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Chaotic Points of Multifunctions

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Abstract. In this paper we will consider various kinds of chaotic points of multifunctions and show their application to the theory of infinite topological games.

1. Introduction and preliminaries

The concept of chaos was introduced into the mathematical vocabulary in 1975 [16]. Since then several different and not equivalent definitions of chaos have been proposed. Survey of these concepts one can find, for example, in [15] or in the monograph [26] for functions on the unit interval. In many cases, various notions of chaos were joined with positive entropy (see e.g. [26]), which has been emphasised particularly in the paper [13, p. 80]: *It is commonly accepted that an evidence of chaos is positivity of topological entropy.* Some heuristic justification of this connection one can find in [4]. In the context of our considerations, it is worth noting that there were also natural investigations regarding the local aspects of chaos—chaotic and entropy points (e.g. [21, 23]), and moreover, problems related to hyperspaces were analyzed (e.g. [9, 13, 25, 29]).

The main goal of this paper is introducing various kinds of chaotic points for multifunctions and later on showing their applications to the theory of infinite topological games. What makes the fundamental difference to the earlier cited papers is that the notions of chaotic points introduced in Section 2 are defined in a way being specific for multifunctions.

Taking into account the connection between entropy and chaos, which has been signaled earlier, we will start our considerations with the notion of e-chaotic point (the prefix "e" comes from the word entropy). Condition (2.1) in the definition of e-chaotic point makes an important difference with respect to considerations form earlier cited papers. The connection to the entropy is established by Theorem 2.4.

The next two notions of l- and u-chaotic points are specific for multifunctions (it is impossible to create their analogues for functions), so that they are brand new. The theorems in Section 2.2 justify the names l- and u-chaotic point.

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Contrary to the above mentioned considerations, DC1 points were defined in an analogous way as in the case of functions [23]. But also in this case we will extend our considerations by introducing the new concept of m-DC1 point, which is unique for the multifunctions. These definitions will be supplemented with the basic properties of the defined notions which, among other things, explain why we relate them to the concept of chaos.

Section 3 concerns the applications of introduced notions to the theory of infinite topological games. A brief historical overview has been presented in the introduction of Section 3. The most known Banach–Mazur game has been mentioned and supplementary literature has been indicated. The essence of research in the theory of infinite topological games was often concentrated on the problem of winning strategies existence (e.g. in the case of the above mentioned Banach–Mazur game, see [20, 24]). We will follow this trend in our considerations.

Throughout the paper we will use the standard notation. The set of all real numbers and positive (nonnegative) integers will be denoted by $\mathbb{R}$ and $\mathbb{N}$ ($\mathbb{N}_0$), respectively. We will use the classic notations of intervals, but additionally let $[\![m, n]\!] = [m, n] \cap \mathbb{N}_0$. In order to simplify the reasoning, when we write $\log x$, we understand that the base of the logarithm is always the number 2.

If A is a subset of some metric space X, then by $\operatorname{Int}(A)$, $\overline{A}$ and $\#(A)$, we will denote the interior, closure and cardinality of a set A, respectively. In order to avoid complex notations when using logarithms, we assume the notation $\#_1(A) = \#(A)$, if $A \neq \emptyset$ and $\#_1(A) = 1$, if $A = \emptyset$. The open ball of radius $r > 0$ centered at x_0 will be denoted by $B(x_0, r)$.

Let $\{K_n\}$ be a sequence of nonempty subsets of a metric space X. We shall say that the sequence $\{K_n\}$ has the extension property if for any i, j and any continuous function $f\colon A \to K_j$, where $A \subset K_i$ is a closed set, one can find a continuous function $f_\star\colon K_i \to K_j$ which is an extension of f, i.e., $f_\star \restriction A = f$.

The following assumption will be needed throughout the paper: the symbol X will always stand for the topological manifold equipped with the metric ϱ (including the unit interval $[0, 1]$ with the natural metric). An m-dimensional manifold with boundary [14] is a nonempty compact metric space (X, ϱ) such that every point $q \in X$ has a neighborhood U that is homeomorphic to an open subset of the m-dimensional upper half space $\mathbb{H}^m = \{(x_1, \ldots, x_m) \in \mathbb{R}^m : x_m \geq 0\}$. The dimension of a manifold X will be denoted by $\dim(X)$. For our considerations, the issues related to the base of manifolds at point (cf. [21]) are of particular importance. We will use the symbol $\mathbb{B}(X)$ to denote the set of all closed submanifolds K of X such that $\dim(K) = \dim(X)$. For $x_0 \in X$, we consider a sequence $\{K_n\}_{n=0}^{\infty} \subset \mathbb{B}(X)$ of connected submanifolds, called a *base at the point* x_0 (we will denote it by $\mathbb{B}(x_0)$), satisfying the following properties:

(M1) $x_0 \in K_n$ for $n \in \mathbb{N}$;

(M2) $K_{n+1} \subset \operatorname{Int}(K_n)$ for $n \in \mathbb{N}$;

(M3) $\lim_{n\to\infty} \operatorname{diam}(K_n) = 0$;

(M4) the sequence $\{K_n\}_{n=1}^{\infty}$ has the extension property.

Following the notation used in [13] by the symbol 2^X, we denote the hyperspace of all nonempty compact subsets of X. Let us denote by 2^X_{uc} the family of all uncountable compact subsets of X. We may endow 2^X with the Hausdorff metric defined in the following way:

$$\varrho_H(A,B) = \max\left(\sup_{a\in A} \varrho(a,B),\ \sup_{b\in B} \varrho(b,A)\right) \quad \text{for } A, B \in 2^X,$$

where $\varrho(x,C) = \inf_{c\in C} \varrho(x,c)$.

Let us note the lemma useful in further considerations.

Lemma 1.1. *Let $A = \{x_0\} \cup C$, $B = \{y_0\} \cup C$, where x_0, y_0 and C are such that $\varrho(x_0, y_0) < \min(\varrho(x_0, C), \varrho(y_0, C))$. Then $\varrho_H(A,B) = \varrho(x_0, y_0)$.*

We will continue to consider 2^X both as a family of compact subsets of X and as a space equipped with the Hausdorff metric (the distinction will be clear from the context). According to [13, Theorem 1], we may infer that $(2^X, \varrho_H)$ is a compact metric space. Let us note, that subscript H always stands for an object in $(2^X, \varrho_H)$, especially $B_H(A,\varepsilon)$ will denote an open ball of radius $\varepsilon > 0$ centered at $A \in 2^X$.

Following the notation used in [22], if $\emptyset \neq A \subset X$, then we will denote $\mathrm{d}(A) = \{B \in 2^X \setminus \{\emptyset\} : B \subset A\}$. Obviously, if A is a closed set then $\mathrm{d}(A)$ is also a closed set in $(2^X, \varrho_H)$.

Our considerations will mainly concern multifunctions $\zeta \colon X \multimap X$, and we will always assume that the multifunctions values are nonempty and closed subsets of X.

A multifunction $\zeta \colon X \multimap X$ is upper semicontinuous at $x_0 \in X$ if for every open set $W \subset X$ such that $\zeta(x_0) \subset W$, there exists an open set $U \subset X$ such that $x_0 \in U$ and $\zeta(U) \subset W$. A multifunction ζ is upper semicontinuous (u.s.c.) if it is upper semicontinuous at every point of X. It is commonly known (see for instance [11, Theorem 1.1.1]) that a multifunction ζ is u.s.c. iff $\zeta_-^{-1}(F)$ is a closed set for every closed set $F \subset Y$.

A multifunction $\zeta \colon X \multimap X$ is lower semicontinuous at $x_0 \in X$ if for every open set $W \subset X$ such that $\zeta(x_0) \cap W \neq \emptyset$, there exists an open set $U \subset X$ such that $x_0 \in U$ and $\zeta(x) \cap W \neq \emptyset$ for all $x \in U$. A multifunction ζ is lower semicontinuous (l.s.c.) if it is lower semicontinuous at every point of X. A multifunction which is both upper and lower semicontinuous is said to be continuous.

Let $\zeta\colon X \multimap X$ be a multifunctions and let $B \subset X$. Of course, then one can consider small preimage and complete preimage of a set B defined in the usual way $\zeta_+^{-1}(B) = \{x \in X : \zeta(x) \subset B\}$ and $\zeta_-^{-1}(B) = \{x \in X : \zeta(x) \cap B \neq \emptyset\}$ (e.g. [11]). Moreover, $\zeta(A) = \bigcup_{a\in A} \zeta(a)$ and if $\zeta_1, \zeta_2\colon X \multimap X$, then $\zeta_2 \circ \zeta_1(x) = \zeta_2(\zeta_1(x))$. Obviously, if $f\colon X \to X$ and $\zeta\colon X \multimap X$, then $f \circ \zeta(x) = f(\zeta(x))$ and $\zeta \circ f(x) = \zeta(f(x))$. We shall say that multifunctions $\zeta_1, \zeta_2\colon X \multimap X$ are conjugate if there exists a homeomorphism $\varphi\colon X \to X$ such that $\varphi \circ \zeta_1 = \zeta_2 \circ \varphi$. Let $\zeta_1, \zeta_2\colon X \multimap X$. We say that ζ_1 is inserted in ζ_2 if $\zeta_1(x) \subset \zeta_2(x)$ for each $x \in X$.

Let $\zeta\colon X \to X$ be a function or $\zeta\colon X \multimap X$ be a multifunction. We will say that a set A ζ-covers (essential ζ-covers) a set B, which we write as $A \underset{\zeta}{\to} B$ ($A \overset{e}{\underset{\zeta}{\to}} B$), if $B \subset \zeta(A)$ ($B \subsetneq \zeta(A)$). A point x_0 will be called a (strongly) fixed point of a multifunction $\zeta\colon X \multimap X$ if $x_0 \in \zeta(x_0)$ ($x_0 \in \operatorname{Int}(\zeta(x_0))$), which we can write as $x_0 \in \operatorname{Fix}(\zeta)$ ($x_0 \in \operatorname{Fix}_s(\zeta)$).

In many papers, the authors have investigated the interrelationships between the dynamics of functions, multifunctions and mappings (e.g. [10, 17, 25]). We will relate some issues to the following situations.

Let $f\colon X \to X$. Then one can consider a multifunction $\zeta_f\colon X \multimap X$ defined in the following way: $\zeta_f(x) = \{f(x)\}$.

Now, let $\zeta\colon X \multimap X$ be a multifunction. Then

- if $\zeta(A)$ is a closed set for each closed set A, then we may consider a function $\overline{\zeta}\colon 2^X \to 2^X$ defined by the formula $\overline{\zeta}(A) = \zeta(A)$;
- one can consider a multifunction $\widehat{\zeta}\colon 2^X \multimap 2^X$ such that $\widehat{\zeta}(A) = \mathrm{d}(\zeta(A))$.

A function $\mathrm{sel}_\zeta\colon X \to X$ is said to be a selection of ζ if $\mathrm{sel}_\zeta(x) \in \zeta(x)$ for all $x \in X$. A multifunction $\zeta\colon X \multimap X$ is s-continuous if ζ has a continuous selection sel_ζ.

If $f\colon X \to \mathbb{R}$ is a nonnegative function, then a multifunction $\Gamma_f\colon X \multimap X$, defined by the formula $\Gamma_f(x) = [0, f(x)]$, is called canonical multifunction associated with the function f [3]. It is well known that if f is upper (resp. lower) semicontinuous, then Γ_f is u.s.c. (resp. l.s.c.) (e.g. [3]).

Now, we will assume that ζ and ζ_i ($i \in \mathbb{N}$) are either functions $\zeta\colon X \to X$ and $\zeta_i\colon X \to X$ or multifunctions $\zeta\colon X \multimap X$ and $\zeta_i\colon X \multimap X$. If $\zeta\colon X \to X$, then we put $\zeta^0(x) = x$. If $\zeta\colon X \multimap X$, then we put $\zeta^0(x) = \{x\}$. Moreover let

$$(\zeta_{1,\infty}) = \{\zeta_i\}_{i=1}^{\infty} \quad \text{and} \quad \zeta_1^n = \zeta_n \circ \cdots \circ \zeta_1 \ \text{ for } n \in \mathbb{N}.$$

A pair $(X, (\zeta_{1,\infty}))$ where $\zeta_i\colon X \to X$ or $\zeta_i\colon X \multimap X$ ($i = 1, 2, \ldots$) is called nonautonomous dynamical system and it is denoted by $(\zeta_{1,\infty})$. If $\zeta_i = \zeta$ for $i \in \mathbb{N}$, then the dynamical system $(\zeta_{1,\infty})$ is called autonomous and it is denoted by (ζ).

A nonautonomous dynamical system of multifunctions $\Omega = (X, (\zeta_{1,\infty}))$ (where $(\zeta_{1,\infty}) = \{\zeta_n\}_{n=1}^{\infty}$) is called cone if ζ_{n+1} is inserted in ζ_n for $n \in \mathbb{N}$. A multifunction $\kappa\colon X \multimap X$ is said to be a vertex of a cone Ω if κ is inserted in ζ_n for $n \in \mathbb{N}$. It is easily seen that, by our assumption connected with X, each cone has a vertex. A finite sequence $(\zeta_1, \ldots, \zeta_m)$ of multifunctions is called frustum if ζ_{n+1} is inserted in ζ_n for $n = 1, 2, \ldots, m$.

Our considerations connected with functions will be related, among others, with local aspects of entropy. Let us recall the definitions useful for further consideration (including the case of discontinuous functions following [6]).

Let $f\colon X \to X$ be a function, $\varepsilon > 0$, $n \in \mathbb{N}$ and $Y \subset X$. A set $E \subset Y$ is an (n, ε)-separated set in Y if for each different points $x, y \in E$, there is $j \in [\![0, n-1]\!]$ such that $\varrho(f^j(x), f^j(y)) > \varepsilon$. Let $S(f, n, \varepsilon)$ denote an (n, ε)-separated set with the maximal possible number of points and $s_n(f, Y, \varepsilon)$ its cardinality. The topological entropy of a function f (more precisely: of a dynamical system (f) generated by the function f) on the set Y is a number

$$h(f, Y) = \lim_{\varepsilon \to 0+} \limsup_{n \to \infty} \frac{1}{n} \log s_n(f, Y, \varepsilon).$$

If $Y = X$, then we will write briefly $h(f)$.

Considering the local aspects of dynamical systems of functions, following [30], we assume that x_0 is an entropy point of f if $h(f, B(x_0, \varepsilon)) > 0$ for any $\varepsilon > 0$.

