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Distance (Signless) Laplacian Eigenvalues of k-uniform Hypergraphs

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Abstract. The distance (signless) Laplacian eigenvalues of a connected hypergraph are the eigenvalues of its distance (signless) Laplacian matrix. For all n-vertex k-uniform hypertrees, we determine the k-uniform hypertree with minimum second largest distance (signless) Laplacian eigenvalue. For all n-vertex k-uniform unicyclic hypergraphs, we obtain the k-uniform unicyclic hypergraph with minimum largest distance (signless) Laplacian eigenvalue, and the k-uniform unicyclic hypergraph with minimum second largest distance Laplacian eigenvalue.

1. Introduction

Let $G = (V(G), E(G))$ be an n-vertex m-edge hypergraph, where $V(G) = \{u_1, u_2, \dots, u_n\}$, $E(G) = \{e_1, e_2, \dots, e_m\}$ and $e_i \subseteq V(G)$ for every $i \in \{1, 2, \dots, m\}$. If $k \geq 2$ and every edge $e \in E(G)$ satisfies $|e| = k$, then G is a k-uniform hypergraph. Let $w, u \in V(G)$. If there is some edge $e \in E(G)$ satisfying $\{w, u\} \subseteq e$, then u is a neighbour of w. Let $N_G(w) = \{u \in V(G) : u \text{ is a neighbour of } w\}$ and $E_G(w) = \{e \in E(G) : w \in e\}$. The degree of w in G is $d_G(w) = |E_G(w)|$. For $e = \{u_1, \dots, u_k\} \in E(G)$, if $d_G(u_1) \geq 2$ and $d_G(u_i) = 1$ for every $i \in \{2, \dots, k\}$, then e is a pendent edge of G at u_1.

Let $P = (u_0, e_1, u_1, \dots, u_{p-1}, e_p, u_p)$ be a sequence of vertices and edges in a hypergraph G. If $\{u_{i-1}, u_i\} \subseteq e_i$, and $u_{i-1} \neq u_i$ for each $i \in \{1, 2, \dots, p\}$, then P is called a walk of length p connecting u_0 and u_p in G. If all vertices u_i are pairwise distinct and all edges e_i are pairwise distinct, then the walk P is called a path. If all vertices u_i are pairwise distinct except $u_0 = u_p$, all edges e_i are pairwise distinct and $p \geq 2$, then the walk P is called a cycle. For any $w, u \in V(G)$, if w and u are connected by a path, then G is a connected hypergraph.

Let G be an n-vertex m-edge k-uniform connected hypergraph. If G contains no cycles, then G is called a k-uniform hypertree. Note that such hypertree G satisfies

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$n = m(k-1)+1$. If G has exactly one cycle, then G is called a k-uniform unicyclic hypergraph. Note that such unicyclic hypergraph G satisfies $n = m(k-1)$.

Let T be a k-uniform hypertree. If there exists a vertex $w \in V(T)$ satisfying $w \in e$ for every edge $e \in E(T)$, then we say T is a hyperstar, and w is the centre of T. We use $S_{n,k}$ to denote the n-vertex k-uniform hyperstar.

Let $U = (V(U), E(U))$ be a k-uniform unicyclic hypergraph, where $V(U) = \{u_1, u_2, \ldots, u_n\}$ and $E(U) = \{e_1, e_2, \ldots, e_m\}$. If $e_i = \{u_{(i-1)(k-1)+1}, \ldots, u_{(i-1)(k-1)+k}\}$ for each $i \in \{1, 2, \ldots, m\}$ and $u_{(m-1)(k-1)+k} = u_1$, then U is called a k-uniform loose cycle. We use $C_{n,k}$ to denote the n-vertex k-uniform loose cycle.

Let U be a k-uniform unicyclic hypergraph that contains $C_{gk-g,k}$ as an induced subhypergraph, where $k \geq 3$ and $g \geq 2$. We label the vertices of $C_{gk-g,k}$ as above. Let $H_1, \ldots, H_{gk-g}$ be the $gk-g$ components of $U - E(C_{gk-g,k})$ with $u_i \in V(H_i)$ for each $i \in \{1, \ldots, gk-g\}$ (it is possible that some H_i consists of a single vertex u_i), and we denote U by $C^k_{gk-g}(H_1, \ldots, H_{gk-g})$. In particular, if $H_i = S_{t_i(k-1)+1,k}$ with $t_i \geq 0$ for some $i \in \{1, \ldots, gk-g\}$, then we use $C^k_{gk-g}(H_1, \ldots, t_i, \ldots, H_{gk-g})$ to denote U. If $H_i = S_{t_i(k-1)+1,k}$ with $t_i = 0$ for some $i \in \{1, \ldots, gk-g\}$, then U is also denoted by $C^k_{gk-g}(H_1, \ldots, u_i, \ldots, H_{gk-g})$.

Let G be an n-vertex connected hypergraph and $w, u \in V(G)$. Suppose that P is a shortest path that connects w and u in G. The length of P is the distance $d_G(w, u)$ between w and u. We define $d_G(w, w) = 0$. The diameter $d = d(G)$ of G is $d = \max\{d_G(w, u) : w, u \in V(G)\}$. The distance matrix of G is an $n \times n$ matrix index by $V(G)$, whose (w, u)-entry is $d_G(w, u)$. For $w \in V(G)$, the transmission of w is defined as $\mathrm{Tr}_G(w) = \sum_{u \in V(G)} d_G(w, u)$. Let $\mathrm{Tr}_{\max}(G) = \max\{\mathrm{Tr}_G(w) : w \in V(G)\}$ be the maximum vertex transmission. If $\mathrm{Tr}_G(w) = r$ (r is a real number) for all $w \in V(G)$, then G is transmission regular. The Wiener index of G is defined as $W(G) = \sum_{\{w,u\} \subseteq V(G)} d_G(w, u) = \frac{1}{2} \sum_{w \in V(G)} \mathrm{Tr}_G(w)$.

Let $\mathrm{Tr}(G) = \mathrm{diag}(\mathrm{Tr}_G(w) : w \in V(G))$. The distance Laplacian matrix of a connected hypergraph G is $\mathcal{L}(G) = \mathrm{Tr}(G) - D(G)$. Let $\partial_1(G), \partial_2(G), \ldots, \partial_n(G)$ be the eigenvalues of $\mathcal{L}(G)$, which are called the distance Laplacian eigenvalues of G and satisfy $\partial_1(G) \geq \partial_2(G) \geq \cdots \geq \partial_n(G)$. The distance signless Laplacian matrix of a connected hypergraph G is $\mathcal{Q}(G) = \mathrm{Tr}(G) + D(G)$. Let $q_1(G), q_2(G), \ldots, q_n(G)$ be the eigenvalues of $\mathcal{Q}(G)$, which are called the distance signless Laplacian eigenvalues of G and satisfy $q_1(G) \geq q_2(G) \geq \cdots \geq q_n(G)$.

In addition, let $J_{n\times m}$ be the $n \times m$ all-one matrix and I_n be the identity matrix of order n. In particular, $1_n = J_{n\times 1}$ and $J_n = J_{n\times n}$. Let A be a real $n \times n$ symmetric matrix and $\lambda_1(A) \geq \lambda_2(A) \geq \cdots \geq \lambda_n(A)$ be its eigenvalues.

The use of distance matrix was arisen in a wide range of areas. For example, Balaban, Ciubotariu and Medeleanu [4] proposed to use the largest distance eigenvalue of graphs as a molecular descriptor, which can be used to investigate the boiling points of alkanes and also to infer the extent of branching. There are many results about the distance eigenvalues of graphs, see [2, 8, 14, 16, 23]. Recently, some scholars paid attention to the distance eigenvalues of hypergraphs, see [9–12, 15, 18, 20–22]. In [1], Aouchiche and Hansen defined the distance (signless) Laplacian eigenvalues of ordinary graphs, and we refer to [3, 6, 7, 19] for more results. In [13], Lin, Zhou and Wang obtained some extremal k-uniform hypergraphs whose distance (signless) Laplacian spectral radius are minimum or maximum.

