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Eakin–Nagata–Eisenbud Theorem for Right S-Noetherian Rings

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Abstract. The Eakin–Nagata theorem examines the condition that the Noetherian property passes through each other between subrings and extension rings in 1968. Later, a noncommutative version of Eakin–Nagata theorem was also proved. Lam called this version Eakin–Nagata–Eisenbud theorem. In addition, Anderson and Dumitrescu introduced the S-Noetherian concept which is an extended notion of the Noetherian property on commutative rings in 2002. In this paper, we consider the S-variant of Eakin–Nagata–Eisenbud theorem for general rings by using S-Noetherian modules. We also show that every right S-Noetherian domain is right Ore, which is embedded into a division ring. For a right S-Noetherian ring, we obtain sufficient conditions for its right ring of fractions to be right S-Noetherian or right Noetherian. As applications, the S-variant of Eakin–Nagata–Eisenbud theorem is applied to the composite polynomial, composite power series and composite skew polynomial rings.

1. Introduction

The notion of Noetherian rings has been an important tool in the arsenal of algebraists because of their applications to many areas of mathematics. For commutative rings, the Noetherian property can be characterized by using prime ideals. It is well known as Cohen theorem that if R is a commutative ring with unity in which every prime ideal is finitely generated, then R is a Noetherian ring. Also, the Noetherian property in commutative rings can be ascent and descent under an additional condition between subrings and extension rings. Let R be a subring of a commutative ring E. Then as stated in [27, p. 158] in 1958 (or see [13, p. 54, Ex. 15]), it was shown that

(EN1) If R is Noetherian and E is finitely generated as an R-module, then E is a Noetherian ring.

Later, Eakin [7] and Nagata [22] in 1968 independently proved the following result, so-called *Eakin–Nagata theorem* (EN2):

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(EN2) If E is Noetherian and E is finitely generated as an R-module, then R is a Noetherian ring.

For the case of noncommutative rings, while (EN1) still holds true (see [10, Corollary 1.5]), (EN2) does not hold true in general as shown in the last example in [12]. However, Eisenbud discovered an alternative proof of (EN2) in [8], which makes use of injective modules and yields a noncommutative generalization. Thus Lam mentioned that a noncommutative version of Eakin–Nagata theorem is called *Eakin–Nagata–Eisenbud theorem* (ENE) in [16, Theorem 3.98]: Let R be a subring of a ring E with the same unity. Then

(ENE1) If R is right Noetherian and E is finitely generated as a right R-module, then E is a right Noetherian ring.

(ENE2) If E is right Noetherian and E is finitely generated as a right R-module with $E = e_1R + \cdots + e_kR$ and $e_ir = re_i$ for all $r \in R$, $i = 1, \ldots, k$, then R is a right Noetherian ring.

In addition, Formanek and Jategaonkar in 1974 showed that (ENE2) still holds on the weakened condition $e_iR = Re_i$ instead of $e_ir = re_i$ for all $r \in R$ (see [9, Theorem 4]).

On the other hand, the notion of (commutative) S-Noetherian rings was introduced by Anderson and Dumitrescu who proved the S-variant of Eakin–Nagata theorem on commutative rings in 2002 [4, Corollary 7]. Then it was further studied in [1, 3, 14, 17–19, 25]. While the research of noncommutative S-Noetherian rings is started recently by a few authors, they found some valuable results. Especially, the S-variant of (ENE1) is proved by Baeck, Lee and Lim [5, Lemma 2.14(6)]. Also, the S-variant of Cohen theorem is shown by Bilgin, Reyes and Tekir [6, Theorem 2.2]. Recall that a submodule N of a right R-module M is called *S-finite* if $Ns \subseteq F \subseteq N$ for an element $s \in S$ and a finitely generated submodule F of M, where S is a multiplicative subset of R. M is *S-Noetherian* if every submodule is S-finite. A ring R is called *right S-Noetherian* if R_R is S-Noetherian. Clearly, every Noetherian module is always S-Noetherian.

Inspired by the above two concepts, in this paper we study the conditions which allow for the right S-Noetherian property to transfer back and forth between subrings and extension rings, which is called the S-variant of Eakin–Nagata–Eisenbud theorem. This is a generalization of a ring version of [9, Theorem 4].

After the introduction and some preliminary background, our focus in Section 2 is on showing the S-variant of the Eakin–Nagata–Eisenbud theorem. First, we provide an alternative proof for the S-variant of (ENE1) by using module theoretic notion (see Theorem 2.3 and Corollary 2.4), that is, the S-Noetherian property passes through the ring extension under an additional condition. For the converse, we define the S-finite normalizing ring extension (see Definition 1.1(3)). When E is an S-finite normalizing ring extension of a ring R, we prove that if E is a right S-Noetherian ring and M is S-finite as a right E-module, then M is S-Noetherian as a right R-module (see Theorem 2.6), which shows that one of our main results holds true for rings as a corollary (The S-variant of (ENE2), Corollary 2.7). Finally, combining two above results, we obtain the S-variant of Eakin–Nagata–Eisenbud theorem for noncommutative rings (see Theorem 2.9).

In Section 3, we consider Ore localizations of right S-Noetherian rings. We prove that every right S-Noetherian domain is a right Ore domain (see Proposition 3.2), which is an extension of the well-known result that every right Noetherian domain is right Ore [26, Chapter II, Proposition 1.7]. Also, for a right denominator subset T of a right S-Noetherian ring R, we provide sufficient conditions for the ring RT^{-1} to be right S-Noetherian or right Noetherian (see Theorems 3.4 and 3.6). In addition, for an S-finite normalizing ring extension E of a ring R, we obtain sufficient conditions for a ring ET^{-1} to be right Noetherian when R is a right S-Noetherian ring, and another sufficient conditions for a ring RT^{-1} to be right Noetherian when E is a right S-Noetherian ring (see Corollary 3.8).

Lastly, we in Section 4 apply the S-variant of Eakin–Nagata–Eisenbud theorem to composite polynomial, composite power series and composite skew polynomial rings. More precisely, we introduce the concept of a right S-stationary chain of rings, and we give equivalent conditions for composite polynomial, power series and skew polynomial rings to be right S-Noetherian (see Theorems 4.4 and 4.9 and Corollary 4.16).

Throughout this paper, all rings are associative rings with unity and all modules are unitary right R-modules. We denote $[N : L] = \{r \in R \mid Lr \subseteq N\}$ for nonempty subsets N and L of a module M. $\mathbb{N}$, $\mathbb{N}_0$, $\mathbb{Z}$ and $\mathbb{Q}$ stand for the sets of natural numbers, nonnegative integers, integers and rational numbers, respectively. The following definitions will be used in various results of this paper.

Definition 1.1. Let R be a subring of a ring E and let S be a multiplicative subset of R.

(1) E is called a *finite ring extension* of R if there exist $e_i \in E$ such that $E = e_1R + \cdots + e_kR$ with $e_1 = 1_R = 1_E$.

(2) (see [20, p. 289]) E is called a *finite normalizing ring extension* of R if there exist $e_i \in E$ such that $E = e_1R + \cdots + e_kR$ with $e_1 = 1_R = 1_E$ and $e_iR = Re_i$ for all $i = 1, \dots, k$.

(3) E is called an *S-finite normalizing ring extension* of R if there exist $s \in S$ and $e_i \in E$ such that $Es \subseteq e_1R + \cdots + e_kR \subseteq E$ with $e_1 = 1_R = 1_E$ and $e_iR = Re_i$ for all $i = 1, \dots, k$.

2. The S-variant of Eakin–Nagata–Eisenbud theorem

Our first result in this section gives an alternative proof of the S-variant of (ENE1) by using the concept of completely prime ideals. Recall that a proper (two-sided) ideal P of a ring R is called a *completely prime ideal* if for any $a, b \in R$, $ab \in P$ implies that $a \in P$ or $b \in P$.

