J}(t, x, u(\cdot)).$$ □

To validate the concepts and achievements presented in the previous sections, two numerical examples are presented in the next section.

4. Numerical example: Application of ISHJB to portfolio control problem and finance

Portfolio control problems are very common and their related models are studied in [27] and their references therein. Also, there exist many problems in the investment portfolio structure and control such that a stochastic differential equation forms their state-space which results in a stochastic optimal control problem. Also, in some research in this area, we usually observe a general structure in which the total wealth at time $t \geq 0$ is denoted by $x(t) = (x_1(t), \ldots, x_n(t))'$ as the state variable of problem. Moreover, $u(t) = (u_1(t), \ldots, u_m(t))'$ is investor's portfolio which $u_i(t)$ denotes the total market value of the investor's wealth in the i-th bond or stock and the n-dimensional standard Wiener process $w(t) = (w_n(t), \ldots, w_n(t))'$ describes the volatility of prices of i-th risk financial asset over $[0, T]$. Furthermore, in the continuous time problems, the cost function of these problems can be stated as the following control quality criterion

$$J = \mathbb{E}\left[x'(T) N x(T) + \int_0^T \left(x'(t) Q x(t) + u'(t) R u(t) \right) dt \right],$$

where Q, R and N are weighting matrices with appropriate dimensions. Also, x' and u' are transpose of vector x and u, respectively (see [5, 27]).

In this section, to represent the uncertainty, we consider the cost function of portfolio control problems as the interval-valued, highlighted in all our previous arguments. We present two examples of investment portfolio control problems in which the weighting matrix Q is an interval. Therefore, we have the interval cost function. Here, the following examples of the stochastic optimal control problem with the interval cost function illustrate Theorem 3.2.

Example 4.1 (Interval stochastic linear regulator problem)**.** By above description, we consider which the system is governed by the following stochastic differential equation:

$$dx(t) = \left(\frac{3}{2}x(t) + u(t)\right) dt + \sqrt{8}\, dw(t) \quad \text{and} \quad x(0) = 0.5,$$

where $w(t)$ is a Wiener process [7]. Also, we consider the following interval cost function in which $\widetilde{Q} = [3/2, 2]$:

$$\mathcal{J} = \left[\frac{1}{2}\mathbb{E}\left(x^2(1) + \int_0^1 \left(3x^2(t) + \frac{3}{2}u^2(t)\right) dt\right), \frac{1}{2}\mathbb{E}\left(x^2(1) + \int_0^1 \left(4x^2(t) + \frac{3}{2}u^2(t)\right) dt\right)\right],$$

$$I^L(s, X(s), u(s)) = \frac{1}{2}x^2(1) + \frac{1}{2}\int_0^1 \left(\frac{3}{2}x^2(t) + u^2(t)\right) dt,$$

$$I^U(s, X(s), u(s)) = \frac{1}{2}x^2(1) + \frac{1}{2}\int_0^1 (4x^2(t) + u^2(t))\, dt,$$

$$V^L(s, X(s), u(s)) = \inf_{u\in U}\left\{\left(\frac{3}{2}x + v\right)V_x^L + 4V_{xx}^L + \frac{3}{2}x^2 + \frac{1}{4}(v)^2\right\},$$

where $\mathcal{J} = \mathcal{J}(t, X(t), u(\cdot))$. We obtain the minimum of optimal control variable with choice of $v = -V_x^L$. Consequently, the lower HJB equation is as follows:

$$V_t^L + \frac{3}{2}xV_x^L + 4V_{xx}^L - \frac{1}{4}(V_x^L)^2 + \frac{3}{2}x^2 = 0 \quad \text{and} \quad V^L(1, x) = \frac{1}{2}x^2.$$

Similarly, the upper HJB equation is as follows:

$$V_t^U + \frac{3}{2}xV_x^U + 4V_{xx}^U - \frac{1}{4}(V_x^U)^2 + 2x^2 = 0 \quad \text{and} \quad V^U(1, x) = \frac{1}{2}x^2.$$

