

On the Syzygies of Ample Line Bundles on Fake Projective Planes

Sui-Chung Ng* and Sai-Kee Yeung

Abstract. Our goal is to study the syzygies of the projective embeddings defined by the ample line bundles on a fake projective plane S . The syzygies are studied in terms of the property N_p . For various kinds of ample line bundles, we give explicit lower bounds for their powers above which the property N_p is satisfied.

1. Introduction

Studying the very-ampleness, projective normality and syzygies of the ample line bundles on a complex surface is natural and has a long history. The main purpose of this paper is to give some detailed estimates for fake projective planes.

There are two natural questions for a projective algebraic manifold M of complex dimension n equipped with an ample line bundle L . The first is to study the smallest multiple of L that gives a projective embedding of the manifold, and the second is to study the syzygy relative to such an embedding. For the first question, the Fujita Conjecture states that $K_M \otimes L^k$ is very ample if $k \geq n + 2$. For the second question, there is the Mukai Conjecture which states that $K_M \otimes L^{n+2+p}$ satisfies the N_p property. For surfaces, Fujita Conjecture is proved by Reider [16]. The Mukai Conjecture is mostly open.

Fake projective planes are interesting surfaces to study among smooth surfaces of general type since they have the smallest Euler number (viz., 3) among all such surfaces and have been classified in [3, 13, 14]. However, there is still no good way to describe such surfaces directly from classical algebraic geometry in general. For a fake projective plane S , K_S^3 is very ample from a well-known result of Bombieri and also Reider [16]. We also know that K_S has no sections. The natural question of whether K_S^2 is very ample or not is not settled, even though there are some examples which are true. Hence one would expect that sharp results for the syzygy of S may be difficult to achieve. The interest

Received December 25, 2020; Accepted September 5, 2021.

Communicated by Jungkai Alfred Chen.

2020 *Mathematics Subject Classification*. 13D02, 14C20, 14J29.

Key words and phrases. fake projective planes, pluricanonical mappings, syzygies.

The first author was partially supported by National Natural Science Foundation of China (Grant No. 11501205) and Science and Technology Commission of Shanghai Municipality (Grant No. 13dz2260400).

The second author was partially supported by grants from the National Science Foundation, USA.

*Corresponding author.

about syzygy is that it would provide information about the pluricanonical ring of S . As a consequence of the current work, it turns out that we have pretty good information about the syzygy, cf. Corollary 1.2, even though it is still short of Mukai Conjecture. The results of this paper are close to the optimal ones with the currently available techniques. The difficulty to achieve a sharper result is related to the difficulty regarding the existence of exceptional collection of objects, cf. Conjecture 5.1. Sharper estimates are obtained for some special fake projective planes as explained in Theorem 5.3 and Remark 5.4 following it in §5.

The syzygy of a given projective embedding can be described using the so-called property N_p , where p is a non-negative integer. For the details of its definition, we refer the reader to [4]. Here, we simply recall that the property N_0 corresponds to projective normality and the property N_1 corresponds to projective normality together with the condition that the homogeneous ideal of the embedded subvariety is generated by quadratic polynomials. The present paper gives effective results for the lower bounds of the powers of a given ample line bundle satisfying the property N_p on a fake projective plane. The reader may refer to [18, 19] for basic geometric facts about fake projective planes.

For the general study of syzygies related to surfaces of general type, the reader can see, for example, the works of Banagere–Hanumanthu [1], Gallego–Purnaprajna [6], Hwang–To [8], Purnaprajna [15] and the very recent work of Niu [12]. Here we remark that unlike the previous results found in the literature for surfaces of general type, which are pertaining to base-point-free line bundles and are expressed in terms of some sort of “regularity” related the relevant line bundles, we established in this paper explicit and concrete numbers for a given arbitrary ample line bundle, which is not necessarily base-point-free, and in particular, including the canonical bundle (which has no non-trivial section).

Our first main result is regarding an arbitrary ample line bundle on a fake projective plane.

Theorem 1.1. *Let A and N be line bundles on a fake projective plane S such that A is ample and N is nef. Then, $A^{6(p+3)} \otimes N$ satisfies the property N_p , where $p \geq 0$.*

It follows easily from the definition that any fake projective plane S is of Picard number 1. Moreover, it is known (see for example [19]) that the canonical line bundle can be written as $K_S = G^3 \otimes \tau$, where G is an ample generator G of $\text{Pic}(S)$ and τ is some torsion line bundle. Thus, we have the following immediate corollary of Theorem 1.1.

Corollary 1.2. *Let S be a fake projective plane. Then K_S^m satisfies the property N_p if $m \geq 2p + 6$, where $p \geq 0$. In particular, K_S^m is projectively normal if $m \geq 6$.*

Remark 1.3. We have also obtained results for smaller values of m , but those require various additional efforts. In fact, we will see that K_S^{2p+2} (resp. K_S^{2p+4}) also satisfies the property N_p for $p \geq 1$ (resp. $p \geq 0$), as a consequence of Theorem 3.6. In particular, K_S^4 is also projectively normal. Furthermore, motivated by the existence problem for exceptional collections for objects in the derived category of coherent sheaves on a fake projective plane S , if we assume that for any ample generator G of $\text{Pic}(S)$ and any torsion line bundle δ , we have $h^0(S, G^2 \otimes \delta) = 0$, a vanishing statement which is a variant of a conjecture mentioned in [5], then we will be able to show that that K_S^{2p+5} also satisfies the property N_p for $p \geq 0$ (Section 5). Such a conjecture is known to be true for fake projective planes with more than three automorphisms. As an intermediate step of the proof, we also obtained some vanishing or almost vanishing results concerning the torsion line bundles of a fake projective plane, which are of some independent interests.

Proposition 1.4. *Let δ be a torsion line bundle on a fake projective plane S and G an ample generator of $\text{Pic}(S)$. Then*

- (a) $h^1(S, \delta) = 0$;
- (b) $h^0(S, G \otimes \delta) \leq 1$, $h^0(S, G^2 \otimes \delta) \leq 2$ and $h^0(S, G^3 \otimes \delta) \leq 1$.

Finally, we mention the following theorem for an arbitrary base-point-free line bundle on a fake projective plane, which will be established in Section 4.

Theorem 1.5. *Let S be a fake projective plane and B be a base-point-free line bundle on S . Then, B^m is projectively normal if $m \geq 2$. Moreover, for $p \geq 1$, B^m satisfies the property N_p if $m \geq p + 1$.*

This follows from a vanishing result on the unramified cover of each fake projective plane defined by the commutator subgroup of its fundamental group.

The results in this paper are established through the well known cohomology criterion given by Green [7]. The details of Green’s criterion will be recalled in Section 2. To show that the cohomology groups appearing in Green’s theorem vanish, our main tool is the classical result of Castelnuovo–Mumford on the surjectivity of certain tensor product maps on various spaces of global sections, which has been used already in the literature, in particular in the work of Gallego–Purnaprajna [6] for surfaces under various assumptions and for the property N_p , $p \geq 1$. Our goal is to provide as sharp an estimate as possible for fake projective planes, especially for lower values of p . Careful analysis is needed to make sure that there is no numerical gap in the range of possible powers of the line bundle satisfying the theorems of this paper, (see Remark 3.4). We have also made an effort to make the the presentation reasonably self-contained.

2. Cohomological criterion for the property N_p

We will first recall in this section the cohomological criterion for the property N_p obtained by Green [7].