Lemma 2.1. *Let R be a ring, S a multiplicative subset of R and M an S-finite R-module. If R is a right S-Noetherian ring and N is a submodule of M which is maximal among all non-S-finite submodules of M, then $[N : M]$ is a completely prime ideal of R which is disjoint from S.*

Proof. Let $P = [N : M]$. Then P is a proper ideal of R. Suppose that P is not a completely prime ideal of R. Then there exist $a, b \in R \setminus P$ such that $ab \in P$. Since $a \notin P$, there exists $m \in M \setminus N$ such that $ma \notin N$. By the maximality of N, $N + maR$ is S-finite; so we can find an element $s_1 \in S$ and a finitely generated submodule N_1 of N such that $(N + maR)s_1 \subseteq N_1 + maR \subseteq N + maR$. Set $I = \{r \in R \mid mar \in N\}$. Then I is a right ideal of R containing P and b. Since R is right S-Noetherian, there exist $s_2 \in S$ and $\ell_1, \ldots, \ell_k \in I$ such that $Is_2 \subseteq \ell_1 R + \cdots + \ell_k R \subseteq I$.

Now, let $n \in N$ be arbitrary. Then $ns_1 = n' + max$ for some $n' \in N_1$ and $x \in R$; so $max = ns_1 - n' \in N$, which indicates that $x \in I$. Thus $xs_2 = \ell_1 y_1 + \cdots + \ell_k y_k$ for some $y_1, \ldots, y_k \in R$. Hence we have

$$\begin{aligned} ns_1 s_2 &= n's_2 + maxs_2 = n's_2 + ma\ell_1 y_1 + \cdots + ma\ell_k y_k \\ &\in N_1 + ma\ell_1 R + \cdots + ma\ell_k R. \end{aligned}$$

Since n was arbitrarily chosen in N and $ma\ell_j \in N$ for all $j = 1, \ldots, k$, we obtain

$$Ns_1 s_2 \subseteq N_1 + ma\ell_1 R + \cdots + ma\ell_k R \subseteq N,$$

which shows that N is S-finite, a contradiction to the hypothesis. Thus P is a completely prime ideal of R.

Moreover, suppose that P is not disjoint from S. Take $s_1' \in P \cap S$. Then $Ms_1' \subseteq N$. Since M is S-finite, there exist $s_2' \in S$ and $m_1, \ldots, m_t \in M$ such that $Ms_2' \subseteq m_1 R + \cdots + m_t R$. Therefore we have $Ns_2' s_1' \subseteq (m_1 R + \cdots + m_t R)s_1' = m_1 Rs_1' + \cdots + m_t Rs_1' \subseteq m_1 P + \cdots + m_t P$. Since R is a right S-Noetherian ring, there exist $s_3' \in S$ and $p_1, \ldots, p_\ell \in P$ such that $Ps_3' \subseteq p_1 R + \cdots + p_\ell R \subseteq P$. Note that $m_i p_j \in N$ for all $i = 1, \ldots, t$ and $j = 1, \ldots, \ell$. Hence we obtain

$$\begin{aligned} Ns_2' s_1' s_3' &\subseteq (m_1 P + \cdots + m_t P)s_3' \\ &= m_1 Ps_3' + \cdots + m_t Ps_3' \end{aligned}$$

$$\begin{aligned}&\subseteq m_1(p_1R+\cdots+p_\ell R)+\cdots+m_t(p_1R+\cdots+p_\ell R)\\&=m_1p_1R+\cdots+m_ip_jR+\cdots+m_tp_\ell R\\&\subseteq N,\end{aligned}$$

which indicates that N is S-finite, a contradiction. Thus $P\cap S=\emptyset$. □

While the next proposition appears in [5, Lemma 2.14(5)], we give an alternative proof by using Lemma 2.1.

Proposition 2.2. *Let R be a ring, S a multiplicative subset of R and M a right R-module. If R is right S-Noetherian and M is S-finite, then M is S-Noetherian.*

Proof. Suppose to the contrary that M is not S-Noetherian. Let $\mathcal{F}$ be the set of non-S-finite submodules of M. Then $\mathcal{F}$ is a nonempty partially ordered set under inclusion. Let $\{L_\alpha\}_{\alpha\in\Lambda}$ be a chain in $\mathcal{F}$ and let $L=\bigcup_{\alpha\in\Lambda}L_\alpha$. We claim that L is not S-finite: Suppose that L is S-finite. Then there exists an element $s\in S$ and a finitely generated submodule G of L such that $Ls\subseteq G$. Since G is finitely generated, $G\subseteq L_\beta$ for some $\beta\in\Lambda$; so $L_\beta s\subseteq G\subseteq L_\beta$. Thus L_β is S-finite, a contradiction, proving the claim. Clearly, L is an upper bound of the chain $\{L_\alpha\}_{\alpha\in\Lambda}$. Thus by Zorn's lemma, we can find a maximal element in $\mathcal{F}$, say N.

Let $P=[N:M]$. Then by Lemma 2.1, P is a completely prime ideal of R which is disjoint from S. Since M is S-finite, there exists an element $w\in S$ and a finitely generated submodule F of M such that $Mw\subseteq F$; so we have $P=[N:M]\subseteq[N:F]\subseteq[N:Mw]=(P:w)$, where $(P:w):=\{r\in R\mid wr\in P\}$. Since $w\notin P$ and P is completely prime, $(P:w)=P$; so we have

$$P=[N:M]=[N:F]=[N:Mw]=(P:w).$$

Write $F=f_1R+\cdots+f_tR$ for some $f_1,\ldots,f_t\in F$. Then $P=[N:f_1R]\cap\cdots\cap[N:f_tR]$. Since P is a proper ideal of R, $f_\ell\notin N$ for some $\ell\in\{1,\ldots,t\}$. By the maximality of N, $N+f_\ell R$ is S-finite; so we can find an element $s_1\in S$ and a finitely generated submodule N_1 of N such that $(N+f_\ell R)s_1\subseteq N_1+f_\ell R\subseteq N+f_\ell R$. Since R is right S-Noetherian, there exist $s_2\in S$ and $t_1,\ldots,t_v\in R$ such that $[N:f_\ell]s_2\subseteq t_1R+\cdots+t_vR\subseteq[N:f_\ell]$.

Now, let $n\in N$ be arbitrary. Then we have $ns_1=n'+f_\ell x$ for some $n'\in N_1$ and $x\in R$. Note that $f_\ell x=ns_1-n'\in N$; so $x\in[N:f_\ell]$. Therefore we can find $y_1,\ldots,y_v\in R$ such that $xs_2=t_1y_1+\cdots+t_vy_v$. Hence we have

$$\begin{aligned}ns_1s_2&=n's_2+f_\ell xs_2=n's_2+f_\ell t_1y_1+\cdots+f_\ell t_vy_v\\&\in N_1+f_\ell t_1R+\cdots+f_\ell t_vR.\end{aligned}$$

Since n was arbitrarily chosen in N and $f_\ell t_j \in N$ for all $j \in \{1, \dots, v\}$, we obtain

$$Ns_1 s_2 \subseteq N_1 + f_\ell t_1 R + \cdots + f_\ell t_v R \subseteq N,$$

which shows that N is S-finite, a contradiction to the fact that N is not S-finite. Thus M is S-Noetherian. □

Theorem 2.3. *Let E be a ring extension of R, M a right E-module and S a multiplicative subset of R. If M is an S-Noetherian as a right R-module, then M is S-Noetherian as a right E-module. In particular, if R is right S-Noetherian and M is S-finite as a right R-module, then M is S-Noetherian as a right E-module.*

Proof. Let N be a right E-submodule of M. Then N is a right R-module; so N is S-finite. Therefore there exist $s \in S$ and $n_1, \dots, n_k \in N$ such that

$$Ns \subseteq n_1 R + \cdots + n_k R \subseteq n_1 E + \cdots + n_k E \subseteq N.$$

Hence N is S-finite as a right E-module. Thus M is an S-Noetherian right E-module. The last statement follows directly from Proposition 2.2. □

The next corollary is the S-variant of (ENE1).

