Log Canonical Thresholds on Burniat Surfaces with $K^2 = 6$ via Pluricanonical Divisors

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Abstract. Let $S$ be a Burniat surface with $K^2_S = 6$ and $\varphi$ be the bicanonical map of $S$. In this paper we show optimal lower bounds of log canonical thresholds of members of pluricanonical sublinear systems of $S$ via Klein group $G$ induced by $\varphi$. Indeed, for a positive even integer $m$, the log canonical threshold of members of an invariant (resp. anti-invariant) part of $|mK_S|$ is greater than or equal to $1/(2m)$ (resp. $1/(2m - 2)$). For a positive odd integer $m$, the log canonical threshold of members of an invariant (resp. anti-invariant) part of $|mK_S|$ is greater than or equal to $1/(2m - 5)$ (resp. $1/(2m)$). The inequalities are all optimal.

1. Introduction

Let $X$ be a variety and $p \in X$ be a smooth point. And let $D$ be an effective Cartier divisor on $X$. The log canonical threshold or the complex singularity exponent of $D$ at $p$ is the number

$$\text{lct}_p(X, D) := \sup \{ c \in \mathbb{Q} \mid |f|^{-c} \text{ is locally } L^2 \text{ near } p \},$$

where $f$ is a local defining equation of $D$ at $p$. In [7] we have the following inequalities

$$\frac{1}{\text{mult}_p(D)} \leq \text{lct}_p(X, D) \leq \frac{\dim X}{\text{mult}_p(D)},$$

and the log canonical threshold of $D$ at $p$ is equal to the absolute value of the largest root of the Bernstein–Sato polynomial of $f$.

The log canonical threshold can be formally defined for log pairs (cf. [7, 8.2 Proposition]). Let $X$ be a normal variety with at worst log canonical singularities, $Z$ be a closed subvariety of $X$ and $D$ be an effective $\mathbb{Q}$-Cartier divisor on $X$. The log canonical threshold of $D$ along $Z$ on $X$ is the number

$$\text{lct}_Z(X, D) := \sup \{ c \in \mathbb{Q} \mid (X, cD) \text{ is log canonical in an open neighborhood of } Z \}.$$

For simplicity, we put $\text{lct}(X, D) = \text{lct}_X(X, D)$.

We have the following invariant for every polarised pair $(X, \mathcal{L})$.

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Definition 1.1. Let $X$ be a normal variety with at worst log canonical singularities, and $\mathcal{L}$ be an ample $\mathbb{Q}$-Cartier divisor on $X$. The global log canonical threshold of a pair $(X, \mathcal{L})$ is the number
\[
glct(X, \mathcal{L}) := \inf \{ \lct(X, D) \mid D \text{ is an effective } \mathbb{Q}\text{-Cartier divisor on } X, \mathbb{Q}\text{-linearly equivalent to } \mathcal{L} \}.
\]

Chen, Chen and Jiang [5] proved the Noether inequality for projective 3-folds of general type. They use the global log canonical threshold of a surface of general type with $p_g = 2$ and $K^2 = 1$ via its ample canonical divisor (see the appendix by Kollár in [5]).

The authors in [6] showed that the global log canonical threshold of a Burniat surface with $K^2 = 6$ via its ample canonical divisor is $1/2$, where the Burniat surface is a minimal surface of general type with $p_g = 0$ and $K^2 = 6$.

In this paper, we give optimal lower bounds of log canonical thresholds of members of pluricanonical sublinear systems via Klein group induced by the bicanonical map of a Burniat surface with $K^2 = 6$.

Let $S$ be a Burniat surface with $K_S^2 = 6$ (see [1, 2, 8–10]). The bicanonical map $\varphi$ of $S$ has an image, a del Pezzo surface $\Sigma$ of degree 6 in $\mathbb{P}^6$ which is a blow-up $\rho : \Sigma \to \mathbb{P}^2$ at three point $p_1$, $p_2$, $p_3$ in general position. Denote by $e_i$ the $(-1)$-curve corresponding to $p_i$, by $e'_i$ the strict transform of the line passing through the two points $p_j$ and $p_k$ by $\rho$, and by $m^i_l$ the strict transform of a general line passing through the point $p_i$ by $\rho$ for each $\{i, j, k\} = \{1, 2, 3\}$ and $l = 1, 2$. Then $\varphi$ is a bidouble covering map over $\Sigma$ with a branch divisor $B := B_1 + B_2 + B_3$ satisfying $2L_i \sim B_j + B_k$ for a line bundle $L_i$ on $\Sigma$ and $\{i, j, k\} = \{1, 2, 3\}$, where
\[
\begin{align*}
B_1 &= e_1 + e'_1 + m^1_1 + m^2_2, \\
B_2 &= e_2 + e'_2 + m^1_1 + m^3_2, \\
B_3 &= e_3 + e'_3 + m^1_1 + m^1_2,
\end{align*}
\]
and $\sim$ means the linearly equivalent relation between divisors.

