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Regularity of Commutators of Maximal Operators with Lipschitz Symbols

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Abstract. This paper is devoted to studying Sobolev regularity properties of commutators of Hardy–Littlewood maximal operator and its fractional case with Lipschitz symbols, both in the global and local case. Some new pointwise estimates for the weak gradients of the above commutators will be established. As applications, some bounds for the above commutators on the Sobolev spaces will be obtained.

1. Introduction

1.1. Background

The regularity theory of maximal operators is an active topic of current research. A driving question related to this theory is whether a given maximal operator improves, preserves or destroys the a priori regularity of an initial datum f. In 1997, Kinnunen [16] first studied the Sobolev regularity for the Hardy–Littlewood maximal operator

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy,$$

where $B(x,r)$ is the open ball in $\mathbb{R}^n$ centered at x with radius r and $|B(x,r)|$ denotes its volume, and showed that M is bounded on the first order Sobolev spaces $W^{1,p}(\mathbb{R}^n)$ for $1 < p \leq \infty$, where

$$W^{1,p}(\mathbb{R}^n) := \bigl\{ f\colon \mathbb{R}^n \to \mathbb{R} \mid \|f\|_{1,p} := \|f\|_p + \|\nabla f\|_p < \infty \bigr\},$$

where $\|f\|_{W^{1,p}(\mathbb{R}^n)} := \|f\|_{1,p}$, $\|f\|_p := \|f\|_{L^p(\mathbb{R}^n)}$ and $\nabla f = (D_1 f, \ldots, D_n f)$ is the weak gradient of f. Later on, more and more scholars devoted to extending Kinnunen's result to various variants (see [7, 17, 19, 23]). Since the derivative of maximal function has no sublinearity, the continuity of $M\colon W^{1,p}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ for $1 < p < \infty$ is affirmatively a nontrivial issue. This question was first answered by Luiro [27] and later extended to the local case in [28] and to the multilinear fractional case in [22]. The endpoint regularity

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properties of maximal operators can be found in [2, 5, 6, 8, 29]. Other interesting works of regularity theory are [20, 21, 24, 28] for the boundedness for maximal operator on the fractional Sobolev spaces, Triebel–Lizorkin spaces and Besov spaces as well as [25, 26] for the regularity properties for the commutators of maximal operators.

Recently, Liu, Xue and Zhang [26] studied the regularity properties for the commutators of Hardy–Littlewood maximal function. To be more precise, let b be a locally integral function defined on $\mathbb{R}^n$, the commutators of the Hardy–Littlewood maximal operator is defined by

$$[b,M](f)(x) = b(x)Mf(x) - M(bf)(x), \quad x \in \mathbb{R}^n.$$

The maximal commutator of M with b is defined as

$$\mathfrak{M}_b f(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |b(x)-b(y)||f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

Liu, Xue and Zhang [26] established the following result.

Theorem 1.1. [26] *Let $1 < p_1, p_2, p < \infty$ and $1/p = 1/p_1 + 1/p_2$. If $b \in W^{1,p_2}(\mathbb{R}^n)$, then the map $[b,M]\colon W^{1,p_1}(\mathbb{R}^n) \to W^{1,p}(\mathbb{R}^n)$ is bounded and continuous. Moreover, if $f \in W^{1,p_1}(\mathbb{R}^n)$, then*

$$\|[b,M](f)\|_{1,p} \le C_{n,p_1,p_2} \|b\|_{1,p_2} \|f\|_{1,p_1}.$$

The above boundedness also holds for $\mathfrak{M}_b$.

Very recently, Liu and Xi [25] extended Theorem 1.1 to the fractional case. Let us introduce the commutators of fractional maximal operator.

Definition 1.2. Let $0 \le \alpha < n$ and M_α be the fractional maximal operator on $\mathbb{R}^n$, i.e., for $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$,

$$M_\alpha f(x) = \sup_{r>0} \frac{r^\alpha}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

For a locally integral function b defined on $\mathbb{R}^n$, the commutator of fractional maximal operator M_α with b is defined by

$$[b,M_\alpha](f)(x) = b(x)M_\alpha f(x) - M_\alpha(bf)(x), \quad x \in \mathbb{R}^n.$$

The fractional maximal commutator of M_α with b is defined as

$$\mathfrak{M}_{b,\alpha} f(x) = \sup_{r>0} \frac{r^\alpha}{|B(x,r)|} \int_{B(x,r)} |b(x)-b(y)||f(y)|\, dy, \quad x \in \mathbb{R}^n.$$

Clearly, $M_\alpha = M$, $[b,M_\alpha] = [b,M]$ and $\mathfrak{M}_{b,\alpha} = \mathfrak{M}_b$ when $\alpha = 0$.

We now introduce the main results of [25] as follows.

Theorem 1.3. [25] *Let $1 < p_1, p_2, p, p_1p_2/(p_1+p_2) < \infty$, $0 \le \alpha < n/p_1$ and $1/q = 1/p_1 + 1/p_2 - \alpha/n$. If $b \in W^{1,p_2}(\mathbb{R}^n)$, then the map $[b, M_\alpha]\colon W^{1,p_1}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$ is bounded. In particular, if $f \in W^{1,p_1}(\mathbb{R}^n)$, then*

$$|D_i[b, M_\alpha](f)(x)| \le |b(x)| M_\alpha(D_i f)(x) + M_\alpha(bD_i f)(x) + |D_i b(x)| M_\alpha f(x) + M_\alpha(D_i b f)(x)$$

for almost every $x \in \mathbb{R}^n$ and $i = 1, 2, \ldots, n$. Moreover,

$$\|[b, M_\alpha](f)\|_{1,q} \le C_{\alpha,n,p_1,p_2} \|b\|_{1,p_2} \|f\|_{1,p_1}.$$

The same conclusions hold for $\mathfrak{M}_{b,\alpha}$.

In this paper we continue to focus on the Sobolev regularity properties for the commutators of Hardy–Littlewood maximal operator and its fractional version. More precisely, we shall establish some new results on the Sobolev regularity properties of the above commutators with Lipschitz symbols, both in the global and local case.

1.2. The global case

In 1990, Milman and Schonbek [30] first introduced the commutator of maximal operator and established the L^p ($1 < p < \infty$) bounds of $[b, M_{\mathcal{C}}]$ when $b \ge 0$ and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Here $M_{\mathcal{C}}$ is the Hardy–Littlewood maximal operator associated to cubes. Subsequently, Bastero, Milman and Ruiz [3] improved the above result by removing the restrictive condition $b \ge 0$. It was shown in [4] that the operator $[b, \widetilde{M_{\mathcal{C}}}]$ can be used in studying the product of a function in $H^1(\mathbb{R}^n)$ and a function in $\mathrm{BMO}(\mathbb{R}^n)$. Later on, the $L^p \to L^q$ bounds for the commutators of fractional maximal operator have been studied by many authors (see [9, 12, 32]). The maximal commutator was first studied by García-Cuerva, Harboure, Segovia and Torrea [10] who showed that the maximal commutator of $M_{\mathcal{C}}$ with b is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$ if and only if $b \in \mathrm{BMO}(\mathbb{R}^n)$. Other interesting papers related to this topic can be consulted [1, 31], among others. It should be pointed out that the corresponding results also hold for $[b, M]$, $[b, M_\alpha]$, $\mathfrak{M}_b$ or $\mathfrak{M}_{b,\alpha}$, which is based on the fact that the Hardy–Littlewood maximal operator associated to balls has same properties as the Hardy–Littlewood maximal operator associated to cubes.

From Theorems 1.1 and 1.3, one sees that, the assumptions that these symbols b belong to Sobolev spaces guarantee certain Sobolev regularity for the commutators of the Hardy–Littlewood maximal operator and its fractional version. A natural question is the following

Question 1.4. Do the commutators $[b, M]$, $[b, M_\alpha]$, $\mathfrak{M}_b$ and $\mathfrak{M}_{b,\alpha}$ have somewhat Sobolev regularity properties when the symbols b are not Sobolev functions?

Question 1.4 is one of main motivations in this paper. In order to formulate our main results, let us introduce Lipschitz space.

Definition 1.5 (Lipschitz space)**.** The *homogeneous* Lipschitz space $Lip(\mathbb{R}^n)$ is defined by

$$Lip(\mathbb{R}^n) := \{f\colon \mathbb{R}^n \to \mathbb{C} \text{ continuous} : \|f\|_{Lip(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{Lip(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n} \sup_{h\in\mathbb{R}^n\setminus\{0\}} \frac{|f(x+h)-f(x)|}{|h|} < \infty.$$

The *inhomogeneous* Lipschitz space $\mathrm{Lip}(\mathbb{R}^n)$ is given by

$$\mathrm{Lip}(\mathbb{R}^n) := \{f\colon \mathbb{R}^n \to \mathbb{C} \text{ continuous} : \|f\|_{\mathrm{Lip}(\mathbb{R}^n)} < \infty\},$$

where

$$\|f\|_{\mathrm{Lip}(\mathbb{R}^n)} := \|f\|_\infty + \|f\|_{Lip(\mathbb{R}^n)} < \infty.$$

Remark 1.6. Let $b \in \mathrm{Lip}(\mathbb{R}^n)$. Then the weak partial derivatives $D_i b$, $i = 1, \ldots, n$, exist almost everywhere. Moreover, it holds that

$$D_i b(x) = \lim_{h\to 0} \frac{b(x+he_i)-b(x)}{h} \tag{1.1}$$

and

$$|D_i b(x)| \le \|b\|_{Lip(\mathbb{R}^n)} \tag{1.2}$$

for almost every $x \in \mathbb{R}^n$. Here $e_l = (0, \ldots, 0, i, 0, \ldots, 0)$ is the canonical i-th base vector in $\mathbb{R}^n$ for $i = 1, \ldots, n$.

To see (1.1) and (1.2), let us fix $i = 1, \ldots, n$. Since b is Lipschitz continuous, then by Rademacher's theorem, we know that b is differentiable almost everywhere. Then the partial derivatives $D_i b$ exists almost everywhere and (1.1) holds. For almost every $x \in \mathbb{R}^n$, we get by (1.1) that

$$|D_i b(x)| = \left|\lim_{h\to 0} \frac{b(x+he_i)-b(x)}{h}\right| \le \lim_{h\to 0} \frac{|b(x+he_i)-b(x)|}{|h|} \le \|b\|_{Lip(\mathbb{R}^n)},$$

which gives (1.2). It follows from (1.2) that

$$|\nabla b(x)| \le \sqrt{n}\|b\|_{Lip(\mathbb{R}^n)} \tag{1.3}$$

for almost every $x \in \mathbb{R}^n$.

We also list some comments on $[b, M_\alpha]$ and $\mathfrak{M}_{b,\alpha}$, which are useful for our aim.

Remark 1.7. Let $1 < p < \infty$, $0 \le \alpha < n/p$ and $1/q = 1/p - \alpha/n$. If $b \in L^\infty(\mathbb{R}^n)$, then the following are valid:

(i) If $f \in L^p(\mathbb{R}^n)$, then

$$\|M_\alpha f\|_q \leq A_{\alpha,n,p}\|f\|_p, \tag{1.4}$$

where $A_{\alpha,n,p} := \|M_\alpha\|_{L^p(\mathbb{R}^n)\to L^q(\mathbb{R}^n)}$. By (1.4) and the sublinearity of M_α, we see that $M_\alpha\colon L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is continuous.

(ii) The operator $[b, M_\alpha]$ is neither positive nor sublinear. Applying Hölder's inequality and (1.4), one has

$$\|[b, M_\alpha](f)\|_q \leq 2A_{\alpha,n,p}\|b\|_\infty\|f\|_p. \tag{1.5}$$

On the other hand, it is easy to see that

$$|[b, M_\alpha](f) - [b, M_\alpha](g)| \leq |b|M_\alpha(f - g) + M_\alpha(b(f - g)),$$

which together with (1.4) implies that the map $[b, M_\alpha]\colon L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is continuous.

(iii) The operator $\mathfrak{M}_{b,\alpha}$ is positive and sublinear. Observe that

$$\mathfrak{M}_{b,\alpha}f(x) \leq |b(x)|M_\alpha f(x) + M_\alpha(bf)(x), \quad x \in \mathbb{R}^n. \tag{1.6}$$

Inequality (1.6) together with Hölder's inequality and (1.4) implies that

$$\|\mathfrak{M}_{b,\alpha}f\|_q \leq 2A_{\alpha,n,p}\|b\|_\infty\|f\|_p. \tag{1.7}$$

Moreover, by the sublinearity and boundedness for $\mathfrak{M}_{b,\alpha}$, we see that the map $\mathfrak{M}_{b,\alpha}\colon L^p(\mathbb{R}^n) \to L^q(\mathbb{R}^n)$ is continuous.