Corollary 2.4. (see [5, Lemma 2.14(6)]) *Let E be a ring extension of a ring R and S be a multiplicative subset of R. If R is a right S-Noetherian ring and E is an S-finite R-module, then E is a right S-Noetherian ring.*

There is an example that (EN2) does not hold on noncommutative rings in [9, p. 181]. Also, this example shows that the S-variant of (EN2) does not hold on noncommutative rings as follows.

Example 2.5. (see [9, p. 181]) Let $R = \left[\begin{smallmatrix} \mathbb{Q} & \mathbb{Q} \\ 0 & \mathbb{Z} \end{smallmatrix}\right]$ and $E = \begin{bmatrix} \mathbb{Q} & \mathbb{Q} \\ \mathbb{Q} & \mathbb{Q} \end{bmatrix}$ be rings. Then E is a ring extension of R and E is generated by $\left[\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} 0 & 0 \\ 1 & 0 \end{smallmatrix}\right]$ as a right R-module. It is clear that E is right Noetherian, but R is not right Noetherian (see [15, Corollary 1.23]). Thus, if $S_1 = \{\left[\begin{smallmatrix} 0 & 0 \\ 0 & 1 \end{smallmatrix}\right]\}$, then E is right S_1-Noetherian, but R is not right S_1-Noetherian. Note that R is a right S_2-Noetherian ring with $S_2 = \{\left[\begin{smallmatrix} 1 & 0 \\ 0 & 0 \end{smallmatrix}\right]\}$.

Now, we concentrate on the other direction of the S-variant of Eakin–Nagata–Eisenbud theorem for general rings. Next, we provide the main theorem by using S-Noetherian modules. As corollaries, it yields the S-variant of (ENE2) and a noncommutative version of Eakin–Nagata theorem. One can also obtain an alternative proof of a noncommutative version of Eakin–Nagata theorem without using injective modules.

Theorem 2.6. *Let E be an S-finite normalizing ring extension of R where S is a multiplicative subset of R. If E is a right S-Noetherian ring and M is S-finite as a right E-module, then M is S-Noetherian as a right R-module.*

Proof. Since E is an S-finite normalizing ring extension of R, there exist $s \in S$ and $e_i \in E$ such that $Es \subseteq e_1R + \cdots + e_kR \subseteq E$ with $e_1 = 1_R = 1_E$ and $e_iR = Re_i$ for all $i = 1, \ldots, k$ (see Definition 1.1(3)). Also, by Proposition 2.2, M is an S-Noetherian E-module; so it is easy to see that M is S-finite as an R-module. Suppose to the contrary that M is not an S-Noetherian R-module. Then there exists a proper R-submodule of M which is not S-finite. Let N be an R-submodule of M which is maximal among non-S-finite R-submodules of M. Then (after reordering e_i with fixed e_1 if necessary) there exists the smallest integer $2 \leq t_1 \leq k$ such that $Ne_i \subseteq N$ and $Ne_j \nsubseteq N$ for all $1 \leq i < t_1 \leq j \leq k$. So there is $m_1 \in N$ such that $m_1e_{t_1} \notin N$. By the maximality of N, $N + m_1e_{t_1}R$ is S-finite. Therefore there exists an element $s_1' \in S$ and a finitely generated submodule N_1 of N such that

$$\big(N + m_1e_{t_1}R\big)s_1' \subseteq N_1 + m_1e_{t_1}R \subseteq N + m_1e_{t_1}R.$$

Thus for any $n \in N$, we have $ns_1' = n_1 + m_1e_{t_1}x_{t_1}$ where $n_1 \in N_1$ and $x_{t_1} \in R$. Then $m_1e_{t_1}x_{t_1} = ns_1' - n_1 \in N$.

Now, let $I_{t_1} = \big\{\ell^{(t_1)} \in R \mid m_1e_{t_1}\ell^{(t_1)} \in N\big\}$. Note that $x_{t_1} \in I_{t_1}$. Since $I_{t_1}E$ is a right ideal of E, $I_{t_1}E$ is S-finite; so we obtain

$$I_{t_1}Es_1 \subseteq \sum_{u=1}^{p_1} \ell_u^{(t_1)}E \subseteq I_{t_1}E$$

for some $s_1 \in S$, $p_1 \in \mathbb{N}$ and $\ell_u^{(t_1)} \in I_{t_1}$, $u = 1, \ldots, p_1$. Therefore since $x_{t_1} \in I_{t_1}E$, we have

$$ns_1's_1 = n_1s_1 + m_1e_{t_1}(x_{t_1}s_1) = n_1s_1 + m_1e_{t_1}\left(\sum_{u=1}^{p_1} \ell_u^{(t_1)}y_u^{(t_1)}\right)$$

for some $y_u^{(t_1)} \in E$. Since $e_{t_1}R = Re_{t_1}$, there exists $\ell_u'^{(t_1)} \in R$ such that $e_{t_1}\ell_u^{(t_1)} = \ell_u'^{(t_1)}e_{t_1}$ for each $1 \leq u \leq p_1$. Thus we obtain

$$\begin{aligned} ns_1's_1s &= n_1s_1s + m_1\sum_{u=1}^{p_1}\big(\ell_u'^{(t_1)}\big(e_{t_1}y_u^{(t_1)}\big)s\big) \\ &= n_1s_1s + m_1\sum_{u=1}^{p_1}\big(\ell_u'^{(t_1)}\big(e_1x_u^{(1,t_1)} + \cdots + e_kx_u^{(k,t_1)}\big)\big) \end{aligned}$$

for some $x_u^{(1,t_1)}, \ldots, x_u^{(k,t_1)} \in R$, because $e_{t_1}y_u^{(t_1)} \in E$, which leads us to that

$$\alpha_1 := ns_1's_1s - n_1s_1s - m_1e_1\sum_{u=1}^{p_1}\ell_u'^{(t_1)}x_u^{(1,t_1)} - \cdots - m_1e_{t_1}\sum_{u=1}^{p_1}\ell_u^{(t_1)}x_u^{(t_1,t_1)}$$

$$\begin{aligned}
&= m_1 e_{t_1+1} \sum_{u=1}^{p_1} \ell_u^{(t_1+1,t_1)} x_u^{(t_1+1,t_1)} + \cdots + m_1 e_k \sum_{u=1}^{p_1} \ell_u^{(k,t_1)} x_u^{(k,t_1)} \\
&\in N
\end{aligned}$$

for some $\ell_u^{(t_1+1,t_1)}, \ldots, \ell_u^{(k,t_1)} \in R$ with

$$\ell_u'^{(t_1)} e_{t_1+1} = e_{t_1+1} \ell_u^{(t_1+1,t_1)}, \quad \ldots, \quad \ell_u'^{(t_1)} e_k = e_k \ell_u^{(k,t_1)}.$$

If $m_1 e_j \sum_{u=1}^{p_1} \ell_u^{(j,t_1)} \in N$ for each $t_1 + 1 \leq j \leq k$, we have

$$Ns_1' s_1 s \subseteq N_1 + m_1 e_1 \sum_{u=1}^{p_1} \left(\ell_u'^{(t_1)} R\right) + \cdots + m_1 e_k \sum_{u=1}^{p_1} \left(\ell_u^{(k,t_1)} R\right) \subseteq N,$$

a contradiction to the fact that N is not S-finite.