Therefore, without obtaining of HJB's solutions, based on Theorem 3.2 and Corollary 3.3, we have

$$\mathcal{V}_x^M = \lambda_1 V_x^L + \lambda_2 V_x^U, \quad \mathcal{V}_{xx}^M = \lambda_1 V_{xx}^L + \lambda_2 V_{xx}^U,$$

$$V^M = \mathcal{V}_x^M + \mathcal{V}_{xx}^M + \left(\lambda_1 (V_x^L)^2 + \lambda_2 (V_x^U)^2\right) + \frac{7}{2}x^2, \tag{4.1}$$

where $0 \le \lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 + \lambda_2 = 1$. The solution of ISHJB equation (4.1) is obtained by separation of variables with $\mathcal{V}^M(t, x) = P(t)x^2 + K(t)$, where $P(t)$ and $K(t)$ are given by

$$P(t) = \frac{4}{1 + 3e^{8t-8}} - \frac{1}{2} \quad \text{and} \quad K(t) = 8\ln\left(\frac{1}{4}(1 + 3e^{8t-8})\right) - 56t + 56.$$

Theorem 3.2 and Corollary 3.3 are satisfied with $\lambda_1 = \lambda_2 = 1/2$. The graphs of value function

$$\mathcal{V}^M(t,x) = \left(\frac{4}{1+3e^{8t-8}} - \frac{1}{2}\right)x^2 + 8\ln\left(\frac{1}{4}(1+3e^{8t-8})\right) - 56t + 56$$

and optimal control variable

$$u^M(t,x) = -\frac{\partial \mathcal{V}^M(t,x)}{\partial x} = -2\left(\frac{4}{1+3e^{8t-8}} - \frac{1}{2}\right)x$$

are given in Figure 4.1.

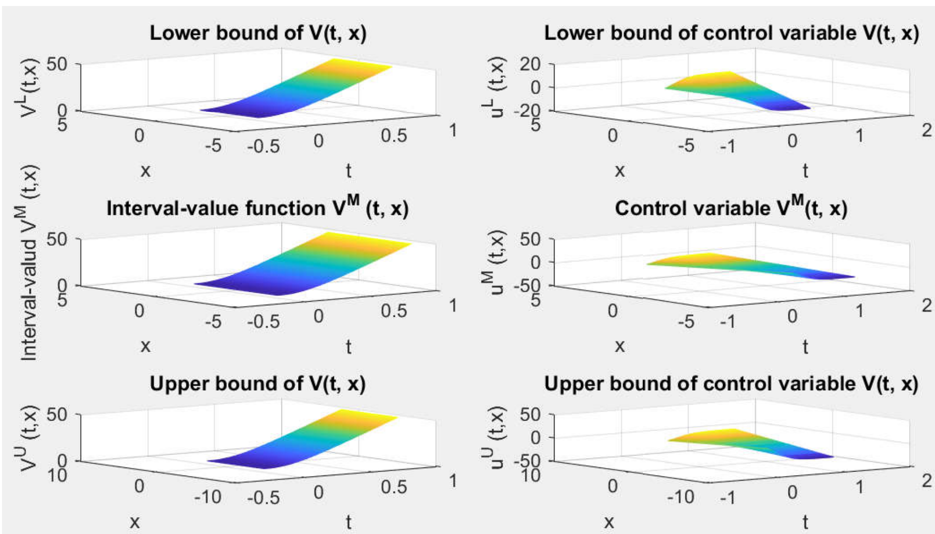

Figure 4.1: The graph of $\mathcal{V}^M(t,x)$ and corresponding control variable $u^M(t,x)$.

Also, The exact solution for the interval valued objective function is

$$\mathcal{J}^* = \mathcal{V}^M(0, 0.5) = 45.79168679.$$

Also, graph of approximate solutions of trajectory $x(t)$ and its simulation of corresponding SDE, state-space equation, by using of Euler–Maruyama, is illustrated in Figure 4.2.

Example 4.2 (Interval voluntary provision of public goods problem). In this example, we consider the dynamic voluntary provision of public goods [23] and develop it to the interval mentioned concepts. Consider stochastic differential equation:

$$dx(t) = (-x(t) + u(t))\,dt + \sqrt{2}x(t)\,dw(t) \quad \text{and} \quad x(0) = \sqrt{2}$$