For $i = 1, 2, 3$, we note $\varphi^*(B_i) = 2R_i$ for some divisor $R_i$ ramified by $\varphi$, and denote by $G$ the Klein group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \{\text{Id}_S, \sigma_1, \sigma_2, \sigma_3\}$ induced by $\varphi$ such that $R_i$ is the divisorial fixed part of $\sigma_i$.

For a positive integer $m$, the natural action of the group $G$ splits the set of global sections of the pluricanonical divisor $mK_S$ of $S$ into eigen spaces via the characters of $G$:
\[
H^0(S, mK_S) = H^0(S, mK_S)^{\text{inv}} \oplus \bigoplus_{i=1}^{3} H^0(S, mK_S)^{\chi_i},
\]
where $\chi_i$ is a character of $G$ such that $\chi_i(\sigma_j) = \delta_{ij}$ for $i, j \in \{1, 2, 3\}$. Then the pluricanonical linear system $|mK_S|$ for a positive integer $m$ contains an invariant part $|mK_S|^0$. 
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(resp. an anti-invariant part \(|mK_S|_i\)) that consists of zeros of sections of \(H^0(S, mK_S)^{\text{inv}}\) (resp. \(H^0(S, mK_S)^{\chi_i}\)) for \(i = 1, 2, 3\), that is,

\[|mK_S| \supseteq |mK_S|_0 \cup \bigcup_{i=1}^{3} |mK_S|_i.\]

We consider the log canonical threshold of members of the invariant and anti-invariant parts of the complete linear system \(|mK_S|\), where \(m\) is a positive integer. To calculate the log canonical threshold, we use the following representation of pluricanonical linear systems for a bidouble covering map \(\varphi: S \to \Sigma\). Denote by \(R\) the ramification divisor \(R_1 + R_2 + R_3\) of \(\varphi\).

**Proposition 1.2.** (cf. [10, Proposition 1.6]) For a positive integer \(n\) and each \(i = 1, 2, 3\) with \(\{i, j, k\} = \{1, 2, 3\}\),

(i) \(|2nK_S|_0 = \varphi^*|n(2K_\Sigma + B)|\) and \(|2nK_S|_i = R_j + R_k + |\varphi^*(n(2K_\Sigma + B) - L_i)|\);

(ii) \(|(2n + 1)K_S|_0 = R + |\varphi^*((2n + 1)K_\Sigma + nB)|\) and \(|(2n + 1)K_S|_i = R_i + |\varphi^*((2n + 1)K_\Sigma + nB + L_i)|\).

We apply \(B \sim -3K_\Sigma\) to Proposition 1.2 and obtain log canonical thresholds of members of the pluricanonical sublinear systems of Burniat surfaces \(S\) with \(K_S^2 = 6\) via the Klein group induced by the bicanonical map of \(\varphi\) as follows.

**Theorem 1.3** (Main theorem). Let \(S\) be a Burniat surface with \(K_S^2 = 6\). Then for a positive integer \(n\) and each \(i = 1, 2, 3\),

(i) if \(D_0 \in |2nK_S|_0\) and \(D_i \in |2nK_S|_i\),

\[\text{lct}(S, D_0) \geq \frac{1}{4n} \quad \text{and} \quad \text{lct}(S, D_i) \geq \frac{1}{4n-2};\]

(ii) if \(D'_0 \in |(2n + 1)K_S|_0\) and \(D'_i \in |(2n + 1)K_S|_i\),

\[\text{lct}(S, D'_0) \geq \frac{1}{4n-3} \quad \text{and} \quad \text{lct}(S, D'_i) \geq \frac{1}{4n+2}.\]

Moreover the inequalities are optimal.