Let X be a complex projective manifold and L be an ample and base-point-free line bundle on X . Denote by $\mathcal{H}^0(X, L)$ the trivial vector bundle on X whose fibers are $H^0(X, L)$. Then we have the following exact sequence of vector bundles

$$0 \rightarrow M_L \rightarrow \mathcal{H}^0(X, L) \rightarrow L \rightarrow 0,$$

where the fiber of M_L at $x \in X$ is the subspace in $H^0(X, L)$ consisting of the sections vanishing at x . Using the theory of Koszul cohomology, Green [7] obtained the following

Theorem 2.1 (Green). *The line bundle L satisfies the property N_p if*

$$H^1(X, \wedge^r M_L \otimes L^s) = 0$$

for $0 \leq r \leq p + 1$ and $s \geq 1$.

Over a field of characteristic zero, $\wedge^r M_L \subset M_L^{\otimes r}$ is a direct summand and therefore it is a common practice to simply verify the vanishing of $H^1(X, M_L^{\otimes r} \otimes L^s)$ for studying the property N_p . The main tool for us to achieve this is the following theorem, pertaining to the so-called *Castelnuovo–Mumford regularity* (cf. [11]).

Theorem 2.2 (Castelnuovo–Mumford). *Let E be a base-point-free line bundle on a projective variety M and \mathcal{F} be a coherent sheaf on M . Suppose that $H^i(M, \mathcal{F} \otimes E^{-i}) = 0$ for $i \geq 1$. Then the multiplication mapping*

$$H^0(M, \mathcal{F} \otimes E^j) \otimes H^0(M, E) \rightarrow H^0(M, \mathcal{F} \otimes E^{j+1})$$

is surjective for all $j \geq 0$.

3. Vanishing theorems and proofs of the main theorems

In what follows, we let S be a fake projective plane. We recall some known facts about fake projective planes, which can be found in the survey articles by Rémy [18] and Yeung [19]. First of all, the Picard number of S is 1 and we will fix an ample generator G of $\text{Pic}(S)$. We have then $G \cdot G = 1$. Moreover, if K_S denotes the canonical line bundle, then $K_S = G^3 \otimes \tau$, for some torsion line bundle τ .

Lemma 3.1. *Let A be an ample line bundle and σ be a torsion line bundle on S . Then, $H^1(S, A^k \otimes \sigma) = H^2(S, A^k \otimes \sigma) = 0$ for $k \geq 4$.*

Proof. Since A is ample and the Picard number of S is 1, we can write $A = G^m \otimes \epsilon$ for some $m \in \mathbb{N}^+$ and some torsion line bundle ϵ . Then, for $i = 1, 2$ and $k \geq 4$,

$$H^i(S, A^k \otimes \sigma) = H^i(S, K_S \otimes G^{mk-3} \otimes \tau^{-1} \otimes \epsilon^k \otimes \sigma) = 0$$

by Kodaira Vanishing Theorem as G^{mk-3} is ample. \square

Proposition 3.2. *Let B be an ample line bundle on S such that $B \otimes K_S^{-2}$ is nef, or equivalently, $B = G^m \otimes \sigma$ for some $m \geq 6$ and some torsion line bundle σ . Then B^k is base-point-free for $k \geq 1$ and $H^1(S, B^k) = H^2(S, B^k) = 0$ for $k \geq 0$.*

Proof. By hypotheses, we can write $B^k = K_S \otimes L$, where $L = G^{m'} \otimes \sigma'$ for some $m' \geq 3$ and some torsion line bundle σ' . Since $L \cdot L = (m')^2 \geq 9$, using the classical results of Reider [16], we see that B^k is base-point-free. For $k \geq 1$, the vanishing of $H^1(S, B^k)$ and $H^2(S, B^k)$ follows directly from Lemma 3.1, while for $k = 0$, it follows from the fact that $H^1(S, \mathcal{O}_S) = 0$ and $H^2(S, \mathcal{O}_S) \cong H^0(S, K_S) = 0$ for any fake projective plane. \square

In the rest of this section, we will establish two vanishing theorems for the cohomology groups appearing in Green's criterion for the property N_p . As mentioned before, our main tool is the theorem of Castelnuovo–Mumford (Theorem 2.2). The basic ideas behind our methods are taken from the work of Gallego–Purnaprajna [6]. For base-point-free line bundles on S , we streamline the arguments in [6] and adapt to the present context for fake projective planes. Geometric properties of fake projective planes are utilized to make sure that conditions for the Castelnuovo–Mumford theorem are satisfied.

From Proposition 3.2, if B is an ample line bundle such that $B \otimes K_S^{-2}$ is nef, then B^k is base-point-free for $k \geq 1$. Thus, we have following exact sequence of vector bundles

$$0 \rightarrow M_{B^k} \rightarrow \mathcal{H}^0(S, B^k) \rightarrow B^k \rightarrow 0.$$

Tensoring with $M_{B^k}^{\otimes(r-1)} \otimes B^\ell$ for $r \geq 1$ and $\ell \geq 0$, we have

$$(3.1) \quad 0 \rightarrow M_{B^k}^{\otimes r} \otimes B^\ell \rightarrow \mathcal{H}^0(S, B^k) \otimes M_{B^k}^{\otimes(r-1)} \otimes B^\ell \rightarrow M_{B^k}^{\otimes(r-1)} \otimes B^{k+\ell} \rightarrow 0$$

for $k, r \geq 1$ and $\ell \geq 0$.

Theorem 3.3. *Let B be an ample line bundle on a fake projective plane S such that $B \otimes K_S^{-2}$ is nef. Let $k \geq 1$ and $\ell \geq 0$. Then,*

- (i) $H^1(S, M_{B^k} \otimes B^\ell) = 0$ if $k + \ell \geq 3$;
- (ii) $H^1(S, M_{B^k}^{\otimes r} \otimes B^\ell) = 0$ if $k + \ell \geq r + 2$ and $\ell \geq r \geq 2$;
- (iii) $H^2(S, M_{B^k}^{\otimes r} \otimes B^\ell) = 0$ if $\ell + 1 \geq r \geq 1$.

Proof. We first show that $H^1(S, M_{B^k} \otimes B^\ell) = 0$ if $k + \ell \geq 3$. In order to simplify the notations, in the proof we will skip the reference to S when writing the cohomology groups.

From the long exact sequence associated to (3.1), we get the exact sequence

$$\cdots \rightarrow H^0(\mathcal{H}^0(S, B^k) \otimes B^\ell) \rightarrow H^0(B^{k+\ell}) \rightarrow H^1(M_{B^k} \otimes B^\ell) \rightarrow H^1(\mathcal{H}^0(S, B^k) \otimes B^\ell) \rightarrow \cdots$$

As $H^1(\mathcal{H}^0(S, B^k) \otimes B^\ell) \cong H^0(B^k) \otimes H^1(B^\ell)$, it vanishes by Proposition 3.2. Thus, $H^1(M_{B^k} \otimes B^\ell)$ vanishes if the map $H^0(\mathcal{H}^0(S, B^k) \otimes B^\ell) \rightarrow H^0(B^{k+\ell})$ is surjective. Equivalently, we need the surjectivity for the map

$$(3.2) \quad H^0(B^k) \otimes H^0(B^\ell) \rightarrow H^0(B^{k+\ell})$$

since $H^0(\mathcal{H}^0(S, B^k) \otimes B^\ell) \cong H^0(B^k) \otimes H^0(B^\ell)$. Consider the natural maps

$$H^0(B) \otimes \cdots \otimes H^0(B) \otimes H^0(B^\ell) \xrightarrow{p} H^0(B^k) \otimes H^0(B^\ell) \xrightarrow{q} H^0(B^{k+\ell}).$$