Again, if not, then (after reordering on $\{e_{t_1+1}, \ldots, e_k\}$ if necessary) there exists the smallest integer $t_1 + 1 \leq t_2 \leq k$ such that

$$m_1 \sum_{u=1}^{p_1} \ell_u'^{(t_1)} e_i = m_1 e_i \sum_{u=1}^{p_1} \ell_u^{(i,t_1)} \in N \quad \text{and} \quad m_1 \sum_{u=1}^{p_1} \ell_u'^{(t_1)} e_j = m_1 e_j \sum_{u=1}^{p_1} \ell_u^{(j,t_1)} \notin N$$

for all $t_1 \leq i < t_2 \leq j \leq k$. Set $m_2 = m_1 \sum_{u=1}^{p_1} \ell_u'^{(t_1)} \in N$. Then $m_2 e_{t_2} \notin N$. By the maximality of N, $N + m_2 e_{t_2} R$ is S-finite. Therefore there exists an element $s_2' \in S$ and a finitely generated submodule N_2 of N such that

$$\left(N + m_2 e_{t_2} R\right) s_2' \subseteq N_2 + m_2 e_{t_2} R \subseteq N + m_2 e_{t_2} R.$$

Thus for any $n \in N$, we have $ns_2' = n_2 + m_2 e_{t_2} x_{t_2}$, which implies that $m_2 e_{t_2} x_{t_2} \in N$ for some $n_2 \in N_2$ and $x_{t_2} \in R$. Now, let $I_{t_2} = \left\{\ell^{(t_2)} \in R \mid m_2 e_{t_2} \ell^{(t_2)} \in N\right\}$. Note that $x_{t_2} \in I_{t_2}$. Since $I_{t_2} E$ is S-finite, we obtain

$$I_{t_2} E s_2 \subseteq \sum_{u=1}^{p_2} \ell_u^{(t_2)} E \subseteq I_{t_2} E$$

for some $s_2 \in S$, $p_2 \in \mathbb{N}$ and $\ell_u^{(t_2)} \in I_{t_2}$, $u = 1, \ldots, p_2$. Therefore for α_1 in (1), we have

$$\alpha_1 s_2' s_2 = n_2 s_2 + m_2 e_{t_2} (x_{t_2} s_2) = n_2 s_2 + m_2 e_{t_2} \left(\sum_{u=1}^{p_2} \ell_u^{(t_2)} y_u^{(t_2)}\right)$$

for some $y_u^{(t_2)} \in E$. Thus we obtain

$$\begin{aligned}
\alpha_1 s_2' s_2 s &= n_2 s_2 s + m_2 \sum_{u=1}^{p_2} \left(\ell_u'^{(t_2)} \left(e_{t_2} y_u^{(t_2)}\right) s\right) \\
&= n_2 s_2 s + m_2 \sum_{u=1}^{p_2} \ell_u'^{(t_2)} \left(e_1 x_u^{(1,t_2)} + \cdots + e_k x_u^{(k,t_2)}\right)
\end{aligned}$$

for some $\ell_u^{\prime(t_2)}, x_u^{(1,t_2)}, \ldots, x_u^{(k,t_2)} \in R$ with $e_{t_2}\ell_u^{(t_2)} = \ell_u^{\prime(t_2)} e_{t_2}$ for each $1 \le u \le p_2$. By the construction, we obtain $m_2e_1, \ldots, m_2e_{t_1}, \ldots, m_2e_{t_2-1} \in N$ which leads us to that

$$\begin{aligned}\alpha_2 &:= \alpha_1 s_2' s_2 s - n_2 s_2 s - m_2 e_1 \sum_{u=1}^{p_2} \ell_u^{\prime(t_2)} x_u^{(1,t_2)} - \cdots - m_2 e_{t_2} \sum_{u=1}^{p_2} \ell_u^{(t_2)} x_u^{(t_2,t_2)} \\ &= m_2 e_{t_2+1} \sum_{u=1}^{p_2} \ell_u^{(t_2+1,t_2)} x_u^{(t_2+1,t_2)} + \cdots + m_2 e_k \sum_{u=1}^{p_2} \ell_u^{(k,t_2)} x_u^{(k,t_2)} \\ &\in N\end{aligned}$$

for some $\ell_u^{(t_2+1,t_2)}, \ldots, \ell_u^{(k,t_2)} \in R$ with $\ell_u^{\prime(t_2)} e_{t_2+1} = e_{t_2+1}\ell_u^{(t_2+1,t_2)}, \ldots, \ell_u^{\prime(t_2)} e_k = e_k \ell_u^{(k,t_2)}$.

If $m_2 e_j \sum_{u=1}^{p_2} \ell_u^{(j,t_2)} \in N$ for each $t_2+1 \le j \le k$, we have

$$\begin{aligned}N s_1' s_1 s s_2' s_2 s \subseteq\ & N_1 + N_2 + m_1 R + m_1 e_2 R + \cdots + m_1 e_{t_1-1} R \\ & + m_1 e_{t_1} \sum_{u=1}^{p_1} \left(\ell_u^{(t_1)} R\right) + \cdots + m_1 e_{t_2-1} \sum_{u=1}^{p_1} \left(\ell_u^{(t_2-1,t_1)} R\right) \\ & + m_2 e_{t_2} \sum_{u=1}^{p_2} \left(\ell_u^{(t_2)} R\right) + \cdots + m_2 e_k \sum_{u=1}^{p_2} \left(\ell_u^{(k,t_2)} R\right) \\ \subseteq\ & N,\end{aligned}$$

a contradiction to the fact that N is not S-finite. If not, then we continue this process again. After finite steps, we can reach that

$$\begin{aligned}\alpha_{w-1} s_w' s_w s &= n_w s_w s + m_w \sum_{u=1}^{p_w} \left(\ell_u^{\prime(t_w)} \left(e_{t_w} y_u^{(t_w)}\right) s\right) \\ &= n_w s_w s + m_w \sum_{u=1}^{p_w} \left(\ell_u^{\prime(t_w)} \left(e_1 x_u^{(1,t_w)} + \cdots + e_k x_u^{(k,t_w)}\right)\right)\end{aligned}$$

for $n_w \in N_w$, where N_w is a finitely generated submodule of N, and

$$m_w e_1 \in m_1 R \subseteq N, \quad m_w e_2 \in m_1 e_2 R \subseteq N, \quad \ldots, \quad m_w e_{t_1-1} \in m_1 e_{t_1-1} R \subseteq N,$$

$$m_w e_{t_1} = m_1 e_{t_1} \left(\sum_{u=1}^{p_1} \ell_u^{(t_1)}\right) \cdots \left(\sum_{u=1}^{p_w} \ell_u^{(t_1,t_w)}\right) \in m_1 e_{t_1} \sum_{u=1}^{p_1} \ell_u^{(t_1)} R \subseteq N,$$

$$\ldots\ldots,$$

$$m_w e_{t_2} \in m_2 e_{t_2} \sum_{u=1}^{p_2} \ell_u^{(t_2)} R \subseteq N,$$

$$\ldots\ldots,$$

$$m_w e_{t_w} \sum_{u=1}^{p_w} \ell_u^{(t_w)} \in m_w e_{t_w} \sum_{u=1}^{p_w} \ell_u^{(t_w)} R \subseteq N, \quad \ldots, \quad m_w e_k \sum_{u=1}^{p_w} \ell_u^{(k,t_w)} \in m_w e_k \sum_{u=1}^{p_w} \ell_u^{(k,t_w)} R \subseteq N.$$

Hence we finally have

$$\begin{aligned} Ns_1's_1s\cdots s_w's_ws \subseteq\ & N_1+N_2+\cdots+N_w+m_1R+\cdots+m_1e_{t_1-1}R \\ & +m_1e_{t_1}\sum_{u=1}^{p_1}\ell_u^{(t_1)}R+\cdots+m_1e_{t_2-1}\sum_{u=1}^{p_1}\ell_u^{(t_2-1,t_1)}R \\ & +m_2e_{t_1}R+\cdots+m_2e_{t_2-1}R \\ & +m_2e_{t_2}\sum_{u=1}^{p_2}\ell_u^{(t_2)}R+\cdots+m_2e_{t_3-1}\sum_{u=1}^{p_2}\ell_u^{(t_3-1,t_2)}R+\cdots \\ & +m_we_{t_{w-1}}R+\cdots+m_we_{t_w-1}R \\ & +m_we_{t_w}\sum_{u=1}^{p_w}\ell_u^{(t_w)}R+\cdots+m_we_k\sum_{u=1}^{p_w}\ell_u^{(k,t_w)}R \\ \subseteq\ & N. \end{aligned}$$

This is a contradiction to the fact that N is not S-finite. Consequently, M is also S-Noetherian as an R-module. □

Comparing Theorem 2.6 to [9, Theorem 4] (If E is a finite normalizing ring extension of R and M is a Noetherian E-module, then M is a Noetherian R-module), we can naturally ask the question: *Let E be an S-finite normalizing ring extension of R. If M is an S-Noetherian E-module, then is M an S-Noetherian R-module?* as the S-variant of [9, Theorem 4] for modules. We did not have any clue for this question so far. However, for the ring case, when $M = E$ is a ring, it is shown that Theorem 2.6 is a generalization of one direction of [9, Theorem 4]. (If E is a finite normalizing ring extension of a ring R and E is a right Noetherian ring, then E is Noetherian as a right R-module.) The next corollary is one of our main results, which is called the S-variant of (ENE2).