**Remark 1.4.** Since \(|2K_S|_i = \emptyset\) for all \(i = 1, 2, 3\) (see [9, Proposition 3.1]), we actually have \(\text{lct}(S, D_i) \geq 1/(4n - 2)\) for any \(D_i \in |2nK_S|_i\) when an integer \(n \geq 2\) in Theorem 1.3(i).
Corollary 1.5. Let \( S \) be a Burniat surface with \( K_S^2 = 6 \). Then for a positive integer \( n \) and each \( i = 1, 2, 3 \),

(i) if \( D_i \in |2nK_S|_i \),

\[
\lct(S, D_i) > \frac{1}{4n};
\]

(ii) if \( D'_0 \in |(2n + 1)K_S|_0 \),

\[
\lct(S, D'_0) > \frac{1}{4n + 2}.
\]

Remark 1.6. Corollary 1.5(i) is \[6\], Proposition 5.2.\]

Since

\[
glct(S, K_S) = \frac{1}{2}
\]

(see \[6\], Theorem 1.3), we obtain

Corollary 1.7. Let \( S \) be a Burniat surface with \( K_S^2 = 6 \). For any positive even (resp. odd) integer \( m \), if a divisor \( D \) is in the linear system \( |mK_S| \) such that \( glct(S, K_S) = \lct(S, \frac{1}{m}D) \), then the divisor \( D \) is not in the anti-invariant parts \( |mK_S|_i \) (resp. the invariant part \( |mK_S|_0 \)) for \( i = 1, 2, 3 \).

Proof. We get the result by Corollary 1.5. \[\square\]

2. Preliminaries

Let \( X \) be a normal variety with at worst log canonical singularities. Note that \( \sim_\mathbb{Q} \) means the \( \mathbb{Q} \)-linearly equivalent relation.

Lemma 2.1. Let \( \mathcal{N}_0 \sim_\mathbb{Q} A \) be an effective \( \mathbb{Q} \)-Cartier divisor on \( X \) such that the log pair \( (X, \mathcal{N}_0) \) is not log canonical at a point \( p \). And let \( \mathcal{N} \sim_\mathbb{Q} A \) be an effective \( \mathbb{Q} \)-Cartier divisor on \( X \) such that the log pair \( (X, \mathcal{N}) \) is log canonical at the point \( p \). Then there is an effective \( \mathbb{Q} \)-Cartier divisor \( \mathcal{N}' \sim_\mathbb{Q} A \) on \( X \) such that at least one component of \( \mathcal{N} \) is not contained in the support of \( \mathcal{N}' \) and the log pair \( (X, \mathcal{N}') \) is not log canonical at the point \( p \).

Proof. See \[4\], Remark 2.22. \[\square\]

The following is used for a non log canonical pair at some smooth point.

Lemma 2.2. (cf. \[7\], 8.10 Lemma) Let \( D \) be an effective \( \mathbb{Q} \)-Cartier divisor on \( X \). If the log pair \( (X, D) \) is not log canonical at some smooth point \( p \), then the inequality

\[
\text{mult}_p(D) > 1
\]

holds.
3. Proof of the main theorem

We remark that for $i = 1, 2, 3$ and $j = 1, 2$,

$$E_i^2 = E_i'^2 = -1, \quad K_S \cdot E_i = K_S \cdot E_i' = 1, \quad M_j^2 = 0 \quad \text{and} \quad K_S \cdot M_j^i = 2,$$

where $\varphi^*(e_i) = 2E_i$, $\varphi^*(e_i') = 2E_i'$ and $\varphi^*(m_j^i) = 2M_j^i$.

3.1. Even pluricanonical linear system

For a positive integer $n$, the complete linear system $|2nK_S|$ contains the invariant part $|2nK_S|_0$ and the anti-invariant parts $|2nK_S|_i$ with $i = 1, 2, 3$, that is,

$$|2nK_S| \supseteq \bigcup_{i=0}^{3} |2nK_S|_i.$$

3.1.1. Invariant part

In [6] we have

$$\text{gltc}(S, 2K_S) = \text{lct}(S, \overline{D}_0) = \frac{1}{4}$$

for some divisor $\overline{D}_0 \in |2K_S|$. For example, $\overline{D}_0 := 2E_1 + 4E_3 + 2E_1' + 4E_2'$, then

$$\text{lct}(S, D_0) \geq \frac{1}{4n}$$

for any $D_0 \in |2nK_S|_0$ and the inequality is optimal.