We see that it suffices to have the surjectivity for $q \circ p$, which is in turn implied by the surjectivity of all the mappings

$$H^0(B) \otimes H^0(B^{m-1+\ell}) \rightarrow H^0(B^{m+\ell})$$

for $1 \leq m \leq k$. We now apply the result of Castelnuovo–Mumford (Theorem 2.2). Thus, the above maps are surjective if

$$H^1(B^{\ell-1}) = H^2(B^{\ell-2}) = 0.$$

By Proposition 3.2, these are satisfied if $\ell \geq 2$. (Here we remark that all the higher cohomology groups appeared in Castelnuovo–Mumford Theorem vanish because $\dim(S) = 2$.) Thus, we see that for $k \geq 1$, $H^1(S, M_{B^k} \otimes B^\ell) = 0$ if $\ell \geq 2$. Since the roles of k and ℓ are symmetric for the mapping in (3.2), it follows that for $k \geq 2$, we also have $H^1(S, M_{B^k} \otimes B^\ell) = 0$ for $\ell = 1$. We hence deduce that $H^1(S, M_{B^k} \otimes B^\ell) = 0$ if $k + \ell \geq 3$. Note that we can also allow $\ell = 0$ here as the map in (3.2) is surjective for $\ell = 0$. We have thus proven (i).

To prove (ii) and (iii), we will need to first establish $H^2(M_{B^k} \otimes B^\ell) = 0$ for $\ell \geq 0$ and $H^2(M_{B^k}^{\otimes 2} \otimes B^\ell) = 0$ for $\ell \geq 1$. We proceed further along the long exact sequence associated to (3.1),

$$\cdots \rightarrow H^1(M_{B^k}^{\otimes(r-1)} \otimes B^{k+\ell}) \rightarrow H^2(M_{B^k}^{\otimes r} \otimes B^\ell) \rightarrow H^2(\mathcal{H}^0(S, B^k) \otimes M_{B^k}^{\otimes(r-1)} \otimes B^\ell) \rightarrow \cdots$$

Here in the sequence, for $r = 1$, we see that the first term $H^1(B^{k+\ell})$ is zero by Proposition 3.2 and for the last term, we have $H^2(\mathcal{H}^0(S, B^k) \otimes B^\ell) \cong H^0(B^k) \otimes H^2(B^\ell) =$

0. It now follows that $H^2(M_{B^k} \otimes B^\ell) = 0$ for $\ell \geq 0$. For $r = 2$, the first term $H^1(M_{B^k} \otimes B^{k+\ell})$ vanishes if $2k + \ell \geq 3$ by (i). For the last term,

$$H^2(\mathcal{H}^0(S, B^k) \otimes M_{B^k} \otimes B^\ell) \cong H^0(B^k) \otimes H^2(M_{B^k} \otimes B^\ell) = 0$$

if $\ell \geq 0$. Therefore, we have $H^2(M_{B^k}^{\otimes 2} \otimes B^\ell) = 0$ if $\ell \geq 1$.

Now we prove the (ii) and (iii) for $r \geq 2$ by induction on r .

Consider now the case $r = 2$. From the long exact sequence associated to (3.1) above, we have the exact sequence

$$\begin{aligned} \cdots \rightarrow H^0(\mathcal{H}^0(S, B^k) \otimes M_{B^k} \otimes B^\ell) &\rightarrow H^0(M_{B^k} \otimes B^{k+\ell}) \\ \rightarrow H^1(M_{B^k}^{\otimes 2} \otimes B^\ell) &\rightarrow H^1(\mathcal{H}^0(S, B^k) \otimes M_{B^k} \otimes B^\ell) \rightarrow \cdots \end{aligned}$$

The last term

$$H^1(\mathcal{H}^0(S, B^k) \otimes M_{B^k} \otimes B^\ell) \cong H^0(B^k) \otimes H^1(M_{B^k} \otimes B^\ell)$$

vanishes when $k + \ell \geq 3$. Hence, for $k + \ell \geq 3$, the cohomology $H^1(M_{B^k}^{\otimes 2} \otimes B^\ell)$ vanishes if

$$H^0(\mathcal{H}^0(S, B^k) \otimes M_{B^k} \otimes B^\ell) \rightarrow H^0(M_{B^k} \otimes B^{k+\ell})$$

is surjective, which is equivalent to the surjectivity of

$$H^0(B^k) \otimes H^0(M_{B^k} \otimes B^\ell) \rightarrow H^0(M_{B^k} \otimes B^{k+\ell})$$

since

$$H^0(\mathcal{H}^0(S, B^k) \otimes M_{B^k} \otimes B^\ell) \cong H^0(B^k) \otimes H^0(M_{B^k} \otimes B^\ell).$$

From the Castelnuovo–Mumford Theorem again, as in the case for $r = 1$, the above map is surjective if we have

$$H^1(M_{B^k} \otimes B^{\ell-1}) = H^2(M_{B^k} \otimes B^{\ell-2}) = 0.$$

By what we have proven previously, these are satisfied if $\ell \geq 2$ and $k + \ell \geq 4$.

Suppose now $r \geq 3$. From (3.1), we have the exact sequence

$$\begin{aligned} \cdots \rightarrow H^0(\mathcal{H}^0(S, B^k) \otimes M_{B^k}^{\otimes(r-1)} \otimes B^\ell) &\rightarrow H^0(M_{B^k}^{\otimes(r-1)} \otimes B^{k+\ell}) \\ \rightarrow H^1(M_{B^k}^{\otimes r} \otimes B^\ell) &\rightarrow H^1(\mathcal{H}^0(S, B^k) \otimes M_{B^k}^{\otimes(r-1)} \otimes B^\ell) \rightarrow \cdots \end{aligned}$$

By the induction hypothesis, the last term

$$H^1(\mathcal{H}^0(S, B^k) \otimes M_{B^k}^{\otimes(r-1)} \otimes B^\ell) \cong H^0(B^k) \otimes H^1(M_{B^k}^{\otimes(r-1)} \otimes B^\ell)$$

vanishes when $k + \ell \geq r$ and $\ell \geq r$. Hence, under the same conditions on k, ℓ, r , the cohomology $H^1(M_{B^k}^{\otimes r} \otimes B^\ell)$ vanishes if

$$H^0(\mathcal{H}^0(S, B^k) \otimes M_{B^k}^{\otimes(r-1)} \otimes B^\ell) \rightarrow H^0(M_{B^k}^{\otimes(r-1)} \otimes B^{k+\ell})$$

is surjective, which is equivalent to the surjectivity of

$$H^0(B^k) \otimes H^0(M_{B^k}^{\otimes(r-1)} \otimes B^\ell) \rightarrow H^0(M_{B^k}^{\otimes(r-1)} \otimes B^{k+\ell})$$

since

$$H^0(\mathcal{H}^0(S, B^k) \otimes M_{B^k}^{\otimes(r-1)} \otimes B^\ell) \cong H^0(B^k) \otimes H^0(M_{B^k}^{\otimes(r-1)} \otimes B^\ell).$$

We argue similarly with the Castelnuovo–Mumford Theorem. Thus, the above map is surjective if we have

$$H^1(M_{B^k}^{\otimes(r-1)} \otimes B^{\ell-1}) = H^2(M_{B^k}^{\otimes(r-1)} \otimes B^{\ell-2}) = 0.$$

The induction hypothesis implies that these conditions are satisfied if $k + (\ell - 1) \geq (r - 1) + 2$ and $\ell - 1 \geq r - 1$ and $\ell - 2 \geq (r - 1) - 1$. In particular, it suffices to have $k + \ell \geq r + 2$ and $\ell \geq r$.