Corollary 2.7 (The S-variant of (ENE2))**.** *Let E be an S-finite normalizing ring extension of R. If E is a right S-Noetherian ring, then R is a right S-Noetherian ring.*

Proof. By Theorem 2.6, E is S-Noetherian as a right R-module. If I is a right ideal of R, then I is S-finite since I is a right R-submodule of E. Thus R is a right S-Noetherian ring. □

When $S = \{1\}$ in Corollary 2.7, we directly obtain the following result which is our motivation.

Corollary 2.8. (see [16, Theorem 3.98], (ENE2)) *Let E be a finite ring extension of R with $e_ir = re_i$ for all $r \in R$, $i = 1, \ldots, k$. If E is a right Noetherian ring, then R is a right Noetherian ring.*

By combining Corollaries 2.4 and 2.7, we obtain the *S-variant of Eakin–Nagata–Eisenbud theorem* for general rings.

Theorem 2.9. *Let E be an S-finite normalizing ring extension of R. Then R is a right S-Noetherian ring if and only if E is a right S-Noetherian ring.*

We conclude this section with the result on the S-variant of Cohen theorem. While the S-variant of Cohen theorem was proved in [6], we are successful to show the S-variant of Cohen theorem without any condition on a multiplicative subset S. Recall that a proper right ideal P of a ring R is a *completely prime right ideal* of R if for any $a, b \in R$, $ab \in P$ and $aP \subseteq P$ imply that $a \in P$ or $b \in P$ (see [23, Definition 2.1]). Clearly, a completely prime ideal is a completely prime right ideal.

Lemma 2.10. *Let S be a multiplicative subset of a ring R. If P is a right ideal of R maximal among non-S-finite right ideals of R, then P is a completely prime right ideal of R.*

Proof. The proof is similar to that of Lemma 2.1. □

Theorem 2.11 (The S-variant of Cohen theorem)**.** *Let S be a multiplicative subset of a ring R. Then the following statements are equivalent.*

(1) *R is a right S-Noetherian ring.*

(2) *Every completely prime right ideal of R is S-finite.*

Proof. (1) $\Rightarrow$ (2). This implication is obvious.

(2) $\Rightarrow$ (1). Suppose to the contrary that R is not a right S-Noetherian ring. Then there exists a non-S-finite right ideal of R. Let $\mathcal{T}$ be the set of non-S-finite right ideals of R. A similar argument as in the proof of Proposition 2.2 shows that $\mathcal{T}$ has a maximal element, say P. By Lemma 2.10, P is a completely prime right ideal of R. This contradicts our assumption. Thus R is a right S-Noetherian ring. □

3. Ore localizations of right S-Noetherian rings

In this section, we always assume that T is a multiplicative subset of R such that $1 \in T$ and $0 \notin T$. T is called a *right denominator set* if (1) for any $a \in R$ and $t \in T$, $aT \cap tR \neq \emptyset$ (i.e., T is *right permutable* or *right Ore*) and (2) for any $a \in R$, $ta = 0$ for some $t \in T$ implies $at' = 0$ for some $t' \in T$ (i.e., T is *right reversible*). A left denominator set can be defined similarly and a denominator set is a left and right denominator set. From now on, T always means a right denominator set of a ring R.

Lemma 3.1. [16, Theorem 10.6] *A ring R has a right ring of fractions with respect to T, denoted by RT^{-1}, if and only if T is a right denominator set.*

As shown in [26, Chapter II, Proposition 1.7], any right Noetherian domain is right Ore. Now, we extend this result to a right S-Noetherian domain as follows.

Proposition 3.2. *Every right S-Noetherian domain is a right Ore domain.*

Proof. Let a and b be nonzero elements of a right S-Noetherian domain R. For each $i \in \mathbb{N} \cup \{0\}$, consider a right ideal $A_i = bR + abR + \cdots + a^i bR$ of R. Then we can get a chain of right ideals of R, $A_0 \subseteq A_1 \subseteq A_2 \subseteq \cdots$. Since R is right S-Noetherian, there exist the smallest integer n and $s \in S$ such that $A_i s \subseteq A_n$ from [6, Theorem 2.3]. Hence $A_{n+1}s \subseteq A_n$ and so $a^{n+1}bs = bc_0 + abc_1 + \cdots + a^n bc_n$ for some $c_0, \dots, c_n \in R$. If $c_i \neq 0$ with $c_0 = \cdots = c_{i-1} = 0$, then $a^i(bc_i + abc_{i+1} + \cdots + a^{n-i}bc_n - a^{n-i+1}bs) = 0$. Since R is a domain, $bc_i + abc_{i+1} + \cdots + a^{n-i}bc_n - a^{n-i+1}bs = 0$. This gives

$$bc_i = a(-bc_{i+1} - \cdots - a^{n-i-1}bc_n + a^{n-i}bs) \neq 0.$$

Thus $aR \cap bR \neq \{0\}$. □

Corollary 3.3. *Every right S-Noetherian domain is embedded into a division ring.*

It is known that RT^{-1} is a Noetherian ring for a commutative Noetherian ring R, and RT^{-1} is also right Noetherian for a right Noetherian ring R, where T is a right denominator subset of a ring R [24, Proposition 3.1.13]. For the case of a commutative S-Noetherian domain R, RT^{-1} is S-Noetherian [17, Lemma 1]. In the next two theorems, we consider an Ore localization of a right S-Noetherian ring R, that is, we obtain sufficient conditions for the ring RT^{-1} to be right S-Noetherian or right Noetherian.

Theorem 3.4. *Let T be a right denominator set in a ring R and let S be a multiplicative subset of R with $sT = Ts$ for all $s \in S$. If R is right S-Noetherian, then RT^{-1} is a right S-Noetherian ring.*

Proof. Clearly, $S' := \left\{\frac{s}{1} \mid s \in S\right\}$ is a multiplicative subset of RT^{-1}. We can identify $S' = S$. Let $\mathcal{B}$ be a right ideal of RT^{-1}. Then $A = \mathcal{B} \cap R$ is a right ideal of R with $A(RT^{-1}) = AT^{-1} = \mathcal{B}$ (see [21, p. 49, Proposition (iii)]). Since R is right S-Noetherian, $As \subseteq f_1R + \cdots + f_kR \subseteq A$ for some $s \in S$ and $f_1, \dots, f_k \in A$. Thus we have

$$\mathcal{B}s = AT^{-1}s = AsT^{-1} \subseteq f_1RT^{-1} + \cdots + f_kRT^{-1} \subseteq \mathcal{B}.$$

Hence RT^{-1} is a right S-Noetherian ring. □

Corollary 3.5. *Let S, T be multiplicative subsets of a commutative ring R. If R is an S-Noetherian ring, then RT^{-1} is an S-Noetherian ring.*