3.1.2. Anti-invariant parts

To show

$$\text{lct}(S, D_i) \geq \frac{1}{4n - 2}$$

for any $D_i \in |2nK_S|_i$, we need the following lemma.

**Lemma 3.1.** [6 Lemma 4.1] Let $\psi: X \to Y$ be a bidouble covering map between a normal variety $X$ and a smooth variety $Y$ branched along an effective divisor $B$ on $Y$, and $D$ be an effective $\mathbb{Q}$-Cartier divisor on $X$. Then

$$(X, D) \text{ is log canonical if } \left( Y, \psi(D) + \frac{1}{2}B \right) \text{ is log canonical.}$$

We deal with an integer $n \geq 2$ by Remark 1.4. Suppose that $\text{lct}(S, D_i) < 1/(4n - 2)$. Then the log pair $(S, \frac{1}{4n-2}D_i)$ is not log canonical at some point $p$. By Lemma 2.2,

$$\text{mult}_p(D_i) > 4n - 2.$$
We put an effective divisor $d_i := \varphi(D_i)$ on $\Sigma$. Then
\[
\left( \Sigma, \frac{1}{4n-2}d_i + \frac{1}{2}B \right)
\] is not log canonical at a point $\varphi(p)$ on $\Sigma$ by Lemma 3.1.

We consider the case $\varphi(p) \notin B_1 \cup B_2 \cup B_3$. Then $(\Sigma, \frac{1}{4n-2}d_i)$ is not log canonical at $\varphi(p)$ which implies
\[
\text{glct}(\Sigma, d_i) < \frac{1}{4n-2}.
\]
However, it contradicts because $d_i \sim Q -nK_\Sigma$ and $\text{glct}(\Sigma, \Delta) \geq 1/2$ for any effective $Q$-Cartier divisor $\Delta \sim Q -K_\Sigma$ since $\Sigma$ is a nonsingular del Pezzo surface of degree 6 (see [3, Theorem 1.7]). Thus $\varphi(p) \in B_1 \cup B_2 \cup B_3$.

By Proposition 1.2, we have an effective $Q$-Cartier divisor $D_i - (R_j + R_k)$ for $\{i, j, k\} = \{1, 2, 3\}$. We may deal with $i = 1$.

The case $p \in E_1 \cap E'_2$. We have
\[
D_1 = \alpha_1 E_1 + \alpha_2 E_2 + \alpha'_3 E'_3 + \Omega,
\]
where rational numbers $\alpha_1 \geq 0$ and $\alpha_2, \alpha'_3 \geq 1$, and $E_1, E_2, E'_3 \not\subset \text{Supp}(\Omega)$ with an effective $Q$-Cartier divisor $\Omega$ (denote by Supp($\Omega$) the support of $\Omega$). Since $p \notin E_2 \cup E'_3$, the log pair $(S, \frac{1}{4n-2}(D_1 - \alpha_2 E_2 - \alpha'_3 E'_3))$ is not log canonical at the point $p$.

Suppose $\alpha_1 = 0$, and then $2n = D_1 \cdot E_1 \geq \text{mult}_p(D_1) \cdot \text{mult}_p(E_1) > 4n - 2$ which is a contradiction. So $\alpha_1 \neq 0$.

Since $D_1 - (R_2 + R_3)$ is effective,
\[
\Omega \cdot M^l_1 \geq (M^3_1 + M^2_2) \cdot M^l_1 = 2.
\]
Thus $4n = D_1 \cdot M^l_1 = \alpha_1 + \Omega \cdot M^l_1$. implies $4n - 2 \geq \alpha_1$, and so
\[
\frac{\alpha_1}{4n-2} \leq 1.
\]
We have a pair $(S, E_1 + \frac{1}{4n-2}\Omega)$ is not log canonical at $p$. By the inversion of adjunction formula,
\[
\text{the pair } \left( E_1, \frac{1}{4n-2}\Omega \big|_{E_1} \right) \text{ is not log canonical at } p.
\]
This implies that
\[
2n + \alpha_1 - \alpha'_3 = (D_1 - \alpha_1 E_1 - \alpha_2 E_2 - \alpha'_3 E'_3) \cdot E_1 > 4n - 2.
\]
On the other hand, since $D_1 - (R_2 + R_3)$ is effective,
\[
2n = D_1 \cdot E'_3 = \alpha_1 + \alpha_2 - \alpha'_3 + \Omega \cdot E'_3 \geq \alpha_1 + \alpha_2 - \alpha'_3 + (M^3_1 + M^2_2) \cdot E'_3 = \alpha_1 + \alpha_2 - \alpha'_3 + 2.
\]
Hence
\[ \alpha_2 < 0 \]
which is a contradiction.