In the literature on dynamical systems, various methods are considered to facilitate the conclusion that a given function has a positive entropy. Similar notions of turbulent functions (e.g. [6]) and functions having a horseshoe are used for this purpose. The name horseshoe for interval maps was given by Misiurewicz [18] (earlier this concept was considered by A. N. Sharkovsky under the name L-scheme). For our purposes, let us assume that a horseshoe for the function f is a family of disjoint closed sets $J_1, \ldots, J_n$ $(n \geq 2)$ such that $f(J_i) \supset J_1 \cup \cdots \cup J_n$ for all $i \in [\![1, n]\!]$. It is commonly known that the existence of a horseshoe implies that the topological entropy is positive.

In 1994, B. Schweizer and J. Smítal [27] introduced the notion: "distributional chaos" for autonomous dynamical systems. The generalization of this definition into a nonautonomous case was introduced by J. Dvořáková [8]. We will be based on this definition, naturally transferring it to the case of multifunctions (using in this case the Hausdorff metric), so it will be convenient to adopt the following agreement:

- if we consider $(\zeta_{1,\infty})$ consisting of functions then we put $\sigma = \varrho$;
- if we consider $(\zeta_{1,\infty})$ consisting of multifunctions then we put $\sigma = \varrho_H$.

Let $x, y \in X$ and $(\zeta_{1,\infty})$ be a dynamical system consisting either of functions or multifunctions and $t > 0$. Then

$$\Phi_{x,y}^{(\zeta_{1,\infty})}(t) = \liminf_{n \to \infty} \frac{1}{n} \#\big(\{j \in [\![0, n-1]\!] : \sigma(\zeta_1^j(x), \zeta_1^j(y)) < t\}\big)$$

is called lower distribution function of x, y for $(\zeta_{1,\infty})$ and

$$\Phi^{*(\zeta_{1,\infty})}_{x,y}(t) = \limsup_{n\to\infty} \frac{1}{n}\#\big(\{j \in [\![0, n-1]\!] : \sigma(\zeta_1^j(x), \zeta_1^j(y)) < t\}\big)$$

is called upper distribution function of x, y for $(\zeta_{1,\infty})$.

Let $x, y \in X$. We shall say that a pair (x, y) is distributionally chaotic of type 1 (D1 for short) for a dynamical system $(\zeta_{1,\infty})$ if $\Phi^{*(\zeta_{1,\infty})}_{x,y}(t) = 1$ for any $t > 0$ and there exists $t_0 > 0$ such that $\Phi^{(\zeta_{1,\infty})}_{x,y}(t_0) = 0$. A set $A \subset X$ is called distributionally scrambled set of type 1 (DS-set for brevity) for a dynamical system $(\zeta_{1,\infty})$ if $\#(A) > 1$ and for each $x, y \in A$ such that $x \neq y$ the pair (x, y) is D1 for this system. A dynamical system $(\zeta_{1,\infty})$ is distributionally chaotic (of type 1), DC1 for brevity, if there exists an uncountable DS-set for this system.

We call an autonomous dynamical system (ζ) consisting of multifunctions m-distributionally chaotic (m-DC1 for brevity) if there exists an uncountable DS-set A such that $\#(\zeta(x)) > 1$ for $x \in A$. For brevity, we will say that ζ is m-DC1 multifunction.

2. Various kinds of chaotic points

2.1. e-chaotic points

The entropy of multifunction can be considered in various approaches. For example, in the paper [22], the concept of a generalized entropy was introduced, which presents a common approach of the entropy for functions and multifunctions. This concept (relating to generalized topological spaces) of entropy coincides with the one given in [1] and [2, Part 4.4]. For our considerations, however, we will accept some natural simplifications for research related to manifolds. Since we will focus on analysis of the local properties, including the local aspects of chaos and entropy, we will only write it in relation to the neighbourhoods of some point.

Let $Y \subset X$. We shall use the symbol $\mathsf{S}(\zeta, Y)$ to represent a family of all finite sequences $(L_1, L_2, \ldots, L_m)$ such that $L_i \in 2^X_{uc}$, $L_i \subset Y$, $L_i \cap L_j = \emptyset$ $(i, j \in \{1, 2, \ldots, m\}$, $i \neq j)$ and

$$\text{if } i, j \in \{1, 2, \ldots, m\} \text{ and } i \neq j, \text{ then there exist } x \in L_i \text{ and } y \in L_j \text{ such that } \zeta(x) \neq \zeta(y). \tag{2.1}$$

Condition (2.1) was added to avoid "pathological" situations where e.g. $\zeta(x) = X$. Then the multifunction ζ would have a character similar to the constant function (all values are the same), so the entropy values should be zero. It would be inconsistent with the further considerations.

Let $\zeta\colon X \to X$ be a function or $\zeta\colon X \multimap X$ be a multifunction. Let $L = (L_1, L_2, \ldots, L_m) \in \mathsf{S}(\zeta, Y)$. A path (with the length k) of the form $L_{p_1} \underset{\zeta}{\to} L_{p_2} \underset{\zeta}{\to} \cdots \underset{\zeta}{\to} L_{p_k} \underset{\zeta}{\to} L_{p_1}$, where $p_i \in \{1, 2, \ldots, m\}$ for $i \in \{1, \ldots, k\}$ will be denoted by $\mathcal{P}(\zeta, L_{p_1}, \ldots, L_{p_k})$. The symbol $\mathsf{P}_k(L, \zeta)$ $(k = 1, 2, \ldots)$ will stand for the set of all paths $\mathcal{P}(\zeta, L_{p_1}, \ldots, L_{p_k})$.

According to the considerations connected with focal entropy points (e.g. [12]) we will accept the following notions and definitions. An entropy of a sequence $L = (L_1, \ldots, L_m) \in \mathsf{S}(\zeta, Y)$ with respect to ζ is the number

$$H_\zeta(L) = \limsup_{k \to \infty} \frac{1}{k} \log(\#_1(\mathsf{P}_k(L, \zeta))).$$

If $Y \subset X$ is an open set, then we will write $H_\zeta(Y) = \sup\{H_\zeta(L) : L \in \mathsf{S}(\zeta, Y)\}$.

The following notion is analogous to that used in [21] for functions. A density of entropy of multifunction ζ at the point x_0 is the number

$$h(\zeta, x_0) = \inf\{H_\zeta(Y) : \ Y \text{ is an open neighbourhood of } x_0\}.$$

Following remark contained in [13] (mentioned in the introduction) linking chaos with positive entropy we assume the succeeding definition: A point x_0 is (strong) e-chaotic point of multifunction ζ if $h(\zeta, x_0) > 0$ $(h(\zeta, x_0) = +\infty)$. In this context, Theorem 2.4 will justify the use of the phrase "chaotic" in this definition.

The following theorem will show the simultaneous existence of (strong) e-chaotic point for conjugate multifunctions (the classic proof for this type of theorems is omitted).

Theorem 2.1. *If multifunctions $\zeta_1, \zeta_2 \colon X \multimap X$ are conjugate (via homeomorphism φ), then x_0 is (strong) e-chaotic point of ζ_1 if and only if $\varphi(x_0)$ is (strong) e-chaotic point of ζ_2.*

As we have already mentioned in the introduction, in many papers the issues related to multifunctions are confronted with the properties of the functions $\overline{\zeta}$ and the multifunctions $\widehat{\zeta}$. In pursuit of this goal, we will first prove a useful lemma (the proof is immediate).

Lemma 2.2. *For any $x \in X$, $A, B \in 2^X$ and $\varepsilon > 0$ the following implications take place*

(a) *if $A \subset B(x, \varepsilon)$, then $\mathrm{d}(A) \subset B_H(\{x\}, 2\varepsilon)$;*

(b) *if $\mathrm{d}(A) \subset B_H(\{x\}, \varepsilon)$, then $A \subset B(x, \varepsilon)$;*

(c) *if $\varrho(A, B) > \varepsilon$, then $\varrho_H(\mathrm{d}(A), \mathrm{d}(B)) \geq \varepsilon$.*

The next theorem will describe the relationship between the possession of a (strong) e-chaotic point of u.s.c. multifunction ζ and the associated multifunction $\widehat{\zeta}$.

Theorem 2.3. *Let multifunction $\zeta \colon X \multimap X$ be u.s.c. If x_0 is a (strong) e-chaotic point of ζ, then $\{x_0\}$ is a (strong) e-chaotic point of $\widehat{\zeta}$.*

Proof. We will show that $\{x_0\}$ is an e-chaotic point of $\widehat{\zeta}$. The consideration in the case of a strong e-chaotic point is analogous.

Fix $\eta > 0$. Since x_0 is an e-chaotic point of ζ, $h(\zeta, x_0) = \alpha > 0$, so in particular $H_\zeta(B(x_0, \eta/2)) \geq \alpha$. This means that there exists $L_\eta = (L_1, \ldots, L_m) \in \mathsf{S}(\zeta, B(x_0, \eta/2))$ such that

$$H_\zeta(L_\eta) = \limsup_{k\to\infty} \frac{1}{k} \log(\#_1(\mathsf{P}_k(L_\eta, \zeta))) > \frac{\alpha}{2}. \tag{2.2}$$

Let us consider $\widehat{L}_\eta = (\mathrm{d}(L_1), \ldots, \mathrm{d}(L_m))$. First we shall show

$$\widehat{L}_\eta \in \mathsf{S}(\widehat{\zeta}, B_H(\{x_0\}, \eta)). \tag{2.3}$$

Note that $\mathrm{d}(L_i)$ are uncountable closed sets in 2^X. Of course, $\mathrm{d}(L_i) \cap \mathrm{d}(L_j) = \emptyset$ for $i \neq j$.

Since $L_\eta \in \mathsf{S}(\zeta, B(x_0, \eta/2))$, then $L_i \subset B(x_0, \eta/2)$, which by Lemma 2.2(a) allows to conclude that $d(L_i) \subset B_H(\{x_0\}, \eta)$.

The proof of (2.3) is completed by showing that

(2.4) if $i, j \in \{1, 2, \ldots, m\}$ and $i \neq j$, then there exist $A \in \mathrm{d}(L_i)$ and $B \in \mathrm{d}(L_j)$ such that $\widehat{\zeta}(A) \neq \widehat{\zeta}(B)$.

Let $i, j \in \{1, \ldots, m\}$ and $i \neq j$. Since L_η satisfies the condition (2.1), then there exist $x \in L_i$ and $y \in L_j$ such that $\zeta(x) \neq \zeta(y)$. There is no loss of generality in assuming $t \in \zeta(x) \setminus \zeta(y)$ and let us put $A = \{x\} \in \mathrm{d}(L_i)$ and $B = \{y\} \in \mathrm{d}(L_j)$. Then $\widehat{\zeta}(A) = \mathrm{d}(\zeta(x))$ and $\widehat{\zeta}(B) = \mathrm{d}(\zeta(y))$. Obviously, $\{t\} \in \widehat{\zeta}(A) \setminus \widehat{\zeta}(B)$. The implication (2.4) has been proved and thus also the proof of (2.3) is finished.

Applying (2.2), we can assert that there exists a sequence $\{k_n\}$ of positive integers such that

$$\lim_{n\to\infty} \frac{1}{k_n} \log(\#_1(\mathsf{P}_{k_n}(L_\eta, \zeta))) > \frac{\alpha}{2}. \tag{2.5}$$

We shall prove that

$$\#_1(\mathsf{P}_{k_n}(L_\eta, \zeta)) \leq \#_1(\mathsf{P}_{k_n}(\widehat{L_\eta}, \widehat{\zeta})). \tag{2.6}$$

Let us fix $n \in \mathbb{N}$ and a path $\mathcal{P}(\zeta, L_{p_1}, \ldots, L_{p_{k_n}})$, where $p_i \in \{1, \ldots, m\}$ for $i = 1, \ldots, k_n$. We will now show

$$\mathrm{d}(L_{p_1}) \underset{\widehat{\zeta}}{\to} \cdots \underset{\widehat{\zeta}}{\to} \mathrm{d}(L_{p_{k_n}}) \underset{\widehat{\zeta}}{\to} \mathrm{d}(L_{p_1}). \tag{2.7}$$

For this purpose we prove that

(2.8) if $L_{p_i} \underset{\zeta}{\to} L_{p_j}$, then $\mathrm{d}(L_{p_i}) \underset{\widehat{\zeta}}{\to} \mathrm{d}(L_{p_j})$ for $i \in \{1, \ldots, k_n\}$ and $j = (i \bmod k_n) + 1$.

So, let $A \in \mathrm{d}(L_{p_j})$. Then

$$A \in \mathrm{d}(\zeta(L_{p_i})) = \widehat{\zeta}(L_{p_i}). \tag{2.9}$$

Let us consider a set $T_A = \zeta_-^{-1}(A) \cap L_{p_i}$. According to the remark after the definition of u.s.c., it follows that $\zeta_-^{-1}(A)$ is a closed set, and consequently $T_A \in \mathrm{d}(L_{p_i})$.

We shall now show that

$$A \subset \zeta(T_A). \tag{2.10}$$

Let us fix $a \in A$. From (2.9), we obtain $A \subset \zeta(L_{p_i})$ and consequently there exists $x_a \in L_{p_i}$ such that $a \in \zeta(x_a)$. From this we conclude that $x_a \in T_A$ and so $a \in \zeta(T_A)$.

According to our assumption that $A \in \mathrm{d}(L_{p_j})$, it follows that A is a closed set which means that $A \in \mathrm{d}(\zeta(T_A))$. Arbitrariness of $A \in \mathrm{d}(L_{p_j})$ gives $\mathrm{d}(L_{p_j}) \subset \bigcup_{T \in \mathrm{d}(L_{p_i})} \mathrm{d}(\zeta(T)) = \bigcup_{T \in \mathrm{d}(L_{p_i})} \widehat{\zeta}(T) = \widehat{\zeta}(\mathrm{d}(L_{p_i}))$. This finishes the proof of (2.8) and thus we have also proved (2.7).

Applying (2.7), we may define a function $\xi\colon \mathsf{P}_{k_n}(L_\eta, \zeta) \to \mathsf{P}_{k_n}(\widehat{L}_\eta, \widehat{\zeta})$ in the following way: $\xi(\mathcal{P}(\zeta, L_{p_1}, \dots, L_{p_{k_n}})) = \mathcal{P}(\widehat{\zeta}, \mathrm{d}(L_{p_1}), \dots, \mathrm{d}(L_{p_{k_n}}))$, $p_i \in \{1, \dots, m\}$, $i = 1, \dots, k_n$.