In this paper, for all n-vertex k-uniform hypertrees, we determine the k-uniform hypertree with minimum second largest distance (signless) Laplacian eigenvalue. For n-vertex k-uniform unicyclic hypergraphs, we obtain the k-uniform unicyclic hypergraphs with minimum largest distance (signless) Laplacian eigenvalue and minimum second largest distance Laplacian eigenvalue, respectively.

2. Preliminaries

Let G be an n-vertex connected k-uniform hypergraph and let $V(G) = \{u_1, u_2, \ldots, u_n\}$ be its vertex set. Let $x = (x_{u_1}, x_{u_2}, \ldots, x_{u_n})^T \in \mathbb{R}^n$. We can also view x as a function $x\colon V(G) \to \mathbb{R}$ such that $x(u_i) = x_{u_i}$ for every $i \in \{1, 2, \ldots, n\}$. We have

$$x^T \mathcal{L}(G) x = \sum_{\{w,u\} \subseteq V(G)} d_G(w,u)(x_w - x_u)^2$$

and

$$x^T \mathcal{Q}(G) x = \sum_{\{w,u\} \subseteq V(G)} d_G(w,u)(x_w + x_u)^2.$$

Lemma 2.1. [5] *Let B be a real $n \times n$ symmetric matrix. If B' is a $t \times t$ principal submatrix of B and $t \leq n$, then*

$$\lambda_{j+n-t}(B) \leq \lambda_j(B') \leq \lambda_j(B), \quad 1 \leq j \leq t.$$

Lemma 2.2. [17] *Let $C = (c_{ij})$ be a complex matrix of order n. Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_p$ are its distinct eigenvalues. Then*

$$\{\lambda_1, \lambda_2, \ldots, \lambda_p\} \subset \bigcup_{i=1}^{n} \left\{ z : |z - c_{ii}| \leq \sum_{j \neq i} |c_{ij}| \right\}.$$

The following result is obtained by Lemma 2.2 and analogous arguments as the proof of Theorem 2.2 in [3].

Lemma 2.3. *For any n-vertex connected hypergraph G, $\partial_n(G) = 0$ with multiplicity 1.*

Lemma 2.4. [24] *Let U be an n-vertex m-edge k-uniform unicyclic hypergraph, where $m = \frac{n}{k-1} \geq 4$ and $U \ncong C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2})$. Then*

$$\begin{aligned} W(U) &\geq W(C_{2k-2}^k(m-3, u_2, \ldots, u_{k-1}, 1, u_{k+1}, \ldots, u_{2k-2})) \\ &> W(C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2})), \end{aligned}$$

where $W(C_{2k-2}^k(m-3, u_2, \ldots, u_{k-1}, 1, u_{k+1}, \ldots, u_{2k-2})) = n^2 - 2n + 6k - 2 + \frac{nk}{2} - 3k^2$.

Lemma 2.5. *Given an n-vertex nontrivial connected hypergraph G, we have*

$$q_1(G) \geq \frac{4W(G)}{n},$$

and equality if and only if G is a transmission regular hypergraph.

Proof. Let $x = (x_{u_1}, x_{u_2}, \ldots, x_{u_n})^T \in \mathbb{R}^n$ be a unit vector and there exists an index i satisfying $x_{u_i} \geq 0$. By the Rayleigh's principle,

$$q_1(G) \geq x^T \mathcal{Q}(G) x.$$

In particular, let $z = \frac{1}{\sqrt{n}}(1, 1, \ldots, 1)^T$. We have

$$q_1(G) \geq z^T \mathcal{Q}(G) z = \sum_{\{w,u\} \subseteq V(G)} d_G(w, u) \left(\frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} \right)^2 = \frac{4W(G)}{n},$$

and equality if and only if G is a transmission regular hypergraph. □

By Lemma 2.2, we obtain the following result.

Lemma 2.6. *If G is an n-vertex nontrivial connected hypergraph, then*

$$q_1(G) \leq 2\,\mathrm{Tr}_{\max}(G).$$

Lemma 2.7. *Let $k \geq 3$ and G be an n-vertex k-uniform hypergraph. If $e_1, e_2, \ldots, e_\ell$ are pendent edges at u, then all vertices in $(e_1 \cup e_2 \cup \cdots \cup e_\ell) \setminus \{u\}$ have the same transmission, say* Tr. *Moreover, $\mathcal{L}(G)$ has* $\mathrm{Tr}+1$ *as an eigenvalue and its multiplicity is at least $(k-2)\ell$.*

Proof. Let $u_i \in e_i \setminus \{u\}$ for each $i \in \{1, 2, \ldots, \ell\}$. Then

$$\mathrm{Tr}_G(u_i) = (k-1) + 2(\ell-1)(k-1) + \sum_{w \in V(G) \setminus \{e_1, e_2, \ldots, e_\ell\}} (d_G(w, u) + 1).$$

Thus all vertices in $(e_1 \cup e_2 \cup \cdots \cup e_\ell) \setminus \{u\}$ have the same transmission, say Tr.

Let $A = (\mathrm{Tr}+1)I_n - \mathcal{L}(G)$. For all $i \in \{1, 2, \ldots, \ell\}$, the rows of A indexed by the vertices $e_i \setminus \{u\}$ are identical. Hence, $\mathcal{L}(G)$ has $\mathrm{Tr}+1$ as an eigenvalue and its multiplicity is at least $(k-2)\ell$. □

3. The second largest distance (signless) Laplacian eigenvalue of k-uniform hypertrees

Lemma 3.1. [13] *The eigenvalues of* $\mathcal{L}(S_{n,k})$ *are* $2n-1$ *(multiplicity* $m-1$*),* $2n-k$ *(multiplicity* $m(k-2)$*),* n*,* 0*, where* $m=\frac{n-1}{k-1}$ *and* $k\geq 2$*.*

Let $k\geq 3$ and T be an n-vertex k-uniform hypertree with diameter d. Suppose that $P=(u_0,e_1,u_1,\ldots,u_{d-1},e_d,u_d)$ is a diametrical path of T. For each $w_i\in V(P)$, the nontrivial component of $T-E(P)$ that contains w_i is denoted by T_{w_i} and let $n_i=|V(T_{w_i})|$, where $1\leq i\leq s$ and $s\leq (d-2)(k-1)+1$. Suppose that T' is obtained from T by transforming every T_{w_i} into a k-uniform hyperstar $S_{n_i,k}$ with centre w_i.

Lemma 3.2. *If* T*,* P *and* T' *are as described above, then*

$$\max\{\mathrm{Tr}_T(u_0),\mathrm{Tr}_T(u_d)\}>\frac{1}{2}(n-1)(d+2)-d(k-1)+\frac{1}{2}d.$$

Proof. If $v\in V(P)$, then $d_T(u_0,v)=d_{T'}(u_0,v)$. If $v\notin V(P)$, then $d_T(u_0,v)\geq d_{T'}(u_0,v)$. Thus

$$\mathrm{Tr}_T(u_0)=\sum_{v\in V(T)}d_T(u_0,v)\geq\sum_{v\in V(T')}d_{T'}(u_0,v)=\mathrm{Tr}_{T'}(u_0).$$

Similarly, $\mathrm{Tr}_T(u_d)\geq\mathrm{Tr}_{T'}(u_d)$. So $\max\{\mathrm{Tr}_T(u_0),\mathrm{Tr}_T(u_d)\}\geq\max\{\mathrm{Tr}_{T'}(u_0),\mathrm{Tr}_{T'}(u_d)\}$.