*The case* \( p \in E_1 \setminus (E_2' \cup E_3') \). We have
\[ D_1 = \alpha_1 E_1 + \alpha_2' E_2' + \alpha_3' E_3' + \Omega, \]
where rational numbers \( \alpha_1 \geq 0 \) and \( \alpha_2', \alpha_3' \geq 1 \), and \( E_1, E_2', E_3' \not\subseteq \text{Supp}(\Omega) \) with an effective \( \mathbb{Q} \)-Cartier divisor \( \Omega \). Then
\[ 2n = D_1 \cdot E_3' = \alpha_1 - \alpha_3' + \Omega \cdot E_3' \geq \alpha_1 - \alpha_3' + (E_2 + M_1^3 + M_2^3) \cdot E_3' = \alpha_1 - \alpha_3' + 3. \]
And since
\[ 4n = D_1 \cdot M_1^1 = \alpha_1 + \Omega \cdot M_1^1 \geq \alpha_1 + (M_1^3 + M_2^3) \cdot M_1^1 = \alpha_1 + 2, \]
we obtain
\[ 2n + \alpha_1 - \alpha_2' - \alpha_3' = (D_1 - \alpha_1 E_1 - \alpha_2' E_2' - \alpha_3' E_3') \cdot E_1 > 4n - 2 \]
by the inversion of adjunction formula. Hence
\[ \alpha_2' < -1 \]
which is a contradiction.

*The case* \( p \in M_1^1 \setminus (E_1 \cup E_1') \). The log pair
\[ \left( S, \frac{1}{4n - 2} (M_1^1 + M_1^3 + M_2^3 + D) \right) \]
is not log canonical at the point \( p \), where \( D_1 \sim R_2 + R_3 + D \) for some \( D \in |\varphi^*(-nK_\Sigma - L_1)| \) by Proposition 1.2(i). We have
\[ D = \alpha M_1^2 + \Delta \]
where a rational number \( \alpha \geq 0 \) and \( M_1^3 \not\subseteq \text{Supp}(\Delta) \) with an effective \( \mathbb{Q} \)-Cartier divisor \( \Delta \). By using a general member \( M \) of the linear system \( |2M_1^2| \) such that \( M \not\subseteq \text{Supp}(D) \),
\[ 8n - 12 = D \cdot M \geq \alpha M_1^3 \cdot \overline{M} = 2\alpha. \]
Thus we can use the inversion of adjunction formula. So the log pair
\[ \left( M_1^3, \frac{1}{4n - 2} (M_1^1 + M_2^3 + \Delta) \right) \]
is not log canonical at \( p \). Then
\[ 1 + (4n - 4) = (M_1^1 + M_2^3 + \Delta) \cdot M_1^3 \geq \text{mult}_p \left( (M_1^1 + M_2^3 + \Delta)|_{M_1^3} \right) > 4n - 2 \]
which is a contradiction.

We can induce a contradiction by using a similar argument like the above cases for each point of \( R \). Therefore for all cases \( i = 1, 2, 3 \),

\[
lct(S, D_i) \geq \frac{1}{4n-2} \quad \text{for any } D_i \in |2nK_S|_i.
\]

And the inequality is optimal because \( \lct_p(S, \bar{D}_i) = 1/(4n-2) \) for

\[
\bar{D}_i := R_{i+1} + R_{i+2} + 2((2n-1)E'_i + (n-2)E'_{i+1} + (2n-3)E_{i+1} + nE_{i+2}) \in |2nK_S|
\]

and

\[
p \in E'_i \setminus (E_{i+1} \cup E_{i+2} \cup M_{i1} \cup M_{i2}),
\]

where the index \( i \in \{1, 2, 3\} \) is considered as modulo 3.