To the end of the proof of (2.6) it is sufficient to show that ξ is one-to-one. So, let us suppose that $\mathcal{P}(\zeta, L_{p_1}, \dots, L_{p_{k_n}}), \mathcal{P}(\zeta, L_{q_1}, \dots, L_{q_{k_n}}) \in \mathsf{P}_{k_n}(L_\eta, \zeta)$ are different paths. Then there exists i_0 such that $L_{p_{i_0}} \neq L_{q_{i_0}}$, which in fact means that $L_{p_{i_0}} \cap L_{q_{i_0}} = \emptyset$. We thus get $\mathrm{d}(L_{p_{i_0}}) \neq \mathrm{d}(L_{q_{i_0}})$, which proves $\xi(\mathcal{P}(\zeta, L_{p_1}, \dots, L_{p_{k_n}})) \neq \xi(\mathcal{P}(\zeta, L_{q_1}, \dots, L_{q_{k_n}}))$. Consequently (2.6) is proved.

On account of (2.6) and (2.5), we have

$$\limsup_{n\to\infty} \frac{1}{k_n} \log(\#_1(\mathsf{P}_{k_n}(\widehat{L_\eta}, \widehat{\zeta}))) > \frac{\alpha}{2}.$$

Consequently $H_{\widehat{\zeta}}(\widehat{L}_\eta) > \alpha/2$ which (taking into account the arbitrariness of η) allows us to conclude that $h(\widehat{\zeta}, \{x_0\}) > 0$. □

Let us now return to the previously mentioned justification for the use of the term "chaotic" in the name e-chaotic point (in the context of the quotation from paper [13]).

Theorem 2.4. *Let $f\colon X \to X$ be a continuous function. If x_0 is an e-chaotic point of the multifunction ζ_f, then*

(a) *x_0 is an entropy point of the function f (and thus $h(f) > 0$);*

(b) *$\{x_0\}$ is an entropy point of the function $\overline{\zeta}_f$ (and thus $h(\overline{\zeta}_f) > 0$).*

Before going to the proof, it is worth noting that in view of the assumptions of the above theorem, we can consider a function $\overline{\zeta}_f$ because $\zeta_f(F)$ is a closed set for any closed set F.

Proof of Theorem 2.4. Let $\sigma > 0$. Following [6] let us assume the notations

$$\overline{s}_f(\varepsilon, \sigma) = \limsup_{k\to\infty} \frac{1}{k} \log(s_k(f, B(x_0, \sigma), \varepsilon)),$$
$$\overline{s}_{\overline{\zeta}_f}(\varepsilon, \sigma) = \limsup_{k\to\infty} \frac{1}{k} \log(s_k(\overline{\zeta}_f, B_H(\{x_0\}, \sigma), \varepsilon)).$$

After accepting these notations, we should prove that

$$h(f, B(x_0, \sigma)) = \lim_{\varepsilon\to 0+} \overline{s}_f(\varepsilon, \sigma) > 0, \tag{2.11}$$

$$h(\overline{\zeta}_f, B(\{x_0\}, \sigma)) = \lim_{\varepsilon\to 0+} \overline{s}_{\overline{\zeta}_f}(\varepsilon, \sigma) > 0. \tag{2.12}$$

In order to prove (2.11) and (2.12), we will show that

there exist $\beta > 0$ and $\varepsilon_0 > 0$ such that $\overline{s}_f(\varepsilon, \sigma) \geq \beta$ and $\overline{s}_{\overline{\zeta}_f}(\varepsilon, \sigma) \geq \beta$ for each $\varepsilon \in (0, \varepsilon_0)$.

Since considerations related to many fragments of this proof are similar to the reasonings contained in the previous proofs, these fragments will be written briefly.

Since x_0 is an e-chaotic point of multifunction ζ_f, then there exists $L_\sigma = (L_1, \ldots, L_m) \in \mathsf{S}(\zeta_f, B(x_0, \sigma/2))$ such that

$$H_{\zeta_f}(L_\sigma) = \limsup_{k\to\infty} \frac{1}{k} \log(\#_1(\mathsf{P}_k(L_\sigma, \zeta_f))) \geq \alpha.$$

Set $\varepsilon_0 = \frac{1}{2}\min\{\varrho(L_i, L_j) : i, j \in \{1, \ldots, m\}, i \neq j\}$ and $\beta = \alpha/2$. We deduce that there exists $\{k_n\} \subset \mathbb{N}$ such that

$$\frac{1}{k_n} \log(\#_1(\mathsf{P}_{k_n}(L_\sigma, \zeta_f))) > \beta. \tag{2.13}$$

We now turn to the proof of (a). Note that if we have the path $\mathcal{P}(\zeta_f, L_{p_1}, \ldots, L_{p_{k_n}})$, then $\mathcal{P}(f, L_{p_1}, \ldots, L_{p_{k_n}})$ is also a path.

Now let $\xi_{\zeta_f} \colon \mathsf{P}_{k_n}(L_\sigma, \zeta_f) \to \mathsf{P}_{k_n}(L_\sigma, f)$ be a function defined in the following way: $\xi_{\zeta_f}(\mathcal{P}(\zeta_f, L_{p_1}, \ldots, L_{p_{k_n}})) = \mathcal{P}(f, L_{p_1}, \ldots, L_{p_{k_n}})$. It follows easily that ξ_{ζ_f} is one-to-one, and so $\#_1(\mathsf{P}_{k_n}(L_\sigma, \zeta_f)) \leq \#_1(\mathsf{P}_{k_n}(L_\sigma, f))$. In the next step of the proof we shall prove

$$s_{k_n}(f, B(x_0, \sigma), \varepsilon) \geq \#_1(\mathsf{P}_{k_n}(L_\sigma, f)). \tag{2.14}$$

Let us fix a path $\mathcal{P}(f, L_{p_1}, \ldots, L_{p_{k_n}}) \in \mathsf{P}_{k_n}(L_\sigma, f)$. Then one can find $x_{p_1,\ldots,p_{k_n}} \in L_{p_1}$ such that $f^j(x_{p_1,\ldots,p_{k_n}}) \in L_{p_{(j \bmod k_n)+1}}$ for $j \in \{0, \ldots, k_n\}$. In this way one can define a function ψ which each path $\mathcal{P}(f, L_1, \ldots, L_{k_n})$ assigns an element $x_{p_1,\ldots,p_{k_n}}$ having the above-mentioned property.

One should remark that if $\mathcal{P}(f, L_{p_1}, \ldots, L_{p_{k_n}}), \mathcal{P}(f, L_{q_1}, \ldots, L_{q_{k_n}}) \in \mathsf{P}_{k_n}(L_\sigma, f)$ and $\mathcal{P}(f, L_{p_1}, \ldots, L_{p_{k_n}}) \neq \mathcal{P}(f, L_{q_1}, \ldots, L_{q_{k_n}})$, then there exists $j_0 \in \{1, \ldots, k_n\}$ such that $L_{p_{j_0}} \cap L_{q_{j_0}} \neq \emptyset$ and consequently

$$\varrho(f^{j_0-1}(x_{p_1,\ldots,p_{k_n}}), f^{j_0-1}(x_{q_1,\ldots,q_{k_n}})) > \varepsilon_0.$$

From these considerations it follows that $\{x_{p_1,\ldots,p_{k_n}} : \mathcal{P}(f, L_{p_1}, \ldots, L_{p_{k_n}}) \in \mathsf{P}_{k_n}(L_\sigma, \zeta_f)\}$ is the (k_n, ε)-separated set contained in $B(x_0, \sigma)$ for $\varepsilon \in (0, \varepsilon_0)$. We have thus proved (2.14). From (2.14) and (2.13) it follows

$$\frac{1}{k_n} \log(s_{k_n}(f, B(x_0, \sigma))) > \beta \quad \text{for } \varepsilon \in (0, \varepsilon_0)$$

and consequently $\overline{s}_f(\varepsilon, \sigma) \geq \beta$, which proves (2.11) and thus proof (a) has been completed.

Now we will show item (b) of our theorem.

Since $L_\sigma = (L_1, \ldots, L_m) \in \mathsf{S}(\zeta_f, B(x_0, \sigma/2))$, we have $L_i \subset B(x_0, \sigma/2)$, $i \in \{1, \ldots, m\}$. Consequently, by Lemma 2.2(a), we have $\mathrm{d}(L_i) \subset B_H(\{x_0\}, \sigma)$, $i \in \{1, \ldots, m\}$, and from Lemma 2.2(c) it may be concluded that

$$\text{if } i, j \in \{1, \ldots, m\} \text{ and } i \neq j, \text{ then } \varrho_H(\mathrm{d}(L_i), \mathrm{d}(L_j)) \geq \varepsilon_0.$$

Now, we will prove that if $\mathcal{P}(\zeta_f, L_1, \ldots, L_{p_{k_n}}) \in \mathsf{P}_{k_n}(L_\sigma, \zeta_f)$, then

$$\mathrm{d}(L_{p_1}) \underset{\overline{\zeta}_f}{\longrightarrow} \cdots \underset{\overline{\zeta}_f}{\longrightarrow} \mathrm{d}(L_{p_{k_n}}) \underset{\overline{\zeta}_f}{\longrightarrow} \mathrm{d}(L_{p_1}). \tag{2.15}$$

For this purpose we need to show

$$\mathrm{d}(L_{p_i}) \underset{\overline{\zeta}_f}{\longrightarrow} \mathrm{d}(L_{p_j}) \quad \text{for } i \in \{1, \ldots, k_n\} \text{ and } j = (i \bmod k_n) + 1. \tag{2.16}$$

Let us fix $i \in \{1, \ldots, k_n\}$. Note that $\zeta_{f-}^{-1}(K) = \zeta_{f+}^{-1}(K)$ for each $K \subset X$.

So let $A \in \mathrm{d}(L_{p_j})$ and put $Z_A = \zeta_{f+}^{-1}(A) \cap L_{p_i} \neq \emptyset$. Let us observe that

$$Z_A \text{ is a closed set.} \tag{2.17}$$

For this, it is sufficient to show that $\zeta_{f+}^{-1}(A)$ is a closed set. A short calculation shows that $\zeta_{f+}^{-1}(A) = \{z \in X : \zeta_f(z) \subset A\} = \{z \in X : f(z) \in A\} = f^{-1}(A)$. Obviously $f^{-1}(A)$ is a closed set because $A \in \mathrm{d}(L_{p_j})$ and f is a continuous function. This completes our argument for (2.17).

From (2.17), closedness of L_{p_i} and $\emptyset \neq Z_A \subset L_{p_i}$, it follows that

$$Z_A \in \mathrm{d}(L_{p_i}). \tag{2.18}$$

In order to prove (2.16) (and thus (2.15)), by virtue of (2.18) it is sufficient to show that

$$A = \overline{\zeta}_f(Z_A). \tag{2.19}$$

As in the proof of (2.10), one can show that $A \subset \zeta_f(Z_A)$. Conversely, let $\alpha \in \zeta_f(Z_A) = \zeta_f(\zeta_{f+}^{-1}(A) \cap L_{p_i})$. So there exists $x \in \zeta_{f+}^{-1}(A) \cap L_{p_i}$ such that $\alpha \in \zeta_f(x)$. Of course, $\zeta_f(x) \subset A$, which means that $\alpha \in A$ and we have $A \supset \zeta_f(Z_A)$. Consequently, $A = \zeta_f(Z_A)$.

In order to complete the proof of (2.19) note that $A = \zeta_f(Z_A) = \overline{\zeta}_f(Z_A)$. According to (2.18), we conclude that $A \in \overline{\zeta}_f(\mathrm{d}(L_{p_i}))$, which finishes the proof of (2.16), and thus also (2.15). Moreover, note that the sequence $(\mathrm{d}(L_{p_1}), \dots, \mathrm{d}(L_{p_{k_n}}))$ fulfils the condition (2.1) with respect $\overline{\zeta}_f$.

Therefore we now move on to the proof of (2.12). On account of (2.15), each path $\mathcal{P}(\zeta_f, L_{p_1}, \dots, L_{p_{k_n}}) \in \mathsf{P}_{k_n}(L_\sigma, \zeta_f)$ can be assigned with a path $\mathcal{P}(\overline{\zeta_f}, \mathrm{d}(L_{p_1}), \dots, \mathrm{d}(L_{p_{k_n}}))$ $\in \mathsf{P}_{k_n}(\overline{L_\sigma}, \overline{\zeta_f})$, where $\overline{L_\sigma} = (\mathrm{d}(L_{p_1}), \dots, \mathrm{d}(L_{p_{k_n}}))$. Of course, this assignment is one-to-one. From this we conclude that $\#_1(\mathsf{P}_{k_n}(L_\sigma, \zeta_f)) \le \#_1(\mathsf{P}_{k_n}(\overline{L_\sigma}, \overline{\zeta}_f))$ and by (2.13) we have

$$\frac{1}{k_n} \log(\#_1(\mathsf{P}_{k_n}(\overline{L_\sigma}, \overline{\zeta}_f))) > \beta.$$

The inequality $s_{k_n}(\overline{\zeta}_f, B_H(\{x_0\}, \varepsilon)) \ge \#(\mathsf{P}_{k_n}\overline{L_\sigma}, \overline{\zeta}_f)$ can be proved just as (2.14) and completing the proof of (2.12) may be the same as in the case of (2.11) (obviously, e.g. we choose $F_{p_1,\dots,p_{k_n}} \in \mathrm{d}(L_{p_1})$ instead of $x_{p_1,\dots,p_{k_n}} \in L_{p_1}$, etc.). □

2.2. l- and u-chaotic points

In Subsections 2.1 and 2.3, we consider the concepts of chaotic points of multifunctions that are in some way analogous to some approaches regarding functions. In this subsection, however, we will consider the approach characteristic only for multifunctions. We will begin by adopting the following definition.

Let $\zeta\colon X \multimap X$. A point $x_0 \in \mathrm{Fix}_s(\zeta)$ is called l-chaotic point of ζ if there exists an open neighbourhood U of x_0 such that $\{x\} \underset{\zeta}{\xrightarrow{e}} \zeta(x_0)$ for $x \in U \setminus \{x_0\}$.

Theorem 2.6 will explain the meaning of the prefix l and Theorem 2.5 will justify the use of the name *chaotic point* (in the context of the results contained in Section 2.1 and earlier-cited statement from [13]).