Now we shall provide a positive answer to Question 1.4 by the following theorem.

Theorem 1.8. *Let $1 < p < \infty$, $0 \leq \alpha < n/p$ and $1/q = 1/p - \alpha/n$. If $b \in \mathrm{Lip}(\mathbb{R}^n)$, then the map $[b, M_\alpha]\colon W^{1,p}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$ is bounded and continuous. In particular, if $f \in W^{1,p}(\mathbb{R}^n)$, then for each $i \in \{1, \ldots, n\}$ and almost every $x \in \mathbb{R}^n$,*

$$|\nabla([b, M_\alpha](f))(x)| \leq 2\sqrt{n}\|b\|_{Lip(\mathbb{R}^n)}M_\alpha f(x) + 2\|b\|_\infty M_\alpha|\nabla f|(x). \tag{1.8}$$

Consequently,

$$\|[b, M_\alpha](f)\|_{1,q} \leq 2A_{\alpha,n,p}(n + 1)\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{1,p}, \tag{1.9}$$

where $A_{\alpha,n,p}$ is given as in (1.4)*. Inequalities* (1.8) *and* (1.9) *also hold for $\mathfrak{M}_{b,\alpha}$.*

Remark 1.9. It is unknown that whether the map $\mathfrak{M}_{b,\alpha}\colon W^{1,p}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$ is continuous under the same conditions in Theorem 1.8, which is certainly an interesting issue.

We would like to remark that the proof of Theorem 1.8 for $[b, M_\alpha]$ is based on Lemma 2.1 and some known regularity results on the Hardy–Littlewood maximal operator and its fractional version (see Lemma 2.4). The main ingredients of proving Theorem 1.8 for $\mathfrak{M}_{b,\alpha}$ are the equivalent characterization of Sobolev spaces (see (2.1)), the characterization of the product of a function in $W^{1,p}(\mathbb{R}^n)$ and a function in $\mathrm{Lip}(\mathbb{R}^n)$ (see Lemma 2.1) and the difference estimates for $\mathfrak{M}_{b,\alpha} f$ (see (3.5)).

1.3. The local case

Let us recall the definitions of the commutators of local fractional maximal operator.

Definition 1.10. Let Ω be a subdomain in $\mathbb{R}^n$ and $0 \le \alpha < n$. We denote by $M_{\alpha,\Omega}$ the local fractional maximal operator on Ω, i.e., for $f \in L^1_{\mathrm{loc}}(\Omega)$,

$$M_{\alpha,\Omega} f(x) = \sup_{0<r<\mathrm{dist}(x,\Omega^c)} \frac{r^\alpha}{|B(x,r)|} \int_{B(x,r)} |f(y)|\, dy, \quad x \in \Omega,$$

where $\Omega^c = \mathbb{R}^n \setminus \Omega$. For a locally integrable function b defined on Ω, we define the commutator of local fractional maximal operator $[b, M_{\alpha,\Omega}]$ by

$$[b, M_{\alpha,\Omega}](f)(x) = b(x) M_{\alpha,\Omega} f(x) - M_{\alpha,\Omega}(bf)(x), \quad x \in \Omega.$$

The fractional maximal commutator of $M_{\alpha,\Omega}$ with b is defined by

$$\mathfrak{M}_{b,\alpha,\Omega} f(x) = \sup_{0<r<\mathrm{dist}(x,\Omega^c)} \frac{r^\alpha}{|B(x,r)|} \int_{B(x,r)} |b(x) - b(y)||f(y)|\, dy, \quad x \in \Omega.$$

It is clear that $[b, M_{\alpha,\Omega}] = [b, M_\alpha]$ and $\mathfrak{M}_{b,\alpha,\Omega} = \mathfrak{M}_{b,\alpha}$ when $\Omega = \mathbb{R}^n$. When $\alpha = 0$, we denote $[b, M_{\alpha,\Omega}] = [b, M_\Omega]$ and $\mathfrak{M}_{b,\alpha,\Omega} = \mathfrak{M}_{b,\Omega}$, the operator $M_{\alpha,\Omega}$ reduces to the usual local Hardy–Littlewood maximal operator M_Ω.

The following are some basic properties for $[b, M_{\alpha,\Omega}]$ and $\mathfrak{M}_{b,\alpha,\Omega}$, which are useful for our proofs.

Remark 1.11. Let $1 < p < \infty$, $0 \le \alpha < n/p$ and $1/q = 1/p - \alpha/n$. Assume that $b \in L^\infty(\Omega)$. The following facts are valid:

(i) If $f \in L^p(\Omega)$, then

$$\|M_{\alpha,\Omega} f\|_{q,\Omega} \le C_{\alpha,n,p} \|f\|_{p,\Omega}. \tag{1.10}$$

By (1.10) and the sublinearity of $M_{\alpha,\Omega}$, we see that $M_{\alpha,\Omega}\colon L^p(\Omega) \to L^q(\Omega)$ is continuous.

(ii) The operator $[b, M_{\alpha,\Omega}]$ is neither positive nor sublinear. Applying Hölder's inequality and (1.10), one may get

$$\|[b, M_{\alpha,\Omega}](f)\|_{q,\Omega} \leq C_{\alpha,n,p} \|b\|_{\infty,\Omega} \|f\|_{p,\Omega}. \tag{1.11}$$

It is not difficult to see that

$$|[b, M_{\alpha,\Omega}](f) - [b, M_{\alpha,\Omega}](g)| \leq |b| M_{\alpha,\Omega}(f - g) + M_{\alpha,\Omega}(b(f - g)),$$

which together with (1.10) implies that $[b, M_{\alpha,\Omega}] \colon L^p(\Omega) \to L^q(\Omega)$ is continuous.

(iii) The operator $\mathfrak{M}_{b,\alpha,\Omega}$ is positive and sublinear. Observe that

$$\mathfrak{M}_{b,\alpha,\Omega} f(x) \leq |b(x)| M_{\alpha,\Omega} f(x) + M_{\alpha,\Omega}(bf)(x), \quad x \in \Omega. \tag{1.12}$$

Inequality (1.12) together with Hölder's inequality and (1.10) implies that

$$\|\mathfrak{M}_{b,\alpha,\Omega} f\|_{q,\Omega} \leq C_{\alpha,n,p} \|b\|_{\infty,\Omega} \|f\|_{p,\Omega}. \tag{1.13}$$

Moreover, we get by the sublinearity and boundedness for $\mathfrak{M}_{b,\alpha,\Omega}$ that the map $\mathfrak{M}_{b,\alpha,\Omega} \colon L^p(\Omega) \to L^q(\Omega)$ is continuous.

The Sobolev regularity for maximal operators in local setting has been studied by many authors. The first work was due to Kinnunen and Lindqvist [17] who proved that the map $M_\Omega \colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is bounded for all $1 < p \leq \infty$ (see also [13]). Here $W^{1,p}(\Omega)$ is the first order Sobolev space on Ω, which is defined in the same way as $W^{1,p}(\mathbb{R}^n)$, but with $\mathbb{R}^n$ replaced by Ω. For simplicity, we denote

$$\|f\|_{1,p,\Omega} := \|f\|_{W^{1,p}(\Omega)}, \quad \|f\|_{p,\Omega} := \|f\|_{L^p(\Omega)}.$$

Later on, the main result of [17] was extended by many authors (see [14, 15, 25, 28]). Particularly, Liu and Xi [25] established the Sobolev regularity of $[b, M_{\alpha,\Omega}]$ and $\mathfrak{M}_{b,\alpha,\Omega}$. The main results of [25] can be listed as follows:

Theorem 1.12. [25]

(i) *Let $1 < p_1, p_2, p_1 p_2/(p_1 + p_2) < \infty$, $1 < p_1 < n$, $1 \leq \alpha < n/p_1$ and $1/p = 1/p_1 + 1/p_2 - (\alpha - 1)/n$. Assume that $|\Omega| < \infty$ and $b \in W^{1,p_2}(\Omega)$, then the map $[b, M_{\alpha,\Omega}] \colon W^{1,p_1}(\Omega) \to W^{1,p}(\Omega)$ is bounded. Moreover, if $f \in W^{1,p_1}(\Omega)$, then*

$$\|[b, M_{\alpha,\Omega}](f)\|_{1,p,\Omega} \leq C_{\alpha,n,p_1,p_2,\Omega} \|b\|_{1,p_2,\Omega} \|f\|_{1,p_1,\Omega}.$$

(ii) *Let $1 < p_1, p_2, p_1 p_2/(p_1 + p_2) < \infty$, $1 < p_1 < n$, $1 \leq \alpha < n/p_1$ and $1/p = 1/p_1 + 1/p_2 - \alpha/n$. Assume that $b \in W^{1,p_2}(\Omega)$ and Ω admits a p_1-Sobolev embedding, then the map $[b, M_{\alpha,\Omega}] \colon W^{1,p_1}(\Omega) \to W^{1,p}(\Omega)$ is bounded. Moreover, if $f \in W^{1,p_1}(\Omega)$, then*

$$\|[b, M_{\alpha,\Omega}](f)\|_{1,p,\Omega} \leq C_{\alpha,n,p_1,p_2} \|b\|_{1,p_2,\Omega} \|f\|_{1,p_1,\Omega}.$$

(iii) *Let* $1 < p_1, p_2 < \infty$, $n/(n-1) < p_1p_2/(p_1+p_2) < \infty$, $1 \leq \alpha < n/p_1$ *and* $1/p = 1/p_1 + 1/p_2 - (\alpha-1)/n$. *Assume that* $|\Omega| < \infty$ *and* $b \in W^{1,p_2}(\Omega)$, *then the map* $[b, M_{\alpha,\Omega}]\colon L^{p_1}(\Omega) \to W^{1,p}(\Omega)$ *is bounded. Moreover, if* $f \in L^{p_1}(\Omega)$, *then*

$$\|[b, M_{\alpha,\Omega}](f)\|_{1,p,\Omega} \leq C_{\alpha,n,p_1,p_2,\Omega}\|b\|_{1,p_2,\Omega}\|f\|_{p_1,\Omega}.$$

The above conclusions hold for $\mathfrak{M}_{b,\alpha,\Omega}$.

Based on Theorems 1.8 and 1.12, it is natural to ask the following question:

Question 1.13. What happens when we consider the Sobolev regularity properties for $[b, M_{\alpha,\Omega}]$ and $\mathfrak{M}_{b,\alpha,\Omega}$ when b belongs to local Lipschitz space?

Question 1.13 is another one of main motivations in this paper. Before establishing the rest results, let us introduce local Lipschitz space.

Definition 1.14 (Local Lipschitz space)**.** The *homogeneous* local Lipschitz space $Lip(\Omega)$ is defined as

$$Lip(\Omega) := \big\{f\colon \Omega \to \mathbb{C} \text{ continuous} : \|f\|_{Lip(\Omega)} < \infty\big\},$$

where

$$\|f\|_{Lip(\Omega)} := \sup_{x\in\Omega} \sup_{h\in\Omega\setminus\{0\}} \frac{|f(x+h)-f(x)|}{|h|} < \infty.$$

The inhomogeneous Lipschitz space $\mathrm{Lip}(\Omega)$ is given by

$$\mathrm{Lip}(\Omega) := \big\{f\colon \Omega \to \mathbb{C} \text{ continuous} : \|f\|_{\mathrm{Lip}(\Omega)} < \infty\big\},$$

where

$$\|f\|_{\mathrm{Lip}(\Omega)} := \|f\|_{\infty,\Omega} + \|f\|_{Lip(\Omega)} < \infty.$$

Remark 1.15. Let $b \in \mathrm{Lip}(\Omega)$. Then the weak partial derivatives D_ib, $i = 1, \ldots, n$, exist almost everywhere. Moreover, it holds that

$$D_ib(x) = \lim_{h\to 0} \frac{b(x+he_i)-b(x)}{h}$$

and

$$|D_ib(x)| \leq \|b\|_{Lip(\Omega)} \tag{1.14}$$

for almost every $x \in \Omega$. By (1.14), we get

$$|\nabla b(x)| \leq \sqrt{n}\|b\|_{Lip(\Omega)} \tag{1.15}$$

for almost every $x \in \Omega$.

The rest of main results can be listed as follows:

Theorem 1.16. *Let* $b \in \mathrm{Lip}(\Omega)$.