In T', for $v\in V(P)$, we have

$$\begin{aligned}d_{T'}(u_0,u_i)+d_{T'}(u_i,u_d)&=d &&\text{for } i=0,1,\ldots,d,\\ d_{T'}(u_0,v)+d_{T'}(v,u_d)&=d+1 &&\text{for } v\in e_i\setminus\{u_{i-1},u_i\}\text{ and } i=1,\ldots,d,\end{aligned}$$

and for $v\notin V(P)$, we have

$$d_{T'}(u_0,v)+d_{T'}(v,u_d)\geq d+2.$$

Thus

$$\begin{aligned}\mathrm{Tr}_{T'}(u_0)+\mathrm{Tr}_{T'}(u_d)&\geq(n-d(k-1)-1)(d+2)+d(d+1)(k-2)+d(d+1)\\&>(n-d(k-1)-1)(d+2)+d^2(k-2)+d(d+1)\\&=(n-1)(d+2)-2d(k-1)+d.\end{aligned}$$

Therefore,

$$\begin{aligned}\max\{\mathrm{Tr}_T(u_0),\mathrm{Tr}_T(u_d)\}&\geq\max\{\mathrm{Tr}_{T'}(u_0),\mathrm{Tr}_{T'}(u_d)\}\geq\frac{1}{2}\big(\mathrm{Tr}_{T'}(u_0)+\mathrm{Tr}_{T'}(u_d)\big)\\&>\frac{1}{2}(n-1)(d+2)-d(k-1)+\frac{1}{2}d.\end{aligned}$$

□

Theorem 3.3. *Let $k \geq 3$, $n \geq 5(k-1)+1$, and T be an n-vertex k-uniform hypertree. Then $\partial_2(T) \geq 2n-1$, with equality if and only if $T \cong S_{n,k}$.*

Proof. Let T be an n-vertex k-uniform hypertree with diameter d. Assume that $P = (u_0, e_1, u_1, \ldots, u_{d-1}, e_d, u_d)$ is a diametrical path of T.

By Lemma 3.1, we have $\partial_2(S_{n,k}) = 2n-1$ for $n \geq 5(k-1)+1$. So it suffices to prove $\partial_2(T) > 2n-1$ for $T \ncong S_{n,k}$ and $n \geq 5(k-1)+1$. If $T \ncong S_{n,k}$, then we have $d \geq 3$. We next consider two cases.

Case 1*:* $d = 3$*.* Suppose that there is a vertex with degree at least 3 and there are at least two vertices each with degree at least 2. Without loss of generality, let $d_T(u_1) \geq 3$. By Lemma 2.7, $\mathrm{Tr}_T(u_0)+1$ is an eigenvalue of $\mathcal{L}(T)$ and its multiplicity is at least $(k-2)(d_T(u_1)-1) \geq 2$. Suppose that there are at least three vertices each with degree 2 and all the other vertices have degree 1. Without loss of generality, let $d_T(u_1) = 2$ and $d_T(u_2) = 2$. Obviously, we have $\mathrm{Tr}_T(u_0) = \mathrm{Tr}_T(u_3)$. By Lemma 2.7, $\mathrm{Tr}_T(u_0)+1$ is an eigenvalue of $\mathcal{L}(T)$ and its multiplicity is at least $(k-2) \geq 1$, and $\mathrm{Tr}_T(u_3)+1$ is an eigenvalue of $\mathcal{L}(T)$ and its multiplicity is at least $(k-2) \geq 1$.

Since

$$\begin{aligned}\mathrm{Tr}_T(u_0) &= (k-1) + 2(k-1) + 3(k-1) + \sum_{v \in V(T)\setminus V(P)} d_T(v,u_0) \\ &\geq (k-1) + 2(k-1) + 3(k-1) + 3(k-1) + 2(n-4(k-1)-1) \\ &= 2n+k-3,\end{aligned}$$

we have $\partial_2(T) \geq 2n+k-2 \geq 2n+1 > 2n-1$.

Suppose that there are exactly two vertices each with degree at least 3 and all the other vertices have degree 1. Similarly as above, we have $\partial_2(T) > 2n-1$.

Suppose that there are exactly one vertex with degree at least 3 and exactly one vertex with degree 2, and all the other vertices have degree 1. Without loss of generality, let $d_T(u_1) \geq 3$ and $d_T(u_2) = 2$. Let $v \in e_3 \setminus \{u_2, u_3\}$. Then we consider the 2×2 principal submatrix of $\mathcal{L}(T)$, denoted by M, indexed by vertices v and u_3, where

$$M = \begin{pmatrix} \mathrm{Tr}_T(v) & -1 \\ -1 & \mathrm{Tr}_T(u_3) \end{pmatrix}.$$

Note that $\mathrm{Tr}_T(v) = \mathrm{Tr}_T(u_3)$ and

$$\mathrm{Tr}_T(v) = (k-1) + 2(k-1) + 3(n-2(k-1)-1) = 3n-3k.$$

By Lemma 2.1, we have $\partial_2(T) \geq \lambda_2(M) = \mathrm{Tr}_T(v) - 1$. Recall that $n \geq 5(k-1)+1$ and $k \geq 3$, so $3n-3k-1 > 2n-1$. Thus $\partial_2(T) > 2n-1$.

Case 2: $d \geq 4$. Let $u \in e_1 \setminus \{u_0, u_1\}$. Since e_1 is a pendent edge at u_1, we have $\mathrm{Tr}_T(u_0) = \mathrm{Tr}_T(u)$. Then we consider the 2×2 principal submatrix of $\mathcal{L}(T)$, denoted by M', indexed by vertices u_0 and u, where

$$M' = \begin{pmatrix} \mathrm{Tr}_T(u_0) & -1 \\ -1 & \mathrm{Tr}_T(u) \end{pmatrix}.$$

Without loss of generality, let $\mathrm{Tr}_T(u_0) \geq \mathrm{Tr}_T(u_d)$. By Lemma 3.2, we have $\mathrm{Tr}_T(u_0) > \frac{1}{2}(n-1)(d+2) - d(k-1) + \frac{1}{2}d$. By Lemma 2.1, we have $\partial_2(T) \geq \lambda_2(M') = \mathrm{Tr}_T(u_0) - 1$.

Since $d \geq 4$ and $n \geq d(k-1)+1$, we have

$$\lambda_2(M') > \frac{1}{2}nd + n - d(k-1) - 2 \geq 3n - d(k-1) - 2 \geq 2n - 1.$$

Thus $\partial_2(T) > 2n-1$. □

Lemma 3.4. *The eigenvalues of $\mathcal{Q}(S_{n,k})$ are $2n-k-2$ (multiplicity $m(k-2)$), $2n-2k-1$ (multiplicity $m-1$), $\frac{5n-2k-4-\sqrt{9n^2-12nk-8n+4k(k+2)}}{2}$, and $\frac{5n-2k-4+\sqrt{9n^2-12nk-8n+4k(k+2)}}{2}$, where $m = \frac{n-1}{k-1}$ and $k \geq 2$.*