Theorem 2.5. *Let $\zeta\colon X \multimap X$. If x_0 is an l-chaotic point of ζ, then x_0 is an e-chaotic point of ζ.*

Proof. We have to show that $h(\zeta, x_0) > 0$. Let us fix an arbitrary neighbourhood Y of x_0. From the definition of l-chaotic point it follows that there exists an open neighbourhood U of x_0 such that $\{x\} \underset{\zeta}{\xrightarrow{e}} \zeta(x_0)$ for $x \in U \setminus \{x_0\}$ and, moreover, we have $x_0 \in \mathrm{Fix}_s(\zeta)$.

Put $Z = Y \cap U \cap \mathrm{Int}(\zeta(x_0))$ and let $F_1, F_2 \subset Z$ be uncountable closed sets such that $x_0 \in F_1$ and $F_1 \cap F_2 = \emptyset$. It is easily seen that $L = (F_1, F_2) \in \mathsf{S}(\zeta, Z)$.

Fix $k \in \mathbb{N}$ and consider $\mathsf{P}_k(L, \zeta)$. We shall show that

$$\#_1(\mathsf{P}_k(L,\zeta)) = 2^k. \tag{2.20}$$

An easy verification shows that $\zeta(F_i) \supset F_1 \cup F_2$ for $i = 1, 2$. This means that $F_i \underset{\zeta}{\to} F_j$ for $i, j \in \{1, 2\}$. Hence we directly obtain (2.20).

Obviously, $H_\zeta(L) = \limsup_{k\to\infty} \frac{1}{k} \log(\#_1(\mathsf{P}_k(L,\zeta))) = \log 2$ and consequently $H_\zeta(Y) \geq H_\zeta(Z) \geq H_\zeta(L) = \log 2$, which according to the arbitrariness of Y gives $h(\zeta, x_0) \geq \log 2 > 0$. □

As it has been already signaled in the introduction the following theorem explains the meaning of the prefix l.

Theorem 2.6. *Let $\zeta \colon X \multimap X$ be a multifunction. If x_0 is an l-chaotic point of ζ, then ζ is lower semicontinuous at x_0.*

Proof. When x_0 is an isolated point the proof is trivial. So, we may assume that $\mathrm{Int}(\{x_0\}) = \emptyset$. Let W be an open set such that $\zeta(x_0) \cap W \neq \emptyset$. Since for some open neighbourhood V of x_0, we have $\{x\} \overset{e}{\underset{\zeta}{\to}} \zeta(x_0)$ for $x \in V \setminus \{x_0\}$, then $\zeta(x) \cap W \neq \emptyset$ for $x \in V$. □

It is not difficult to verify that the above theorem cannot be strengthened by the demand that x_0 is a continuity point of the multifunction. Now, we consider the relationship with the multifunction $\widehat{\zeta}$.

Theorem 2.7. *Let $\zeta \colon X \multimap X$ be a multifunction. If x_0 is an l-chaotic point of ζ, then $\{x_0\}$ is an l-chaotic point of $\widehat{\zeta}$.*

Proof. Our proof starts with the observation that

$$\widehat{\zeta}(\{x\}) = \mathrm{d}(\zeta(\{x\})) = \mathrm{d}(\zeta(x)) \quad \text{for an arbitrary } x \in X. \tag{2.21}$$

First, we shall show that

$$\{x_0\} \in \mathrm{Fix}_s(\widehat{\zeta}). \tag{2.22}$$

Since $x_0 \in \mathrm{Fix}_s(\zeta)$, there exists $\delta > 0$ such that $B(x_0, \delta) \subset \zeta(x_0)$. In order to get inclusion (2.22), it is convenient to show $B_H(\{x_0\}, \delta) \subset \widehat{\zeta}(\{x_0\})$. So, let $P \in B_H(\{x_0\}, \delta)$. Then $\mathrm{d}(P) \subset B_H(\{x_0\}, \delta)$ and next $P \subset B(x_0, \delta) \subset \zeta(x_0)$ follows from Lemma 2.2(b). Taking into account (2.21), we infer that $P \in \widehat{\zeta}(\{x_0\})$. Arbitrariness of the choice of $P \in B_H(\{x_0\}, \delta)$ proves that $B_H(\{x_0\}, \delta) \subset \widehat{\zeta}(\{x_0\})$ and consequently we have (2.22).

Our assumptions guarantee the existence of $\varepsilon > 0$ such that $\{x\} \underset{\zeta}{\overset{e}{\to}} \zeta(x_0)$ for an arbitrary $x \in B(x_0, \varepsilon) \setminus \{x_0\}$, and consequently

$$\text{for any } x \in B(x_0, \varepsilon) \setminus \{x_0\}, \text{ there exists an element } y \text{ such that } \{y\} \in \mathrm{d}(\zeta(x)) \setminus \mathrm{d}(\zeta(x_0)). \tag{2.23}$$

Now, we shall prove that

$$Q \underset{\widehat{\zeta}}{\overset{e}{\to}} \widehat{\zeta}(\{x_0\}) \quad \text{for each nonempty } Q \in B_H(\{x_0\}, \varepsilon) \setminus \{\{x_0\}\}.$$

Lemma 2.2(b) gives $Q \subset B(x_0, \varepsilon)$. Let $x_Q \in Q$. From (2.21) it may be concluded that

$$\widehat{\zeta}(\{x_0\}) = \mathrm{d}(\zeta(x_0)) \subset \mathrm{d}(\zeta(x_Q)) \subset \mathrm{d}(\zeta(Q)) = \widehat{\zeta}(Q).$$

What is left is to show that $\widehat{\zeta}(\{x_0\}) \neq \widehat{\zeta}(Q)$. It is easily seen that $Q \neq \{x_0\}$, and so there exists $q \in Q \setminus \{x_0\}$. Obviously, $q \in B(x_0, \varepsilon) \setminus \{x_0\}$, which, on the basis of (2.23) allows us to conclude that there exists q_1 such that $\{q_1\} \in \mathrm{d}(\zeta(q)) \setminus \mathrm{d}(\zeta(x_0))$. From (2.21), we obtain $\{q_1\} \in \widehat{\zeta}(\{q\}) \setminus \widehat{\zeta}(\{x_0\})$, thereby $\{q_1\} \in \widehat{\zeta}(Q) \setminus \widehat{\zeta}(\{x_0\})$. □

Following the definition of l-chaotic point, a dual definition of u-chaotic point can be formulated.

Let $\zeta\colon X \multimap X$. A point $x_0 \in \mathrm{Fix}_s(\zeta)$ is called u-chaotic point of ζ if there exists an open neighbourhood U of x_0 such that $\{x_0\} \underset{\zeta}{\overset{e}{\to}} \zeta(x)$ for $x \in U \setminus \{x_0\}$. It is not difficult to prove the theorem analogous to Theorem 2.6.

Theorem 2.8. *Let $\zeta\colon X \multimap X$ be a multifunction. If x_0 is an u-chaotic point of ζ, then ζ is upper semicontinuous at x_0.*

Despite the analogy between definitions of l-chaotic point and u-chaotic point, both concepts are significantly different. They are not only mutually independent (which is easy to see), but they also have different properties. For example, for the u-chaotic points the theorem analogous to Theorem 2.5 nor does theorem analogous to Theorem 2.7 hold. For a counterexample (in both cases), consider a canonical multifunction Γ_f, where $f\colon [0,1] \to [0,1]$ is defined in the following way: $f(x) = 0$ for $x \in [0,1] \setminus \{1/2\}$ and $f(1/2) = 1$.

Although there is no complete analogy to Theorem 2.5, in the case of canonical multifunction, some analogous theorem can be obtained.

Theorem 2.9. *Let $f\colon [0,1] \to [0,1]$ be a continuous function and let $x_0 \in X$ be a u-chaotic point of the canonical function $\Gamma_f\colon X \multimap X$ associated with the function f, then x_0 is an e-chaotic point of Γ_f.*

Proof. We will carry out the proof assuming that $x_0 \neq 1$ (in the case $x_0 = 1$ the reasoning is analogous). We have to show that density of entropy of Γ_f at x_0 is positive, i.e., $h(\Gamma_f, x_0) > 0$.

Let us fix an arbitrary open neighbourhood Y of x_0. Moreover, it follows easily that there exists $\sigma > 0$ such that

$$[x_0, x_0 + \sigma] \subset \Gamma_f(x_0). \tag{2.24}$$

We shall show that there exists an open neighbourhood V of x_0 such that

$$(x_0 - \sigma/2, x_0 + \sigma/2) \cap [0, 1] \subset \Gamma_f(x) \quad \text{for each } x \in V. \tag{2.25}$$

According to the property of canonical function, it follows that Γ_f is l.s.c. at x_0. Let $W = (x_0 + \sigma/2, 1]$. On account of (2.24), we have $\Gamma_f(x_0) \cap W \neq \emptyset$, and so there exists an open neighbourhood V of x_0 such that $\Gamma_f(x) \cap W \neq \emptyset$ for each $x \in V$. Taking into account the definition of Γ_f, we conclude that condition (2.25) is fulfilled.

Let us also consider an open neighbourhood U of x_0 such that $\{x_0\} \xrightarrow[\Gamma_f]{e} \Gamma_f(x)$ for any $x \in U \setminus \{x_0\}$. Finally, let us examine open neighbourhood $Z = Y \cap U \cap V \cap (x_0 - \sigma/2, x_0 + \sigma/2) \cap [0, 1]$ of x_0 and let $F_1, F_2 \subset Z$ be uncountable closed sets such that $x_0 \in F_1$ and $F_1 \cap F_2 = \emptyset$. Considering $L = (F_1, F_2) \in \mathsf{S}(\Gamma_f, Z)$, by reasoning analogous to the proof of Theorem 2.5, it can be proved that $\Gamma_f(F_i) \supset F_1 \cup F_2$ for $i = 1, 2$ and consequently we have $\#_1(\mathsf{P}_k(L, \Gamma_f)) = 2^k$ for any $k \in \mathbb{N}$. Thus $H_{\Gamma_f}(L) = \limsup_{k\to\infty} \frac{1}{k} \log(\#_1(\mathsf{P}_k(L, \Gamma_f))) = \log 2$ and next $H_{\Gamma_f}(Y) \geq H_{\Gamma_f}(Z) \geq H_{\Gamma_f}(L) = \log 2$, which gives (according to arbitrariness of Y) $h(\Gamma_f, x_0) \geq \log 2 > 0$. $\square$

Note also remark, important for further considerations.

Remark 2.10. Let $\zeta\colon X \multimap X$. If $x_0 \in X$ be an isolated point, then x_0 is an l-chaotic and u-chaotic point of ζ if and only if $x_0 \in \mathrm{Fix}(\zeta)$.

2.3. Distributionally chaotic point

The analysis of the behaviour of some functions and multifunctions allows to notice that there exist points around which “uncountable distributionally scrambled sets” are focused. This observation leads us to analysing local aspects of “distibutional chaos”, which become the aim of paper [23] (in the context of functions). Minor modifications (to the concept of envelope) allow us to transfer the definitions from the paper [23] to the case of multifunction.

Let $\zeta\colon X \multimap X$ be a multifunction. We shall say that $x_0 \in X$ is a DC1 point of the dynamical system (ζ) if for any $\varepsilon > 0$, there exists an uncountable set S being a DS-set for the dynamical system (ζ) such that there are $n \in \mathbb{N}$ and a closed set $A \supset S$ fulfilling the condition $\zeta^{i\cdot n}(A) \cap B(x_0,\varepsilon) \neq \emptyset$ for $i \in \mathbb{N}_0$. The set A described above will be called *(n,ε)-envelope of the set S.*

Deeper relations between DC1 points and e-chaotic or l-chaotic point can be inferred from Theorem 2.11 and from the considerations contained in Section 3. Below we will consider only an example of the multifunction $\zeta\colon [0,1] \multimap [0,1]$ for which $x_0 = 1/2$ is an l-chaotic point (thus, by Theorem 2.5 is also an e-chaotic point), which is not a DC1 point.

Let $f\colon [0,1] \to [0,1]$ be a continuous function such that $f(0) = 1 = f(1)$, $f(1/2) = 3/4$ and f is linear on each interval $[0,1/2]$ and $[1/2,1]$. Let us consider canonical multifunction $\Gamma_f\colon [0,1] \multimap [0,1]$ associated with the function f. Set $x_0 = 1/2$. It follows immediately that x_0 is an l-chaotic point of Γ_f. Note that $\Gamma_f^2(x) = [0,1]$ for any $x \in [0,1]$. Thus for arbitrary $x,y \in [0,1]$ and $j \geq 2$, we have $\varrho_H(\Gamma_f^j(x),\Gamma_f^j(y)) = 0$. Consequently there is no DS-set for Γ_f.

In the context of the example above, the following statement seems to be interesting.

Theorem 2.11. *Let $\zeta_0\colon [0,1] \multimap [0,1]$ and let $x_0 \in (0,1)$ be an l-chaotic point of ζ_0. Then there exists an m-DC1 multifunction $\zeta_1\colon X \multimap X$ inserted in ζ_0 and such that x_0 is a DC1 point of ζ_1 and it is not an e-chaotic point of ζ_1 (so all the more, according to Theorem* 2.5*, it is not an l-chaotic point).*

Proof. According to our assumption there exists $\delta_1 > 0$ such that

$$[x_0-\delta, x_0+\delta] \subset \zeta_0(x_0) \cap (0,1) \subset \zeta_0(x) \quad \text{for } x \in [x_0-\delta, x_0+\delta]. \tag{2.26}$$

Let $\{a_n\}$, $\{b_n\}$ be sequences converging to x_0 and such that

$$x_0 < \dots < a_n < b_n < \dots < a_2 < b_2 < a_1 < b_1 < x_0 + \frac{\delta}{3}.$$

Taking into account, for example, the considerations contained in [26] it is easy to check that there exists a continuous DC1 function $\tau_n\colon [a_n,b_n] \to [a_n,b_n]$ such that $\tau(a_n) = a_n$, $\tau(b_n) = b_n$, $n \in \mathbb{N}$. This means that

(2.27)

there exists an uncountable distributionally scrambled set $S_n \subset [a_n,b_n]$ of τ_n for $n \in \mathbb{N}$.