(i) *Let* $1 < p < \infty$. *Then the map* $[b, M_\Omega]\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ *is bounded and continuous. If* $f \in W^{1,p}(\Omega)$, *then*

$$|\nabla[b, M_\Omega](f)(x)| \le 4\|b\|_{\infty,\Omega} M_\Omega|\nabla f|(x) + 3\sqrt{n}\|b\|_{Lip(\Omega)} M_\Omega f(x) \tag{1.16}$$

for almost every $x \in \Omega$. *Consequently,*

$$\|[b, M_\Omega](f)\|_{1,p,\Omega} \le C_{n,p}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega}. \tag{1.17}$$

(ii) *Let* $p \in (1, n)$, $\alpha \in [1, n/p)$ *and* $q = np/(n - (\alpha - 1)p)$. *Assume that* $|\Omega| < \infty$, *then* $[b, M_{\alpha,\Omega}]\colon W^{1,p}(\Omega) \to W^{1,q}(\Omega)$ *is bounded and continuous. In particular, if* $f \in W^{1,p}(\Omega)$, *then*

$$\begin{aligned}&|\nabla[b, M_{\alpha,\Omega}](f)(x)| \\ &\le 3\sqrt{n}\|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) + 4\|b\|_{\infty,\Omega} M_{\alpha,\Omega}|\nabla f|(x) + 2\alpha\|b\|_{\infty,\Omega} M_{\alpha-1,\Omega} f(x)\end{aligned} \tag{1.18}$$

for almost every $x \in \Omega$. *Consequently,*

$$\|[b, M_{\alpha,\Omega}](f)\|_{1,q,\Omega} \le C_{\alpha,n,p,|\Omega|}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega}. \tag{1.19}$$

(iii) *Let* $p \in (1, n)$, $\alpha \in [1, n/p)$ *and* $q = pn/(n - \alpha p)$. *If* $f \in W^{1,p}(\Omega)$ *and* Ω *admits a* p-*Sobolev embedding, i.e.,* $\|f\|_{\widetilde{p},\Omega} \le C_p\|f\|_{1,p,\Omega}$ *with* $\widetilde{p} = np/(n-p)$, *then* $[b, M_{\alpha,\Omega}](f) \in W^{1,q}(\Omega)$. *Moreover,*

$$\begin{aligned}&|\nabla[b, M_{\alpha,\Omega}](f)(x)| \\ &\le 3\sqrt{n}\|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) + 4\|b\|_{\infty,\Omega} M_{\alpha,\Omega}|\nabla f|(x) + 2\alpha\|b\|_{\infty,\Omega} M_{\alpha-1,\Omega} f(x)\end{aligned} \tag{1.20}$$

for almost every $x \in \Omega$. *Consequently,*

$$\|[b, M_{\alpha,\Omega}](f)\|_{1,q,\Omega} \le C_{\alpha,n,p}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega}. \tag{1.21}$$

(iv) *Let* $p \in (n/(n-1), \infty)$, $\alpha \in [1, \min\{(n-1)/p, n - 2n/((n-1)p)\} + 1)$, $q = np/(n - (\alpha - 1)p)$ *and* $|\Omega| < \infty$. *If* $f \in L^p(\Omega)$, *then*

$$\begin{aligned}&|\nabla[b, M_{\alpha,\Omega}](f)(x)| \\ &\le \sqrt{n}\|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) + C_n\|b\|_{\infty,\Omega}\big(M_{\alpha-1,\Omega} f(x) + \mathcal{S}_{\alpha-1,\Omega} f(x)\big)\end{aligned} \tag{1.22}$$

for almost every $x \in \Omega$. *Moreover,*

$$\|[b, M_{\alpha,\Omega}](f)\|_{1,q,\Omega} \le C_{\alpha,n,p,|\Omega|}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{p,\Omega}. \tag{1.23}$$

Here $\mathcal{S}_{\alpha,\Omega}$ is the local spherical maximal operator, i.e.,

$$\mathcal{S}_{\alpha,\Omega}f(x) = \sup_{0<r<\operatorname{dist}(x,\Omega^c)} \frac{r^\alpha}{|\partial B(x,r)|} \int_{\partial B(x,r)} |f(y)| \, d\mathcal{H}^{n-1}(y),$$

where $d\mathcal{H}^{n-1}$ is the normalized $(n-1)$-dimensional Hausdorff measure.

Theorem 1.17. *Let $b \in \operatorname{Lip}(\Omega)$.*

(i) *Let $1 < p < \infty$. Then the map $\mathfrak{M}_{b,\Omega}\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is bounded. If $f \in W^{1,p}(\Omega)$, then*

$$|\nabla \mathfrak{M}_{b,\Omega} f(x)| \le 3\sqrt{n} \|b\|_{Lip(\Omega)} M_\Omega f(x) + 2\mathfrak{M}_{b,\Omega}|\nabla f|(x) \tag{1.24}$$

for almost every $x \in \Omega$. Consequently,

$$\|\mathfrak{M}_{b,\Omega} f\|_{1,p,\Omega} \le C_{n,p} \|b\|_{\operatorname{Lip}(\Omega)} \|f\|_{1,p,\Omega}. \tag{1.25}$$

(ii) *Let $p \in (1,n)$, $\alpha \in [1, n/p)$ and $q = np/(n-(\alpha-1)p)$. Assume that $|\Omega| < \infty$, then $\mathfrak{M}_{b,\alpha,\Omega}\colon W^{1,p}(\Omega) \to W^{1,q}(\Omega)$ is bounded. In particular, if $f \in W^{1,p}(\Omega)$, then*

$$|\nabla \mathfrak{M}_{b,\alpha,\Omega} f(x)| \le \alpha \mathfrak{M}_{b,\alpha-1,\Omega} f(x) + 3\sqrt{n} \|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) + 2\mathfrak{M}_{b,\alpha,\Omega}|\nabla f|(x) \tag{1.26}$$

for almost every $x \in \Omega$. Consequently,

$$\|\mathfrak{M}_{b,\alpha,\Omega} f\|_{1,q,\Omega} \le C_{\alpha,n,p,|\Omega|} \|b\|_{\operatorname{Lip}(\Omega)} \|f\|_{1,p,\Omega}. \tag{1.27}$$

(iii) *Let $p \in (1,n)$, $\alpha \in [1, n/p)$ and $q = pn/(n-\alpha p)$. If $f \in W^{1,p}(\Omega)$ and Ω admits a p-Sobolev embedding, i.e., $\|f\|_{\widetilde{p},\Omega} \le C_p \|f\|_{1,p,\Omega}$ with $\widetilde{p} = np/(n-p)$, then $\mathfrak{M}_{b,\alpha,\Omega} f \in W^{1,q}(\Omega)$. Moreover,*

$$|\nabla \mathfrak{M}_{b,\alpha,\Omega} f(x)| \le \alpha \mathfrak{M}_{b,\alpha-1,\Omega} f(x) + 2\mathfrak{M}_{b,\alpha,\Omega}|\nabla f|(x) + 2\sqrt{n} \|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) \tag{1.28}$$

for almost every $x \in \Omega$. Consequently,

$$\|\mathfrak{M}_{b,\alpha,\Omega} f\|_{1,q,\Omega} \le C_{\alpha,n,p} \|b\|_{\operatorname{Lip}(\Omega)} \|f\|_{1,p,\Omega}. \tag{1.29}$$

(iv) *Let $p \in (n/(n-1), \infty)$, $\alpha \in [1, \min\{(n-1)/p, n-2n/((n-1)p)\}+1)$ and $|\Omega| < \infty$. If $f \in L^p(\Omega)$, then*

$$\begin{aligned} &|\nabla \mathfrak{M}_{b,\alpha,\Omega} f(x)| \\ &\le (n-\alpha) \mathfrak{M}_{b,\alpha-1,\Omega} f(x) + \sqrt{n} \|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) + 2n \|b\|_{\infty,\Omega} \mathcal{S}_{\alpha-1,\Omega} f(x) \end{aligned} \tag{1.30}$$

for almost every $x \in \Omega$. Moreover,

$$\|\mathfrak{M}_{b,\alpha,\Omega} f\|_{1,q,\Omega} \le C_{\alpha,n,p,|\Omega|} \|b\|_{\operatorname{Lip}(\Omega)} \|f\|_{p,\Omega}, \tag{1.31}$$

where $q = np/(n-(\alpha-1)p)$.

The proof of Theorem 1.16 is based on Lemmas 2.5 and 2.6 and a characterization of the product of a function in $W^{1,p}(\Omega)$ and a function in $\mathrm{Lip}(\Omega)$ (see Lemma 2.3). The proof of Theorem 1.17 is motivated by the idea in [14, 15, 25]. However, some new techniques and more refined analyses are needed in the proof of Theorem 1.17.

Finally, we shall show that the above commutators preserve the zero boundary values in Sobolev's sense. Recall that the Sobolev space $W_0^{1,p}(\Omega)$ with zero boundary values with $1 \leq p < \infty$, is defined as the completion of $\mathcal{C}_0^\infty(\Omega)$ with respect to the Sobolev norm. In 1998, Kinnunen and Lindqvist [17] first proved that M_Ω is bounded on $W_0^{1,p}(\Omega)$ with $1 < p < \infty$. Later on, Heikkinen, Kinnunen, Korvenpää and Tuominen [15] proved that $M_{\alpha,\Omega}$ is bounded from $L^p(\Omega)$ to $W_0^{1,q}(\Omega)$ for $p > n/(n-1)$, $1 \leq \alpha < n/p$ and $q = np/(n-(\alpha-1)p)$ by assuming that $|\Omega| < \infty$. Recently, by assuming that $|\Omega| < \infty$, Hart, Liu and Xue [14] established the boundedness for $M_{\alpha,\Omega}\colon W^{1,p}(\Omega) \to W_0^{1,q}(\Omega)$ for $1 < p, q < \infty$, $q = np/(n-(\alpha-1)p)$, $1 \leq \alpha < n/p+1$.

As direct applications of Theorems 1.16 and 1.17, the following conclusions are valid.

Corollary 1.18. *Let $b \in \mathrm{Lip}(\Omega)$.*

(i) *Let $1 < p < \infty$. If $f \in W_0^{1,p}(\Omega)$, then $[b, M_\Omega](f) \in W_0^{1,p}(\Omega)$.*

(ii) *Let $p \in (1,n)$, $\alpha \in [1, n/p)$ and $q = np/(n-(\alpha-1)p)$. Assume that $|\Omega| < \infty$, then $[b, M_{\alpha,\Omega}]\colon W^{1,p}(\Omega) \to W_0^{1,q}(\Omega)$ is bounded.*

(iii) *Let $p \in (1,n)$, $\alpha \in [1, n/p)$ and $q = pn/(n-\alpha p)$. If $f \in W^{1,p}(\Omega)$ and Ω admits a p-Sobolev embedding, then the map $[b, M_{\alpha,\Omega}]\colon W^{1,p}(\Omega) \to W_0^{1,q}(\Omega)$ is bounded.*

(iv) *Let $p \in (n/(n-1), \infty)$, $\alpha \in [1, \min\{(n-1)/p, n-2n/((n-1)p)\}+1)$, $q = np/(n-(\alpha-1)p)$ and $|\Omega| < \infty$, then the map $[b, M_{\alpha,\Omega}]\colon L^p(\Omega) \to W_0^{1,q}(\Omega)$ is bounded.*

The same conclusions hold for the operator $\mathfrak{M}_{b,\alpha,\Omega}$.

1.4. Outline of this paper

This paper will be organized as follows. Section 2 contains some auxiliary notations and lemmas, which paly key roles in the proofs of Theorems 1.8, 1.16 and 1.17. In Section 3 we shall prove Theorem 1.8. The proofs for Theorems 1.16 and 1.17 will be given in Sections 4 and 5, respectively. Finally, we shall prove Corollary 1.18 in Section 6.

2. Preliminary notations and lemmas

This section is devoted to presenting some notations and lemmas.