Proof. Let $e_1, e_2, \ldots, e_m$ be the pendent edges of $S_{n,k}$ at centre v, where $m = \frac{n-1}{k-1}$. By calculation, we have $\mathrm{Tr}_T(v) = n-1$ and $\mathrm{Tr}_T(u) = 2n-k-1$ for any $u \in V(S_{n,k}) \setminus \{v\}$. We partition $V(S_{n,k})$ into $\{v\} \cup (e_1 \setminus \{v\}) \cup (e_2 \setminus \{v\}) \cup \cdots \cup (e_m \setminus \{v\})$. Then

$$\mathcal{Q}(S_{n,k}) = \begin{pmatrix} n-1 & 1_{k-1}^T & 1_{k-1}^T & \cdots & 1_{k-1}^T \\ 1_{k-1} & aI_{k-1}+J_{k-1} & 2J_{k-1} & \cdots & 2J_{k-1} \\ 1_{k-1} & 2J_{k-1} & aI_{k-1}+J_{k-1} & \cdots & 2J_{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1_{k-1} & 2J_{k-1} & 2J_{k-1} & \cdots & aI_{k-1}+J_{k-1} \end{pmatrix},$$

where $a = 2n-k-2$. Thus

$$\begin{aligned} &|\lambda I_n - \mathcal{Q}(S_{n,k})| \\ &= \begin{vmatrix} \lambda-n+1 & -1_{k-1}^T & -1_{k-1}^T & \cdots & -1_{k-1}^T \\ -1_{k-1} & (\lambda-a)I_{k-1}-J_{k-1} & -2J_{k-1} & \cdots & -2J_{k-1} \\ -1_{k-1} & -2J_{k-1} & (\lambda-a)I_{k-1}-J_{k-1} & \cdots & -2J_{k-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -1_{k-1} & -2J_{k-1} & -2J_{k-1} & \cdots & (\lambda-a)I_{k-1}-J_{k-1} \end{vmatrix} \\ &= (\lambda-a)^{m(k-2)} \begin{vmatrix} \lambda-n+1 & -(k-1)1_m^T \\ -1_m & (\lambda-a+k-1)I_m - 2(k-1)J_m \end{vmatrix} \end{aligned}$$

$$= (\lambda - a)^{m(k-2)}(\lambda - a + k - 1)^{m-1}\begin{vmatrix} \lambda - n + 1 & -(n-1) \\ -1 & \lambda - a - 2(n-1) + k - 1 \end{vmatrix}$$
$$= (\lambda - a)^{m(k-2)}(\lambda - a + k - 1)^{m-1} f(\lambda),$$

where $f(\lambda) = \lambda^2 - (5n - 2k - 4)\lambda + 4n^2 - 2nk - 8n + 2k + 4$. The two roots of $f(\lambda)$ are $\lambda_1 = \frac{5n-2k-4-\sqrt{9n^2-12nk-8n+4k(k+2)}}{2}$ and $\lambda_2 = \frac{5n-2k-4+\sqrt{9n^2-12nk-8n+4k(k+2)}}{2}$.

Hence, the eigenvalues of $\mathcal{Q}(S_{n,k})$ are $2n - k - 2$ (multiplicity $m(k-2)$), $2n - 2k - 1$ (multiplicity $m - 1$), $\frac{5n-2k-4-\sqrt{9n^2-12nk-8n+4k(k+2)}}{2}$, and $\frac{5n-2k-4+\sqrt{9n^2-12nk-8n+4k(k+2)}}{2}$. □

Remark 3.5. By calculation, we have $\frac{5n-2k-4+\sqrt{9n^2-12nk-8n+4k(k+2)}}{2} > 2n - k - 2$. For $m \geq 3$ (i.e., $n \geq 3(k-1)+1$) and $k \geq 2$, we have

$$\begin{aligned} 9n^2 - 12nk - 8n + 4k(k+2) - n^2 &= 4[n(2n - 3k - 2) + k(k+2)] \\ &\geq 4[n(3k - 6) + k(k+2)] > 0. \end{aligned}$$

Thus $2n - k - 2 > \frac{5n-2k-4-\sqrt{9n^2-12nk-8n+4k(k+2)}}{2}$.

Theorem 3.6. *Let $k \geq 3$, $n \geq 3(k-1)+1$, and T be an n-vertex k-uniform hypertree. Then $q_2(T) \geq 2n - k - 2$, with equality if and only if $T \cong S_{n,k}$.*

Proof. Let T be an n-vertex k-uniform hypertree with diameter d. Assume that $P = (u_0, e_1, u_1, \ldots, u_{d-1}, e_d, u_d)$ is a diametrical path of T.

By Lemma 3.4 and Remark 3.5, we have $q_2(S_{n,k}) = 2n - k - 2$. So it suffices to prove $q_2(T) > 2n - k - 2$ for $T \ncong S_{n,k}$ and $n \geq 3(k-1)+1$. If $T \ncong S_{n,k}$, then we have $d \geq 3$. We next consider two cases.

Case 1*:* $d = 3$. Let $v \in e_1 \setminus \{u_0, u_1\}$. Then we consider the 2×2 principal submatrix of $\mathcal{Q}(T)$, denoted by M, indexed by vertices u_0 and v, where

$$M = \begin{pmatrix} \mathrm{Tr}_T(u_0) & 1 \\ 1 & \mathrm{Tr}_T(v) \end{pmatrix}.$$

Note that $\mathrm{Tr}_T(v) = \mathrm{Tr}_T(u_0)$ and

$$\begin{aligned} \mathrm{Tr}_T(u_0) &= (k-1) + 2(k-1) + 3(k-1) + \sum_{u \in V(T) \setminus V(P)} d_T(u, u_0) \\ &\geq (k-1) + 2(k-1) + 3(k-1) + 2(n - 3(k-1) - 1) = 2n - 2. \end{aligned}$$

By Lemma 2.1, we have $q_2(T) \geq \lambda_2(M) = \mathrm{Tr}_T(u_0) - 1$. Since $2n - 3 > 2n - k - 2$ for $k \geq 3$, we have $q_2(T) > 2n - k - 2$.

Case 2: $d \geq 4$. Let $u \in e_1 \setminus \{u_0, u_1\}$. Since e_1 is a pendent edge at v_1, we have $\mathrm{Tr}_T(u_0) = \mathrm{Tr}_T(u)$. Then we consider the 2×2 the principal submatrix of $\mathcal{Q}(T)$, denoted by M', indexed by vertices u_0 and u, where

$$M' = \begin{pmatrix} \mathrm{Tr}_T(u_0) & 1 \\ 1 & \mathrm{Tr}_T(u) \end{pmatrix}.$$

Without loss of generality, let $\mathrm{Tr}_T(u_0) \geq \mathrm{Tr}_T(u_d)$. By Lemma 3.2, we have $\mathrm{Tr}_T(u_0) > \frac{1}{2}(n-1)(d+2) - d(k-1) + \frac{1}{2}d$. By Lemma 2.1, we have $q_2(T) \geq \lambda_2(M') = \mathrm{Tr}_T(u_0) - 1$. Since $d \geq 4$ and $n \geq d(k-1)+1$, we have

$$\lambda_2(M') > \frac{1}{2}nd + n - d(k-1) - 2 \geq 3n - d(k-1) - 2 \geq 2n - 1 > 2n - k - 2.$$

Thus $q_2(T) > 2n - k - 2$. □

4. Distance (signless) Laplacian eigenvalues of k-uniform unicyclic hypergraphs

Lemma 4.1. *The eigenvalues of* $\mathcal{L}(C^k_{2k-2}(m-2, u_2, \ldots, u_{2k-2}))$ *are* 0, $2n-2$, $2n-2k+2$, n, $2n-1$ *(multiplicity* $m-2$*), and* $2n-k$ *(multiplicity* $(m-2)(k-2)+2(k-3)$*), where* $m = \frac{n}{k-1} \geq 3$ *and* $k \geq 3$.