3.2. Odd pluricanonical linear system

For a positive integer \( n \), the complete linear system \(|(2n + 1)K_S|\) contains the invariant part \(|(2n + 1)K_S|_0\) and the anti-invariant parts \(|(2n + 1)K_S|_i\) with \( i = 1, 2, 3 \), that is,

\[
|(2n + 1)K_S| \supseteq \bigcup_{i=0}^3 |(2n + 1)K_S|_i.
\]

3.2.1. Invariant part

We prove that for any \( D'_0 \in |(2n + 1)K_S|_0 \), the log pair \((S, \frac{1}{4n-3}D'_0)\) is log canonical. To obtain a contradiction, we assume that there is a member \( D'_0 \) of \(|(2n + 1)K_S|_0\) such that the log pair \((S, \frac{1}{4n-3}D'_0)\) is not log canonical at some point \( p \). Note that

\[
|(2n + 1)K_S|_0 = R + |2(n - 1)K_S|
\]

(see Proposition 1.2 and apply \( B \sim -3K_S \) and \( K_S \sim_Q \varphi^*(K_\Sigma + \frac{1}{2}B) \)). Thus there is the member \( D' \) of the complete linear system \(|2(n - 1)K_S|\) such that \( D'_0 = R + D' \). Since the global log canonical threshold of the pair \((S, 2(n-1)K_S)\) is \( 1/(4n-4) \) (see \[6\] Theorem 1.3), \( p \) is contained in \( R \). We consider the following cases.

The case \( p \in E_3 \cap E'_1 \). The log pair \((S, \frac{1}{4n-3}(E_3 + E'_1 + D'))\) is not log canonical at the point \( p \). For the effective divisor

\[
N := (4n - 3)E_3 + (4n - 3)E'_1 + (2n - 2)E_2 + (2n - 2)E'_2 \sim E_3 + E'_1 + D',
\]

the log canonical threshold of the log pair \((S, N)\) is \( 1/(4n - 3) \). By Lemma 2.1, there is an effective \( \mathbb{Q} \)-Cartier divisor \( N' \sim \mathbb{Q} N \) such that at least one component of \( N \) is not
contained in the support of \( N' \) and the log pair \((S, \frac{1}{4n-3}N')\) is not log canonical at \( p \). Thus at least one of \( E_2, E_3, E'_1 \) and \( E'_2 \) is not contained in \( \text{Supp}(N') \).

We can represent
\[ N' = \alpha_3 E_3 + \alpha'_1 E'_1 + \Omega, \]
where rational numbers \( \alpha_3, \alpha'_1 \geq 0 \) and \( E_3, E'_1 \not\subset \text{Supp}(\Omega) \) with an effective \( \mathbb{Q} \)-Cartier divisor \( \Omega \).

Suppose \( E_2 \not\subset \text{Supp}(N') \). Then
\[ 2n - 1 = N' \cdot E_2 \geq \alpha'_1 E'_1 \cdot E_2 = \alpha'_1 \]
By the inversion of adjunction formula, the log pair
\[ \left( E'_1, \frac{1}{4n-3}(\alpha_3 E_3 + \Omega) \right|_{E'_1} \]
is not log canonical at \( p \). Thus
\[ (2n - 2) + \alpha'_1 = (\alpha_3 E_3 + \Omega) \cdot E'_1 \geq \text{mult}_p (\left( \alpha_3 E_3 + \Omega \right|_{E'_1}) > 4n - 3 \]
which is a contradiction.

For each case \( E'_2, E_3 \) or \( E'_1 \not\subset \text{Supp}(N') \), we also get a contradiction by using a similar argument as above. We remark that \( E_3 \not\subset \text{Supp}(N') \) (resp. \( E'_1 \not\subset \text{Supp}(N') \)) means \( \alpha_3 = 0 \) (resp. \( \alpha'_1 = 0 \)).