2.1. Preliminary notations

Throughout the paper, the letter $C_{\alpha,\beta}$ denote the positive constants that depend on the parameters α, β. Let $f \in L^p(\mathbb{R}^n)$ with $p \geq 1$. For all $h \in \mathbb{R}$, $|h| > 0$, $y \in \mathbb{R}^n$ and $i = 1, \dots, n$, we define the function $f_{h,i}$ by setting

$$f_{h,i}(x) = \frac{f(x + he_i) - f(x)}{|h|}, \quad x \in \mathbb{R}^n.$$

It is well known that for $p \geq 1$, $f_{h,i} \to D_i f$ in $L^p(\mathbb{R}^n)$ when $h \to 0$ if $f \in W^{1,p}(\mathbb{R}^n)$. For $h \in \mathbb{R}^n$ and any arbitrary functions f defined on $\mathbb{R}^n$, we define the first order difference of f by

$$\Delta_h f(x) := f(x + h) - f(x), \quad x \in \mathbb{R}^n.$$

For $y \in \mathbb{R}^n$, we define the function f_y by $f_y(x) = f(x + y)$. Set

$$G(f; p) = \limsup_{h \to 0} \frac{\|\Delta_h f\|_p}{|h|}.$$

According to [11, Section 7.11], we have

$$u \in W^{1,q}(\mathbb{R}^n),\ 1 < q < \infty \quad \Longleftrightarrow \quad u \in L^q(\mathbb{R}^n) \text{ and } G(u; q) < \infty. \tag{2.1}$$

2.2. Some lemmas

The following result presents a characterization of the product of a function in $W^{1,p}(\mathbb{R}^n)$ and a function in $\mathrm{Lip}(\mathbb{R}^n)$.

Lemma 2.1. *Let $1 < p < \infty$. If $f \in W^{1,p}(\mathbb{R}^n)$ and $b \in \mathrm{Lip}(\mathbb{R}^n)$, then $bf \in W^{1,p}(\mathbb{R}^n)$. Moreover,*

$$D_i(bf) = bD_i f + fD_i b, \quad i = 1, \dots, n \tag{2.2}$$

almost everywhere in $\mathbb{R}^n$. Consequently,

$$\nabla(bf) = b\nabla f + f\nabla b \tag{2.3}$$

almost everywhere in $\mathbb{R}^n$. In particular, it holds that

$$\|bf\|_{1,p} \leq \sqrt{n}\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{1,p}. \tag{2.4}$$

Proof. The proof is similar to that of Lemma 2.1 in [26]. At first, we claim that $bf \in W^{1,p}(\mathbb{R}^n)$. It is easy to see that

$$\|bf\|_p \leq \|b\|_\infty \|f\|_p. \tag{2.5}$$

One can easily check that

$$\Delta_h(bf) = (bf)_h - bf = b_h \Delta_h f + f \Delta_h b \tag{2.6}$$

for any $h \in \mathbb{R}^n$. It follows from (2.6) that

$$G(bf;p) = \limsup_{h\to 0} \frac{\|\Delta_h(bf)\|_p}{|h|} \le \|b\|_\infty G(f;p) + \|b\|_{Lip(\mathbb{R}^n)}\|f\|_p < \infty.$$

This together with (2.1) and (2.5) yields that $bf \in W^{1,p}(\mathbb{R}^n)$.

Next we shall prove (2.2). Fix $i \in \{1,\dots,n\}$. Noting that $f_{h,i} \to D_i f$ and $(bf)_{h,i} \to D_i(bf)$ in $L^p(\mathbb{R}^n)$ when $h \to 0$. This together with (1.1) yields that there exist a sequence of numbers $\{h_k\}$ satisfying $\lim_{k\to\infty} h_k = 0$ and a measurable set E satisfying $|\mathbb{R}^n \setminus E| = 0$ such that $f_{h_k,i}(x) \to D_i f(x)$, $(bf)_{h_k,i}(x) \to D_i(bf)(x)$ and $b_{h_k,i}(x) \to D_i b(x)$ as $k \to \infty$ for all $x \in E$. It is easy to see that $b(h_k e_i + x) \to b(x)$ as $k \to \infty$ for all $x \in E$ since $|b(h_k e_i + x) - b(x)| \le \|b\|_{Lip(\mathbb{R}^n)}|h_k|$.

Therefore, we get from (2.6) that

$$\begin{aligned} D_i(bf)(x) &= \lim_{k\to\infty} (bf)_{h_k,i}(x) \\ &= \lim_{k\to\infty} (b(h_k e_i + x) f_{h_k,i}(x) + b_{h_k,i}(x) f(x)) = b(x) D_i f(x) + f(x) D_i b(x) \end{aligned}$$

for any $x \in E$. This proves (2.2). Equality (2.3) follows easily from (2.2). By (2.3), (2.5), (1.3) and Minkowski's inequality, we have

$$\|bf\|_{1,p} = \|bf\|_p + \|\nabla(bf)\|_p \le \|b\|_\infty \|f\|_p + \|b\nabla f\|_p + \|f\nabla b\|_p \le \sqrt{n}\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{1,p},$$

which gives (2.4) and completes the proof. □

Proposition 2.2. [15, 17] *Let $1 \le p \le \infty$. If $f_k \to f$, $g_k \to g$ weakly in $L^p(\Omega)$ and $f_k \le g_k$ $(k = 1, 2, \dots)$ almost everywhere in Ω, then $f \le g$ almost everywhere in Ω.*

Applying Proposition 2.2, we can get a local version of Lemma 2.1.

Lemma 2.3. *Let $1 < p < \infty$. If $b \in \mathrm{Lip}(\Omega)$ and $f \in W^{1,p}(\Omega)$, then $bf \in W^{1,p}(\Omega)$. Moreover,*

$$\nabla(bf) = b\nabla f + f\nabla b \tag{2.7}$$

almost everywhere in Ω. In particular, it holds that

$$\|bf\|_{1,p,\Omega} \le \sqrt{n}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega}. \tag{2.8}$$

Proof. Since $f \in W^{1,p}(\Omega)$, there exists a sequence $\{\varphi_j\}_{j=1}^{\infty}$ of functions in $W^{1,p}(\Omega) \cap \mathcal{C}^{\infty}(\Omega)$ such that $\varphi_j \to f$ in $W^{1,p}(\Omega)$ as $j \to \infty$. Fix $j \in \mathbb{N}$. Note that b is differentiable almost everywhere in Ω. By Leibniz rule,

$$D_i(\varphi_j b)(x) = D_i\varphi_j(x)b(x) + D_i b(x)\varphi_j(x)$$

for all $i = 1, \ldots, n$ and almost every $x \in \Omega$. Thus we have

$$\nabla(\varphi_j b)(x) = b(x)\nabla\varphi_j(x) + \varphi_j(x)\nabla b(x) \tag{2.9}$$

for almost every $x \in \Omega$. By (2.9), (1.15) and Minkowski's inequality, one finds that

$$\begin{aligned} \|\varphi_j b\|_{1,p,\Omega} &= \|\varphi_j b\|_{p,\Omega} + \|\nabla(\varphi_j b)\|_{p,\Omega} \\ &\leq \|b\|_{\infty,\Omega}\|\varphi_j\|_{p,\Omega} + \|\nabla\varphi_j b\|_{p,\Omega} + \|\nabla b\varphi_j\|_{p,\Omega} \leq \sqrt{n}\|b\|_{\mathrm{Lip}(\Omega)}\|\varphi_j\|_{1,p,\Omega} \end{aligned} \tag{2.10}$$

and

$$\begin{aligned} \|\nabla(\varphi_j b) - (b(\nabla f) + f(\nabla b))\|_{p,\Omega} &\leq \|b\nabla\varphi_j - b\nabla f\|_{p,\Omega} + \|\varphi_j\nabla b - f\nabla b\|_{p,\Omega} \\ &\leq \sqrt{n}\|b\|_{\mathrm{Lip}(\Omega)}\|\varphi_j - f\|_{1,p,\Omega}. \end{aligned} \tag{2.11}$$

It follows from (2.10) and (2.11) that $\{\varphi_j b\}_{j=1}^{\infty}$ is a bounded sequence in $W^{1,p}(\Omega)$ and

$$\nabla(\varphi_j b) \to b\nabla f + f\nabla b \quad \text{in } L^p(\Omega) \text{ as } j \to \infty. \tag{2.12}$$

Since $\varphi_j b \to fb$ in $L^p(\Omega)$ as $j \to \infty$, then by Riesz theorem, there exists a subsequence $\{\varphi_{j_k} b\}_{k=1}^{\infty}$ such that

$$\varphi_{j_k}(x)b(x) \to f(x)b(x) \quad \text{as } k \to \infty$$

for almost every $x \in \Omega$. Consequently, there exists a measurable set E such that $|\Omega \backslash E| = 0$ and

$$\varphi_{j_k}(x)b(x) \to f(x)b(x) \quad \text{as } k \to \infty$$

for every $x \in E$. From the above we can conclude that the Sobolev derivative $\nabla(fb)$ exists almost everywhere in E and that

$$\nabla(\varphi_j b) \to \nabla(fb) \quad \text{weakly in } L^p(E) \text{ as } j \to \infty.$$

Furthermore, the weak gradient $\nabla(fb)$ exists almost everywhere in Ω and

$$\nabla(\varphi_j b) \to \nabla(fb) \quad \text{weakly in } L^p(\Omega) \text{ as } j \to \infty. \tag{2.13}$$

Applying Proposition 2.2 and (2.12), (2.13), we can get (2.7). By (2.7), (1.15) and Minkowski's inequality, one can get

$$\begin{aligned} \|bf\|_{1,p,\Omega} &= \|bf\|_{p,\Omega} + \|\nabla(bf)\|_{p,\Omega} \\ &\leq \|b\|_{\infty,\Omega}\|f\|_{p,\Omega} + \big(\|b\|_{\infty,\Omega}\|\nabla f\|_{p,\Omega} + \sqrt{n}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{p,\Omega}\big) \\ &\leq \sqrt{n}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega}, \end{aligned}$$

which gives (2.8). $\square$

In order to prove Theorem 1.8 for the commutator $[b, M_\alpha]$, we need the following known results.

Lemma 2.4. [16,19,22,27] *Let $1 < p < \infty$, $0 \le \alpha < n/p$ and $q = np/(n-\alpha p)$. Then the map $M_\alpha\colon W^{1,p}(\mathbb{R}^n) \to W^{1,q}(\mathbb{R}^n)$ is bounded and continuous. Moreover, if $f \in W^{1,p}(\mathbb{R}^n)$, then*

$$|D_i M_\alpha f(x)| \le M_\alpha D_i f(x), \quad i = 1, 2, \ldots, n$$

for almost every $x \in \mathbb{R}^n$. Consequently,

$$\|M_\alpha f\|_{1,q} \le C_{p,q,n} \|f\|_{1,p}.$$

The boundedness part and pointwise estimates in Lemma 2.4 for $\alpha = 0$ (resp., $0 < \alpha < n$) follows from [16, Theorem 1.4] (resp., [19, Theorem 2.1] and [19, Remark 2.2]). The continuity part in Lemma 2.4 for $\alpha = 0$ (resp., $0 < \alpha < n$) follows from [27, Theorem 4.1] (resp., [22, Remark 1]).

The following known results are the main ingredients of proving Theorem 1.16.

Lemma 2.5. (i) [17] *Let $1 < p < \infty$. Then the map $M_\Omega\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is bounded. Moreover, if $f \in W^{1,p}(\Omega)$, then*

$$|\nabla M_\Omega f(x)| \le 2 M_\Omega |\nabla f|(x)$$

for almost every $x \in \Omega$.

(ii) [28] *The map $M_\Omega\colon W^{1,p}(\Omega) \to W^{1,p}(\Omega)$ is continuous.*

Lemma 2.6. (i) [15] *Let $p \in (1,n)$, $\alpha \in [1, n/p)$ and $q = np/(n-(\alpha-1)p)$. If $f \in W^{1,p}(\Omega)$ and $|\Omega| < \infty$, then*

$$|\nabla M_{\alpha,\Omega} f(x)| \le 2 M_{\alpha,\Omega} |\nabla f|(x) + \alpha M_{\alpha-1,\Omega} f(x)$$

for almost every $x \in \Omega$. Consequently,

$$\|M_{\alpha,\Omega} f\|_{1,q,\Omega} \le C_{\alpha,n,p,\Omega} \|f\|_{1,p,\Omega}.$$

(ii) [14] *Let $p \in (1,n)$, $\alpha \in [1, n/p)$ and $q = np/(n-(\alpha-1)p)$. Assume that $|\Omega| < \infty$, then the map $M_{\alpha,\Omega}\colon W^{1,p}(\Omega) \to W^{1,q}(\Omega)$ is continuous.*

(iii) [14] *Let $p \in (1,n)$, $\alpha \in [1, n/p)$ and $q = pn/(n-\alpha p)$. If $f \in W^{1,p}(\Omega)$ and Ω admits a p-Sobolev embedding, i.e., $\|f\|_{\widetilde{p},\Omega} \le C_p \|f\|_{1,p,\Omega}$ with $\widetilde{p} = np/(n-p)$, then $M_{\alpha,\Omega} f \in W^{1,q}(\Omega)$. Moreover,*

$$|\nabla M_{\alpha,\Omega} f(x)| \le 2 M_{\alpha,\Omega} |\nabla f|(x) + \alpha M_{\alpha-1,\Omega} f(x)$$

for almost every $x \in \Omega$. Consequently,

$$\|M_{\alpha,\Omega} f\|_{1,q,\Omega} \le C_{\alpha,n,p} \|f\|_{1,p,\Omega}.$$

(iv) [15] *Let* $p \in (n/(n-1), \infty)$ *and* $\alpha \in [1, \min\{(n-1)/p, n-2n/((n-1)p)\}+1)$. *If* $f \in L^p(\Omega)$, *then*

$$|\nabla M_{\alpha,\Omega} f(x)| \le C_n\big(M_{\alpha-1,\Omega} f(x) + \mathcal{S}_{\alpha-1,\Omega} f(x)\big)$$

for almost every $x \in \Omega$. *If in addition* $|\Omega| < \infty$, *then*

$$\|M_{\alpha,\Omega} f\|_{1,q,\Omega} \le C_{\alpha,n,p,\Omega}\|f\|_{1,p,\Omega},$$

where $q = np/(n-(\alpha-1)p)$.