Proof. Let $U = C^k_{2k-2}(m-2, u_2, \ldots, u_{2k-2})$ and $f_1, f_2, \ldots, f_{m-2}$ be the pendent edges of $C^k_{2k-2}(m-2, u_2, \ldots, u_{2k-2})$ at u_1. It is easy to see that $a := \mathrm{Tr}_U(u_1) = n-1$, $c := \mathrm{Tr}_U(u_k) = 2n-2k+1$, $b := \mathrm{Tr}_U(u) = 2n-k-1$ for any $u \notin \{u_1, u_k\}$. Then we have $a = \frac{b+k-1}{2}$ and $c = b-k+2$. We partition $V(C^k_{2k-2}(m-2, u_2, \ldots, u_{2k-2}))$ into $\{u_1\} \cup (e_1 \setminus \{u_1, u_k\}) \cup \{u_k\} \cup (e_2 \setminus \{u_1, u_k\}) \cup (f_1 \setminus \{u_1\}) \cup \cdots \cup (f_{m-2} \setminus \{u_1\})$. Then

$$\mathcal{L}(C^k_{2k-2}(m-2, u_2, \ldots, u_{2k-2})) = \begin{pmatrix} a & -1^T_{k-2} & -1 & -1^T_{k-2} & -1^T_{k-1} & -1^T_{k-1} & \cdots & -1^T_{k-1} \\ -1_{k-2} & M & -1_{k-2} & -2J_{k-2} & N & N & \cdots & N \\ -1 & -1^T_{k-2} & c & -1^T_{k-2} & -21^T_{k-1} & -21^T_{k-1} & \cdots & -21^T_{k-1} \\ -1_{k-2} & -2J_{k-2} & -1_{k-2} & M & N & N & \cdots & N \\ -1_{k-1} & N^T & -21_{k-1} & N^T & P & -2J_{k-1} & \cdots & -2J_{k-1} \\ -1_{k-1} & N^T & -21_{k-1} & N^T & -2J_{k-1} & P & \cdots & -2J_{k-1} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ -1_{k-1} & N^T & -21_{k-1} & N^T & -2J_{k-1} & -2J_{k-1} & \cdots & P \end{pmatrix},$$

where $M=(b+1)I_{k-2}-J_{k-2}$, $P=(b+1)I_{k-1}-J_{k-1}$ and $N=-2J_{(k-2)\times(k-1)}$. Thus

$$\begin{aligned}
&|\lambda-\mathcal{L}(C^k_{2k-2}(m-2,u_2,\ldots,u_{2k-2}))|\\
&=\begin{vmatrix}
\lambda-a & 1^T_{k-2} & 1 & 1^T_{k-2} & 1^T_{k-1} & 1^T_{k-1} & \cdots & 1^T_{k-1}\\
1_{k-2} & \lambda I_{k-2}-M & 1_{k-2} & 2J_{k-2} & -N & -N & \cdots & -N\\
1 & 1^T_{k-2} & \lambda-c & 1^T_{k-2} & 21^T_{k-1} & 21^T_{k-1} & \cdots & 21^T_{k-1}\\
1_{k-2} & 2J_{k-2} & 1_{k-2} & \lambda I_{k-2}-M & -N & -N & \cdots & -N\\
1_{k-1} & -N^T & 21_{k-1} & -N^T & \lambda I_{k-1}-P & 2J_{k-1} & \cdots & 2J_{k-1}\\
1_{k-1} & -N^T & 21_{k-1} & -N^T & 2J_{k-1} & \lambda I_{k-1}-P & \cdots & 2J_{k-1}\\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots\\
1_{k-1} & -N^T & 21_{k-1} & -N^T & 2J_{k-1} & 2J_{k-1} & \cdots & \lambda I_{k-1}-P
\end{vmatrix}\\
&=(\lambda-b-1)^{(m-2)(k-2)+2(k-3)}\\
&\times\begin{vmatrix}
\lambda-a & k-2 & 1 & k-2 & (k-1)1^T_{m-2}\\
1 & \lambda-b+k-3 & 1 & 2(k-2) & 2(k-1)1^T_{m-2}\\
1 & k-2 & \lambda-c & k-2 & 2(k-1)1^T_{m-2}\\
1 & 2(k-2) & 1 & \lambda-b+k-3 & 2(k-1)1^T_{m-2}\\
1_{m-2} & 2(k-2)1_{m-2} & 21_{m-2} & 2(k-2)1_{m-2} & (\lambda-b-k)I_{m-2}+2(k-1)J_{m-2}
\end{vmatrix}\\
&=(\lambda-b-1)^{(m-2)(k-2)+2(k-3)}(\lambda-b-k)^{m-3}g(\lambda),
\end{aligned}$$

where $g(\lambda)=|\lambda I_5-A|$ and

$$A=\begin{pmatrix}
\frac{b+k-1}{2} & 2-k & -1 & 2-k & \frac{3k-5-b}{2}\\
-1 & b-k+3 & -1 & 4-2k & 3k-5-b\\
-1 & 2-k & b-k+2 & 2-k & 3k-5-b\\
-1 & 4-2k & -1 & b-k+3 & 3k-5-b\\
-1 & 4-2k & -2 & 4-2k & 4k-5
\end{pmatrix}.$$

Since

$$\begin{aligned}
g(\lambda)=\lambda^5&-\left(\frac{7}{2}b+\frac{3}{2}k+\frac{5}{2}\right)\lambda^4+\left(\frac{9}{2}b^2+4bk+\frac{13}{2}b-\frac{k^2}{2}+\frac{15}{2}k-2\right)\lambda^3\\
&+\left(\frac{5}{2}b+\frac{3}{2}k-12bk+\frac{bk^2}{2}-\frac{7}{2}b^2k-\frac{11}{2}b^2-\frac{5}{2}b^3-\frac{13}{2}k^2+\frac{3}{2}k^3+\frac{3}{2}\right)\lambda^2\\
&+\left(\frac{b^4}{2}+b^3k+\frac{3}{2}b^3+\frac{9}{2}b^2k-\frac{b^2}{2}-bk^3+\frac{9}{2}bk^2-\frac{3}{2}b-\frac{k^4}{2}+\frac{3}{2}k^3+\frac{k^2}{2}-\frac{3}{2}k\right)\lambda\\
&=\lambda(\lambda-b-k)(\lambda-b-k+1)(\lambda-b+k-3)\left(\lambda-\frac{b+k+1}{2}\right),
\end{aligned}$$

the eigenvalues of $\mathcal{L}(C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2}))$ are 0, $2n-2$, $2n-2k+2$, n, $2n-1$ (multiplicity $m-2$), and $2n-k$ (multiplicity $(m-2)(k-2)+2(k-3)$). □

4.1. The largest distance (signless) Laplacian eigenvalue of k-uniform unicyclic hypergraphs

Theorem 4.2. *Let $k \geq 3$, $n \geq 6(k-1)$, and U be an n-vertex k-uniform unicyclic hypergraph. Then $\partial_1(U) \geq 2n-1$, with equality if and only if $U \cong C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2})$, where $m = \frac{n}{k-1}$.*

Proof. By Lemma 4.1, we have $\partial_1(C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2})) = 2n-1$ for $n \geq 6(k-1)$. So it suffices to prove $\partial_1(U) > 2n-1$ for $U \ncong C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2})$ and $n \geq 6(k-1)$. By Lemma 2.4, we have

$$W(U) \geq n^2 - 2n + 6k - 2 + \frac{nk}{2} - 3k^2 > W(C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2})).$$

Since $\sum_{i=1}^n \partial_i(U) = 2W(U)$, and by Lemma 2.3, we have $(n-1)\partial_1(U) \geq 2W(U)$. Note that $6k^2 - 13k + 6 > 0$ for $k \geq 3$. Hence,

$$\begin{aligned}
\partial_1(U) &\geq \frac{2W(U)}{n-1} \geq \frac{2n^2 - 4n + 12k - 4 + nk - 6k^2}{n-1} \\
&= 2n + (k-2) - \frac{6k^2 - 13k + 6}{n-1} \geq 2n + (k-2) - \frac{6k^2 - 13k + 6}{6k-7} \\
&= 2n + (k-2) - \frac{(6k-7)(k-1) - 1}{6k-7} = 2n - 1 + \frac{1}{6k-7} > 2n - 1.
\end{aligned}$$