Theorem 3.6. *Let T be a right denominator set in a ring R and let S be a multiplicative subset of R with $S \subseteq T$. If R is right S-Noetherian, then RT^{-1} is a right Noetherian ring.*

Proof. Let S', $\mathcal{B}$ and A be as in the proof of Theorem 3.4. Since R is right S-Noetherian, $As \subseteq f_1R + \cdots + f_kR \subseteq A$ for some $s \in S$ and $f_1, \ldots, f_k \in A$. Now, let $\frac{a}{t} \in \mathcal{B}$ be arbitrary. Then we obtain

$$\begin{aligned}\frac{a}{t} = \frac{as}{ts} &= \frac{f_1r_1 + \cdots + f_kr_k}{ts} = \frac{f_1r_1}{ts} + \cdots + \frac{f_kr_k}{ts} \\ &= f_1\frac{r_1}{ts} + \cdots + f_k\frac{r_k}{ts} \in f_1RT^{-1} + \cdots + f_kRT^{-1}.\end{aligned}$$

Hence $\mathcal{B} \subseteq f_1RT^{-1} + \cdots + f_kRT^{-1} \subseteq \mathcal{B}$, which shows that $\mathcal{B}$ is finitely generated. Thus RT^{-1} is right Noetherian. $\square$

The next examples show that the condition "$S \subseteq T$" in Theorem 3.6 is not superfluous, and the converse of Theorem 3.6 is not true, in general.

Example 3.7. (1) Let D be a Noetherian integral domain, $T = D\setminus\{0\}$, $\mathbf{X} = \{X_i \mid i \in \mathbb{N}\}$ a set of indeterminates over D and $R = D[\![\mathbf{X}]\!]/\langle X_iX_j \mid i \neq j\rangle$ the factor ring of $D[\![\mathbf{X}]\!]$ by the ideal $\langle X_iX_j \mid i \neq j\rangle$. Note that T is a right denominator set of R. Fix an $i \in \mathbb{N}$ and a set $S = \{\overline{X_i}^n \mid n \in \mathbb{N}\}$. Then an easy calculation shows that R is an S-Noetherian ring. Consider $RT^{-1} = F[\![\mathbf{X}]\!]/\langle X_iX_j \mid i \neq j\rangle$, where F is a field of fraction of D. Let $\overline{X_i}$ be the image of X_i under RT^{-1}. Then $\langle\overline{X_1}\rangle \subsetneq \langle\overline{X_1}, \overline{X_2}\rangle \subsetneq \cdots$ is an ascending chain of ideals of RT^{-1}; so RT^{-1} is not a Noetherian ring. Note that RT^{-1} is S-Noetherian because $sT = Ts$ for all $s \in S$.

(2) Let D be an integral domain and $\mathbf{X} = \{X_i \mid i \in \mathbb{N}\}$ be a set of indeterminates over D. Let $R = D[\mathbf{X}]$ be the polynomial ring over D. Then RT^{-1} is a division ring and thus Noetherian, where $T = R \setminus \{0\}$. However, R is not an S-Noetherian ring for any multiplicative subset $S \subseteq D \setminus \{0\}$. For an ideal $I = \langle X_1, X_2, \ldots\rangle$ and for any $s \in S$, there is no finitely generated ideal J such that $Is \subseteq J \subseteq I$. Hence R is not S-Noetherian.

From the S-variant of Eakin–Nagata–Eisenbud theorem (see Theorem 2.9) and Theorem 3.6, we have

Corollary 3.8. *Let T be a right denominator set in rings R and E, and let S be a multiplicative subset of R with $S \subseteq T$. If E is an S-finite normalizing ring extension of a ring R, then the following assertions hold true.*

(1) *If R is right S-Noetherian, then ET^{-1} is right Noetherian.*

(2) *If E is right S-Noetherian, then RT^{-1} is right Noetherian.*

Corollary 3.9. *Let R be a right Ore ring and let S be a multiplicative subset of R consisting of regular elements in R. If R is right S-Noetherian, then the classical right ring of quotients of R, $Q^r_{cl}(R)$, is right Noetherian.*

4. Composite polynomial, power series and skew polynomial rings

In this section, as applications of Theorem 2.9, we consider the conditions for composite polynomial, composite power series and composite skew polynomial rings to be right S-Noetherian when the based ring is right S-Noetherian. Take $\mathcal{R} = (R_n)_{n\geq 0}$ an ascending chain of rings with the same unity and X an indeterminate over R. Then $\mathcal{R}[X] := \left\{ \sum_{i=0}^{n} a_i X^i \mid a_i \in R_i \right\}$ and $\mathcal{R}[\![X]\!] := \left\{ \sum_{n=0}^{\infty} a_n X^n \mid a_n \in R_n \right\}$ are rings with unity. We call these *the composite polynomial ring* and *the composite power series ring*, respectively. Ahmed and Sana [2, Definition 2.4] introduced the concept of an S-stationary ascending chain of commutative rings where S is a multiplicative subset of a ring R. First, we study the conditions for an ascending chain of rings to be right S-Noetherian. Now, we define a right S-Noetherian chain of general rings as follows.

Definition 4.1. Let $\mathcal{R} = (R_n)_{n\geq 0}$ be an ascending chain of rings with the same unity and S be a multiplicative subset of R_0. We say that $\mathcal{R}$ is a *right S-Noetherian chain* if it satisfies the following three conditions:

(1) R_0 is a right S-Noetherian ring;

(2) $\mathcal{R}$ is right S-stationary, i.e., there exist $s \in S$ and $n \in \mathbb{N}_0$ such that $R_i s \subseteq R_n$ for all $i \geq n$;

(3) For each positive integer n, R_n is S-finite as a right R_0-module.

Remark 4.2. Since $R_i s \subseteq R_i \subseteq R_n$ for all $i \leq n$ for an ascending chain of rings, we use the condition that there exist $s \in S$ and $n \in \mathbb{N}_0$ such that $R_i s \subseteq R_n$ for all i instead of Definition 4.1(2).

Lemma 4.3. *Let $\mathcal{R} = (R_n)_{n\geq 0}$ be an ascending chain of rings, $E = \bigcup_{n\geq 0} R_n$ and S a multiplicative subset of R_0. If $E[X]$ is an S-finite normalizing ring extension of a ring $\mathcal{R}[X]$, then E is an S-finite normalizing ring extension of R_0.*

Proof. Since $E[X]$ is an S-finite normalizing ring extension of $\mathcal{R}[X]$, there exist $s \in S$ and $e_1, \dots, e_k \in E[X]$ such that $E[X]s \subseteq e_1\mathcal{R}[X] + \cdots + e_k\mathcal{R}[X]$ and $e_i\mathcal{R}[X] = \mathcal{R}[X]e_i$ for every $i = 1, \dots, k$; so for any $a \in E$, we have $as = e_1(0)g_1(0) + \cdots + e_k(0)g_k(0)$ for some $g_1, \dots, g_k \in \mathcal{R}[X]$. Therefore we have $Es \subseteq e_1(0)R_0 + \cdots + e_k(0)R_0$ with $e_i(0)R_0 = R_0 e_i(0)$ for all i. Thus E is an S-finite normalizing ring extension of R_0. $\square$

Recall that for a multiplicative subset S of a ring R, S is said to be *right anti-Archimedean* if $\bigcap_{n=1}^{\infty} s^n R \cap S \neq \emptyset$ for all $s \in S$. Note that if E is a ring extension of R and S is a right anti-Archimedean subset of R, then S is also a right anti-Archimedean subset of E. For each $\ell \in \mathbb{N}_0$, denote $\mathcal{R}_\ell[X] = \left\{ \sum_{i=\ell}^{m} a_i X^{i-\ell} \mid a_i \in R_i \text{ and } m \geq \ell \right\}$.