In order to prove Theorems 1.16 and 1.17, the following result is also needed.

Lemma 2.7. [15] *Let* $n \ge 2$, $p > n/(n-1)$, $0 \le \alpha < \min\{(n-1)/p, n-2n/((n-1)p)\}$. *Then* $\|\mathcal{S}_{\alpha,\Omega} f\|_{q,\Omega} \le C_{\alpha,n,p}\|f\|_{p,\Omega}$.

3. Proof of Theorem 1.8

Throughout this section, we fix $1 < p < \infty$, $0 \le \alpha < n/p$, $1/q = 1/p - \alpha/n$, $f \in W^{1,p}(\mathbb{R}^n)$ and $b \in \mathrm{Lip}(\mathbb{R}^n)$.

3.1. Proof of Theorem 1.8 for $[b, M_\alpha]$

The proof of Theorem 1.8 for $[b, M_\alpha]$ will be divided into two steps:

Step 1: *Proofs of* (1.8) *and* (1.9). Invoking Lemma 2.1 we note that $bf \in W^{1,p}(\mathbb{R}^n)$ and

$$D_i(bf) = bD_i f + fD_i b, \quad i = 1, \ldots, n \tag{3.1}$$

almost everywhere in $\mathbb{R}^n$. By Lemma 2.4, we have that $M_\alpha f \in W^{1,q}(\mathbb{R}^n)$. This together with Lemma 2.1, (3.1) and Lemma 2.4 implies that

$$\begin{aligned}
&|D_i([b, M_\alpha](f))(x)| \\
&\le |D_i(bM_\alpha f)(x)| + |D_i M_\alpha(bf)(x)| \\
&\le |D_i b(x)|M_\alpha f(x) + |b(x)||D_i M_\alpha f(x)| + M_\alpha(D_i(bf))(x) \\
&\le |D_i b(x)|M_\alpha f(x) + |b(x)||M_\alpha D_i f(x)| + M_\alpha(D_i bf)(x) + M_\alpha(D_i fb)(x)
\end{aligned} \tag{3.2}$$

for any $i = 1, \ldots, n$ and almost every $x \in \mathbb{R}^n$. By (3.2) and the arguments similar to those used to derive (2.4) in [16], we have

$$|\nabla([b, M_\alpha](f))(x)| \le |\nabla b(x)|M_\alpha f(x) + |b(x)|M_\alpha|\nabla f|(x) + M_\alpha(|\nabla b|f)(x) + M_\alpha(|\nabla f|b)(x)$$

for almost every $x \in \mathbb{R}^n$. This together with (1.3) leads to

$$|\nabla([b, M_\alpha](f))(x)| \le 2\sqrt{n}\|b\|_{Lip(\mathbb{R}^n)} M_\alpha f(x) + 2\|b\|_\infty M_\alpha|\nabla f|(x)$$

for almost every $x \in \mathbb{R}^n$. This proves (1.8). By (1.8), (1.4), (1.5) and Minkowski's inequality, we have

$$\begin{aligned}\|[b, M_\alpha](f)\|_{1,q} &= \|[b, M_\alpha](f)\|_q + \|\nabla[b, M_\alpha](f)\|_q \\ &\le 2A_{\alpha,n,p}\|b\|_\infty\|f\|_p + 2\sqrt{n}\|b\|_{Lip(\mathbb{R}^n)}\|M_\alpha f\|_q + 2\|b\|_{L^\infty(\mathbb{R}^n)}\|M_\alpha|\nabla f|\|_q \\ &\le 2A_{\alpha,n,p}\sqrt{n}\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{1,p}.\end{aligned}$$

This proves (1.9).

Step 2: Proof of the continuity part. Let $f_j \to f$ in $W^{1,p}(\mathbb{R}^n)$ as $j \to \infty$. By Lemma 2.1 we see that $bf_j \to bf$ in $W^{1,p}(\mathbb{R}^n)$ as $j \to \infty$. This together with the continuity part of Lemma 2.4 implies that $M_\alpha(bf_j) \to M_\alpha(bf)$ in $W^{1,q}(\mathbb{R}^n)$ as $j \to \infty$. Moreover, $M_\alpha f_j \to M_\alpha f$ in $W^{1,q}(\mathbb{R}^n)$ as $j \to \infty$. By Lemma 2.1 again, we have that $bM_\alpha f_j \to bM_\alpha f$ in $W^{1,q}(\mathbb{R}^n)$ as $j \to \infty$. Thus, $[b, M_\alpha](f_j) \to [b, M_\alpha](f)$ in $W^{1,q}(\mathbb{R}^n)$ as $j \to \infty$.

3.2. Proof of Theorem 1.8 for $\mathfrak{M}_{b,\alpha}$

We divide the proof of Theorem 1.8 for $\mathfrak{M}_{b,\alpha}$ into three steps:

Step 1: Proof of $\mathfrak{M}_{b,\alpha}f \in W^{1,q}(\mathbb{R}^n)$. Fix $x, h \in \mathbb{R}^n$, we can write

$$(\mathfrak{M}_{b,\alpha}f)_h(x) = \sup_{r>0}\frac{r^\alpha}{|B(x,r)|}\int_{B(x,r)}|b_h(x) - b_h(y)||f_h(y)|\,dy,$$

which leads to

$$\begin{aligned}&|(\mathfrak{M}_{b,\alpha}f)_h(x) - \mathfrak{M}_{b,\alpha}f(x)| \\ &\le \sup_{r>0}\frac{r^\alpha}{|B(x,r)|}\int_{B(x,r)}|(b_h(x) - b_h(y))f_h(y) - (b(x) - b(y))f(y)|\,dy.\end{aligned} \tag{3.3}$$

Observe that

$$\begin{aligned}&(b_h(x) - b_h(y))f_h(y) - (b(x) - b(y))f(y) \\ &= (b_h(x) - b_h(y))(f_h(y) - f(y)) + (b_h(x) - b(x))f(y) - (b_h(y) - b(y))f(y),\end{aligned}$$

which together with (3.3) and (1.6) implies that

$$\begin{aligned}|\Delta_h(\mathfrak{M}_{b,\alpha}f)(x)| &\le \mathfrak{M}_{b_h,\alpha}(f_h - f)(x) + |b_h(x) - b(x)|M_\alpha f(x) + M_\alpha((b_h - b)f)(x) \\ &\le |b(x+h)|M_\alpha(\Delta_h f)(x) + |\Delta_h b(x)|M_\alpha f(x) + M_\alpha(\Delta_h b\Delta_h f)(x) \\ &\quad + M_\alpha(\Delta_h bf)(x) + M_\alpha(\Delta_h fb)(x)\end{aligned} \tag{3.4}$$

for any $h, x \in \mathbb{R}^n$. By (3.4), (1.4) and Minkowski's inequality, one has

$$\begin{aligned}\|\Delta_h(\mathfrak{M}_{b,\alpha}f)\|_q &\le \|b(\cdot + h)M_\alpha(\Delta_h f)\|_q + \|\Delta_h bM_\alpha f\|_q + \|M_\alpha(\Delta_h b\Delta_h f)\|_q \\ &\quad + \|M_\alpha(\Delta_h bf)\|_q + \|M_\alpha(\Delta_h fb)\|_q \\ &\le \|b\|_\infty\|M_\alpha(\Delta_h f)\|_q + \|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|M_\alpha f\|_q|h| \\ &\quad + A_{\alpha,p,n}\big(\|\Delta_h b\Delta_h f\|_p + \|\Delta_h bf\|_p + \|\Delta_h fb\|_p\big) \\ &\le 4A_{\alpha,p,n}\big(\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|\Delta_h f\|_p + \|b\|_{Lip(\mathbb{R}^n)}\|f\|_p|h|\big),\end{aligned} \tag{3.5}$$

which combining with $G(f;p) < \infty$ leads to

$$G(\mathfrak{M}_{b,\alpha}f;q) = \limsup_{h\to 0} \frac{\|\Delta_h(\mathfrak{M}_{b,\alpha}f)\|_q}{|h|} \le 4A_{p,q}\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}(G(f;p) + \|f\|_p) < \infty. \tag{3.6}$$

On the other hand, by (1.7) we known that $\mathfrak{M}_{b,\alpha}f \in L^q(\mathbb{R}^n)$. This together with (3.6) and (2.1) implies that $\mathfrak{M}_{b,\alpha}f \in W^{1,q}(\mathbb{R}^n)$.

Step 2: Estimate for $\nabla\mathfrak{M}_{b,\alpha}(f)$. We want to show that

$$|\nabla(\mathfrak{M}_{b,\alpha}f)(x)| \le 2\sqrt{n}\|b\|_{Lip(\mathbb{R}^n)}M_\alpha f(x) + 2\|b\|_\infty M_\alpha|\nabla f|(x) \tag{3.7}$$

for almost every $x \in \mathbb{R}^n$. To prove (3.7), it suffices to show that

$$|D_i(\mathfrak{M}_{b,\alpha}f)(x)| \le |D_ib(x)|M_\alpha f(x) + |b(x)|M_\alpha D_if(x) + M_\alpha(D_ibf)(x) + M_\alpha(D_ifb)(x) \tag{3.8}$$

for any $i = 1,\dots,n$ and almost every $x \in \mathbb{R}^n$. Once (3.8) was proved, by (3.8) and using the arguments similar to those used in deriving (2.4) in [16], we have

$$|\nabla(\mathfrak{M}_{b,\alpha}f)(x)| \le |b(x)|M_\alpha|\nabla f|(x) + |\nabla b(x)|M_\alpha f(x) + M_\alpha(|\nabla b|f)(x) + M_\alpha(|\nabla f|b)(x)$$

for almost every $x \in \mathbb{R}^n$, which together with (1.3) leads to (3.7).

Now we prove (3.8). Fix $i \in \{1,\dots,n\}$. Since $f \in W^{1,p}(\mathbb{R}^n)$ and $\mathfrak{M}_{b,\alpha}f \in W^{1,q}(\mathbb{R}^n)$, then $f_{h,i} \to D_if$ in $L^p(\mathbb{R}^n)$ when $h \to 0$ and $([b,M_\alpha](f))_{h,i} \to D_i([b,M_\alpha](f))$ in $L^q(\mathbb{R}^n)$ when $h \to 0$. It is clear that $f_{h,i}b \to D_ifb$ in $L^p(\mathbb{R}^n)$ when $h \to 0$. By Remark 1.7(i), we have that $M_\alpha(f_{h,i}) \to M_\alpha(D_if)$ and $M_\alpha(f_{h,i}b) \to M_\alpha(D_ifb)$ in $L^q(\mathbb{R}^n)$ as $h \to 0$. By (1.2) and (1.3) we have that $|D_ib(x)| \le \|b\|_{Lip(\mathbb{R}^n)}$ and $b_{h,i}(x) \to D_ib(x)$ as $h \to 0$ for almost every $x \in \mathbb{R}^n$. It follows that $b_{h,i}(x)f(x) \to D_ib(x)f(x)$ as $h \to 0$ for almost every $x \in \mathbb{R}^n$. By Fatou lemma, we have that $b_{h,i}f \to D_ibf$ in $L^p(\mathbb{R}^n)$ as $h \to 0$. This together with Remark 1.7(i) implies that $M_\alpha(b_{h,i}f) \to M_\alpha(D_ibf)$ in $L^q(\mathbb{R}^n)$ as $h \to 0$. Therefore, there exist a sequence $\{h_k\}$ satisfying $h_k > 0$ and $\lim_{k\to 0} h_k = 0$ and a measurable set E with $|\mathbb{R}^n \setminus E| = 0$ such that for all $x \in E$,

(i) $(\mathfrak{M}_{b,\alpha}f)_{h_k,i}(x) \to D_i(\mathfrak{M}_{b,\alpha}f)(x)$ as $k \to \infty$ and $b_{h_k,i}(x) \to D_ib(x)$ as $k \to \infty$;

(ii) $M_\alpha(f_{h_k,i})(x) \to M_\alpha(D_if)(x)$ as $k \to \infty$, $M_\alpha(f_{h_k,i}b)(x) \to M_\alpha(D_ifb)(x)$ as $k \to \infty$;

(iii) $M_\alpha(b_{h_k,i}f)(x) \to M_\alpha(D_ibf)(x)$ as $k \to \infty$.