Let us define $\tau_0\colon [x_0, x_0+\delta/3] \to [x_0, x_0+\delta/3]$ in the following way:

$$\tau_0(x) = \begin{cases} x & \text{if } x \in [x_0, x_0+\delta/3] \setminus \bigcup_{n=1}^{\infty} [a_n,b_n], \\ \tau_n(x) & \text{if } x \in [a_n,b_n] \text{ for } n \in \mathbb{N}. \end{cases}$$

Let us finally define a multifunction $\zeta_1\colon [0,1] \multimap [0,1]$ by the formula

$$\zeta_1(x) = \begin{cases} \zeta_0(x) & \text{if } x \notin [x_0-\delta, x_0+\delta/3] \cup [x_0+2\delta/3, x_0+\delta], \\ [x_0-\delta, x_0-\delta/2] & \text{if } x \in [x_0-\delta, x_0), \\ \{\tau_0(x)+2\delta/3\} \cup [x_0-\delta, x_0-\delta/2] & \text{if } x \in [x_0, x_0+\delta/3], \\ \{\tau_0(x-2\delta/3)\} \cup [x_0-\delta, x_0-\delta/2] & \text{if } x \in [x_0+2\delta/3, x_0+\delta]. \end{cases}$$

On account of (2.26), it is easy to see that ζ_1 is inserted in ζ_0.

Detailed calculations show that $\zeta_1^{2j-1}(x) = \{\tau_0^{2j-1}(x) + 2\delta/3\} \cup [x_0 - \delta, x_0 - \delta/2]$ and $\zeta_1^{2j}(x) = \{\tau_0^{2j}(x)\} \cup [x_0 - \delta, x_0 - \delta/2]$ for $j \in \mathbb{N}$ and $x \in [x_0, x_0 + \delta/3]$. Moreover, let us note that $\tau_0^{2j-1}(x) + 2\delta/3 \in [x_0 + 2\delta/3, x_0 + \delta]$ and $\tau_0^{2j}(x) \in [x_0, x_0 + \delta/3]$ for $x \in [x_0, x_0 + \delta/3]$, $j \in \mathbb{N}$.

From the above observations and Lemma 1.1, it may be concluded that

$$\varrho_H(\zeta_1^k(x), \zeta_1^k(y)) = |\tau_0^k(x) - \tau_0^k(y)| \quad \text{for } x, y \in [x_0, x_0 + \delta/3],\ k \in \mathbb{N}. \tag{2.28}$$

We will now show that

$$x_0 \text{ is a DC1 point for } \zeta_1. \tag{2.29}$$

So let $\varepsilon > 0$. Then there exists $n_\varepsilon \in \mathbb{N}$ such that $[a_{n_\varepsilon}, b_{n_\varepsilon}] \subset (x_0 - \varepsilon, x_0 + \varepsilon) \cap (0, 1)$. By (2.27) there exists an uncountable distributionally scrambled set $S_{n_\varepsilon} \subset [a_{n_\varepsilon}, b_{n_\varepsilon}]$ of τ_{n_ε}, and thus for τ_0. Then

$$S_{n_\varepsilon} \text{ is an uncountable distributionally scrambled set of } \zeta_1. \tag{2.30}$$

Let $x, y \in S_{n_\varepsilon}$, $x \neq y$ and $t > 0$. Then, taking into account (2.28), we obtain

$$\begin{aligned}
\Phi_{x,y}^{*(\zeta_1)}(t) &= \limsup_{n\to\infty} \frac{1}{n} \#(\{k \in [\![0, n-1]\!] : \rho_H(\zeta_1^k(x), \zeta_1^k(y)) < t\}) \\
&= \limsup_{n\to\infty} \frac{1}{n} \#(\{k \in [\![0, n-1]\!] : |\tau_0^k(x) - \tau_0^k(y)| < t\}) = 1.
\end{aligned}$$

Since S_{n_ε} is an uncountable distributionally scrambled set of τ_0, there exists $t_0 > 0$ such that

$$\Phi_{x,y}^{(\tau_0)}(t_0) = \limsup_{n\to\infty} \frac{1}{n} \#(\{k \in [\![0, n-1]\!] : |\tau_0^k(x) - \tau_0^k(y)| < t_0\}) = 0.$$

And again, taking into account (2.28), we have

$$\begin{aligned}
\Phi_{x,y}^{(\zeta_1)}(t) &= \limsup_{n\to\infty} \frac{1}{n} \#(\{k \in [\![0, n-1]\!] : \rho_H(\zeta_1^k(x), \zeta_1^k(y)) < t_0\}) \\
&= \limsup_{n\to\infty} \frac{1}{n} \#(\{k \in [\![0, n-1]\!] : |\tau_0^k(x) - \tau_0^k(y)| < t_0\}) = 0,
\end{aligned}$$

which ends the proof of (2.30).

Now note that $[a_{n_\varepsilon}, b_{n_\varepsilon}]$ is an $(n_\varepsilon, \varepsilon)$-envelope of ζ_1. Obviously $S_{n_\varepsilon} \subset [a_{n_\varepsilon}, b_{n_\varepsilon}]$. Let $n = 2$, $i \in \mathbb{N}_0$ and $p \in S_{n_\varepsilon} \subset [a_{n_\varepsilon}, b_{n_\varepsilon}]$. Then $\zeta_1^{2\cdot i}(p) \ni \tau_0^{2\cdot i}(p)$, which means that $\zeta_1^{2\cdot i}([a_{n_\varepsilon}, b_{n_\varepsilon}]) \cap (x_0 - \varepsilon, x_0 + \varepsilon) \cap (0, 1) \neq \emptyset$, which finishes the proof that $[a_{n_\varepsilon}, b_{n_\varepsilon}]$ is a $(n_\varepsilon, \varepsilon)$-envelope of ζ_1. The proof of (2.29) is complete.

Moreover, the set $\zeta_1(x)$ is uncountable for an arbitrary $x \in S_{n_\varepsilon}$, which allows to conclude that ζ_1 is an m-DC1 multifunction.

Note that $\zeta_1([x_0 - \delta, x_0 + \delta/3]) \subset [x_0 - \delta, x_0 - \delta/2] \cup [x_0 + 2\delta/3, x_0 + \delta]$, and so if $Y \subset (x_0 - \delta/4, x_0 + \delta/4)$, then for an arbitrary $L \in \mathsf{S}(\zeta_1, Y)$, we have $\mathsf{P}_k(L) = \emptyset$ for $k \geq 2$. Thus $H_{\zeta_1}(L) = 0$, and by arbitrariness of $L \in \mathsf{S}(\zeta_1, Y)$, it follows that $H_{\zeta_1}(Y) = 0$ and consequently $h(\zeta_1, x_0) = 0$. $\square$

3. Let us play chaos

In many scientific studies, one can find combinations and applications of various approaches of dynamical systems and games, not only in relation to purely mathematical issues (e.g. [7]). In this paper, however, we will combine the issues of infinite topological games and the various kinds of chaotic points of multifunction.

The notion of a finite positional game (with perfect information) was introduced in the monograph [19] (note that combinatorial game was described in the 17th century). However, infinite positional games were considered earlier. This was connected to the famous Banach–Mazur Game which appeared for the first time in the famous Scottish Book (Problem No. 43, S. Mazur entry). A number of outstanding mathematicians can be associated with the problems of this game. For some historical overview, see e.g. [24,28]. Of course, considerations about infinite topological games are also applied to multifunctions (e.g. survey [5]).

The main goal of this chapter is to show the application of the various notions of chaotic points to infinite topological games.

We begin by describing the two types of games l-MG (lower multifunctions game) and u-MG (upper multifunctions game). A finite (but greater than 1) number of players participate in the game. We will assume in this section that there are k players: $\pi_1, \pi_2, \ldots, \pi_k$, $k \in \mathbb{N} \setminus \{1\}$. The starting point for all games is a multifunction $\zeta_0(x) = X$ for $x \in X$.

In the game l-MG (u-MG), the players choose s-continuous, m-DC1 multifunction having l-chaotic point (u-chaotic point) in such a way that all the functions selected so far form a frustum. Players choose in order: $\pi_1, \pi_2, \ldots, \pi_k, \pi_1, \pi_2, \ldots, \pi_k, \pi_1$, etc. Since this is an infinite game, we will get a nonautonomous dynamical system being a cone and thus the set of vertices of created cone is nonempty.

To each player π_i ($i \in [\![1, k]\!]$), there is assigned a sentence $s(\pi_i)$ describing the properties of a certain vertex of created cone, determining his winnings. Obviously, we can take care of some elegance related to the description of our game by adopting an assumption: "if there is a vertex for which $s(\pi_{i_0})$ is true then there is no vertex with the properties $s(\pi_i)$ for $i \in [\![1, k]\!] \setminus \{i_0\}$". The acceptance or omission of the above assumption means that there may be one or more winners in the game. It will depend on the fact whether $s(\pi_i)$ for different players are mutually exclusive (e.g. $s(\pi_1)$ contradicts $s(\pi_2)$ in the game for two players), or the sentenced may be simultaneously fulfilled for two (or more) players (e.g. if

$s(\pi_1)$ means that during the game a cone with vertex having e-chaotic point is created and $s(\pi_2)$ means the existence of a vertex with l-chaotic point, see Theorem 2.5). Clearly, one can create also a game with no winner. However, as it was indicated in the introduction, we will pay our attention only to the problem of the existence of winning strategy for a fixed player, whose sentence will be connected with the existence of earlier presented chaotic points. Thus, we will only formulate $s(\pi_{i_0})$ for a fixed player π_{i_0} ($i_0 \in [\![1,k]\!]$), assuming by default that certain $s(\pi_i)$ sentences are also defined for the other players.

Let us denote by M_l (M_u) the set of all s-continuous, m-DC1 multifunctions $\zeta\colon X \multimap X$ having l-chaotic point (u-chaotic point). For $i \in [\![1,k]\!]$, let C_l^i (C_u^i) denote the set of all frustums of the form $(\zeta_0,\dots,\zeta_{\omega\cdot k+i-1})$ for $\omega \in \mathbb{N}_0$, consisting of functions belonging to M_l (M_u). In other words, C_l^i (C_u^i) consists of sequences of multifunctions selected in the game l-MG (u-MG) preceding the next choice of π_i, $i \in [\![1,k]\!]$.

The strategy in l-MG (u-MG) for the player π_i ($i \in [\![1,k]\!]$) is a function $\tau_i^l\colon C_l^i \to M_l$ ($\tau_i^u\colon C_u^i \to M_u$). We will say that player π_i ($i \in [\![1,k]\!]$) follows strategy τ_i^l (τ_i^u) if at any step of his play his choice is the value of the function τ_i^l (τ_i^u) for a frustrum consisting of previously selected functions.

Strategy τ_i^l (τ_i^u), for $i \in [\![1,k]\!]$, is called a winning strategy for player π_i if player π_i, following strategy τ_i^l (τ_i^u), regardless of the choices of other players and, obviously, irrespective to the defined winning sentences for other players, will lead to the creation of a cone for which $s(\pi_i)$ is true (i.e., player π_i wins the game regardless of the actions of other players).

Before making the main considerations, let us note some useful lemmas.

Lemma 3.1. (e.g. [21,26]) *A function $f\colon P \to P$, where P is a non-degenerate compact interval, has a positive entropy iff the dynamical system (f) is distributionally chaotic of type* 1.

Lemma 3.2. (cf. [23,27]) *Let $L \subset X$ be an arc, $\phi\colon [0,1] \to L$ be a homeomorphism, $f\colon [0,1] \to [0,1]$ be a continuous function and $g = \phi \circ f \circ \phi^{-1}$. If $S \subset [0,1]$ is an uncountable DS-set for the dynamical system (f), then $\phi(S)$ is an uncountable DS-set for the dynamical system (g).*

Lemma 3.3. *Let $\zeta\colon X \multimap X$ be an s-continuous multifunction and let x_0 be an l-chaotic point of ζ. Then for an arbitrary base $\mathbb{B}(x_0) = \{K_n\}_{n=1}^{\infty}$, there exist $n_0 \in \mathbb{N}$ and continuous selection* sel_ζ *of ζ such that $\mathrm{sel}_\zeta(x) = x$ for $x \in K_{n_0}$.*

Proof. Since x_0 is an l-chaotic point of ζ, we have $x_0 \in \mathrm{Fix}(\zeta)$ and there exists an open neighbourhood U of x_0 such that

$$x_0 \in \mathrm{Int}(\zeta(x_0)) \subset \zeta(x) \quad \text{for each } x \in U. \tag{3.1}$$

Fix $K_{s_0} \in \mathbb{B}(x_0)$ such that

$$K_{s_0} \subset U \cap \mathrm{Int}(\zeta(x_0)). \tag{3.2}$$

Let us denote by s_ζ an arbitrary continuous selection of ζ. Then, by (M2), there is $m_0 \in \mathbb{N}$ such that $m_0 > s_0$, (and so $K_{m_0} \subset \mathrm{Int}(K_{s_0})$) and $s_\zeta(K_{m_0}) \subset K_{s_0} \subset U$. On account of (M2) and (M3), one can find $n_0 \in \mathbb{N}$ such that $K_{n_0} \subset \mathrm{Int}(K_{m_0})$.

Consider the function $s_\zeta^\star \colon K_{n_0} \cup \mathrm{Fr}(K_{m_0}) \to K_{s_0}$ given by

$$s_\zeta^\star(x) = \begin{cases} x & \text{if } x \in K_{n_0}, \\ s_\zeta(x) & \text{if } x \in \mathrm{Fr}(K_{m_0}). \end{cases}$$

Using (M4) one can find a continuous extension $s'_\zeta \colon K_{m_0} \to K_{s_0}$ of $s_\zeta^\star$. From (3.1) and (3.2), it follows that $K_{s_0} \subset \zeta(x)$ for each $x \in K_{s_0}$, and so $s'_\zeta(x) \in \zeta(x)$ for each $x \in K_{m_0}$. Now, we are able to define a function $\mathrm{sel}_\zeta \colon X \to X$ in the following way:

$$\mathrm{sel}_\zeta(x) = \begin{cases} s'_\zeta(x) & \text{if } x \in K_{m_0}, \\ s_\zeta(x) & \text{if } x \notin K_{m_0}. \end{cases}$$

Taking into account the equality $s'_\zeta \restriction \mathrm{Fr}(K_{m_0}) = s_\zeta \restriction \mathrm{Fr}(K_{m_0})$, we conclude that sel_ζ is a continuous function. Obviously sel_ζ is a selection of ζ and $\mathrm{sel}_\zeta(x) = x$ for $x \in K_{n_0}$. □

Remark 3.4. Let us assume the notations as in Lemma 3.3. When analyzing the proof, we notice at once that if $n \geq n_0$, then $\mathrm{sel}_\zeta(x) = x$ for $x \in K_n$.