The case \( p \in E_3 \setminus (E'_1 \cup E'_2) \). The log pair \((S, \frac{1}{4n-3}(E_3 + M^3_1 + M^3_2 + D'))\) is not log canonical at the point \( p \). We have
\[ D' = \alpha_3 E_3 + \alpha'_1 E'_1 + \alpha'_2 E'_2 + \Delta, \]
where rational numbers \( \alpha_3, \alpha'_1, \alpha'_2 \geq 0 \) and \( E_3, E'_1, E'_2 \not\subset \text{Supp}(\Delta) \) with an effective \( \mathbb{Q} \)-Cartier divisor \( \Delta \). Let \( \tilde{M} \) be a general member of the linear system \(|2M^3_1|\) such that \( \tilde{M} \not\subset \text{Supp}(D') \). Then
\[ 8n - 8 = D' \cdot \tilde{M} \geq \alpha_3 E_3 \cdot \tilde{M} = 2\alpha_3 \]
implies that \( 4n - 4 \geq \alpha_3 \). By the inversion of adjunction formula, the log pair
\[ \left( E_3, \frac{1}{4n-3}(M^3_1 + M^3_2 + \Delta) \right|_{E_3} \]
is not log canonical at \( p \). Thus
\[ (2n - 1) + \alpha_3 - \alpha'_1 - \alpha'_2 \geq ((M^3_1 + M^3_2 + \Delta) \cdot E_3)_p \geq \text{mult}_p ((M^3_1 + M^3_2 + \Delta) \big|_{E_3}) > 4n - 3 \]
which implies \( \alpha_3 > (2n - 2) + \alpha'_1 + \alpha'_2 \). Meanwhile, the inequality
\[ (2n - 2) - \alpha_3 + \alpha'_1 = \Delta \cdot E'_1 \geq 0 \]
implies that \((2n - 2) + \alpha' \geq \alpha_3\) which is a contradiction.

The case \(p \in M_1^1 \setminus (E_1 \cup E_1')\). Set \(M := M_1^2 + M_2^2 + M_3^3 + M_4^3\). Then the log pair
\[
\left( S, \frac{1}{4n - 3} (M_1^1 + M + D') \right)
\]
is not log canonical at the point \(p\). We have
\[
D' = \alpha M_1^1 + \Delta,
\]
where a rational number \(\alpha \geq 0\) and \(M_1^1 \not\subset \text{Supp}(\Delta)\) with an effective \(\mathbb{Q}\)-Cartier divisor \(\Delta\).

By using a general member \(\hat{M}\) of the linear system \(|2M_1^1|\) such that \(\hat{M} \not\subset \text{Supp}(D')\),
\[
8n - 8 = D' \cdot \hat{M} \geq \alpha M_1^1 \cdot \hat{M} = 2\alpha
\]
which implies \(4n - 4 \geq \alpha\). By the inversion of adjunction formula, the log pair
\[
\left( M_1^1, \frac{1}{4n - 3} (M + \Delta) \big|_{M_1^1} \right)
\]
is not log canonical at \(p\). Then
\[
1 + (4n - 4) \geq \left( (M + \Delta) \cdot M_1^1 \right)_p \geq \text{mult}_p((M + \Delta)|_{M_1^1}) > 4n - 3
\]
which is a contradiction.

We can induce a contradiction by using a similar argument like the above cases for each point of \(R\). Hence
\[
\text{lct}(S, D'_0) \geq \frac{1}{4n - 3} \quad \text{for any } D'_0 \in |(2n + 1)K_S|_0.
\]
And the inequality is optimal because \(\text{lct}_p(S, D'_0) = 1/(4n - 3)\) for
\[
D'_0 := R + 2(n - 1)(2E_2' + E_3' + 2E_1 + E_3) \in |(2n + 1)K_S|_0
\]
and
\[
p \in E_2' \setminus (E_1 \cup E_3 \cup M_1^2 \cup M_2^2).
\]

3.2.2. Anti-invariant part

For a positive integer \(n\) and \(i = 1, 2, 3\), \(|(2n + 1)K_S|_i\) is represented by
\[
R_i + |\varphi^*((1 - n)K_\Sigma + L_i)|
\]
(see Proposition 1.2 and apply \(B \sim -3K_\Sigma\)).
We may consider for $i = 1$. The divisor
\[
\mathcal{D}_1' := E_1 + E_1' + M_1^2 + M_2^2 + 2((2n+1)E_2' + (n-1)E_1' + nE_1 + 2nE_3)
\]
is in $|(2n+1)K_S|_1$. The log canonical threshold of the log pair $(S, \mathcal{D}_1')$ is $1/(4n + 2)$. Note that the global log canonical threshold of the log pair $(S, K_S)$ is $1/2$ (see [6, Theorem 1.3]). This means that the infimum of the set
\[
\{\text{lct}(S, D_1') \mid D_1' \in |(2n+1)K_S|_1\}
\]
is $1/(4n + 2)$. Thus
\[
\inf\{\text{lct}(S, D_i') \mid D_i' \in |(2n+1)K_S|_i\} = \frac{1}{4n + 2}
\]
for each $i = 1, 2, 3$.

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