It remains to consider $H^2(M_{B^k}^{\otimes r} \otimes B^\ell)$. From the long exact sequence associated to (3.1), we get the exact sequence

$$\dots \rightarrow H^1(M_{B^k}^{\otimes(r-1)} \otimes B^{k+\ell}) \rightarrow H^2(M_{B^k}^{\otimes r} \otimes B^\ell) \rightarrow H^2(\mathcal{H}^0(S, B^k) \otimes M_{B^k}^{\otimes(r-1)} \otimes B^\ell) \rightarrow \dots$$

By the induction hypothesis, we have $H^1(M_{B^k}^{\otimes(r-1)} \otimes B^{k+\ell}) = 0$ for $2k + \ell \geq r + 1$ and $k + \ell \geq r - 1$. Also by the induction hypothesis,

$$H^2(\mathcal{H}^0(S, B^k) \otimes M_{B^k}^{\otimes(r-1)} \otimes B^\ell) \cong H^0(B^k) \otimes H^2(M_{B^k}^{\otimes(r-1)} \otimes B^\ell) = 0$$

if $\ell + 1 \geq r - 1$. Combining all these, it now follows that $H^2(M_{B^k}^{\otimes r} \otimes B^\ell) = 0$ if $\ell + 1 \geq r$. The proof is now complete. \square

Remark 3.4. As in the proof of Proposition 3.2, by applying the result of Reider on a fake projective plane S , it follows easily that for any ample line bundle A , its power A^k is base-point-free for $k \geq 6$ and hence the previous theorem will also imply certain vanishing statements for A . However, if we just simply do the direct translation in this way, there will be “gaps” between the powers of A in which we do not know whether the analogous cohomology groups vanish or not.

In view of the remark above, for the purpose of obtaining a lower bound, above which every power of an arbitrary ample line bundle on S satisfies the property N_p , we improve upon the proof of Theorem 3.3 and obtain the following

Theorem 3.5. *Let A be an ample line bundle and σ, τ be torsion line bundles on a fake projective plane S . For $r \geq 1$ and $k \geq 12$, we have $H^1(S, M_{A^k \otimes \sigma}^{\otimes r} \otimes A^\ell \otimes \tau) = 0$ if $\ell \geq 6(r+2)$, and $H^2(S, M_{A^k \otimes \sigma}^{\otimes r} \otimes A^\ell \otimes \tau) = 0$ if $\ell \geq 6r - 2$.*

Proof. The structure of the proof will be similar to that of Theorem 3.3 and thus we will be brief on certain arguments. We will again prove by induction on r .

Suppose $r = 1$. When $\ell \geq 18$, as $H^1(A^\ell \otimes \tau) = 0$, the cohomology $H^1(M_{A^k \otimes \sigma} \otimes A^\ell \otimes \tau)$ vanishes if the map

$$H^0(A^k \otimes \sigma) \otimes H^0(A^\ell \otimes \tau) \rightarrow H^0(A^{k+\ell} \otimes \sigma \otimes \tau)$$

is surjective.

Since $k \geq 12$, we can write $A^k \otimes \sigma = A^{k_1} \otimes \cdots \otimes A^{k_p} \otimes \sigma$ for some $p \geq 2$ such that $6 \leq k_1 \leq k_2 \leq \cdots \leq k_p$. When $k = 12$ or 13 , such decomposition is unique. If $14 \leq k \leq 17$, we choose $k_1 = 7$ and hence $7 \leq k_2 \leq 10$. If $k \geq 18$, we can always choose our k_j such that $6 = k_1 \leq \cdots \leq k_p \leq 9$ for some $p \geq 3$. Hence, for any $k \geq 12$, we can always choose k_j such that $6 \leq k_1 \leq 7$ and $k_1 \leq \cdots \leq k_p \leq 10$.

By considering the natural maps

$$H^0(A^{k_1}) \otimes \cdots \otimes H^0(A^{k_p} \otimes \sigma) \otimes H^0(A^\ell \otimes \tau) \longrightarrow H^0(A^k \otimes \sigma) \otimes H^0(A^\ell \otimes \tau) \longrightarrow H^0(A^{k+\ell} \otimes \sigma \otimes \tau),$$

we see that it suffices to have the surjectivity for the naturally defined mappings

$$\varphi_j: H^0(A^{k_j}) \otimes H^0(A^{k_1+\cdots+k_{j-1}+\ell} \otimes \tau) \rightarrow H^0(A^{k_1+\cdots+k_j+\ell} \otimes \tau)$$

for $1 \leq j \leq p-1$ and also

$$\varphi_p: H^0(A^{k_p} \otimes \sigma) \otimes H^0(A^{k_1+\cdots+k_{p-1}+\ell} \otimes \tau) \rightarrow H^0(A^{k+\ell} \otimes \sigma \otimes \tau).$$

Since $k_j \geq 6$, every A_{k_j} and also $A_{k_p} \otimes \sigma$ are base-point-free as in the proof of Proposition 3.2. Thus, we can apply the result of Castelnuovo–Mumford and it says that φ_1 is surjective if

$$H^1(A^{\ell-k_1} \otimes \tau) = H^2(A^{\ell-2k_1} \otimes \tau) = 0.$$

By Lemma 3.1, these are true if $\ell \geq 18$ since $6 \leq k_1 \leq 7$. Similarly, for $2 \leq j \leq p-1$, φ_j is surjective if

$$H^1(A^{k_1+\cdots+k_{j-1}+\ell-k_j} \otimes \tau) = H^2(A^{k_1+\cdots+k_{j-1}+\ell-2k_j} \otimes \tau) = 0.$$

These hold true if $\ell \geq 18$ since $k_1 \geq 6$ and $k_j \leq 10$. The surjectivity for φ_p can be seen in the same manner.

For $H^2(M_{A^k \otimes \sigma} \otimes A^\ell \otimes \tau)$, it vanishes for $\ell \geq 4$, which can be seen easily from the long exact sequence associated to (3.1) for $r = 1$,

$$\cdots \rightarrow H^1(A^{k+\ell} \otimes \sigma \otimes \tau) \rightarrow H^2(M_{A^k \otimes \sigma} \otimes A^\ell \otimes \tau) \rightarrow H^2(\mathcal{H}^0(S, A^k \otimes \sigma) \otimes A^\ell \otimes \tau) \rightarrow \cdots.$$

The case for $r = 1$ is now settled.