Before we go on to the basic considerations, let us note some observations that precede the further results. Theorem 3.5 will show that we cannot have a winning strategy when we demand a cone with a vertex of l-chaotic point (the same result can be shown for a u-chaotic point). For this reason, we will weaken our demands limiting our considerations to the closure of the set defined below. Let us denote by $O_l(\zeta)$ ($O_u(\zeta)$) the set of all l-chaotic points (u-chaotic points) of ζ. If we have a dynamical system $(\zeta_{1,\infty}) = \{\zeta_i\}_{i\in\mathbb{N}}$, then we put $\Lambda_l = \bigcup_{n=1}^{\infty} O_l(\zeta_n)$ ($\Lambda_u = \bigcup_{n=1}^{\infty} O_u(\zeta_n)$).

Let us now consider the situation when we have players π_e and π_l in the game l-MG (not necessarily participating in the same game) and to whom the following sentences are assigned:

> $s(\pi_e)$ - as a result of the game there will be created a cone such that there exist a point $x_0 \in \overline{\Lambda_l}$ and a vertex κ such that x_0 is a DC1 point and a strong e-chaotic point of κ.
>
> $s(\pi_l)$ - as a result of the game will be created a cone with the vertex having l-chaotic point.

Theorem 3.5. *Let us assume that* $\dim(X) \geq 2$. *Then*

(l-MG1) *player* π_e *has a winning strategy in the game* l*-MG.*

(l-MG2) *player* π_l *has no winning strategy in the game* l*-MG.*

Proof. The method of the proof was selected in such a way that it was possible to shorten the considerations of both this reasoning and the proof of Theorem 3.6 by referring to some parts of the already presented proof.

To simplify, we will carry out the proof with the additional assumption that π_e makes the second choice (i.e., $\pi_e = \pi_2$). If $\pi_e = \pi_i$ for $i > 2$ the proof is analogous (with an obvious change of indexes). If $\pi_e = \pi_1$, the proof is also similar. The only exception is that in the first step of choosing, the player π_e chooses an arbitrary multifunction fulfilling the rules of l-MG game and at the second step of choosing he starts building a strategy according to the scheme shown below.

In order to define the winning strategy τ_2^l for π_e (let us recall that we assume: $\pi_e = \pi_2$), we must assign to each frustum $(\zeta_0, \ldots, \zeta_{\omega \cdot k+1})$ $(\omega \in \mathbb{N}_0)$ a multifunction $\zeta_{\omega \cdot k+2}$ fulfilling the rules of the game, so that as the game result we obtain a cone for which $s(\pi_e)$ is true.

So let us assume that for some $\omega \in \mathbb{N}_0$, we have frustum $(\zeta_0, \ldots, \zeta_{\omega \cdot k+1}) \in C_l^2$, i.e., we have sequence of multifunctions selected in the game l-MG $(\zeta_0, \ldots, \zeta_{\omega \cdot k+1}) \in C_l^2$ that precede the next choice of π_e. Now we have to define the multifunction $\zeta_{\omega k+2}$ being the value of the function τ_2^l at $(\zeta_0, \ldots, \zeta_{\omega \cdot k+1})$.

According to the rules of our game, the player π_1 chooses (in the previous step) s-continuous, m-DC1 multifunction $\zeta_{\omega k+1}$ having l-chaotic point $x_{\omega k+1}$. Using the property of $\zeta_{\omega k+1}$ (corresponding to the rules of this game), we can conclude that there exists an open set $U_{\omega k+2}$ such that

$$x_{\omega k+1} \in U_{\omega k+2} \subset \zeta_{\omega k+1}(x_{\omega k+1}) \subset \zeta_{\omega k+1}(x) \quad \text{for each } x \in U_{\omega k+2}. \tag{3.3}$$

Obviously, $U_{\omega k+2}$ is not a singleton. Fix $\mathbb{B}(x_{\omega k+1}) = \{K_n\}_{n=1}^{\infty}$ fulfilling the conditions (M1)–(M4) and let $K_{n_{\omega k+2}} \in \mathbb{B}(x_{\omega k+1})$ be such that

$$K_{n_{\omega k+2}} \subset U_{\omega k+2} \quad \text{and} \quad \operatorname{diam}(K_{n_{\omega k+2}}) \leq \frac{1}{2^{\omega k+2}}. \tag{3.4}$$

On account of Lemma 3.3 and Remark 3.4, one can assume that $K_{n_{\omega k+2}}$ has the property: there is a continuous selection $\mathrm{sel}_{\zeta_{\omega k+1}}$ of $\zeta_{\omega k+1}$ such that $\mathrm{sel}_{\zeta_{\omega k+1}}(x) = x$ for $x \in K_{n_{\omega k+2}}$.

So let $z_{\omega k+2} \in \operatorname{Int}(K_{n_{\omega k+2}}) \setminus \{x_{\omega k+1}\}$. Note that $z_{\omega k+2} \in \zeta_{\omega k+1}(x_{\omega k+1})$. Since $K_{n_{\omega k+2}}$ is an arcwise connected set, there exists an arc $L_{\omega k+2} = L(x_{\omega k+1}, z_{\omega k+2}) \subset K_{n_{\omega k+2}}$. From (M2) and (M3), one can deduce that there is $K_{m_{\omega k+2}} \in \mathbb{B}(x_{\omega k+1})$ such that

$$K_{m_{\omega k+2}} \subset \operatorname{Int} K_{n_{\omega k+2}} \subset U_{\omega k+2} \quad \text{and} \quad z_{\omega k+2} \notin K_{m_{\omega k+2}}.$$

Let $L^{\omega k+2} \subset L_{\omega k+2}$ be an arc satisfying the following conditions

$\operatorname{diam}(L^{\omega k+2}) < \varrho(z_{\omega k+2}, L^{\omega k+2})$;

(3.5) there exist an open set $U^{\omega k+2} \subset \operatorname{Int}(K_{n_{\omega k+2}}) \setminus K_{m_{\omega k+2}}$ such that $L^{\omega k+2} \subset U^{\omega k+2}$ and a homeomorhism $\varphi_{\omega k+2} \colon U^{\omega k+2} \xrightarrow{\text{onto}} B(\alpha_{\omega k+2}, r_{\omega k+2}) \subset \mathbb{H}^{\dim(X)}$.

Next, we distinguish $2^{\omega k+2}$ arcs contained in $L^{\omega k+2}$:

$$L_i^{\omega k+2} \subset L^{\omega k+2} \text{ such that } L_{i_1}^{\omega k+2} \cap L_{i_2}^{\omega k+2} = \emptyset \text{ for } i_1, i_2 \in \{1, \dots, 2^{\omega k+2}\},\ i_1 \neq i_2.$$

Of course, $z_{\omega k+2} \notin \bigcup_{i=1}^{2^{\omega k+2}} L_i^{\omega k+2}$. Let $h_i^{\omega k+2} \colon L_i^{\omega k+2} \xrightarrow{\text{onto}} L^{\omega k+2}$ be a homeomorphism for $i \in \{1, \dots, 2^{\omega k+2}\}$ and let $\mu_{\omega k+2} \colon L^{\omega k+2} \to L^{\omega k+2}$ be a continuous function such that $\mu_{\omega k+2} \restriction L_i^{\omega k+2} = h_i^{\omega k+2}$, $i \in \{1, \dots, 2^{\omega k+2}\}$. Now, we define auxiliary continuous function $g'_{\omega k+2} \colon \operatorname{Fr}(K_{m_{\omega k+2}}) \cup L^{\omega k+2} \cup \operatorname{Fr}(K_{n_{\omega k+2}}) \cup \{z_{\omega k+2}\} \to K_{n_{\omega k+2}}$ in the following way:

$$g'_{\omega k+2} = \begin{cases} x & \text{if } x \in \operatorname{Fr}(K_{m_{\omega k+2}}) \cup \{z_{\omega k+2}\}, \\ \mu_{\omega k+2}(x) & \text{if } x \in L^{\omega k+2}. \end{cases}$$

Using (M4), one can infer that there exists continuous function $g_{\omega k+2} \colon K_{n_{\omega k+2}} \to K_{n_{\omega k+2}}$ being continuous extension of $g'_{\omega k+2}$.

Before starting the definition of $\zeta_{\omega k+2}$, we consider sequence of points $\{t_i^{\omega k+2}\}_{i=1}^{\infty} \subset \operatorname{Int}(K_{n_{\omega k+2}}) \setminus K_{m_{\omega k+2}}$ converging to some point $t_0^{\omega k+2} \in \operatorname{Fr}(K_{m_{\omega k+2}})$.

Let $\mathbb{N}_{\omega k+2} = \{n \in \mathbb{N} : n \geq m_{\omega k+2}\}$. Now we are able to define a multifunction $\zeta_{\omega k+2} \colon X \multimap X$ as follows:

$$\zeta_{\omega k+2}(x) = \begin{cases} \zeta_{\omega k+1}(x) \cap \overline{B(\operatorname{sel}_{\zeta_{\omega k+1}}(x), \frac{1}{2^{\omega k+2}})} & \text{if } x \notin K_{n_{\omega k+2}}, \\ \{g_{\omega k+2}(x)\} & \text{if } x \in K_{n_{\omega k+2}} \setminus (\operatorname{Int}(K_{m_{\omega k+2}}) \cup L^{\omega k+2}), \\ \{g_{\omega k+2}(x)\} \cup \{z_{\omega k+2}\} & \text{if } x \in L^{\omega k+2}, \\ K_{m_{\omega k+2}} \cup \{t_i^{\omega k+2} : i \geq n\} & \text{if } x \in \operatorname{Int}(K_n) \setminus \operatorname{Int}(K_{n+1}) \text{ for } n \in \mathbb{N}_{\omega k+2}, \\ K_{m_{\omega k+2}} & \text{if } x = x_{\omega k+1}. \end{cases}$$

First, let us note that from (3.4) we have

$$\operatorname{diam}(\zeta_{\omega k+2}(x)) \leq \frac{1}{2^{\omega k+2}} \quad \text{for each } x \in X. \tag{3.6}$$

We will show that $\zeta_{\omega k+2}$ meets the requirements for this game. Note that by (3.3) and (3.4), we have that $\zeta_{\omega k+2}$ is inserted in $\zeta_{\omega k+1}$. Now, we shall show that $\zeta_{\omega k+2}$ is an s-continuous multifunction. For this purpose let us define a function $\operatorname{sel}_{\zeta_{\omega k+2}} \colon X \to X$ in the following way:

$$\operatorname{sel}_{\zeta_{\omega k+2}}(x) = \begin{cases} x & \text{if } x \in K_{m_{\omega k+2}}, \\ g_{\omega k+2}(x) & \text{if } x \in K_{n_{\omega k+2}} \setminus K_{m_{\omega k+2}}, \\ \operatorname{sel}_{\zeta_{\omega k+1}}(x) & \text{if } x \notin K_{n_{\omega k+2}}. \end{cases}$$

It follows immediately that $\mathrm{sel}_{\zeta_{\omega k+2}}$ is a continuous selection of $\zeta_{\omega k+2}$.

We will now show that

$$x_{\omega k+1} \text{ is an } l\text{-chaotic point of } \zeta_{\omega k+2}. \tag{3.7}$$

Let us first note that

$$x_{\omega k+1} \in \mathrm{Fix}_s(\zeta_{\omega k+2}). \tag{3.8}$$

In fact, according to (M1) we infer $x_{\omega k+1} \in \mathrm{Int}(K_{m_{\omega k+2}}) \subset \zeta_{\omega k+2}(x_{\omega k+1})$, which gives (3.8).

Now, we shall prove that there exists a neighbourhood V of $x_{\omega k+1}$ such that

$$x \xrightarrow[\zeta_{\omega k+2}]{e} \zeta_{\omega k+2}(x_{\omega k+1}) \quad \text{for } x \in V \setminus \{x_{\omega k+1}\}. \tag{3.9}$$

Put $V = \mathrm{Int}(K_{m_{\omega k+2}})$ and let $x \in V \setminus \{x_{\omega k+1}\}$. Then there exists $n_x \in \mathbb{N}_{\omega k+2}$ such that $x \in \mathrm{Int}(K_{n_x}) \setminus \mathrm{Int}(K_{n_x+1})$. Let us observe that $\zeta_{\omega k+2}(x_{\omega k+1}) = K_{m_{\omega k+2}} \subsetneqq K_{m_{\omega k+2}} \cup \{t_n : n \geq n_x\} = \zeta_{\omega k+2}(x)$. The proof of (3.9) is complete. Obviously, (3.8) and (3.9) prove (3.7).

In the next step of the proof, we will show that

$$\zeta_{\omega k+2} \text{ is an } m\text{-DC1 multifunction.} \tag{3.10}$$

Let $\varphi\colon [0,1] \to L^{\omega k+2}$ be a homeomorphism. Put $f = \varphi^{-1} \circ \mu_{\omega k+2} \circ \varphi\colon [0,1] \to [0,1]$. Note that $\varphi^{-1}(L_1^{\omega k+2})$, $\varphi^{-1}(L_2^{\omega k+2})$ are disjoint, closed sets, and moreover, $f(\varphi^{-1}(L_i^{\omega k+2})) = \varphi^{-1}(\mu_{\omega k+2}(L_i^{\omega k+2})) = \varphi^{-1}(L^{\omega k+2}) = [0,1]$, $i \in \{1,2\}$, and so $\varphi^{-1}(L_1^{\omega k+2})$, $\varphi^{-1}(L_2^{\omega k+2})$ form a horseshoe for f and consequently $h(f) > 0$. Taking into account Lemma 3.1, we conclude that dynamical system (f) is distributionally chaotic, and so it has uncountable DS set $M_{\omega k+2}$. By Lemma 3.2, $\varphi(M_{\omega k+2})$ is an uncountable DS set for the dynamical system $(\mu_{\omega k+2})$, and so for each $x, y \in \varphi(M_{\omega k+2})$ $(x \neq y)$ and $t > 0$, we have

$$\Phi^{*(\mu_{\omega k+2})}_{x,y}(t) = \limsup_{n\to\infty} \frac{1}{n} \#(\{j \in [\![0, n-1]\!] : \rho(\mu^j_{\omega k+2}(x), \mu^j_{\omega k+2}(y)) < t\}) = 1 \tag{3.11}$$

and for each $x, y \in \varphi(M_{\omega k+2})$ $(x \neq y)$ there exists $t_0 > 0$ such that

$$\Phi^{(\mu_{\omega k+2})}_{x,y}(t_0) = \liminf_{n\to\infty} \frac{1}{n} \#(\{j \in [\![0, n-1]\!] : \rho(\mu^j_{\omega k+2}(x), \mu^j_{\omega k+2}(y)) < t_0\}) = 0. \tag{3.12}$$

Of course, $\varphi(M_{\omega k+2}) \subset L^{\omega k+2}$. Moreover, we have $\zeta^j_{\omega k+2}(x) = \{\mu^j_{\omega k+2}(x)\} \cup \{z_{\omega k+2}\}$ for $x \in L^{\omega k+2}$ and $j \in \mathbb{N}$.