□

Theorem 4.3. *Let $k \geq 3$, $n \geq 4(k-1)$, and U be an n-vertex k-uniform unicyclic hypergraph. Then $q_1(U) \geq q_1(C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2}))$, with equality if and only if $U \cong C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2})$.*

Proof. Let U be an n-vertex k-uniform unicyclic hypergraph and $U \ncong C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2})$. By Lemmas 2.4, 2.5 and 2.6, we have

$$q_1(U) \geq \frac{4W(U)}{n} \geq 2 \cdot \frac{2n^2 - 4n + 12k - 4 + nk - 6k^2}{n},$$

and

$$q_1(C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2})) \leq 2\,\mathrm{Tr}_{\max}(C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2})) = 2(2n-k-1).$$

Since

$$\begin{aligned}
&\frac{2n^2 - 4n + 12k - 4 + nk - 6k^2}{n} - (2n-k-1) \\
&= 2n - 4 + k - \frac{6k^2 - 12k + 4}{n} - 2n + k + 1 = 2k - 3 - \frac{6k^2 - 12k + 4}{n} \\
&\geq 2k - 3 - \frac{3k^2 - 6k + 2}{2k-2} = \frac{(k-2)^2}{2k-2} > 0,
\end{aligned}$$

we have $q_1(U) > q_1(C_{2k-2}^k(m-2, u_2, \ldots, u_{2k-2}))$. □

4.2. The second largest distance Laplacian eigenvalue of k-uniform unicyclic hypergraphs

Lemma 4.4. *Let $k \geq 3$, $n \geq 5(k-1)$ and U be an n-vertex k-uniform unicyclic hypergraph and $U \ncong C_{n,k}$. Suppose that $P = (v_0, f_1, v_1, \ldots, v_{d-1}, f_d, v_d)$ is a diametrical path of U satisfies f_1 is a pendent edge at v_1 and $d \geq 4$. Then $\partial_2(U) > 2n-1$.*

Proof. Let $w \in f_1 \setminus \{v_0, v_1\}$. Then we consider the 2×2 principal submatrix of $\mathcal{L}(U)$, denoted by M, indexed by vertices v_0 and w, where

$$M = \begin{pmatrix} \mathrm{Tr}_U(v_0) & -1 \\ -1 & \mathrm{Tr}_U(w) \end{pmatrix}.$$

Note that $\mathrm{Tr}_U(v_0) = \mathrm{Tr}_U(w)$. Let t be the length of the cycle of U.

Case 1*: $t > 2$.* In this case,

$$\begin{aligned} \mathrm{Tr}_U(v_0) &\geq (k-1) + 2(k-1) + \cdots + d(k-1) - 1 + 2(n - d(k-1) - 1) \\ &= \frac{d(d+1)}{2}(k-1) + 2n - 2d(k-1) - 3 \\ &= \frac{k-1}{2}(d^2 - 3d) + 2n - 3 \geq 2(k-1) + 2n - 3 \geq 2n+1. \end{aligned}$$

By Lemma 2.1, we have $\partial_2(U) \geq \lambda_2(M) = \mathrm{Tr}_U(v_0) - 1 > 2n-1$.

Case 2*: $t = 2$ and $d \geq 5$.* In this case,

$$\begin{aligned} \mathrm{Tr}_U(v_0) &> (k-1) + 2(k-1) + \cdots + d(k-1) - d + 2(n - d(k-1) - 1) \\ &= \frac{d(d+1)}{2}(k-1) - d + 2n - 2d(k-1) - 2 \\ &= \frac{k-1}{2}(d^2 - 3d) - d + 2n - 2 \geq d(k-2) + 2n - 2 \geq 2n+1. \end{aligned}$$

By Lemma 2.1, we have $\partial_2(U) \geq \lambda_2(M) = \mathrm{Tr}_U(v_0) - 1 > 2n-1$.

Case 3*: $t = 2$ and $d = 4$.* If P contains at most one edge of the cycle of U, then

$$\begin{aligned} \mathrm{Tr}_U(v_0) &\geq (k-1) + 2(k-1) + 3(k-1) + 4(k-1) + 2(n - 4(k-1) - 1) \\ &= 2n + 2(k-1) - 2 > 2n+1. \end{aligned}$$

By Lemma 2.1, we have $\partial_2(U) \geq \lambda_2(M) = \mathrm{Tr}_U(u_0) - 1 > 2n-1$.

If P contains two edges of the cycle of U and $d_U(v_1) = 2$, then

$$\begin{aligned} \mathrm{Tr}_U(v_0) &\geq (k-1) + 2(k-1) + 3(k-1) + 4(k-1) - 4 + 3(n - 4(k-1)) \\ &= 3n - 2(k-1) - 4 \geq 3(k-1) - 4 + 2n > 2n+1. \end{aligned}$$

By Lemma 2.1, we have $\partial_2(U) \geq \lambda_2(M) = \mathrm{Tr}_U(v_0) - 1 > 2n-1$.

If P contains two edges of the cycle of U and $d_U(v_1) \geq 3$, then

$$\begin{aligned}\mathrm{Tr}_U(v_0) &\geq (k-1) + 2(k-1) + 3(k-1) + 4(k-1) - 4 + 2(n - 4(k-1)) \\ &= 2n + 2(k-1) - 4 \geq 2n.\end{aligned}$$

By Lemma 2.7, $\mathrm{Tr}_U(v_0) + 1$ is an eigenvalue of $\mathcal{L}(U)$ and its multiplicity is at least $(k-2)(d_U(v_1) - 1) \geq 2$. Thus $\partial_2(U) \geq 2n + 1 > 2n - 1$. □

Lemma 4.5. *Let $k \geq 3$ and $U \cong C^k_{gk-g}(t_1, t_2, \ldots, t_k, H_{k+1}, H_{k+2}, \ldots, H_{gk-g})$. If there exist pairwise distinct $i, j, z \in \{1, 2, \ldots, k\}$ such that $t_i \geq 2$ and $t_j, t_z \geq 1$, or there exist distinct $i, j \in \{1, 2, \ldots, k\}$ such that $t_i, t_j \geq 2$ and $t_y = 0$ for any $y \in \{1, 2, \ldots, k\} \setminus \{i, j\}$, then $\partial_2(U) > 2n - 1$.*

Proof. First suppose that $t_i \geq 2$ and $t_j, t_z \geq 1$ for pairwise distinct $i, j, z \in \{1, 2, \ldots, k\}$. Let $v \in V(H_i) \setminus \{u_i\}$. By Lemma 2.7, $\mathrm{Tr}_U(v) + 1$ is an eigenvalue of $\mathcal{L}(U)$ and its multiplicity is at least $(k-2)t_i \geq 2$. Since

$$\mathrm{Tr}_U(v) \geq (k-1) + 2(k-1) + 3(k-1) + 3(k-1) + 2(n - 4(k-1) - 1) = 2n + k - 3,$$

we have $\partial_2(U) \geq 2n + k - 2 \geq 2n + 1 > 2n - 1$.