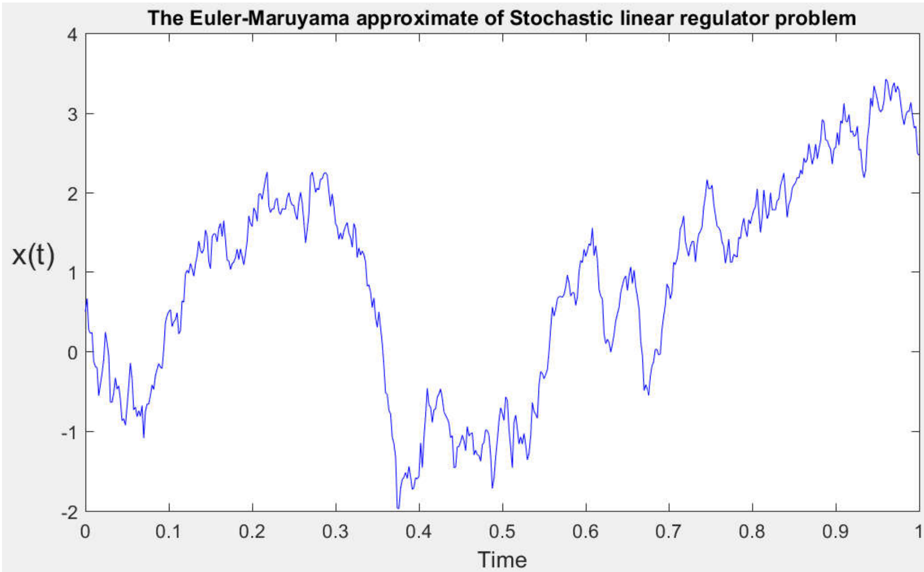

Figure 4.2: Simulation via Euler–Maruyama.

with the following interval cost function in which $\widetilde{Q} = [1/4, 3/4]$ as

$$\mathcal{J} = \left[\mathbb{E}\left(\int_0^1 \left(\frac{1}{4}x^2(t) - \frac{1}{4}u^2(t)\right)\, dt\right), \mathbb{E}\left(\int_0^1 \left(\frac{3}{4}x^2(t) - \frac{1}{4}u^2(t)\right)\, dt\right)\right],$$

$$I^L(s, X(s), u(s)) = \mathbb{E}\left(\int_0^1 \left(\frac{1}{4}x^2(t) - \frac{1}{4}u^2(t)\right)\, dt\right),$$

$$I^U(s, X(s), u(s)) = \mathbb{E}\left(\int_0^1 \left(\frac{3}{4}x^2(t) - \frac{1}{4}u^2(t)\right)\, dt\right),$$

$$V^L(s, X(s), u(s)) = \inf_{u \in U}\left\{(-x + v)V_x^L + x^2 V_{xx}^L + \frac{1}{4}x^2 + \frac{1}{4}(v)^2\right\},$$

where $\mathcal{J} = \mathcal{J}(t, X(t), u(\cdot))$. We obtain the minimum of optimal control variable with choice of $v = V_x^L$. Consequently, the lower HJB equation is as follows:

$$V_t^L - xV_x^L + x^2 V_{xx}^L + \frac{3}{4}(V_x^L)^2 + \frac{1}{4}x^2 = 0 \quad \text{and} \quad V^L(1, x) = 0.$$

Similarly, the upper HJB equation is as follows:

$$V_t^U - xV_x^U + x^2 V_{xx}^U + \frac{3}{4}(V_x^U)^2 + \frac{3}{4}x^2 = 0 \quad \text{and} \quad V^U(1, x) = 0.$$

Therefore, without obtaining of HJB's solutions, based on Theorem 3.2 and Corollary 3.3, we have

$$\begin{aligned} \mathcal{V}_x^M &= \lambda_1 V_x^L + \lambda_2 V_x^U, \quad \mathcal{V}_{xx}^M = \lambda_1 V_{xx}^L + \lambda_2 V_{xx}^U, \\ V^M &= \mathcal{V}_x^M + \mathcal{V}_{xx}^M + \left(\lambda_1 (V_x^L)^2 + \lambda_2 (V_x^U)^2\right) + x^2, \end{aligned} \tag{4.2}$$

where $0 \leq \lambda_1, \lambda_2 \in \mathbb{R}$ such that $\lambda_1 + \lambda_2 = 1$. The solution of ISHJB equation (4.2) is obtained by separation of variables with $\mathcal{V}^M(t,x) = P(t)x^2$, where $P(t)$ is given by