Above facts together with (3.4) will give that

$$\begin{aligned}
&|D_i(\mathfrak{M}_{b,\alpha}f)(x)| \\
&= \left|\lim_{k\to\infty}(\mathfrak{M}_{b,\alpha}f)_{h_k,i}(x)\right|
\end{aligned}$$

$$
\begin{aligned}
&\le \lim_{k\to\infty} \frac{1}{h_k}\big(|b(x+h_ke_i)|M_\alpha(\Delta_{h_ke_i}f)(x) + |\Delta_{h_ke_i}b(x)|M_\alpha f(x)\\
&\qquad\qquad + M_\alpha(\Delta_{h_ke_i}b\Delta_{h_ke_i}f)(x) + M_\alpha(\Delta_{h_ke_i}bf)(x) + M_\alpha(\Delta_{h_ke_i}fb)(x)\big)\\
&\le \lim_{k\to\infty} \big(|b(x+h_ke_i)|M_\alpha(f_{h_k,i})(x) + |b_{h_k,i}(x)|M_\alpha f(x)\\
&\qquad\qquad + M_\alpha(\Delta_{h_ke_i}bf_{h_k,i})(x) + M_\alpha(b_{h_k,i}f)(x) + M_\alpha(f_{h_k,i}b)(x)\big)\\
&\le \lim_{k\to\infty} \big((\|b\|_{Lip(\mathbb{R}^n)}|h_k| + |b(x)|)M_\alpha(f_{h_k,i})(x) + |b_{h_k,i}(x)|M_\alpha f(x)\\
&\qquad\qquad + \|b\|_{Lip(\mathbb{R}^n)}|h_k|M_\alpha(f_{h_k,i})(x) + M_\alpha(b_{h_k,i}f)(x) + M_\alpha(f_{h_k,i}b)(x)\big)\\
&\le |b(x)|M_\alpha(D_if)(x) + |D_ib(x)|M_\alpha f(x) + M_\alpha(D_ibf)(x) + M_\alpha(D_ifb)(x)
\end{aligned}
$$

for all $x \in E$. This proves (3.8).

Step 3*: Proof of the boundedness part.* It follows from (3.7), (1.4), (1.7) and Minkowski's inequality that

$$
\begin{aligned}
\|\mathfrak{M}_{b,\alpha}f\|_{1,q} &= \|\mathfrak{M}_{b,\alpha}f\|_q + \|\nabla\mathfrak{M}_{b,\alpha}f\|_q\\
&\le 2A_{\alpha,n,p}\|b\|_\infty\|f\|_p + 2\sqrt{n}\|b\|_{Lip(\mathbb{R}^n)}\|M_\alpha f\|_q + 2\|b\|_\infty\|M_\alpha|\nabla f|\|_q\\
&\le 2A_{\alpha,n,p}\sqrt{n}\|b\|_{\mathrm{Lip}(\mathbb{R}^n)}\|f\|_{1,p}.
\end{aligned}
$$

This finishes the proof of Theorem 1.8.

4. Proof of Theorem 1.16

The proof of Theorem 1.16 will be divided into four steps:

4.1. Proof of Theorem 1.16(i)

Let $1 < p < \infty$, $f \in W^{1,p}(\Omega)$ and $b \in \mathrm{Lip}(\Omega)$. By Lemma 2.3, we have that $bf \in W^{1,p}(\Omega)$ and

$$
\nabla(bf)(x) = b(x)\nabla f(x) + f(x)\nabla b(x) \tag{4.1}
$$

for almost every $x \in \Omega$. Combining (4.1) with Lemma 2.5(i) and (1.15) implies that

$$
\begin{aligned}
|\nabla M_\Omega(bf)(x)| &\le 2M_\Omega(\nabla(bf))(x) \le 2M_\Omega(b|\nabla f|)(x) + 2M_\Omega(f|\nabla b|)(x)\\
&\le 2\|b\|_{\infty,\Omega}M_\Omega|\nabla f|(x) + 2\sqrt{n}\|b\|_{Lip(\Omega)}M_\Omega f(x)
\end{aligned}
\tag{4.2}
$$

for almost every $x \in \Omega$. On the other hand, by Lemma 2.5(i), we have that $M_\Omega \in W^{1,p}(\Omega)$ and $|\nabla M_\Omega f(x)| \le 2M_\Omega|\nabla f|(x)$ for almost every $x \in \Omega$. These facts together with Lemma 2.3 and (1.15) imply that

$$
\begin{aligned}
|\nabla(bM_\Omega f)(x)| &= |M_\Omega f(x)\nabla b(x) + b(x)\nabla M_\Omega f(x)|\\
&\le |\nabla b(x)|M_\Omega f(x)| + 2|b(x)||M_\Omega|\nabla f|(x)|\\
&\le \sqrt{n}\|b\|_{Lip(\Omega)}M_\Omega f(x) + 2\|b\|_{\infty,\Omega}M_\Omega|\nabla f|(x)
\end{aligned}
\tag{4.3}
$$

for almost every $x \in \Omega$. It follows from (4.2) and (4.3) that

$$|\nabla[b, M_\Omega](f)(x)| \leq 4\|b\|_{\infty,\Omega} M_\Omega|\nabla f|(x) + 3\sqrt{n}\|b\|_{Lip(\Omega)} M_\Omega f(x)$$

for almost every $x \in \Omega$. This proves (1.16). By (1.16), (1.10) and Minkowski's inequality, we have

$$\begin{aligned}\|\nabla[b, M_\Omega](f)\|_{p,\Omega} &\leq 4\|b\|_{\infty,\Omega}\|M_\Omega|\nabla f|\|_{p,\Omega} + 3\sqrt{n}\|b\|_{Lip(\Omega)}\|M_\Omega f\|_{p,\Omega} \\ &\leq C_{n,p}\big(\|b\|_{\infty,\Omega}\|\nabla f\|_{p,\Omega} + \|b\|_{Lip(\Omega)}\|f\|_{p,\Omega}\big),\end{aligned}$$

which together with (1.11) leads to (1.17).

It remains to prove the continuity part. Let $f_j \to f$ in $W^{1,p}(\Omega)$ as $j \to \infty$. By Lemma 2.3, we have that $bf_j \to bf$ in $W^{1,p}(\Omega)$ as $j \to \infty$. This together with Lemma 2.5(ii) implies that $M_\Omega(bf_j) \to M_\Omega(bf)$ in $W^{1,p}(\Omega)$ as $j \to \infty$. On the other hand, by Lemma 2.5(ii) again, one sees that $M_\Omega f_j \to M_\Omega f$ in $W^{1,p}(\Omega)$ as $j \to \infty$. This together with Lemma 2.3 leads to $bM_\Omega f_j \to bM_\Omega f$ in $W^{1,p}(\Omega)$ as $j \to \infty$. Therefore, we have that $[b, M_\Omega](f_j) \to [b, M_\Omega](f)$ in $W^{1,p}(\Omega)$ as $j \to \infty$.

4.2. Proof of Theorem 1.16(ii)

Let $p \in (1, n)$, $\alpha \in [1, n/p)$, $q = np/(n - (\alpha - 1)p)$ and $q_1 = np/(n - \alpha p)$. It is clear that $q < q_1$. Let $f \in W^{1,p}(\Omega)$ and $b \in \mathrm{Lip}(\Omega)$. By Lemma 2.3, we have that $bf \in W^{1,p}(\Omega)$ and $\nabla(bf)(x) = b(x)\nabla f(x) + f(x)\nabla b(x)$ for almost every $x \in \Omega$. This together with Lemma 2.6(i) and (1.15) yields that

$$\begin{aligned}&|\nabla M_{\alpha,\Omega}(bf)(x)| \\ &\leq 2M_{\alpha,\Omega}(|\nabla(bf)|)(x) + \alpha M_{\alpha-1,\Omega}(bf)(x) \\ &\leq 2M_{\alpha,\Omega}(|\nabla b|f)(x) + 2M_{\alpha,\Omega}(|\nabla f|b)(x) + \alpha\|b\|_{\infty,\Omega} M_{\alpha-1,\Omega} f(x) \\ &\leq 2\sqrt{n}\|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) + 2\|b\|_{\infty,\Omega} M_{\alpha,\Omega}|\nabla f|(x) + \alpha\|b\|_{\infty,\Omega} M_{\alpha-1,\Omega} f(x)\end{aligned} \tag{4.4}$$

for almost every $x \in \Omega$.

On the other hand, by Lemmas 2.2, 2.6(i) and (1.15), we have

$$\begin{aligned}&|\nabla(bM_{\alpha,\Omega} f)(x)| \\ &\leq |\nabla b|(x) M_{\alpha,\Omega} f(x) + |b(x)||\nabla M_{\alpha,\Omega} f|(x) \\ &\leq \sqrt{n}\|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) + |b(x)|\big(2M_{\alpha,\Omega}|\nabla f|(x) + \alpha M_{\alpha-1,\Omega} f(x)\big) \\ &\leq \sqrt{n}\|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) + 2\|b\|_{\infty,\Omega} M_{\alpha,\Omega}|\nabla f|(x) + \alpha\|b\|_{\infty,\Omega} M_{\alpha-1,\Omega} f(x))\end{aligned} \tag{4.5}$$

for almost every $x \in \Omega$. In light of (4.4) and (4.5) we would have

$$\begin{aligned}&|\nabla[b, M_{\alpha,\Omega}](f)(x)| \\ &\leq |\nabla(bM_{\alpha,\Omega} f)(x)| + |\nabla M_{\alpha,\Omega}(bf)(x)| \\ &\leq 3\sqrt{n}\|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) + 4\|b\|_{\infty,\Omega} M_{\alpha,\Omega}|\nabla f|(x) + 2\alpha\|b\|_{\infty,\Omega} M_{\alpha-1,\Omega} f(x)\end{aligned}$$

for almost every $x \in \Omega$. This proves (1.18).

By (1.18), (1.10), Minkowski's inequality and Hölder's inequality, we have

$$\begin{aligned}
&\|\nabla[b, M_{\alpha,\Omega}](f)\|_{q,\Omega} \\
&\leq 3\sqrt{n}\|b\|_{Lip(\Omega)}\|M_{\alpha,\Omega}f\|_{q,\Omega} + 4\|b\|_{\infty,\Omega}\|M_{\alpha,\Omega}|\nabla f|\|_{q,\Omega} + 2\alpha\|b\|_{\infty,\Omega}\|M_{\alpha-1,\Omega}f\|_{q,\Omega} \\
&\leq 3\sqrt{n}\|b\|_{Lip(\Omega)}|\Omega|^{1/q-1/q_1}\|M_{\alpha,\Omega}f\|_{q_1,\Omega} + 4\|b\|_{\infty,\Omega}|\Omega|^{1/q-1/q_1}\|M_{\alpha,\Omega}|\nabla f|\|_{q_1,\Omega} \\
&\quad + C_{\alpha,p,n}\|b\|_{\infty,\Omega}\|f\|_{p,\Omega} \\
&\leq C_{\alpha,n,p,|\Omega|}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega},
\end{aligned}$$

which together with (1.11) leads to (1.19).

We now prove the continuity part. Let $f_j \to f$ in $W^{1,p}(\Omega)$ as $j \to \infty$. By Lemma 2.3, we see that $bf_j \to bf$ in $W^{1,p}(\Omega)$ as $j \to \infty$. This together with Lemma 2.6(ii) implies that $M_{\alpha,\Omega}(bf_j) \to M_{\alpha,\Omega}(bf)$ in $W^{1,q}(\Omega)$ as $j \to \infty$. On the other hand, by Lemma 2.6(ii), we have that $M_{\alpha,\Omega}f_j \to M_{\alpha,\Omega}f$ in $W^{1,q}(\Omega)$ as $j \to \infty$. This together with Lemma 2.3 leads to $bM_{\alpha,\Omega}f_j \to bM_{\alpha,\Omega}f$ in $W^{1,q}(\Omega)$ as $j \to \infty$. Therefore, we have that $[b, M_{\alpha,\Omega}](f_j) \to [b, M_{\alpha,\Omega}](f)$ in $W^{1,q}(\Omega)$ as $j \to \infty$.