Theorem 4.4. *Let $\mathcal{R} = (R_n)_{n\geq 0}$ be an ascending chain of rings, $E = \bigcup_{n\geq 0} R_n$, S a right anti-Archimedean subset of R_0 and let $E[X]$ be an S-finite normalizing ring extension of a ring $\mathcal{R}[X]$. Then the following statements are equivalent.*

(1) *$\mathcal{R}[X]$ is a right S-Noetherian ring.*

(2) *$\mathcal{R}$ is a right S-Noetherian chain.*

(3) *R_0 is a right S-Noetherian ring.*

Proof. (1) $\Rightarrow$ (2). Suppose that $\mathcal{R}[X]$ is a right S-Noetherian ring. First, we claim that R_0 is a right S-Noetherian ring: Let I be a right ideal of R_0. Then $I + X\mathcal{R}_1[X]$ is a right ideal of $\mathcal{R}[X]$; so we can find $s \in S$ and $f_1, \ldots, f_n \in I + X\mathcal{R}_1[X]$ such that

$$(I + X\mathcal{R}_1[X])s \subseteq f_1\mathcal{R}[X] + \cdots + f_n\mathcal{R}[X] \subseteq I + X\mathcal{R}_1[X].$$

Hence $Is \subseteq f_1(0)R_0 + \cdots + f_n(0)R_0 \subseteq I$. Thus I is S-finite, which indicates that R_0 is a right S-Noetherian ring.

Second, we claim that $\mathcal{R}$ is right S-stationary: Note that $X\mathcal{R}_1[X]$ is a right ideal of $\mathcal{R}[X]$. Since $\mathcal{R}[X]$ is a right S-Noetherian ring, we can find $s \in S$ and $f_1, \ldots, f_m \in X\mathcal{R}_1[X]$ such that

$$(X\mathcal{R}_1[X])s \subseteq f_1\mathcal{R}[X] + \cdots + f_m\mathcal{R}[X] \subseteq X\mathcal{R}_1[X].$$

For each $i = 1, \ldots, m$, write $f_i = \sum_{j=1}^{n_i} d_{ij}X^j$ because $f_i(0) = 0$. Then

$$f_1\mathcal{R}[X] + \cdots + f_m\mathcal{R}[X] \subseteq \sum_{\substack{1\leq i\leq m\\1\leq j\leq n_i}} d_{ij}X^j\mathcal{R}[X].$$

Set $r = \mathrm{Max}\{n_1, \ldots, n_m\}$. Let $k \geq r+1$ and $a \in R_k$. Then $aX^k \in X\mathcal{R}_1[X]$ and so $(aX^k)s = \sum_{\substack{1\leq i\leq m\\1\leq j\leq n_i}} (d_{ij}X^j)g_{ij}$ for some $g_{11}, \ldots, g_{mn_m} \in \mathcal{R}[X]$. For each $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$, let b_{ijt} be the coefficient of X^t in g_{ij}. Then $as = \sum_{\substack{1\leq i\leq m\\1\leq j\leq n_i}} d_{ij}b_{ij,k-j}$; so we have

$$R_k s \subseteq \sum_{\substack{1\leq i\leq m\\1\leq j\leq n_i}} d_{ij}R_{k-j} \subseteq \sum_{\substack{1\leq i\leq m\\1\leq j\leq n_i}} d_{ij}R_{k-1}.$$

By repeating the same process when $k = r+p$ for all positive integers p, we obtain

$$R_{r+p}s^p \subseteq \sum_{\substack{1\leq i\leq m\\1\leq j\leq n_i}} d_{ij}R_r \subseteq R_r$$

because $d_{ij} \in R_r$ for all $i = 1, \ldots, m$ and $j = 1, \ldots, n_i$. Since S is a right anti-Archimedean subset of R_0, we can choose an element $w \in \bigcap_{n\geq 0} s^nR_0 \cap S$. Then $R_{r+p}w \subseteq R_r$ for all positive integers p. Thus $\mathcal{R}$ is right S-stationary.

Finally, we claim that R_n is S-finite as a right R_0-module for each $n \in \mathbb{N}$: Consider a right ideal $X^n\mathcal{R}_n[X]$ of $\mathcal{R}[X]$. Since $\mathcal{R}[X]$ is a right S-Noetherian ring, $X^n\mathcal{R}_n[X]$ is S-finite. Therefore we can find $s \in S$ and $f_1, \ldots, f_m \in X^n\mathcal{R}_n[X]$ such that

$$(X^n\mathcal{R}_n[X])s \subseteq f_1\mathcal{R}[X] + \cdots + f_m\mathcal{R}[X] \subseteq X^n\mathcal{R}_n[X].$$

For each $i = 1, \ldots, m$, write $f_i = \sum_{j=n}^{k_i} d_{ij}X^j$ where $d_{ij} \in R_j$. For any $b \in R_n$, $(bX^n)s = f_1g_1 + \cdots + f_mg_m$ for some $g_1, \ldots, g_m \in \mathcal{R}[X]$; so $bs = d_{1n}g_1(0) + \cdots + d_{mn}g_m(0)$. Thus $R_ns \subseteq d_{1n}R_0 + \cdots + d_{mn}R_0 \subseteq R_n$. Hence R_n is S-finite as a right R_0-module.

(2) $\Rightarrow$ (3). It is trivial from the definition of a right S-Noetherian chain $\mathcal{R}$.

(3) $\Rightarrow$ (1). Suppose that R_0 is a right S-Noetherian ring. From Lemma 4.3, E is an S-finite normalizing ring extension of R_0, that is, E is an S-finite R_0-module. Thus by Corollary 2.4, E is a right S-Noetherian ring. Since S is a right anti-Archimedean subset of R_0, S is also a right anti-Archimedean subset of E; so $E[X]$ is a right S-Noetherian ring from [5, Corollary 3.3]. From Corollary 2.7, $\mathcal{R}[X]$ is a right S-Noetherian ring. □

If $R_0 = R$ and $R_n = E$ for all $n \geq 1$ in Theorem 4.4, we obtain

Corollary 4.5. *Let $E[X]$ be an S-finite normalizing ring extension of a ring $R + XE[X]$ and S be a right anti-Archimedean subset of R. Then R is a right S-Noetherian ring if and only if so is the composite polynomial ring $R + XE[X]$.*

Corollary 4.6. *Let E be an S-finite normalizing ring extension of R and let S be a right anti-Archimedean subset of R. Then R is a right S-Noetherian ring if and only if so is $E[X]$.*

When $R = R_n$ for all $n \in \mathbb{N}_0$, Theorem 4.4 guarantees that the converse of [5, Corollary 3.3] also holds. Thus we have

Corollary 4.7. *Let S be a right anti-Archimedean subset of a ring R. Then R is a right S-Noetherian ring if and only if so is the polynomial ring $R[X]$.*

We obtain necessary and sufficient conditions for an ascending chain of rings to be right Noetherian as a corollary when $S = \{1\}$ in Theorem 4.4. Note that the concept of a Noetherian chain of commutative rings was first introduced by Haouat [11].

Corollary 4.8. *Let $\mathcal{R} = (R_n)_{n\geq 0}$ be an ascending chain of rings, $E = \bigcup_{n\geq 0} R_n$ and let $E[X]$ be a finite normalizing ring extension of a ring $\mathcal{R}[X]$. Then the following statements are equivalent.*

(1) *$\mathcal{R}[X]$ is a right Noetherian ring.*

(2) *$\mathcal{R}$ is a right Noetherian chain.*

(3) *R_0 is a right Noetherian ring.*

We next give necessary and sufficient conditions for the ring $\mathcal{R}[\![X]\!]$ to be right S-Noetherian. For each nonnegative integer ℓ, let $\mathcal{R}_\ell[\![X]\!] := \{ \sum_{i=\ell}^{\infty} a_i X^{i-\ell} \mid a_i \in R_i \}$.