Suppose now $r \geq 2$. We first consider $H^1(M_{A^k}^{\otimes r} \otimes A^\ell)$. We have the exact sequence

$$\begin{aligned} \cdots \rightarrow H^0(\mathcal{H}^0(S, A^k \otimes \sigma) \otimes M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^\ell \otimes \tau) &\rightarrow H^0(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{k+\ell} \otimes \sigma \otimes \tau) \\ &\rightarrow H^1(M_{A^k \otimes \sigma}^{\otimes r} \otimes A^\ell \otimes \tau) \rightarrow H^1(\mathcal{H}^0(S, A^k \otimes \sigma) \otimes M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^\ell \otimes \tau) \rightarrow \cdots. \end{aligned}$$

By the induction hypothesis, the last term

$$H^1(\mathcal{H}^0(S, A^k \otimes \sigma) \otimes M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^\ell \otimes \tau) \cong H^0(A^k \otimes \sigma) \otimes H^1(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^\ell \otimes \tau)$$

vanishes when $\ell \geq 6(r+1)$. Hence, for $\ell \geq 6(r+1)$, the cohomology group $H^1(M_{A^k \otimes \sigma}^{\otimes r} \otimes A^\ell \otimes \tau)$ vanishes if

$$H^0(A^k \otimes \sigma) \otimes H^0(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^\ell \otimes \tau) \rightarrow H^0(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{k+\ell} \otimes \sigma \otimes \tau)$$

is surjective. As in the case $r = 1$, we can write $A^k \otimes \sigma = A^{k_1} \otimes \cdots \otimes A^{k_p} \otimes \sigma$ for some $p \geq 2$ such that $k_j \geq 6$ for $1 \leq j \leq p$. But for $r \geq 2$, we choose our k_j such that $6 = k_1 \leq k_2 \leq \cdots \leq k_p \leq 11$.

As before, it suffices to have the surjectivity for

$$\phi_j: H^0(A^{k_j}) \otimes H^0(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{k_1+\cdots+k_{j-1}+\ell} \otimes \tau) \rightarrow H^0(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{k_1+\cdots+k_j+\ell} \otimes \tau)$$

for $1 \leq j \leq p-1$ and also

$$\phi_p: H^0(A^{k_p} \otimes \sigma) \otimes H^0(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{k_1+\cdots+k_{p-1}+\ell} \otimes \tau) \rightarrow H^0(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{k+\ell} \otimes \sigma \otimes \tau).$$

By Castelnuovo–Mumford Theorem, ϕ_1 is surjective if

$$H^1(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{\ell-k_1} \otimes \tau) = H^2(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{\ell-2k_1} \otimes \tau) = 0.$$

The induction hypothesis implies that these conditions are satisfied if $\ell - k_1 = \ell - 6 \geq 6(r+1)$ and $\ell - 2k_1 = \ell - 12 \geq 6(r-1) - 2$, which are both true if $\ell \geq 6(r+2)$.

For $2 \leq j \leq p-1$, ϕ_j is surjective if

$$H^1(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{k_1+\cdots+k_{j-1}+\ell-k_j} \otimes \tau) = H^2(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{k_1+\cdots+k_{j-1}+\ell-2k_j} \otimes \tau) = 0.$$

By simple arithmetic, together with the induction hypothesis, it follows that these are satisfied if $\ell - 5 \geq 6(r+1)$ and $\ell - 16 \geq 6(r-1) - 2$. In particular, it is more than sufficient if we have $\ell \geq 6(r+2)$. The surjectivity for ϕ_p is checked similarly.

It remains to check that $H^2(M_{A^k \otimes \sigma}^{\otimes r} \otimes A^\ell \otimes \tau) = 0$ when $\ell \geq 6r - 2$. From the exact sequence

$$\begin{aligned} & \cdots \rightarrow H^1(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{k+\ell} \otimes \sigma \otimes \tau) \\ & \rightarrow H^2(M_{A^k \otimes \sigma}^{\otimes r} \otimes A^\ell \otimes \tau) \rightarrow H^2(\mathcal{H}^0(S, A^k \otimes \sigma) \otimes M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^\ell \otimes \tau) \rightarrow \cdots, \end{aligned}$$

we have, by the induction hypothesis, $H^1(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^{k+\ell} \otimes \sigma \otimes \tau) = 0$ for if $k+\ell \geq 6(r+1)$. Since $k \geq 12$, the inequality is satisfied when $\ell \geq 6r - 2$. Similarly,

$$H^2(\mathcal{H}^0(S, A^k \otimes \sigma) \otimes M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^\ell \otimes \tau) \cong H^0(A^k \otimes \sigma) \otimes H^2(M_{A^k \otimes \sigma}^{\otimes(r-1)} \otimes A^\ell \otimes \tau) = 0$$

if $\ell \geq 6(r-1) - 2$. Thus, the second term $H^2(M_{A^k}^{\otimes r} \otimes A^\ell)$ in the exact sequence above is zero when $\ell \geq 6r - 2$. The proof is now complete. \square

We are now ready to obtain the final result of this section and also give the proof of Theorem 1.1.

Theorem 3.6. *Let B be an ample line bundle on a fake projective plane S such that $B \otimes K_S^{-2}$ is nef, where K_S is the canonical line bundle. Then, B^m is projectively normal (i.e., satisfies the property N_0) if $m \geq 2$. More generally, for $p \geq 1$, B^m satisfies the property N_p if $m \geq p + 1$.*

Proof. As remark earlier, if $B \otimes K_S^{-2}$ is nef, then B^m is base-point-free for $m \geq 1$. Suppose $m \geq 2$. For B^m to satisfy the property N_0 , by Green's criterion, it suffices to have $H^1(S, B^{ms}) = H^1(S, M_{B^m} \otimes B^{ms}) = 0$ for $s \geq 1$. These follow respectively from Proposition 3.2 and Theorem 3.3.

Suppose now $p \geq 1$ and $m \geq p + 1$. To see that B^m satisfies the property N_p , we just need to check $H^1(S, M_{B^m}^{\otimes r} \otimes B^{ms}) = 0$ for $0 \leq r \leq p + 1$ and $s \geq 1$. When $r = 0$, it again follows from Proposition 3.2. For $r = 1$, it follows from Theorem 3.3(i) since $m + ms \geq 2m \geq 2(p + 1) \geq 4$. Finally, for $2 \leq r \leq p + 1$, we have

$$m + ms \geq 2m \geq 2(p + 1) \geq 2r \geq r + 2,$$

and $ms \geq p + 1 \geq r \geq 2$ and hence the result follows from Theorem 3.3(ii). \square

Corollary 3.7. *In addition to the results for pluricanonical bundles given by Corollary 1.2, we also have K_S^{2p+2} satisfies the property N_p for $p \geq 1$. In particular, K_S^{2p+4} satisfies the property N_p for $p \geq 0$.*

Remark 3.8. For an ample and base-point-free line bundle B on a complex projective surface of general type, Gallego–Purnaprajna [6] proved that B^m satisfies the property N_1 if $m \geq 3$; and for $p \geq 2$, B^m satisfies the property N_p if $m \geq p + 1$. The latter result contains in particular our results for $p \geq 2$ in Theorem 3.6 since the line bundles satisfying the hypotheses of Theorem 3.6 are ample and base-point-free.

Proof of Theorem 1.1. Let m be a positive integer such that $m \geq 6(p+3)$. We fix an ample generator G of $\text{Pic}(S)$ and let A and N be an ample and a nef line bundle on S respectively. To show that $A^m \otimes N$ satisfies the property N_p , we will verify for $0 \leq r \leq p+1$ and $s \geq 1$, $H^1(S, M_{A^m \otimes N}^{\otimes r} \otimes (A^m \otimes N)^s) = 0$.