From (3.5) it may be concluded that

$$\varrho_H(\zeta^j_{\omega k+2}(x), \zeta^j_{\omega k+2}(y)) = \varrho(\mu^j_{\omega k+2}(x), \mu^j_{\omega k+2}(x)) \quad \text{for } x, y \in L^{\omega k+2} \text{ and } j \in \mathbb{N}_0.$$

Taking into account (3.11) and (3.12), we have $\Phi^{*(\zeta_{\omega k+2})}_{x,y}(t) = 1$ and $\Phi^{(\zeta_{\omega k+2})}_{x,y}(t_0) = 0$.

We also know that $\#(\zeta_{\omega k+2}(x)) > 1$ for each $x \in \varphi(M_{\omega k+2}) \subset L^{\omega k+2}$. Fact (3.10) has been thus proven. We have shown that $\zeta_{\omega k+2}$ meets all the conditions for multifunction in the l-MG game.

We will now prove that

$$\begin{aligned}&\text{if } \mathrm{sel}_{\zeta_{\omega k+2+j}} \text{ is a continuous selection } \zeta_{\omega k+2+j}, \text{ then}\\ &\mathrm{sel}_{\zeta_{\omega k+2+j}}(x) = g_{\omega k+2}(x) \text{ and } \zeta_{\omega k+2+j}(x) \subset \{g_{\omega\cdot k+2}(x)\} \cup \{z_{\omega k+2}\}\\ &\text{for } x \in L^{\omega k+2} \text{ and } \omega, j \in \mathbb{N}.\end{aligned} \tag{3.13}$$

Let us fix $\overline{x} \in L^{\omega k+2}$ and $\omega, j \in \mathbb{N}$. Moreover, let $\overline{\delta}$ be a positive number such that $z_{\omega k+2} \notin B(\overline{x}, \overline{\delta}) \subset \mathrm{Int}(K_{n_{\omega k+2}}) \setminus K_{m_{\omega k+2}}$. Since $\zeta_{\omega k+2+j}$ is inserted in $\zeta_{\omega k+2}$, we have

$$\mathrm{sel}_{\zeta_{\omega k+2+j}}(x) = g_{\omega k+2}(x) \quad \text{for } x \in K_{n_{\omega k+2}} \setminus (\mathrm{Int}(K_{m_{\omega k+2}}) \cup L^{\omega k+2}). \tag{3.14}$$

Let us assume such notations as in (3.5). Then $\varphi_{\omega k+2}(L^{\omega k+2})$ is an arc contained in $B(\alpha_{\omega k+2}, r_{\omega k+2})$ and $\varphi_{\omega k+2}(\overline{x}) \in \varphi_{\omega k+2}(L^{\omega k+2})$. So, taking into account that $\dim(X) \geq 2$, one can find a sequence $\{\alpha_n\} \subset B(\alpha_{\omega k+2}, r_{\omega k+2}) \setminus \varphi_{\omega k+2}(L^{\omega k+2})$ such that $\alpha_n \to \varphi_{\omega k+2}(\overline{x})$. Set $\overline{x}_{w,n} = \varphi^{-1}_{\omega k+2}(\alpha_n) \in U^{\omega k+2}$. In this way we have determined the sequence $\{\overline{x}_{w,n}\}_{n=1}^{\infty} \subset \mathrm{Int}(K_{n_{\omega k+2}}) \setminus (K_{m_{\omega k+2}} \cup L^{\omega k+2})$ such that $\lim_{n\to\infty} \overline{x}_{w,n} = \overline{x}$. Obviously, $\{\mathrm{sel}_{\zeta_{\omega k+2+j}}(x_{w,n})\}_{n=1}^{\infty}$ tends to $\mathrm{sel}_{\zeta_{\omega k+2+j}}(\overline{x})$. From (3.14), we conclude that

$$\mathrm{sel}_{\zeta_{\omega k+2+j}}(\overline{x}) = \lim_{n\to\infty}(\mathrm{sel}_{\zeta_{\omega k+2+j}}(\overline{x}_{w,n})) = \lim_{n\to\infty} g_{\omega k+2}(\overline{x}_{w,n}) = g_{\omega k+2}(\overline{x}).$$

Consequently (3.13) is proved.

Note further that since $\zeta_{\omega k+2}$ is inserted in any multifunctions ζ_j for $j < \omega k + 2$ ($\omega \in \mathbb{N}$), taking into account (3.13) we have

$$g_{\omega k+2}(x) \in \zeta_j(x) \quad \text{for } x \in L^{\omega k+2},\ j \in \mathbb{N}. \tag{3.15}$$

Putting $\tau^l_2(\zeta_0, \ldots, \zeta_{\omega\cdot k+1}) = \zeta_{\omega k+2}$ for $\omega \in \mathbb{N}_0$ (in the case $\pi_e = \pi_1$, we would consider $\omega \in \mathbb{N}$), we define the strategy for player π_e. Let us show that this is a winning strategy in the game l-MG.

The construction (in subsequent stages) of multifunctions connected with player π_e was based on points $x_1, x_{k+1}, \ldots, x_{\omega k+1}, \ldots$ being l-chaotic points of multifunctions preceding choices of player π_e. In this way one can consider a sequence $\{x_{\omega k+1}\}_{\omega=0}^{\infty}$. According to the compactness of X, there exists an accumulation point x_0 of this sequence. Without restriction of generality, one can assume that

$$\lim_{\omega\to\infty} x_{\omega k+1} = x_0. \tag{3.16}$$

Obviously, $x_0 \in \overline{\Lambda_l}$. As a result of the game, a cone was created. Let us define its vertex κ in the following way:

$$\kappa(x) = \bigcap_{n=1}^{\infty} \zeta_n(x) \quad \text{for } x \in X.$$

Now, we shall prove

$$x_0 \text{ is a strong } e\text{-chaotic point of } \kappa. \tag{3.17}$$

We should therefore show that $h(\kappa, x_0) = +\infty$. Fix $\beta > 0$ and let Y_0 be an arbitrary open neighbourhood of x_0. For this purpose, it suffices to find $L_* \in \mathsf{S}(\kappa, Y_0)$ such that $H_\kappa(L_*) \geq \beta$.

From (3.16) and the methods of defining multifunctions $\zeta_{\omega k+2}$ ($\omega \in \mathbb{N}_0$), one can deduce that there exists $\omega_0 \in \mathbb{N}$ such that $x_{\omega_0 k+1} \in K_{n_{\omega_0 \cdot k+1}} \subset Y_0$ and $\omega_0 k + 1 > \beta$. We see at once that there exists $L_* = (L_1^{\omega_0 \cdot k+2}, \ldots, L_{2^{\omega_0 \cdot k+2}}^{\omega_0 \cdot k+2})$ such that $L_i^{\omega_0 \cdot k+2} \subset L^{\omega_0 \cdot k+2} \subset Y_0$ for $i \in \{1, \ldots, 2^{\omega_0 \cdot k+2}\}$.

From (3.15), we conclude that

$$g_{\omega_0 \cdot k+2}(x) \in \zeta_i(x) \quad \text{for } x \in L^{\omega_0 \cdot k+2} \text{ and } i \in \mathbb{N}, \tag{3.18}$$

and so $\zeta_i(L_i^{\omega_0 \cdot k+2}) \supset L^{\omega_0 \cdot k+2}$ for $i \in \mathbb{N}$, and consequently $\kappa(L_i^{\omega_0 \cdot k+2}) \supset L^{\omega_0 \cdot k+2}$. Obviously, for each $i \neq j$ ($i, j \in \{1, \ldots, 2^{\omega_0 \cdot k+2}\}$), one can find $x_q \in L_i^{\omega_0 \cdot k+2}$, $y_q \in L_j^{\omega_0 \cdot k+2}$ such that $g_{\omega_0 \cdot k+2}(x_q) \neq g_{\omega_0 \cdot k+2}(y_q)$. From (3.13), we also know that $\zeta_i(x) \subset \{g_{\omega_0 \cdot k+2}(x)\} \cup \{z_{\omega_0 \cdot k+2}\}$ for $x \in L^{\omega_0 \cdot k+2}$ and $i \geq \omega_0 \cdot k + 2$. Consequently, $\kappa(x) \subset \{g_{\omega_0 \cdot k+2}(x)\} \cup \{z_{\omega_0 \cdot k+2}\}$ for $x \in L^{\omega_0 \cdot k+2}$. Note that by virtue of the above and (3.18), we have $\kappa(x_q) \neq \kappa(y_q)$. This gives $L_* = (L_1^{\omega_0 \cdot k+2}, \ldots, L_{2^{\omega_0 \cdot k+2}}^{\omega_0 \cdot k+2}) \in \mathsf{S}(\kappa, Y_0)$.

Let $t \in \mathbb{N}$. Then $\#_1(\mathsf{P}_t(L_*, \kappa)) = (2^{\omega_0 \cdot k+2})^t$, and consequently $H_\kappa(L_*) = \omega_0 \cdot k + 2 > \beta$. This finishes the proof of (3.17).

We will now prove that

$$x_0 \text{ is a DC1 point of } \kappa. \tag{3.19}$$

For these considerations it will be convenient to state beforehand

$$\{g_{\omega \cdot k+2}(x)\} = \kappa(x) \quad \text{for } x \in L^{\omega \cdot k+2} \text{ and } \omega \in \mathbb{N}. \tag{3.20}$$

Fix $\omega \in \mathbb{N}$ and $\widehat{x} \in L^{\omega \cdot k+2}$. (3.13) and (3.15) allow to note that

$$g_{\omega \cdot k+2}(\widehat{x}) \in \kappa(\widehat{x}) \subset \{g_{\omega \cdot k+2}(\widehat{x})\} \cup \{z_{\omega \cdot k+2}\}. \tag{3.21}$$

So, let $\widehat{\omega} > \omega$ be a positive integer such that $\frac{1}{2^{\widehat{\omega} \cdot k+2}} < \varrho(z_{\omega k+2}, L^{\omega k+2})$ and $K_{n_{\widehat{\omega} \cdot k+2}} \subset K_{m_{\omega k+2}}$. Then, in view of the construction of multifunctions in building the strategy of player π_e, we have $\zeta_{\widehat{\omega}\cdot k+2}(\widehat{x}) = \zeta_{\widehat{\omega}\cdot k+1}(\widehat{x}) \cap \overline{B(\mathrm{sel}_{\zeta_{\widehat{\omega}\cdot k+2}}, \frac{1}{2^{\widehat{\omega}\cdot k+2}})}$. Taking into account (3.13), (3.21) and $\frac{1}{2^{\widehat{\omega}\cdot k+2}} < \varrho(z_{\omega k+2}, L^{\omega k+2})$, we obtain $\zeta_{\widehat{\omega}k+2}(\widehat{x}) = \{g_{\omega\cdot k+2}(\widehat{x})\}$ and thus the proof of (3.20) is complete.

Let $\varepsilon > 0$. Taking into account (3.16) and (M1), (M3), there exists $\omega_1 \in \mathbb{N}$ such that $x_{\omega_1\cdot k+1} \in K_{n_{\omega_1\cdot k+2}} \subset B(x_0, \varepsilon)$. This means that $L^{\omega_1\cdot k+2} \subset B(x_0, \varepsilon)$.

Similar to the proof of (3.10), one can consider homeomorphism $\widehat{\varphi}\colon [0,1] \to L^{\omega_1\cdot k+2}$ and a set $M_{\omega_1\cdot k+2}$ such that $\widehat{\varphi}(M_{\omega_1\cdot k+2})$ is an uncountable DS set for the dynamical system $(\mu_{\omega_1\cdot k+2})$. This allows us to write equations analogous to (3.11) and (3.12) for each $x, y \in \widehat{\varphi}(M_{\omega_1 k+2})$ $(x \neq y)$ and $t > 0$, we have

$$\Phi^{*(\mu_{\omega_1 k+2})}_{x,y}(t) = \limsup_{n\to\infty} \frac{1}{n} \#(\{j \in [\![0, n-1]\!] : \rho(\mu^j_{\omega_1 k+2}(x), \mu^j_{\omega_1 k+2}(y)) < t\}) = 1 \tag{3.22}$$

and for each $x, y \in \widehat{\varphi}(M_{\omega_1 k+2})$ $(x \neq y)$, there exists $t_0 > 0$ such that

$$\Phi^{(\mu_{\omega_1 k+2})}_{x,y}(t_0) = \liminf_{n\to\infty} \frac{1}{n} \#(\{j \in [\![0, n-1]\!] : \rho(\mu^j_{\omega_1 k+2}(x), \mu^j_{\omega_1 k+2}(y)) < t_0\}) = 0. \tag{3.23}$$

By (3.20), we have

$$\kappa(z) = \{\mu_{\omega_1 k+2}(z)\} \quad \text{for each } z \in L^{\omega_1\cdot k+2}, \tag{3.24}$$

and consequently

$$\varrho_H(\zeta_{\omega_1 k+2}(x), \zeta_{\omega_1 k+2}(y)) = \varrho(\mu_{\omega_1 k+2}(x), \mu_{\omega_1 k+2}(x)) \quad \text{for } x, y \in L^{\omega_1 k+2}.$$

Now note that

$$\kappa^j(x) = \{\mu^j_{\omega_1 k+2}(x)\} \quad \text{for } x \in L^{\omega_1\cdot k+2}. \tag{3.25}$$

Of course, $\varphi(M_{\omega_1 k+2}) \subset L^{\omega k+2}$. From (3.22), (3.23), (3.24) and (3.25), we obtain $\Phi^{*(\zeta_{\omega_1 k+2})}_{x,y}(t) = 1$ and $\Phi^{(\zeta_{\omega_1 k+2})}_{x,y}(t_0) = 0$.