Next suppose that $t_i, t_j \geq 2$ and $t_y = 0$ for $y \in \{1, 2, \ldots, k\} \setminus \{i, j\}$. Similarly as above, we have $\partial_2(U) > 2n - 1$. □

Lemma 4.6. *Let $k \geq 3$, $n \geq 6(k-1)$ and $U \cong C^k_{gk-g}(t_1, t_2, \ldots, t_k, H_{k+1}, H_{k+2}, \ldots, H_{gk-g})$. If there exist distinct $i, j \in \{1, 2, \ldots, k\}$ such that $t_i \geq 2$, $t_j = 1$, and $t_z = 0$ for any $z \in \{1, 2, \ldots, k\} \setminus \{i, j\}$, then $\partial_2(U) > 2n - 1$.*

Proof. Suppose that there exist distinct $i, j \in \{1, 2, \ldots, k\}$ such that $t_i \geq 2$, $t_j = 1$, and $t_z = 0$ for any $z \in \{1, 2, \ldots, k\} \setminus \{i, j\}$. Let $w, v \in V(H_j) \setminus \{u_j\}$. Then we consider the 2×2 principal submatrix of $\mathcal{L}(U)$, denoted by M, indexed by vertices v and w, where

$$M = \begin{pmatrix} \mathrm{Tr}_U(v) & -1 \\ -1 & \mathrm{Tr}_U(w) \end{pmatrix}.$$

Note that $\mathrm{Tr}_U(v) = \mathrm{Tr}_U(w)$ and

$$\mathrm{Tr}_U(v) \geq (k-1) + 2(k-1) + 2(k-1) + 3(n - 3k + 3 - 1) \geq 3n - 4k + 1.$$

By Lemma 2.1, we have $\partial_2(U) \geq \lambda_2(M) = \mathrm{Tr}_U(v) - 1 \geq 3n - 4k > 2n - 1$. □

Lemma 4.7. *Let $k \geq 3$ and $U \cong C^k_{gk-g}(t_1, t_2, \ldots, t_k, H_{k+1}, H_{k+2}, \ldots, H_{gk-k})$. If $t_i \leq 1$ for $1 \leq i \leq k$ and there are at least four vertices each with exactly one pendent edge in e_1, then we have $\partial_2(U) > 2n - 1$.*

Proof. Suppose that $t_i \leq 1$ for $1 \leq i \leq k$ and there are at least four vertices each with exactly one pendent edge. Without loss of generality, we may assume that $t_2 = 1$ and $t_3 = 1$. Let $v \in V(H_2) \setminus \{u_2\}$ and $w \in V(H_3) \setminus \{u_3\}$. By Lemma 2.7, $\mathrm{Tr}_U(v) + 1$ is an eigenvalue of $\mathcal{L}(U)$ and its multiplicity is at least $(k-2) \geq 1$, and $\mathrm{Tr}_U(w) + 1$ is an eigenvalue of $\mathcal{L}(U)$ and its multiplicity is at least $(k-2) \geq 1$. Since $\mathrm{Tr}_U(w) = \mathrm{Tr}_U(v)$ and

$$\mathrm{Tr}_U(v) \geq (k-1) + 2(k-1) + 3(k-1) + 3(k-1) + 2(n-4(k-1)-1) = 2n+k-3,$$

we have $\partial_2(U) \geq 2n+k-2 \geq 2n+1 > 2n-1$. □

Theorem 4.8. *Let $k \geq 3$, $n \geq 7(k-1)$ and U be an n-vertex k-uniform unicyclic hypergraph. Then $\partial_2(U) \geq 2n-1$, with equality if and only if $U \cong C^k_{2k-2}(m-2, u_2, \ldots, u_{2k-2})$, where $m = \frac{n}{k-1}$.*

Proof. By Lemma 4.1, we have $\partial_2(C^k_{2k-2}(m-2, u_2, \ldots, u_{2k-2})) = 2n-1$ for $n \geq 7(k-1)$. So it suffices to prove $\partial_2(U) > 2n-1$ for $U \ncong C^k_{2k-2}(m-2, u_2, \ldots, u_{2k-2})$ and $n \geq 7(k-1)$.

We first consider the case $U \cong C_{n,k}$. Let $s = \left\lfloor \frac{n}{2(k-1)} \right\rfloor$. Let w and v be two vertices in e_1 and e_2 with degree 1, respectively. Then we consider the 2×2 principal submatrix of $\mathcal{L}(U)$, denoted by M, indexed by vertices v and w, where

$$M = \begin{pmatrix} \mathrm{Tr}_U(v) & -2 \\ -2 & \mathrm{Tr}_U(w) \end{pmatrix}.$$

Obviously, we have $\mathrm{Tr}_U(v) = \mathrm{Tr}_U(w)$ and

$$\begin{aligned} \mathrm{Tr}_U(v) &= (k-1) + 2(k-1) + \cdots + s(k-1) \\ &\quad + 2(k-1) + 3(k-1) + \cdots + (s+1)(k-1) - (s+1) \\ &= \frac{s(s+1)}{2}(k-1) + \frac{s(s+3)}{2}(k-1) - (s+1) \\ &= (k-1)(s^2+2s) - (s+1) \\ &\geq 5s(k-1) - (s+1) = 2n + s(k-2) - 1 \geq 2n + s - 1. \end{aligned}$$

By Lemma 2.1, we have $\partial_2(U) \geq \lambda_2(M) = \mathrm{Tr}_U(v) - 2$. Since $s = \left\lfloor \frac{n}{2(k-1)} \right\rfloor \geq 3$, we have $2n + s - 3 \geq 2n$ and thus $\partial_2(U) > 2n-1$.

In the following, suppose that $U \ncong C_{n,k}$ and the diameter of U is d.

Case 1*:* $d \geq 4$. We choose a diametrical path $P = (v_0, f_1, v_1, \ldots, v_{d-1}, f_d, v_d)$ of U such that f_1 is a pendent edge at v_1. By Lemma 4.4, we have $\partial_2(U) > 2n-1$.

Case 2*:* $d = 3$.

Subcase 2.1*:* $U \cong C^k_{2k-2}(t_1, t_2, \ldots, t_k, u_{k+1}, u_{k+2}, \ldots, u_{2k-2})$.

If $U \ncong C^k_{2k-2}(u_1, m-2, u_3, \ldots, u_k, u_{k+1}, u_{k+2}, \ldots, u_{2k-2})$, then by Lemmas 4.5, 4.6 and 4.7, we have $\partial_2(U) > 2n-1$.

If $U \cong C_{2k-2}^k(u_1, m-2, u_3, \ldots, u_k, u_{k+1}, u_{k+2}, \ldots, u_{2k-2})$, then we consider the principal submatrix of $\mathcal{L}(U)$, denoted by N, indexed by $V(H_2) \setminus \{u_2\}$. Obviously, $\mathrm{Tr}_U(w) = 2n-3$ for $w \in V(H_2) \setminus \{u_2\}$. Then

$$N = \begin{pmatrix} (2n-2)I_{k-1} - J_{k-1} & -2J_{k-1} & \cdots & -2J_{k-1} \\ -2J_{k-1} & (2n-2)I_{k-1} - J_{k-1} & \cdots & -2J_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ -2J_{k-1} & -2J_{k-1} & \cdots & (2n-2)I_{k-1} - J_{k-1} \end{pmatrix}.$$

Thus

$$\begin{aligned} &|\lambda I_{n-2k+2} - N| \\ =& \begin{vmatrix} (\lambda-2n+2)I_{k-1} + J_{k-1} & 2J_{k-1} & \cdots & 2J_{k-1} \\ 2J_{k-1} & (\lambda-2n+2)I_{k-1} + J_{k-1} & \cdots & 2J_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 2J_{k-1} & 2J_{k-1} & \cdots & (\lambda-2n+2)I_{k-1} + J_{k-1} \end{vmatrix} \\ =& (\lambda-2n+2)^{(m-2)(k-2)}(\lambda-2n-k+3)^{(m-3)}(\lambda-5k+7). \end{aligned}$$

By Lemma 2.1, we have $\partial_2(U) \geq \lambda_2(N) = 2n+k-3 > 2n-1$.