We have $A^m \otimes N = G^k \otimes \sigma$, for some positive integer $k \geq 6(p+3)$ and some torsion line bundle σ . Thus, $H^1(S, (A^m \otimes N)^s) = H^1(S, G^{ks} \otimes \sigma^s) = 0$ for $s \geq 1$ by Lemma 3.1. Next, for $1 \leq r \leq p+1$ and $s \geq 1$,

$$H^1(S, M_{A^m \otimes N}^{\otimes r} \otimes (A^m \otimes N)^s) = H^1(S, M_{G^k \otimes \sigma}^{\otimes r} \otimes G^{ks} \otimes \sigma^s) = 0$$

by Theorem 3.5 since $k \geq 18$ and $ks \geq k \geq 6(p+3) \geq 6(r+2)$. \square

4. Syzygies for base-point-free line bundles

Let S be a fake projective plane. There is a one-to-one correspondence between set of torsion elements of $H_1(S, \mathbb{Z})$ and the set of torsion line bundles on S , as given by the Universal Coefficient Theorem (cf. [10]). Moreover, a torsion line bundle δ corresponds to a surjective representation $\rho_\delta: \pi_1(S) \rightarrow \mathbb{Z}/d\mathbb{Z}$, where $d \in \mathbb{N}^+$ is the order of δ . The representation ρ_δ necessarily descends to a representation $\rho_\delta^\flat: H_1(S, \mathbb{Z}) \cong \pi_1(S)/[\pi_1(S), \pi_1(S)] \rightarrow \mathbb{Z}/d\mathbb{Z}$. If we let $C_\delta := \ker(\rho_\delta^\flat)$, then $H_1(S, \mathbb{Z})/C_\delta \cong \mathbb{Z}/d\mathbb{Z}$.

Lemma 4.1. *Let M be a compact complex manifold with $b_1(M) = 0$ and δ be a torsion line bundle on M of order d . Let M_δ be the finite unramified covering of M defined by the kernel of the representation $\rho_\delta: \pi_1(M) \rightarrow \mathbb{Z}/d\mathbb{Z}$ associated to δ . Then $h^1(M, \delta) = 0$ if $b_1(M_\delta) = 0$.*

Proof. Let $\pi: M_\delta \rightarrow M$ be the finite unramified covering map. Then $\pi^*\delta$ is the trivial line bundle on M_δ and we have (see [2, p. 55])

$$\pi_*(\mathcal{O}_{M_\delta}) = \bigoplus_{j=0}^{d-1} \mathcal{O}_M(\delta^j).$$

Hence,

$$\frac{1}{2}b_1(M_\delta) = h^1(M_\delta, \mathcal{O}_{M_\delta}) = h^1(M, \pi_*(\mathcal{O}_{M_\delta})) = \sum_{j=0}^{d-1} h^1(M, \delta^j).$$

The lemma now follows. \square

Let S be a fake projective plane and $\Pi \subset \text{PU}(2,1)$ be the lattice associated to S , which is isomorphic to the fundamental group $\pi_1(S)$. By definition, $b_1(S) = 0$ as S is a fake projective plane and it follows that $H_1(S, \mathbb{Z}) \cong \Pi/[\Pi, \Pi]$ is a finite abelian group. Let $\rho: \Pi \rightarrow \Pi/[\Pi, \Pi]$ be the canonical projection. Let C be a subgroup of $\Pi/[\Pi, \Pi]$ and

$p_C: \Pi/[\Pi, \Pi] \rightarrow (\Pi/[\Pi, \Pi])/C$ be the natural projection and $\rho_C = p_C \circ \rho$. Let $S_C = B_{\mathbb{C}}^2/\ker(\rho_C)$ be the finite unramified covering of S associated to $\ker(\rho_C)$, where $B_{\mathbb{C}}^2 \subset \mathbb{C}^2$ is the unit ball.

Theorem 4.2. *For every subgroup C of $\Pi/[\Pi, \Pi]$, we have $b_1(S_C) = 0$.*

Proof. Let S_0 be the unramified covering of S corresponding to the lattice $[\Pi, \Pi]$. As $\ker(\rho) = [\Pi, \Pi]$ and $\ker(\rho) \subset \ker(\rho_C)$, it is clear that S_0 is a finite unramified covering of S_C , which is also a holomorphic isometric covering with respect to the canonical metrics induced from $B_{\mathbb{C}}^2$. In particular, any harmonic one form on $B_{\mathbb{C}}^2/\ker(\rho_C)$ can be pulled back to $B_{\mathbb{C}}^2/\ker(\rho)$ and hence $b_1(S_0) \geq b_1(S_C)$ from Poincaré Duality. Hence it suffices to check that $b_1(S_0) = 0$. Equivalently, we must check that the abelianization $\Pi'/[\Pi', \Pi']$ of $\Pi' = [\Pi, \Pi]$ is finite. This was kindly verified for us by Donald Cartwright using Magma, as follows. There are (see [3]) exactly 50 groups Π 's which are fundamental groups of fake projective planes S . Their names are listed in the file `registerofgps.txt` in the weblink of [3]. Each is a subgroup of a maximal arithmetic subgroup $\bar{\Gamma}$ of $\text{PU}(2, 1)$. Explicit generators and relations for these $\bar{\Gamma}$'s are given in various files of the weblink. For example, see `C20p2/gpc20p2generators_reducesyntax.txt` for the generators and relations for the three $\bar{\Gamma}$'s named $(C_{20}, p = 2, \emptyset)$, $(C_{20}, p = 2, \{3+\})$ and $(C_{20}, p = 2, \{3-\})$. Abstract presentations of these groups $\bar{\Gamma}$ are given in the file `barGammapresentations.txt` of that weblink, where for each of the fake projective planes S named in `registerofgps.txt`, generators s_1, s_2, \dots are given for its fundamental group Π .

Using the data about generators and relations for the lattices involved from the file `finite-cover-b1.txt`, for each of these 50 Π 's, we exhibit the subgroup $\Pi' = [\Pi, \Pi]$ of Π , then verify that $\Pi'/[\Pi', \Pi']$ is finite using Magma's `AbelianQuotientInvariants` routine.

The subgroup Π' was in each case found as follows. Using Magma's `Rewrite` routine, we first find a presentation of Π from that of $\bar{\Gamma}$. Let $n = |\Pi/[\Pi, \Pi]| = |H_1(S, \mathbb{Z})|$. Using Magma's `LowIndexNormalSubgroups`(Π, n) routine, we list the normal subgroups of Π of index at most n . Then we find the unique subgroup in this (sometimes lengthy) list which has index exactly n , and contains all commutators $s_i^{-1}s_j^{-1}s_i s_j$. \square

As discussed at the beginning of this section, for each torsion line bundle δ of order d on a fake projective plane $S = B_{\mathbb{C}}^2/\Pi$, there is a subgroup $C_\delta < H_1(S, \mathbb{Z})$ such that $H_1(S, \mathbb{Z})/C_\delta \cong \mathbb{Z}/d\mathbb{Z}$ and the representation of $\pi_1(S)$ associated to δ is just the composition $\pi_1(S) \rightarrow H_1(S, \mathbb{Z}) \rightarrow H_1(S, \mathbb{Z})/C_\delta$. With the identifications $\pi_1(S) \cong \Pi$ and $H_1(S, \mathbb{Z}) \cong \pi_1(S)/[\pi_1(S), \pi_1(S)] \cong \Pi/[\Pi, \Pi]$, the finite unramified cover M_δ associated to δ described in Lemma 4.1 is just S_C in Theorem 4.2 when $C = C_\delta$. Thus, by combining Lemma 4.1 and Theorem 4.2, we have

Corollary 4.3. *Let S be a fake projective plane and δ be a torsion line bundle on S . Then $h^1(S, \delta) = 0$.*

The following lemma is a consequence of a well known theorem of Remmert–Van de Ven [17, p. 155]. One may also see [9, 15.6.2].