$$P(t) = -\frac{\sqrt{2}}{2}\tan\big(\sqrt{2}(t-1)\big).$$

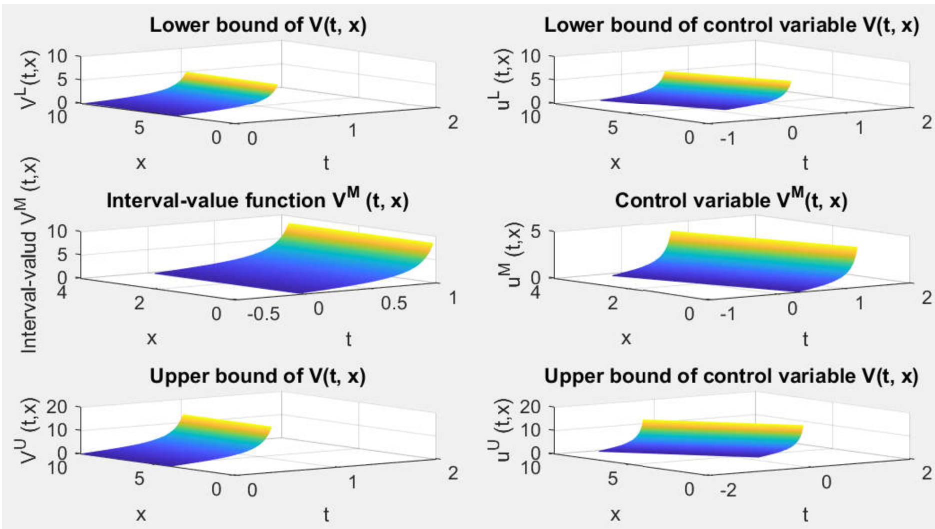

Figure 4.3: The graph of $\mathcal{V}^M(t,x)$ and corresponding control variable $u^M(t,x)$.

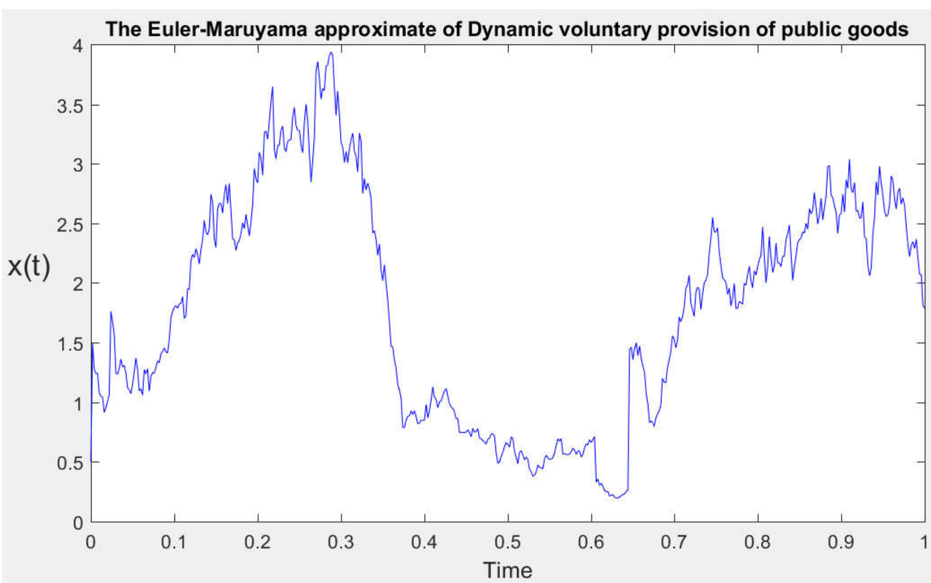

Figure 4.4: Simulation via Euler–Maruyama.

Theorem 3.2 and Corollary 3.3 are satisfied with $\lambda_1 = \lambda_2 = 1/2$. The graphs of value function

$$\mathcal{V}^M(t,x) = -\frac{\sqrt{2}}{2}\tan\big(\sqrt{2}(t-1)\big)x^2$$

and optimal control variable

$$u^M(t,x) = -\sqrt{2}\tan\big(\sqrt{2}(t-1)\big)x$$

are given in Figure 4.3. Also, the exact solution for the interval valued objective function is

$$\mathcal{J}^* = \mathcal{V}^M(0,\sqrt{2}) = 8.957797.$$

Also, graph of approximate solutions of trajectory $x(t)$ and its simulation of corresponding SDE, dynamic voluntary provision of public goods, state-space equation, by using of Euler–Maruyama, is illustrated in Figure 4.4.

5. Conclusion

In this work, a stochastic optimal control problem has been studied. Moreover, to make some flexible decisions in the real world, the cost function has been allowed to be an interval. Then, a minimum principle for the stochastic optimal control problem with interval cost function has introduced. Also, we have obtained an interval stochastic Hamilton–Jacobi–Bellman equations. Since many problems in portfolio investment may present as interval stochastic, then an interval stochastic linear regulator problem and the dynamic voluntary provision of public goods have been simulated and studied by the interval-valued cost functions such that all our mentioned results have been used for solving this.

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