Theorem 4.9. *Let $\mathcal{R} = (R_n)_{n\geq 0}$ be an ascending chain of rings, $E = \bigcup_{n\geq 0} R_n$, S a right anti-Archimedean subset of R_0 consisting of regular elements, and let $E[\![X]\!]$ be an S-finite normalizing ring extension of a ring $\mathcal{R}[\![X]\!]$. If E is right Ore, then the following statements are equivalent.*

(1) *$\mathcal{R}[\![X]\!]$ is a right S-Noetherian ring.*

(2) *$\mathcal{R}$ is a right S-Noetherian chain.*

(3) *R_0 is a right S-Noetherian ring.*

Proof. The proofs of (1) $\Rightarrow$ (2) and (2) $\Rightarrow$ (3) are similar to those in Theorem 4.4. Note that to show that $\mathcal{R}$ is right S-stationary in (1) $\Rightarrow$ (2), we can use a right ideal $\mathcal{I} = \langle \{d_j X^j \mid d_j \in R_j, j \geq 1\} \rangle$ of $\mathcal{R}[\![X]\!]$ instead of a right ideal $X\mathcal{R}_1[X]$ of $\mathcal{R}[X]$.

(3) $\Rightarrow$ (1). Suppose that R_0 is a right S-Noetherian ring. By the similar proof in Lemma 4.3, it is easy to see that E is S-finite as a right R_0-module. Thus E is a right S-Noetherian ring from Corollary 2.4. Since S is also a right anti-Archimedean subset of R consisting of regular elements, $E[\![X]\!]$ is a right S-Noetherian ring by [5, Theorem 3.6]. Therefore from Corollary 2.7, $\mathcal{R}[\![X]\!]$ is a right S-Noetherian ring. $\square$

If $R_0 = R$ and $R_n = E$ for all $n \geq 1$ in Theorem 4.9, we obtain

Corollary 4.10. *Let $E[\![X]\!]$ be an S-finite normalizing ring extension of $R + XE[\![X]\!]$, E be right Ore and S be a right anti-Archimedean subset of R consisting of regular elements. Then R is a right S-Noetherian ring if and only if so is the composite power series ring $R + XE[\![X]\!]$.*

Corollary 4.11. *Let E be an S-finite normalizing ring extension of R, E be right Ore and S be a right anti-Archimedean subset of R consisting of regular elements. Then R is a right S-Noetherian ring if and only if so is $E[\![X]\!]$.*

When $R = R_n$ for all $n \in \mathbb{N}_0$, Theorem 4.9 guarantees that the converse of [5, Theorem 3.6] also holds. Thus we have

Corollary 4.12. *Let R be right Ore and S be a right anti-Archimedean subset of R consisting of regular elements. Then R is a right S-Noetherian ring if and only if so is the power series ring $R[\![X]\!]$.*

In the case of $S = \{1\}$, we have

Corollary 4.13. *Let $\mathcal{R} = (R_n)_{n\geq 0}$ be an ascending chain of rings, $E = \bigcup_{n\geq 0} R_n$ a right Ore ring and $E[\![X]\!]$ be a finite normalizing ring extension of $\mathcal{R}[\![X]\!]$. Then the following statements are equivalent.*

(1) *$\mathcal{R}[\![X]\!]$ is a right Noetherian ring.*

(2) *$\mathcal{R}$ is a right Noetherian chain.*

(3) *R_0 is a right Noetherian ring.*

We conclude this paper with the result on the condition for composite skew polynomial rings to be right S-Noetherian. Let $R \subseteq E$ be an extension of rings and σ be an endomorphism of E. By $E[X;\sigma]$, we mean the *skew polynomial ring* over E (of the endomorphism type), subject to the left skewed constraint $Xa = \sigma(a)X$. If $\mathcal{R} = (R_n)_{n\geq 0}$ is an ascending chain of rings, $E = \bigcup_{n\geq 0} R_n$ and σ is an endomorphism of E with $\sigma(R_n) \subseteq R_n$ for each n, then $\mathcal{R}[X;\sigma] = \left\{ \sum_{i=0}^{n} a_i X^i \mid a_i \in R_i \right\}$ is a ring. We call it the *composite skew polynomial ring*. When σ is the identity map on E, a composite skew polynomial ring is a composite polynomial ring. A multiplicative subset S of a ring R is said to be a *right σ-anti-Archimedean subset for an automorphism σ of R* if $\bigcap_{n\geq 1} \left(\prod_{j=0}^{n-1} \sigma^{-n+j}(s) \right) R \cap S \neq \emptyset$ for all $s \in S$ [5]. Note that if E is a ring extension of R and S is a right σ-anti-Archimedean subset of R, then S is also a right σ-anti-Archimedean subset of E.

Lemma 4.14. *Let $R \subseteq E$ be an extension of rings, σ an automorphism of E such that $\sigma(R) \subseteq R$ and S be a multiplicative subset of R. If $E[X;\sigma]$ is an S-finite normalizing ring extension of a ring $\mathcal{R}[X;\sigma]$, then E is an S-finite normalizing ring extension of R.*

Proof. The proof is similar to that of Lemma 4.3. □

Proposition 4.15. *Let $\mathcal{R} = (R_n)_{n\geq 0}$ be an ascending chain of rings, $E = \bigcup_{n\geq 0} R_n$, S a right σ-anti-Archimedean subset of R_0 for an automorphism σ of E such that $\sigma(R_n) \subseteq R_n$ for each n, and let $E[X;\sigma]$ be an S-finite normalizing ring extension of a ring $\mathcal{R}[X;\sigma]$. Then the following statements are equivalent.*

(1) *$\mathcal{R}[X;\sigma]$ is a right S-Noetherian ring.*

(2) *R_0 is a right S-Noetherian ring.*

Proof. (1) $\Rightarrow$ (2). Let I be a right ideal of R_0. Then $I + \mathcal{R}_1[X;\sigma]X$ is a right ideal of $\mathcal{R}[X;\sigma]$; so we can find $s \in S$ and $f_1, \ldots, f_m \in I + \mathcal{R}_1[X;\sigma]X$ such that

$$(I + \mathcal{R}_1[X;\sigma]X)s \subseteq f_1\mathcal{R}[X;\sigma] + \cdots + f_m\mathcal{R}[X;\sigma] \subseteq I + \mathcal{R}_1[X;\sigma]X.$$

So $Is \subseteq f_1(0)R_0 + \cdots + f_m(0)R_0 \subseteq I$. Thus I is S-finite. Hence R_0 is a right S-Noetherian ring.

(2) $\Rightarrow$ (1). Suppose R_0 is a right S-Noetherian ring. By Lemma 4.14, E is an S-finite normalizing ring extension of R_0, hence E is S-finite as a right R_0-module. Thus E is a right S-Noetherian ring from Corollary 2.4. It is easy to see that S is a right σ-anti-Archimedean subset of E; so $E[X;\sigma]$ is a right S-Noetherian ring from [5, Corollary 3.2(1)]. Therefore from Corollary 2.7, $\mathcal{R}[X;\sigma]$ is a right S-Noetherian ring. $\square$

Corollary 4.16. *Let $R \subseteq E$ be an extension of rings, σ an automorphism of E such that $\sigma(R) \subseteq R$, S a right σ-anti-Archimedean subset of R, and let $E[X;\sigma]$ be an S-finite normalizing ring extension of a ring $R + XE[X;\sigma]$. Then the following statements are equivalent.*

(1) *R is a right S-Noetherian ring.*

(2) *E is a right S-Noetherian ring.*

(3) *$E[X;\sigma]$ is a right S-Noetherian ring.*

(4) *$R + XE[X;\sigma]$ is a right S-Noetherian ring.*

When $E = R$, Corollary 4.16 guarantees that the converse of [5, Corollary 3.2(1)] also holds. Thus we have

Corollary 4.17. *Let σ be an automorphism of a ring R and S a right σ-anti-Archimedean subset of R. Then R is a right S-Noetherian ring if and only if so is the skew polynomial ring $R[X;\sigma]$.*

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