Subcase 2.2: $U \cong C_{2k-2}^k(H_1, u_2, \ldots, u_{2k-2})$, where H_1 is obtained from $e = \{u_1, w_1, \ldots, w_{k-1}\}$ by attaching pendent edges at u_1 and w_i for $1 \leq i \leq k-1$, and there exists w_j with $d_{H_1}(w_j) \geq 2$.

First suppose that there exists w_i with $d_{H_1}(w_i) \geq 3$ for some $1 \leq i \leq k-1$, say $d_{H_1}(w_1) \geq 3$. Let $v \in N_{H_1}(w_1) \setminus e$. By Lemma 2.7, $\mathrm{Tr}_U(v) + 1$ is an eigenvalue of $\mathcal{L}(U)$ and its multiplicity is at least $(k-2)(d_{H_1}(w_1) - 1) \geq 2$. Note that

$$\mathrm{Tr}_U(v) \geq (k-1) + 2(k-1) + 3(k-1) + 3(k-2) + 2(n-4(k-1)) = 2n+k-4.$$

Thus $\partial_2(U) \geq 2n+k-3 \geq 2n > 2n-1$.

Next suppose that $d_{H_1}(w_i) \leq 2$ for $1 \leq i \leq k-1$ and there is a vertex w_j with $d_{H_1}(w_j) = 2$. Let $v, w \in N_{H_1}(w_j) \setminus e$. Then we consider the 2×2 principal submatrix of $\mathcal{L}(U)$, denoted by M^*, indexed by vertices v and w, where

$$M^* = \begin{pmatrix} \mathrm{Tr}_U(v) & -1 \\ -1 & \mathrm{Tr}_U(w) \end{pmatrix}.$$

Note that $\mathrm{Tr}_U(v) = \mathrm{Tr}_U(w)$ and

$$\mathrm{Tr}_U(v) = (k-1) + 2(k-1) + 3(k-1) + 3(k-2) + 3(n-4k+4) = 3n-3k.$$

By Lemma 2.1, we have $\partial_2(U) \geq \lambda_2(M^*) = \mathrm{Tr}_U(v) - 1 = 3n - 3k - 1 > 2n - 1$.

Subcase 2.3: $U \cong C_{3k-3}^k(t_1, t_2, \ldots, t_k, u_{k+1}, u_{k+2}, \ldots, u_{3k-3})$ or $U \cong C_{3k-3}^k(t_1, u_2, \ldots, u_{k-1}, t_k, u_{k+1}, \ldots, u_{2k-2}, t_{2k-1}, u_{2k}, \ldots, u_{3k-3})$.

If $U \notin \{C_{3k-3}^k(m-3, u_2, \ldots, u_k, u_{k+1}, u_{k+2}, \ldots, u_{3k-3}), C_{3k-3}^k(u_1, m-3, u_3, \ldots, u_k, u_{k+1}, u_{k+2}, \ldots, u_{3k-3}), C_{3k-3}^k(t_1, u_2, \ldots, u_{k-1}, t_k, u_{k+1}, \ldots, u_{2k-2}, t_{2k-1}, u_{2k}, \ldots, u_{3k-3})\}$, then by Lemmas 4.5, 4.6 and 4.7, we have $\partial_2(U) > 2n - 1$.

If $U \cong C_{3k-3}^k(u_1, m-3, u_3, \ldots, u_k, u_{k+1}, u_{k+2}, \ldots, v_{3k-3})$, then let v be a pendent vertex in $V(H_2) \setminus \{u_2\}$. By Lemma 2.7, $\mathrm{Tr}_U(v) + 1$ is an eigenvalue of $\mathcal{L}(U)$ and its multiplicity is at least $(k-2)(m-3) \geq 2$. Since

$$\mathrm{Tr}_U(v) = (k-1) + 2(k-1) + 3(k-1) + 3(k-2) + 2(n - 4(k-1)) = 2n + k - 4,$$

we have $\partial_2(U) \geq 2n + k - 3 \geq 2n > 2n - 1$.

If $U \cong C_{3k-3}^k(m-3, u_2, \ldots, u_k, u_{k+1}, u_{k+2}, \ldots, u_{3k-3})$, then we consider the principal submatrix of $\mathcal{L}(U)$, denoted by N', indexed by $V(H_1) \setminus \{u_1\}$. Obviously, $\mathrm{Tr}_U(w) = 2n - 3$ for $w \in V(H_1) \setminus \{u_1\}$. Then

$$N' = \begin{pmatrix} (2n-2)I_{k-1} - J_{k-1} & -2J_{k-1} & \cdots & -2J_{k-1} \\ -2J_{k-1} & (2n-2)I_{k-1} - J_{k-1} & \cdots & -2J_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ -2J_{k-1} & -2J_{k-1} & \cdots & (2n-2)I_{k-1} - J_{k-1} \end{pmatrix}.$$

Thus

$$\begin{aligned} &|\lambda I_{n-3k+3} - N'| \\ &= \begin{vmatrix} (\lambda - 2n + 2)I_{k-1} + J_{k-1} & 2J_{k-1} & \cdots & 2J_{k-1} \\ 2J_{k-1} & (\lambda - 2n + 2)I_{k-1} + J_{k-1} & \cdots & 2J_{k-1} \\ \vdots & \vdots & \ddots & \vdots \\ 2J_{k-1} & 2J_{k-1} & \cdots & (\lambda - 2n + 2)I_{k-1} + J_{k-1} \end{vmatrix} \\ &= (\lambda - 2n + 2)^{(m-3)(k-2)} (\lambda - 2n - k + 3)^{(m-4)} (\lambda - 7k + 9). \end{aligned}$$

By Lemma 2.1, we have $\partial_2(U) \geq \lambda_2(N') = 2n + k - 3 > 2n - 1$.

If $U \cong C_{3k-3}^k(t_1, u_2, \ldots, u_{k-1}, u_k, u_{k+1}, \ldots, u_{2k-2}, t_{2k-1}, u_{2k}, \ldots, u_{3k-3})$, then at least one of $t_1 \geq 2$ and $t_{2k-1} \geq 2$ holds since $n \geq 7(k-1)$. By Lemmas 4.5 and 4.6, we have $\partial_2(U) > 2n - 1$.

Subcase 2.4: $U \cong C_{4k-4}^k(t_1, u_2, \ldots, u_{k-1}, t_k, u_{k+1}, u_{k+2}, \ldots, u_{4k-4})$ or $U \cong C_{4k-4}^k(t_1, u_2, \ldots, u_{k-1}, u_k, u_{k+1}, \ldots, u_{4k-4})$.

If $U \cong C_{4k-4}^k(t_1, u_2, \ldots, u_{k-1}, t_k, u_{k+1}, u_{k+2}, \ldots, v_{4k-4})$, then at least one of $t_1 \geq 2$ and $t_k \geq 2$ holds since $n \geq 7(k-1)$. By Lemmas 4.5 and 4.6, we have $\partial_2(U) > 2n - 1$.

If $U \cong C^k_{4k-4}(t_1, u_2, \ldots, u_{k-1}, u_k, u_{k+1}, \ldots, u_{4k-4})$, then let v be a pendent vertex in $V(H_1) \setminus \{u_1\}$. By Lemma 2.7, $\mathrm{Tr}_U(v) + 1$ is an eigenvalue of $\mathcal{L}(U)$ and its multiplicity is at least $(k-2)(m-4) \geq 2$. Since

$$\mathrm{Tr}_U(v) = (k-1) + 4(k-1) + 3(k-1) + 3(k-2) + 2(n - 5(k-1)) = 2n + k - 4,$$

we have $\partial_2(U) \geq 2n + k - 3 \geq 2n > 2n - 1$. □

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