Lemma 4.4. *Let M be a complex projective manifold and L be a holomorphic line bundle on M such that $h^0(M, L) \geq 1$. Then $h^0(M, L^2) \geq 2h^0(M, L) - 1$.*

Proposition 4.5. *Let S be a fake projective plane and G be an ample generator of $\text{Pic}(S)$. For any torsion line bundle δ on S , $h^0(S, G \otimes \delta) \leq 1$, $h^0(S, G^2 \otimes \delta) \leq 2$ and $h^0(S, G^3 \otimes \delta) \leq 1$.*

Proof. Since $K_S = G^3 \otimes \tau$ for some torsion line bundle τ , using Riemann–Roch Formula with Kodaira Vanishing Theorem, it is easy to get that $h^0(G^4 \otimes \delta^2) = 3$ for any torsion line bundle δ . It now follows from Lemma 4.4 that $h^0(S, G^2 \otimes \delta) \leq 2$, which in turn implies $h^0(S, G \otimes \delta) \leq 1$.

For $h^0(S, G^3 \otimes \delta)$, the result follows again from Riemann–Roch Formula, together with Corollary 4.3 and the fact that $K_S = G^3 \otimes \tau$. \square

Proof of Proposition 1.4. This is now a consequence of Corollary 4.3 and Proposition 4.5. \square

We are now ready to prove Theorem 1.5.

Proof of Theorem 1.5. Let G be an ample generator of $\text{Pic}(S)$. As the Picard number of S is 1, B must be ample, and we can write $B = G^k \otimes \sigma$, for some $k \geq 1$ and some torsion line bundle σ . As B is base-point-free, it is clear from Proposition 4.5 that $k \neq 1, 3$. We claim that $k \neq 2$ neither. Suppose otherwise that $k = 2$. Then by Proposition 4.5, $h^0(S, B) \leq 2$. Since B is base-point-free, we have in fact $h^0(S, B) = 2$. But S is of Picard number 1, the zero divisors of two linearly independent sections of $\Gamma(S, B)$ must have non-trivial intersection, which contradicts base-point-freeness. Hence, $k \geq 4$. Finally, as Lemma 3.1 is true whenever $k \geq 4$, we conclude that Proposition 3.2, Theorems 3.3 and 3.6 are all true for any base-point-free line bundle. \square

5. Refinement with a conjecture on vanishing cohomology

Motivated by the existence problem for exceptional collections for objects in the derived category of coherent sheaves on a fake projective plane S , we formulate the following conjecture, which is a variant of Conjecture 1.1 in [5] and Conjecture 2 in [10].

Conjecture 5.1. *Let S be a fake projective plane and G be an ample generator of $\text{Pic}(S)$. Then, $H^0(S, G^2 \otimes \sigma) = 0$ for any torsion line bundle σ .*

The conjecture above implies a stronger version of Lemma 3.1, as follows:

Proposition 5.2. *If Conjecture 5.1 holds for a fake projective plane S , then for any ample generator G and torsion σ of $\text{Pic}(S)$, we have $H^1(S, G^k \otimes \sigma) = 0$ for $k \geq 1$ and $H^2(S, G^k \otimes \sigma) = 0$ for $k = 1, 2$, or $k \geq 4$.*

Proof. If $H^0(S, G^2 \otimes \sigma) = 0$ for any torsion σ , then we also have $H^0(S, G \otimes \sigma) = 0$ for any torsion σ . Since $K_S = G^3 \otimes \tau$ for some torsion τ , using Serre Duality, we get

$$H^2(S, G \otimes \sigma) = H^2(S, G^2 \otimes \sigma) = 0$$

for any torsion σ . Now, by Riemann–Roch formula, it follows that

$$H^1(S, G \otimes \sigma) = H^1(S, G^2 \otimes \sigma) = 0$$

for any torsion σ .

Finally, $H^1(S, G^3 \otimes \sigma) = H^1(S, \tau \otimes \sigma^{-1}) = 0$ by Serre Duality and Corollary 4.3. (Note that $H^2(S, G^3 \otimes \sigma) = H^0(S, \tau \otimes \sigma^{-1})$ is zero if and only if $\sigma \neq \tau$.) \square

With the help of Proposition 5.2, we are now able to show that for a fake projective plane S satisfying Conjecture 5.1, its pluricanonical line bundle K_S^{2p+5} , which is missed in Corollary 1.2, also satisfies the property N_p . The essential case is the projective normality (i.e., when $p = 0$), which we have singled out in the following theorem.

Theorem 5.3. *Let S be a fake projective plane satisfying Conjecture 5.1. Then K_S^m is projectively normal for $m \geq 4$.*

Proof. By Corollary 1.2, it remains to prove that K_S^5 is projectively normal. Since $K_S = G^3 \otimes \tau$, for some torsion τ , using Green’s criterion (for the property N_0), we just need to show that for any torsion line bundles σ, δ , $H^1(S, M_{G^{15} \otimes \sigma} \otimes G^\ell \otimes \delta) = 0$ for $\ell = 15s$, where $s \in \mathbb{N}^+$. As in the proof of Theorem 3.5, after decomposing as $G^{15} \otimes \sigma = G^7 \otimes G^8 \otimes \sigma$, it suffices to verify that the tensor product maps

$$\begin{aligned} H^0(G^7) \otimes H^0(G^\ell \otimes \delta) &\rightarrow H^0(G^{7+\ell} \otimes \sigma \otimes \delta), \\ H^0(G^8 \otimes \sigma) \otimes H^0(G^{7+\ell} \otimes \delta) &\rightarrow H^0(G^{15+\ell} \otimes \sigma \otimes \delta) \end{aligned}$$

are surjective for $\ell = 15s$, $s \in \mathbb{N}^+$. Using Castelnuovo–Mumford Theorem, it is sufficient to have, respectively,

$$H^1(G^{\ell-7} \otimes \delta) = H^2(G^{\ell-14} \otimes \delta) = 0$$

and

$$H^1(G^{7+\ell-8} \otimes \sigma \otimes \delta) = H^2(G^{7+\ell-16} \otimes \sigma \otimes \delta) = 0.$$

By Proposition 5.2, these conditions hold for $\ell = 15s$, $s \in \mathbb{N}^+$. \square

Remark 5.4. The above argument implies that K_S^m is projective normal for $m \geq 4$ for fake projective planes with cardinality of automorphism group $|\text{Aut}(S)| > 3$, using the facts, cf. [10] and the references therein, that $K_S = G^3$ for some ample line bundle G with $H^0(S, G^2) = 0$ and letting δ and σ be trivial in the above argument.

After settling projective normality, we can now prove that the general property N_p holds for K_S^{2p+5} . We will see that for $p \geq 1$, the argument does not require Conjecture 5.1. The major step is the following vanishing theorem tailored for the purpose.

Proposition 5.5. *Let S be a fake projective plane. Let G be an ample generator and σ , δ be torsions in $\text{Pic}(S)$. For $k \geq 18$ and $r \geq 1$, we have $H^1(S, M_{G^k \otimes \sigma}^{\otimes r} \otimes G^\ell \otimes \delta) = 0$ if $\ell \geq 6(r+1)$.*

Proof. We will prove by induction on r . The proof is similar to that of Theorem 3.5 and will be brief.

Suppose $r = 1$. We observe that $H^1(M_{G^k \otimes \sigma} \otimes G^\ell \otimes \delta) = 0$ if $k \geq 18$ and $\ell \geq 12$. To see this, we just need to note that at the beginning of the proof in Theorem 3.5, the roles of k and ℓ are symmetric and thus we can exchange k and ℓ in the vanishing statement for $r = 1$ in Theorem 3.5. The case for $r = 1$ is now settled.