4.3. Proof of Theorem 1.16(iii)

Let $p \in (1, n)$, $\alpha \in [1, n/p)$ and $q = pn/(n - \alpha p)$. Set $1/\widetilde{p} = 1/p - 1/n$. It is clear that $1/q = 1/\widetilde{p} - (\alpha - 1)/n$. By using Lemma 2.6(iii) and the arguments similar to those used to derive (1.18), one can get (1.20). By (1.20), (1.10) and Minkowski's inequality, we have

$$\begin{aligned}
&\|\nabla[b, M_{\alpha,\Omega}](f)\|_{q,\Omega} \\
&\leq 3\sqrt{n}\|b\|_{Lip(\Omega)}\|M_{\alpha,\Omega}f\|_{q,\Omega} + 4\|b\|_{\infty,\Omega}\|M_{\alpha,\Omega}|\nabla f|\|_{q,\Omega} + 2\alpha\|b\|_{\infty,\Omega}\|M_{\alpha-1,\Omega}f\|_{q,\Omega} \\
&\leq C_{\alpha,n,p}\big(\|b\|_{Lip(\Omega)}\|f\|_{p,\Omega} + \|b\|_{\infty,\Omega}\|\nabla f\|_{p,\Omega} + \|b\|_{\infty,\Omega}\|f\|_{\widetilde{p},\Omega}\big) \\
&\leq C_{\alpha,n,p}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega},
\end{aligned}$$

which together with (1.11) implies (1.21).

4.4. Proof of Theorem 1.16(iv)

Let $p \in (n/(n-1), \infty)$, $\alpha \in [1, \min\{(n-1)/p, n - 2n/((n-1)p)\} + 1)$ and $|\Omega| < \infty$. By Lemma 2.6(iv), we have

$$\begin{aligned}
|\nabla M_{\alpha,\Omega}(bf)(x)| &\leq C_n\big(M_{\alpha-1,\Omega}(bf)(x) + \mathcal{S}_{\alpha-1,\Omega}(bf)(x)\big) \\
&\leq C_n\|b\|_{\infty,\Omega}\big(M_{\alpha-1,\Omega}f(x) + \mathcal{S}_{\alpha-1,\Omega}f(x)\big)
\end{aligned} \tag{4.6}$$

for almost every $x \in \Omega$. By Lemmas 2.6(iv), 2.3 and (1.15), we have

$$\begin{aligned}
|\nabla(bM_{\alpha,\Omega}f)(x)| &\leq |\nabla b(x)|M_{\alpha,\Omega}f(x) + |b(x)||\nabla M_{\alpha,\Omega}f(x)| \\
&\leq \sqrt{n}\|b\|_{Lip(\Omega)}M_{\alpha,\Omega}f(x) + C_n\|b\|_{\infty,\Omega}\big(M_{\alpha-1,\Omega}f(x) + \mathcal{S}_{\alpha-1,\Omega}f(x)\big)
\end{aligned} \tag{4.7}$$

for almost every $x \in \Omega$. Combining (4.6) with (4.7) implies (1.22).

Let $1/q_1 = 1/p - \alpha/n$. Clearly, $q < q_1$. By (1.22), (1.10), Minkowski's inequality, Hölder's inequality and Lemma 2.7, we have

$$\begin{aligned}
&\|\nabla[b, M_{\alpha,\Omega}](f)\|_{q,\Omega}\\
&\leq \sqrt{n}\|b\|_{Lip(\Omega)}\|M_{\alpha,\Omega}f\|_{q,\Omega} + C_n\|b\|_{\infty,\Omega}\big(\|M_{\alpha-1,\Omega}f\|_{q,\Omega} + \|\mathcal{S}_{\alpha-1,\Omega}f\|_{q,\Omega}\big)\\
&\leq \sqrt{n}\|b\|_{Lip(\Omega)}|\Omega|^{1/q-1/q_1}\|M_{\alpha,\Omega}f\|_{q_1,\Omega} + C_{\alpha,n,p}\|b\|_{\infty,\Omega}\|f\|_{p,\Omega}\\
&\leq C_{\alpha,n,p,|\Omega|}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{p,\Omega},
\end{aligned}$$

which together with (1.11) implies (1.23). This completes the proof of Theorem 1.16.

5. Proof of Theorem 1.17

5.1. Preliminary notation and lemmas

For convenience, we set $\delta(x) = \mathrm{dist}(x, \Omega^c)$. It is clear that δ is a Lipschitz function. By Rademacher's theorem, we see that δ is differentiable almost everywhere in Ω. Moreover, $|\nabla\delta(x)| = 1$ for almost every $x \in \Omega$. Let b, f be two suitable functions defined on Ω. For $t \in (0,1)$ and $\alpha \in (0,n)$, we define the function $A_{t,b,\alpha}(f)\colon \Omega \to [-\infty,\infty]$ by

$$A_{t,b,\alpha}(f)(x) = \frac{(t\delta(x))^{\alpha}}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}|b(x)-b(y)|f(y)\,dy.$$

When $\alpha = 0$, we denote $A_{t,b,\alpha} = A_{t,b}$.

In what follows, for any arbitrary functions $F(x,y)$ defined on $\Omega\times\Omega$, we set $\nabla_x F = (D_{1,x}F, \ldots, D_{n,x}F)$, where $D_{i,x}F$ is the i-th weak partial derivative of F in x.

Lemma 5.1. *Let $p \in (1,n)$, $\alpha \in [1,n/p)$ and $q = np/(n-(\alpha-1)p)$. Assume that $|\Omega| < \infty$ and $b \in \mathrm{Lip}(\Omega)$. If $f \in W^{1,p}(\Omega)$, then $A_{t,b,\alpha}(f) \in W^{1,q}(\Omega)$ and*

$$|\nabla A_{t,b,\alpha}(f)(x)| \leq \alpha\mathfrak{M}_{b,\alpha-1,\Omega}f(x) + 3\sqrt{n}\|b\|_{Lip(\Omega)}M_{\alpha,\Omega}f(x) + 2\mathfrak{M}_{b,\alpha,\Omega}|\nabla f|(x) \tag{5.1}$$

for almost every $x \in \Omega$.

Proof. At first we assume that $f \in W^{1,p}(\Omega)\cap\mathcal{C}^{\infty}(\Omega)$. Fix $i = 1, 2, \ldots, n$. By [25, (5.8)], we have

$$\begin{aligned}
&|\nabla A_{t,b,\alpha}(f)(x)|\\
&\leq \alpha\frac{(t\delta(x))^{\alpha-1}}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}|b(x)-b(y)||f(y)|\,dy\\
&\quad + \frac{|\nabla\delta(x)|}{\delta(x)}\frac{(t\delta(x))^{\alpha}}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}|\nabla_y(|b(x)-b(y)|f(y))\cdot(y-x)|\,dy
\end{aligned}$$

$$
\begin{aligned}
&+\frac{(t\delta(x))^\alpha}{|B(x,t\delta(x))|}\Big(\int_{B(x,t\delta(x))}|\nabla_x|b(x)-b(y)|f(y)|\,dy \\
&\qquad+\int_{B(x,t\delta(x))}|\nabla_y(|b(x)-b(y)|f(y))|\,dy\Big) \\
&\leq \alpha\mathfrak{M}_{b,\alpha-1,\Omega}f(x)+\frac{2(t\delta(x))^\alpha}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}|\nabla_y(|b(x)-b(y)|f(y))|\,dy \\
&\quad+\frac{(t\delta(x))^\alpha}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}|\nabla_x|b(x)-b(y)|f(y)|\,dy
\end{aligned}\tag{5.2}
$$

for almost every $x\in\Omega$. Since $b\in\mathrm{Lip}(\Omega)$, then $|b(x)-b(\cdot)|\in\mathrm{Lip}(\Omega)$, $\||b(x)-b(\cdot)|\|_{Lip(\Omega)}\leq\|b\|_{Lip(\Omega)}$ and $\||b(x)-b(\cdot)|\|_{\mathrm{Lip}(\Omega)}\leq 2\|b\|_{\mathrm{Lip}(\Omega)}$. Similarly, we have that $|b(\cdot)-b(y)|\in\mathrm{Lip}(\Omega)$ and $\||b(\cdot)-b(y)|\|_{Lip(\Omega)}\leq\|b\|_{Lip(\Omega)}$. Invoking Lemma 2.3, we have that $|b(x)-b(\cdot)|f(\cdot)\in W^{1,p}(\Omega)$ and

$$
\nabla_y(|b(x)-b(y)|f(y))=f(y)\nabla_y|b(x)-b(y)|+|b(x)-b(y)|\nabla f(y)\tag{5.3}
$$

for almost every $y\in\Omega$. By (1.15) and the fact that $\||b(x)-b(\cdot)|\|_{Lip(\Omega)}\leq\|b\|_{Lip(\Omega)}$, we have that $|\nabla_y|b(x)-b(y)||\leq\sqrt{n}\|b\|_{Lip(\Omega)}$ for all $x\in\Omega$. Similarly, $|\nabla_x|b(x)-b(y)||\leq\sqrt{n}\|b\|_{Lip(\Omega)}$ for all $y\in\Omega$. Then we get from (5.3) that

$$
|\nabla_y(|b(x)-b(y)|f(y))|\leq\sqrt{n}\|b\|_{Lip(\Omega)}|f(y)|+|b(x)-b(y)||\nabla f(y)|\tag{5.4}
$$

for almost every $y\in\Omega$. By (5.4), (5.2) and the fact that $|\nabla_x|b(x)-b(y)||\leq\sqrt{n}\|b\|_{Lip(\Omega)}$, we have

$$
\begin{aligned}
&|\nabla A_{t,b,\alpha}(f)(x)| \\
&\leq \alpha\mathfrak{M}_{b,\alpha-1,\Omega}f(x)+\sqrt{n}\|b\|_{Lip(\Omega)}\frac{(t\delta(x))^\alpha}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}|f(y)|\,dy \\
&\quad+\frac{2(t\delta(x))^\alpha}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}\big(|b(x)-b(y)||\nabla f(y)|+\sqrt{n}\|b\|_{Lip(\Omega)}|f(y)|\big)\,dy \\
&\leq \alpha\mathfrak{M}_{b,\alpha-1,\Omega}f(x)+3\sqrt{n}\|b\|_{Lip(\Omega)}M_{\alpha,\Omega}f(x)+2\mathfrak{M}_{b,\alpha,\Omega}|\nabla f|(x)
\end{aligned}\tag{5.5}
$$

for almost every $x\in\Omega$. This proves (5.1) for $f\in W^{1,p}(\Omega)\cap\mathcal{C}^\infty(\Omega)$.