The proof of (3.19) is completed by showing that $K_{n_{\omega_1\cdot k+2}}$ is an $(1,\varepsilon)$-envelope of $\widehat{\varphi}(M_{\omega_1 k+2})$. By the rules of building a strategy for player π_e, we have $\zeta_{\omega_1\cdot k+2}(K_{n_{\omega_1\cdot k+2}}) \subset K_{n_{\omega_1\cdot k+2}}$. Since κ is inserted in $\zeta_{\omega_1\cdot k+2}$, so also $\kappa(K_{n_{\omega_1\cdot k+2}}) \subset K_{n_{\omega_1\cdot k+2}}$ and consequently $\kappa^i(K_{n_{\omega_1\cdot k+2}}) \subset K_{n_{\omega_1\cdot k+2}}$ for $i \in \mathbb{N}$.

We have completed the proof of part (l-MG1) of our theorem.

Now let us consider the part (l-MG2) of the theorem. For the proof, it is sufficient to note that the use of the strategy for π_e by any of the other players will create a situation where the sentence $s(\pi_l)$ with respect to the created cone will not be true. Suppose, contrary to our claim, that there exists a vertex $\kappa^\star$ of the cone that was created as a result of the game such that $\kappa^\star$ has l-chaotic point $x_\star$. Then $x_\star \in \mathrm{Int}(\kappa^\star(x_\star))$. Let $y_\star \in \kappa^\star(x_\star)$ be such a point that $x_\star \neq y_\star$. Denote $\varepsilon_\star = \varrho(x_\star, y_\star) > 0$. On the other hand, according to the strategy for π_e one can conclude that there exists $\omega_* \in \mathbb{N}$ such that $\frac{1}{2^{\omega_* k+2}} < \varepsilon_\star$. Then, by (3.6), $\operatorname{diam}(\zeta_{\omega_* k+2}(x_*)) < \varepsilon_\star$, and so $\operatorname{diam}(\kappa_*(x_*)) < \varepsilon_\star$, which contradicts the fact that $\varrho(x_*, y_*) = \varepsilon_\star$. □

Now, let us go to the game u-MG and player π_u to whom the following statement is assigned.

> $s(\pi_u)$ - as a result of the game there will be created a cone such that there exist a point $x_0 \in \overline{\Lambda_u}$ and a vertex κ such that x_0 is DC1 point and a strong e-chaotic point of κ.

Theorem 3.6. *Let us assume that* $\dim(X) \geq 2$. *Then player* π_u *has a winning strategy in the game u-MG if and only if* π_u *chooses first.*

Proof. Necessity. Suppose, contrary to our claim, that π_u does not choose first. Let us fix an element $x_1 \in X$ and the base $\mathbb{B}(x_1) = \{K_n\}_{n=1}^{\infty}$ fulfilling the conditions (M1)–(M4). Consider $z_1 \in \operatorname{Int}(K_1) \setminus K_2 \neq \emptyset$ and an arc $L_1 = L(x_1, z_1) \subset K_1$. Let L^1 be a subarc of the arc L_1 such that

$$\begin{aligned}&\operatorname{diam}(L^1) < \varrho(z_1, L^1);\\ &\text{there exist an open set } U^1 \text{ such that } L^1 \subset U^1 \subset \operatorname{Int}(K_1) \setminus K_2\\ &\text{and a homeomorhism } \varphi\colon U^1 \xrightarrow{\text{onto}} B(\alpha_1, r_1) \subset \mathbb{H}^{\dim(X)}.\end{aligned}$$

As in the proof of item (l-MG1) of Theorem 3.5, one can distinguish two subarcs of the arc L^1:

$$L_1^1, L_2^1 \subset L^1 \text{ such that } L_1^1 \cap L_2^1 = \emptyset.$$

Continuing the analogy with the proof of Theorem 3.5, we observe that $z_1 \notin L_1^1 \cup L_2^1$ and there exist homeomorphisms $h_i\colon L_i^1 \xrightarrow{\text{onto}} L^1$ ($i \in \{1, 2\}$). Let $\mu_1\colon L^1 \to L^1$ be a continuous function such that $\mu_1 \restriction L_i = h_i$, $i \in \{1, 2\}$. Next one can define continuous function $g'\colon K_2 \cup L^1 \cup \operatorname{Fr}(K_1) \cup \{z_1\} \to K_1$ as follows:

$$g'(x) = \begin{cases} x & \text{if } x \in K_2 \cup \operatorname{Fr}(K_1) \cup \{z_1\},\\ \mu_1(x) & \text{if } x \in L^1.\end{cases}$$

Obviously, there exists a continuous function $g\colon K_1 \to K_1$ such that $g \restriction K_2 \cup L^1 \cup \operatorname{Fr}(K_1) \cup \{z_1\} = g'$. Finally, we define a multifunction $\zeta_1\colon X \multimap X$ in the following way:

$$\zeta_1(x) = \begin{cases} \{g(x)\} & \text{if } x \in K_1 \setminus (L^1 \cup \{x_1\}),\\ \{g(x)\} \cup \{z_1\} & \text{if } x \in L^1,\\ \{x\} & \text{if } x \notin K_1,\\ K_1 & \text{if } x = x_1.\end{cases}$$

The simple considerations lead us to the conclusion that ζ_1 meets all the requirements connected with multifunctions in this game.

Obviously x_1 is the only one u-chaotic point of ζ_1. Consequently, all multifunctions in this game will only have one u-chaotic point and it is x_1.

As the result of this game, there will be created a cone such that $\overline{\Lambda_u} = \{x_1\}$. Moreover, it is easily seen that $\zeta_i(x) = \{x\}$ for each $x \in K_2 \setminus \{x_1\}$ and $i \in \mathbb{N}$. Thus, if κ is an arbitrary vertex of created cone, then $\kappa(x) = \{x\}$ for any $x \in \mathrm{Int}(K_2) \setminus \{x_1\}$, and consequently x_1 is neither a strong e-chaotic point of κ nor a DC1 point of κ.

Sufficiency. Let us assume that $\pi_u = \pi_1$. To show that π_u has a winning strategy we define a multifunction $\zeta_1 = \tau_1^u((\zeta_0))$ and we will describe further choices of the player π_e in such a way, that regardless of the choices of other players we obtain a cone such that $s(\pi_u)$ is true.

Let us fix $x_1 \in X$ and let $\mathbb{B}(x_1) = \{K_n\}_{n=1}^{\infty}$ fulfill conditions (M1)–(M4). Next we choose $z_n \in \mathrm{Int}(K_n) \setminus K_{n+1}$ for each $n \in \mathbb{N}$ and consider an arc $L_n = L(x_1, z_n) \subset K_n$, $n \in \mathbb{N}$. Let us further distinguish subarcs L^n of L_n such that

$$
\begin{aligned}
&\mathrm{diam}(L^n) < \varrho(z_n, L^n);\\
&\text{there exist an open set } U^n \text{ such that } L^n \subset U^n \subset \mathrm{Int}(K_n) \setminus K_{n+1}\\
&\text{and a homeomorhism } \varphi_n \colon U^n \xrightarrow{\text{onto}} B(\alpha_n, r_n) \subset \mathbb{H}^{\dim(X)}.
\end{aligned}
$$

Continuing this procedure, one can distinguish 2^n subarcs $L_i^n \subset L^n$ ($i \in \{1, 2, \ldots, 2^n\}$) in such a way that

$$L_{i_1}^n \cap L_{i_2}^n = \emptyset \quad \text{for } i_1 \neq i_2,\ i_1, i_2 \in \{1, 2, \ldots, 2^n\}.$$

Obviously, $z_n \notin \bigcup_{i=1}^{2^n} L_i^n$ for $n \in \mathbb{N}$. Now again as in the proof of Theorem 3.5, let $h_i^n \colon L_i^n \xrightarrow{\text{onto}} L^n$ be a homeomorphism ($i \in \{1, 2, \ldots, 2^n\}$, $n \in \mathbb{N}$) and let $\mu_n \colon L^n \to L^n$ be a continuous function such that $\mu_n \restriction L_i^n = h_i^n$. In a similar way as in the proof of Theorem 3.5, one can introduce a continuous function $g_n \colon K_n \to K_n$ such that for each $n \in \mathbb{N}$,

$$
g_n \restriction K_{n+1} \cup L^n \cup \mathrm{Fr}(K_n) \cup \{z_n\}(x) = \begin{cases} x & \text{if } x \in K_{n+1} \cup \mathrm{Fr}(K_n) \cup \{z_n\},\\ \mu_n(x) & \text{if } x \in L^n. \end{cases}
$$

Finally, we define a multifunction $\zeta_1 \colon X \multimap X$ in the following way:

$$
\zeta_1(x) = \begin{cases} \{g_n(x)\} & \text{if } x \in K_n \setminus (K_{n+1} \cup L^n),\ n \in \mathbb{N},\\ \{g_n(x)\} \cup \{z_n\} & \text{if } x \in L^n,\ n \in \mathbb{N},\\ \{x\} & \text{if } x \notin K_1,\\ K_1 & \text{if } x = x_1. \end{cases}
$$

We will show that ζ_1 meets the conditions for multifunction in the game u-MG.

It is clear that ζ_1 is inserted in ζ_0. Moreover, we see at once that

$$\mathrm{sel}_{\zeta_1}(x) = \begin{cases} x & \text{if } x \in \{x_1\} \cup (X \setminus K_1), \\ g_n(x) & \text{if } x \in K_n \setminus K_{n+1},\ n \in \mathbb{N} \end{cases}$$

is a continuous selection of ζ_1 and x_1 is the only u-chaotic point of ζ_1.

As in the case of reasoning carried out in the proof of (3.10), we may prove (ζ_1) is an m-DC1 multifunction.

Now players will choose in order: $\pi_2, \ldots, \pi_k, \pi_u = \pi_{k+1}, \pi_{k+2}, \ldots$. Let us denote the selected multifunctions $\zeta_2, \ldots, \zeta_k, \zeta_{k+1}, \zeta_{k+2}, \ldots$. Of course, they meet the conditions of the game u-MG. Note that in this case

$$\zeta_i(x) = \begin{cases} \{x\} & \text{if } x \notin K_1, \\ \{g_n(x)\} & \text{if } x \in K_n \setminus (K_{n+1} \cup L^n) \text{ and } i, n \in \mathbb{N}. \end{cases} \tag{3.26}$$

On account of (3.26), in a similar way to (3.13) one can show that

(3.27) if sel_{ζ_i} is a continuous selection ζ_i, then $\mathrm{sel}_{\zeta_i}(x) = g_n(x)$ for $x \in L^n$ and $i \in \mathbb{N}$.

So let us assume that in the step $k+1$, player π_u chose the multifunction $\zeta_{k+1} \colon X \multimap X$ defined by the formula

$$\zeta_{k+1}(x) = \begin{cases} \zeta_k(x) & \text{if } x \notin L^1, \\ \{g_1(x)\} & \text{if } x \in L^1. \end{cases}$$

It follows easily that ζ_{k+1} meets the requirements for this game.

In general, for $\omega \in \mathbb{N}$, we have $\tau_1^u((\zeta_0, \zeta_1, \ldots, \zeta_{\omega k})) = \zeta_{\omega k+1}$, where

$$\zeta_{\omega k+1}(x) = \begin{cases} \zeta_{\omega k}(x) & \text{if } x \notin L^\omega, \\ \{g_\omega(x)\} & \text{if } x \in L^\omega. \end{cases}$$

We shall show that τ_1^u is a winning strategy in game for the player π_e in the game u-MG.

As a result of the game some cone was created. Let us define its vertex κ in the following way:

$$\kappa(x) = \bigcap_{n=1}^{\infty} \zeta_n(x).$$

First note that the form of ζ_1 and the rules of the game u-MG cause that x_1 is the only one u-chaotic point of ζ_n for $n \in \mathbb{N}$. Thus $x_1 \in \overline{\Lambda_u}$. Now, we will prove that

$$x_1 \text{ is a strong } e\text{-chaotic point of } \kappa. \tag{3.28}$$

We should therefore show that $h(\kappa, x_1) = +\infty$. Fix $\sigma > 0$ and let Z be an arbitrary neighbourhood of x_1. Let us choose $n_1 \in \mathbb{N}$ such that $K_{n_1} \subset Z$ and $n_1 > \sigma$. From (3.27), one can infer that $g_{n_1}(x) \in \kappa(x)$ for $x \in L^{n_1}$, and hence $\kappa(L_i^{n_1}) \supset L^{n_1}$ for $i \in \{1, 2, \ldots, 2^{n_1}\}$. Obviously, for each $i_1 \neq i_2$, $i_1, i_2 \in \{1, 2, \ldots, 2^{n_1}\}$, one can find $y_1 \in L_{i_1}^{n_1}$, $y_2 \in L_{i_2}^{n_1}$ such that $g_{n_1}(y_1) \neq g_{n_1}(y_2)$ and $g_{n_1}(y_1) \in \zeta_i(y_1) \subset \{g_{n_1}(y_1)\} \cup \{z_{n_1}\}$ and $g_{n_1}(y_2) \in \zeta_i(y_2) \subset \{g_{n_1}(y_2)\} \cup \{z_{n_1}\}$ for $i \in \mathbb{N}$, which means that $\kappa(y_1) \neq \kappa(y_2)$. So, we have $L_u = (L_1^{n_1}, \ldots, L_{2^{n_1}}^{n_1}) \in \mathsf{S}(\kappa, Z)$.

Let $d \in \mathbb{N}$. Then $\#_1(P_d(L_u, \kappa)) = (2^{n_1})^d$ and consequently $H_\kappa(L_u) = 2^{n_1} > \sigma$. From the above inequality, the claim (3.28) easily follows.

It will thus be sufficient to prove that

$$x_1 \text{ is a DC1 point of } \kappa.$$

For this, let us first note that the construction of multifunction $\zeta_{\omega k+1} = \tau_1^u((\zeta_0, \ldots, \zeta_{\omega k}))$ shows that

$$\kappa(x) = \{g_\omega(x)\} = \{g'_\omega(x)\} = \{\mu_\omega(x)\} \quad \text{for } x \in L^\omega \text{ and } \omega \in \mathbb{N}.$$

Similar to (3.19), excluding (3.20) from consideration, it can be shown that there is DS-set $M_n \subset L^n$ ($n \in \mathbb{N}$) for dynamical system (κ) and hence L^n is a suitable envelope of M_n. □

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