Suppose $r = 2$. Since $k \geq 18$, we have a decomposition $G^k \otimes \sigma = G^{k_1} \otimes \cdots \otimes G^{k_p} \otimes \sigma$ for some $p \geq 3$ and $6 = k_1 \leq \cdots \leq k_p \leq 9$.

To show that $H^1(M_{G^k \otimes \sigma}^{\otimes r} \otimes G^\ell \otimes \delta) = 0$, it suffices to verify that $H^1(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^\ell \otimes \delta) = 0$ and the surjectivity for the following mappings:

$$\varphi_j: H^0(G^{k_j}) \otimes H^0(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^{k_1+\cdots+k_{j-1}+\ell} \otimes \delta) \rightarrow H^0(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^{k_1+\cdots+k_j+\ell} \otimes \delta)$$

for $1 \leq j \leq p-1$ and

$$\varphi_p: H^0(G^{k_p} \otimes \sigma) \otimes H^0(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^{k_1+\cdots+k_{p-1}+\ell} \otimes \delta) \rightarrow H^0(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^{k+\ell} \otimes \sigma \otimes \delta).$$

Now, together with Castelnuovo–Mumford Theorem, we conclude that the cohomology $H^1(S, M_{G^k \otimes \sigma}^{\otimes r} \otimes G^\ell \otimes \delta)$ vanishes if we have

$$\begin{aligned} H^1(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^\ell \otimes \delta) &= 0, \\ H^1(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^{\ell-k_1} \otimes \delta) &= H^2(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^{\ell-2k_1} \otimes \delta) = 0, \\ H^1(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^{k_1+\cdots+k_{j-1}+\ell-k_j} \otimes \delta) &= H^2(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^{k_1+\cdots+k_{j-1}+\ell-2k_j} \otimes \delta) = 0 \end{aligned}$$

for $2 \leq j \leq p - 1$ and

$$H^1(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^{k+\ell-2k_p} \otimes \sigma \otimes \delta) = H^2(M_{G^k \otimes \sigma}^{\otimes(r-1)} \otimes G^{k+\ell-3k_p} \otimes \sigma \otimes \delta) = 0.$$

These all hold if $k \geq 18$ and $\ell - 6 \geq 6r$, by the induction hypothesis and Theorem 3.5 (for the vanishing of H^2). The proof is now complete. \square

Theorem 5.6. *Let S be a fake projective plane satisfying Conjecture 5.1. Then K_S^m satisfies the property N_p if $m \geq 2p + 4$.*

Proof. By Corollary 1.2, it remains to check that K_S^{2p+5} satisfies the property N_p . We have already shown that K_S^5 satisfies the property N_0 in Theorem 5.3 and we now let $p \geq 1$. Let $L_p = K_S^{2p+5}$. According to Green's criterion, we just need to check that

$$H^1(S, M_{L_p}^{\otimes r} \otimes L_p^s) = 0$$

for $0 \leq r \leq p + 1$ and $s \geq 1$.

Let G be an ample generator of $\text{Pic}(S)$ and then we can write

$$L_p = K_S^{2p+5} = G^{6p+15} \otimes \delta$$

for some torsion line bundle δ . For $p \geq 1$, the desired vanishing statement follows from Proposition 5.5 since $6p + 15 > 18$ and

$$(6p + 15)s \geq 6p + 15 > 6(p + 2) \geq 6(r + 1). \quad \square$$

Acknowledgments

It is a pleasure for the authors to thank Donald Cartwright for providing computer data files for the proof of Theorem 4.2 which leads to Theorem 1.5 and thank Lawrence Ein and Jun-Muk Hwang for fruitful discussions and providing very useful references. They would also like to thank the anonymous referees for carefully reading the paper and giving helpful suggestions.

References

- [1] P. Banagere and K. Hanumanthu, *Syzygies of surfaces of general type*, *Geom. Dedicata*. **167** (2013), 123–149.
- [2] W. P. Barth, K. Hulek, C. A. M. Peters and A. Van de Ven, *Compact Complex Surfaces*, Second edition, *Ergebnisse der Mathematik und ihrer Grenzgebiete. 3. Folge. A Series of Modern Surveys in Mathematics* **4**, Springer-Verlag, Berlin, 2004.

- [3] D. I. Cartwright and T. Steger, *Enumeration of the 50 fake projective planes*, C. R. Math. Acad. Sci. Paris **348** (2010), no. 1-2, 11–13.
- [4] L. Ein and R. Lazarsfeld, *Syzygies and Koszul cohomology of smooth projective varieties of arbitrary dimension*, Invent. Math. **111** (1993), no. 1, 51–67.
- [5] S. Galkin, L. Katzarkov, A. Mellit and E. Shinder, *Derived categories of Keum’s fake projective planes*, Adv. Math. **278** (2015), 238–253.
- [6] F. J. Gallego and B. P. Purnaprajna, *Projective normality and syzygies of algebraic surfaces*, J. Reine Angew. Math. **506** (1999), 145–180.
- [7] M. L. Green, *Koszul cohomology and the geometry of projective varieties II*, J. Differential Geom. **20** (1984), no. 1, 279–289.
- [8] J.-M. Hwang and W.-K. To, *Syzygies of compact complex hyperbolic manifolds*, J. Algebraic Geom. **22** (2013), no. 1, 175–200.
- [9] J. Kollár, *Shafarevich Maps and Automorphic Forms*, Princeton University Press, Princeton, NJ, 1995.
- [10] C.-J. Lai and S.-K. Yeung, *Exceptional collection of objects on some fake projective planes*, To appear in Int. Math. Res. Not. <https://doi.org/10.1093/imrn/rnab186>
- [11] R. Lazarsfeld, *Positivity in Algebraic Geometry I*, Springer-Verlag, Berlin, 2004.
- [12] W. Niu, *On syzygies of Calabi–Yau varieties and varieties of general type*, Adv. Math. **343** (2019), 756–788.
- [13] G. Prasad and S.-K. Yeung, *Fake projective planes*, Invent. Math. **168** (2007), no. 2, 321–370.
- [14] ———, *Addendum to “Fake projective planes”*, Invent. Math. **182** (2010), no. 1, 213–227.
- [15] B. P. Purnaprajna, *Some results on surfaces of general type*, Canad. J. Math. **57** (2005), no. 4, 724–749.
- [16] I. Reider, *Vector bundles of rank 2 and linear systems on algebraic surfaces*, Ann. of Math. (2) **127** (1988), no. 2, 309–316.
- [17] R. Remmert and A. J. H. M. Van de Ven, *Zur Funktionentheorie homogener komplexer Mannigfaltigkeiten*, Topology **2** (1963), 137–157.

- [18] R. Rémy, *Covolume des groupes S -arithmétiques et faux plans projectifs*, Astérisque No. **326** (2009), Exp. No. 984, 83–129.
- [19] S.-K. Yeung, *Classification of fake projective planes*, in: *Handbook of Geometric Analysis*, No. 2, 391–431, Adv. Lect. Math. (ALM) **13**, Int. Press, Somerville, MA, 2010.

Sui-Chung Ng

School of Mathematical Sciences, Shanghai Key Laboratory of PMMP, East China Normal University, Minhang District, Shanghai, China

E-mail address: `scng@math.ecnu.edu.cn`

Sai-Kee Yeung

Mathematics Department, Purdue University, West Lafayette, IN 47907, USA

E-mail address: `yeung@math.purdue.edu`