Next we complete the rest of proof by an approximation argument. Assume that $f\in W^{1,p}(\Omega)$ for some $p\in(1,n)$. There exists a sequence of functions $\{\varphi_j\}_{j=1}^\infty$ in $W^{1,p}(\Omega)\cap\mathcal{C}^\infty(\Omega)$ such that $\varphi_j\to f$ in $W^{1,p}(\Omega)$ as $j\to\infty$. By Hölder's inequality, we have

$$
\begin{aligned}
|A_{t,b,\alpha}(\varphi_j)(x)-A_{t,b,\alpha}(f)(x)|&\leq\frac{(t\delta(x))^\alpha}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}|b(x)-b(y)||\varphi_j(y)-f(y)|\,dy \\
&\leq\big(|b(x)|+\|b\|_{\infty,\Omega}\big)\|\varphi_j-f\|_{p,\Omega}\frac{(t\delta(x))^\alpha}{|B(x,t\delta(x))|^{1/p}},
\end{aligned}
$$

which leads to

$$\lim_{j\to\infty} A_{t,b,\alpha}(\varphi_j)(x) = A_{t,b,\alpha}(f)(x)$$

for all $x \in \Omega$. It was shown that

$$|\nabla A_{t,b,\alpha}(\varphi_j)(x)| \le \alpha \mathfrak{M}_{b,\alpha-1,\Omega}\varphi_j(x) + 3\sqrt{n}\|b\|_{Lip(\Omega)} M_{\alpha,\Omega}\varphi_j(x) + 2\mathfrak{M}_{b,\alpha,\Omega}|\nabla \varphi_j|(x) \tag{5.6}$$

for almost every $x \in \Omega$. Let $q_1 = np/(n-\alpha p)$. It is clear that $q < q_1$. By (1.15), (1.13), (5.6), Hölder's inequality and Minkowski's inequality, we have

$$\begin{aligned}
&\|\nabla A_{t,b,\alpha}(\varphi_j)\|_{q,\Omega} \\
&\le \alpha\|\mathfrak{M}_{b,\alpha-1,\Omega}\varphi_j\|_{q,\Omega} + 3\sqrt{n}\|b\|_{Lip(\Omega)}\|M_{\alpha,\Omega}\varphi_j\|_{q,\Omega} + 2\|\mathfrak{M}_{b,\alpha,\Omega}|\nabla\varphi_j|\|_{q,\Omega} \\
&\le C_{\alpha,n,p}\|b\|_{\infty,\Omega}\|\varphi_j\|_{p,\Omega} + 3\sqrt{n}\|b\|_{Lip(\Omega)}|\Omega|^{1/q-1/q_1}\|M_{\alpha,\Omega}\varphi_j\|_{q_1,\Omega} \\
&\quad + 2|\Omega|^{1/q-1/q_1}\|\mathfrak{M}_{b,\alpha,\Omega}|\nabla\varphi_j|\|_{q_1,\Omega} \\
&\le C_{\alpha,n,p,|\Omega|}\|b\|_{\mathrm{Lip}(\Omega)}\|\varphi_j\|_{1,p,\Omega}.
\end{aligned} \tag{5.7}$$

This yields that $\{|\nabla A_{t,b,\alpha}(\varphi_j)|\}_{j=1}^{\infty}$ is a bounded sequence in $L^q(\Omega)$. By the fact that $A_{t,b,\alpha}(\varphi_j)(x) \to A_{t,b,\alpha}(f)(x)$ as $j \to \infty$ for almost every $x \in \Omega$, we have that the Sobolev derivative $\nabla A_{t,b,\alpha}(f)$ exists almost everywhere in Ω and there exists a subsequence $\{\nabla A_{t,b,\alpha}(\varphi_{j_\ell})\}_{\ell=1}^{\infty}$ of $\{\nabla A_{t,b,\alpha}(\varphi_j)\}_{j=1}^{\infty}$ such that

$$\nabla A_{t,b,\alpha}(\varphi_{j_\ell}) \to \nabla A_{t,b,\alpha}(f) \quad \text{weakly in } L^q(\Omega) \text{ as } \ell \to \infty. \tag{5.8}$$

On the other hand, we get by Remark 1.11(iii) that $\mathfrak{M}_{b,\alpha-1,\Omega}\varphi_j \to \mathfrak{M}_{b,\alpha-1,\Omega}f$ in $L^q(\Omega)$ and $\mathfrak{M}_{b,\alpha,\Omega}|\nabla\varphi_j| \to \mathfrak{M}_{b,\alpha,\Omega}|\nabla f|$ and $M_{\alpha,\Omega}\varphi_j \to M_{\alpha,\Omega}f$ in $L^{q_1}(\Omega)$ as $j \to \infty$. Hence, by Hölder's inequality, we have that $\mathfrak{M}_{b,\alpha,\Omega}|\nabla\varphi_j| \to \mathfrak{M}_{b,\alpha,\Omega}|\nabla f|$ and $M_{\alpha,\Omega}\varphi_j \to M_{\alpha,\Omega}f$ in $L^q(\Omega)$ as $j \to \infty$. For convenience, we set

$$\mathfrak{h}_\ell := \alpha\mathfrak{M}_{b,\alpha-1,\Omega}\varphi_{j_\ell}(x) + 3\sqrt{n}\|b\|_{Lip(\Omega)}M_{\alpha,\Omega}\varphi_{j_\ell}(x) + 2\mathfrak{M}_{b,\alpha,\Omega}|\nabla\varphi_{j_\ell}|(x).$$

It was proved that

$$\mathfrak{h}_\ell \to \alpha\mathfrak{M}_{b,\alpha-1,\Omega}f + 3\sqrt{n}\|b\|_{Lip(\Omega)}M_{\alpha,\Omega}f + 2\mathfrak{M}_{b,\alpha,\Omega}|\nabla f| \quad \text{in } L^q(\Omega) \text{ as } \ell \to \infty. \tag{5.9}$$

Combining (5.9) with (5.6), (5.8) and Proposition 2.2 yields (5.1). □

Lemma 5.2. *Let $1 < p < \infty$ and $b \in \mathrm{Lip}(\Omega)$. If $f \in W^{1,p}(\Omega)$, then $A_{t,b}(f) \in W^{1,p}(\Omega)$ and*

$$|\nabla A_{t,b}(f)(x)| \le 3\sqrt{n}\|b\|_{Lip(\Omega)}M_\Omega f(x) + 2\mathfrak{M}_{b,\Omega}|\nabla f|(x) \tag{5.10}$$

for almost every $x \in \Omega$.

Proof. At first we assume that $f \in W^{1,p}(\Omega) \cap \mathcal{C}^{\infty}(\Omega)$. By (5.2) and the arguments similar to those used in deriving (5.5),

$$|\nabla A_{t,b}(f)(x)| \leq 3\sqrt{n}\|b\|_{Lip(\Omega)} M_{\Omega} f(x) + 2\mathfrak{M}_{b,\Omega}|\nabla f|(x)$$

for almost every $x \in \Omega$. This proves (5.10) for the case $f \in W^{1,p}(\Omega) \cap \mathcal{C}^{\infty}(\Omega)$. The rest of the proof follows from the arguments similar to those used in the proof of Lemma 5.1. We omit the details. □

Lemma 5.3. *Let $p \in (1,n)$, $\alpha \in [1, n/p)$ and $q = pn/(n-\alpha p)$. Let Ω admit a p-Sobolev embedding and $b \in \mathrm{Lip}(\Omega)$. If $f \in W^{1,p}(\Omega)$, then $A_{t,b,\alpha}(f) \in W^{1,q}(\Omega)$ and*

$$|\nabla A_{t,b,\alpha}(f)(x)| \leq \alpha \mathfrak{M}_{b,\alpha-1,\Omega} f(x) + 2\mathfrak{M}_{b,\alpha,\Omega}|\nabla f|(x) + 3\sqrt{n}\|b\|_{Lip(\Omega)} M_{\alpha,\Omega} f(x) \tag{5.11}$$

for almost every $x \in \Omega$.

Proof. By (5.5), we known that (5.11) holds for all $f \in W^{1,p}(\Omega) \cap \mathcal{C}^{\infty}(\Omega)$. The rest of the proof follows from an approximation argument. Assume that $f \in W^{1,p}(\Omega)$ for some $p \in (1,n)$. There exists a sequence of functions $\{\varphi_j\}_{j=1}^{\infty}$ in $W^{1,p}(\Omega) \cap \mathcal{C}^{\infty}(\Omega)$ such that $\varphi_j \to f$ in $W^{1,p}(\Omega)$ as $j \to \infty$. It was known that

$$\lim_{j\to\infty} A_{t,b,\alpha}(\varphi_j)(x) = A_{t,b,\alpha}(f)(x)$$

for all $x \in \Omega$ and

$$|\nabla A_{t,b,\alpha}(\varphi_j)(x)| \leq \alpha \mathfrak{M}_{b,\alpha-1,\Omega}\varphi_j(x) + 3\sqrt{n}\|b\|_{Lip(\Omega)} M_{\alpha,\Omega}\varphi_j(x) + 2\mathfrak{M}_{b,\alpha,\Omega}|\nabla \varphi_j|(x) \tag{5.12}$$

for almost every $x \in \Omega$.

Let $1/\widetilde{p} = 1/p - 1/n$. Clearly, $1/q = 1/\widetilde{p} - (\alpha-1)/n$. Since Ω admits a p-Sobolev embedding, then

$$\|u\|_{\widetilde{p},\Omega} \leq C_{p,n}\|u\|_{1,p,\Omega}, \quad \forall\, u \in L^p(\Omega). \tag{5.13}$$

By (1.10), (1.13), (5.12), (5.13) and Minkowski's inequality, we have

$$\begin{aligned}
&\|\nabla A_{t,b,\alpha}(\varphi_j)\|_{q,\Omega} \\
&\leq \alpha\|\mathfrak{M}_{b,\alpha-1,\Omega}\varphi_j\|_{q,\Omega} + 3\sqrt{n}\|b\|_{Lip(\Omega)}\|M_{\alpha,\Omega}\varphi_j\|_{q,\Omega} + 2\|\mathfrak{M}_{b,\alpha,\Omega}|\nabla\varphi_j|\|_{q,\Omega} \\
&\leq C_{\alpha,n,p}\|b\|_{\infty,\Omega}\big(\|\varphi_j\|_{\widetilde{p},\Omega} + \|\nabla\varphi_j\|_{p,\Omega}\big) + C_{\alpha,n,p}\|b\|_{Lip(\Omega)}\|\varphi_j\|_{p,\Omega} \\
&\leq C_{\alpha,n,p}\|b\|_{\mathrm{Lip}(\Omega)}\|\varphi_j\|_{1,p,\Omega}.
\end{aligned} \tag{5.14}$$

On the other hand, we get by (5.13) that $\varphi_j \to f$ in $L^{\widetilde{p}}(\Omega)$ as $j \to \infty$. Then by Remark 1.11, we have that $\mathfrak{M}_{b,\alpha,\Omega}|\nabla\varphi_j| \to \mathfrak{M}_{b,\alpha,\Omega}|\nabla f|$, $M_{\alpha,\Omega}\varphi_j \to M_{\alpha,\Omega} f$ and $\mathfrak{M}_{b,\alpha-1,\Omega}\varphi_j \to \mathfrak{M}_{b,\alpha-1,\Omega} f$ in $L^q(\Omega)$ as $j \to \infty$. The rest of the proof follows from the arguments similar to those used in the proof of Lemma 5.1. We omit the details. □

Lemma 5.4. *Let $p \in (n/(n-1), \infty)$, $\alpha \in [1, \min\{(n-1)/p, n-2n/((n-1)p)\}+1)$, $q = np/(n-(\alpha-1)p)$ and $|\Omega| < \infty$. If $b \in \mathrm{Lip}(\Omega)$ and $f \in L^{p_1}(\Omega)$, then $A_{t,b}(f) \in W^{1,q}(\Omega)$ and*

$$\begin{aligned}|\nabla A_{t,b,\alpha}(f)(x)| \leq (n-\alpha)\mathfrak{M}_{b,\alpha-1,\Omega}f(x) + \sqrt{n}\|b\|_{Lip(\Omega)}M_{\alpha,\Omega}f(x)\\ + 2n\|b\|_{\infty,\Omega}\mathcal{S}_{\alpha-1,\Omega}f(x)\end{aligned}$$

for almost every $x \in \Omega$.

Proof. Let $f \in L^p(\Omega) \cap \mathcal{C}^\infty(\Omega)$. It was shown in the proof of [25, Lemma 5.3] that

$$\begin{aligned}|\nabla A_{t,b,\alpha}(f)(x)| \leq{}& (n-\alpha)\frac{|\nabla\delta(x)|}{|\delta(x)|}\frac{(t\delta(x))^\alpha}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}|b(x)-b(y)||f(y)|\,dy\\ &+ \frac{(t\delta(x))^\alpha}{|B(x,t\delta(x))|}\bigg(\int_{B(x,t\delta(x))}|\nabla_x|b(x)-b(y)|||f(y)|\,dy\\ &\quad+\int_{\partial B(x,t\delta(x))}|b(x)-b(y)||f(y)||\nu(y)|\,d\mathcal{H}^{n-1}(y)\\ &\quad+t\int_{\partial B(x,t\delta(x))}|b(x)-b(y)||f(y)|\,d\mathcal{H}^{n-1}(y)|\nabla\delta(x)|\bigg)\end{aligned}$$

for almost every $x \in \Omega$, where $\nu(y) = (y-x)/(t\delta(x))$. This together with the fact that $|\nabla_x|b(x)-b(y)|| \leq \sqrt{n}\|b\|_{Lip(\Omega)}$ implies that

$$\begin{aligned}&|\nabla A_{t,b}(f)(x)|\\ &\leq (n-\alpha)\frac{(t\delta(x))^{\alpha-1}}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}|b(x)-b(y)||f(y)|\,dy\\ &\quad+\sqrt{n}\|b\|_{Lip(\Omega)}\frac{(t\delta(x))^\alpha}{|B(x,t\delta(x))|}\int_{B(x,t\delta(x))}|f(y)|\,dy\\ &\quad+\frac{n(t\delta(x))^{\alpha-1}}{|\partial B(x,t\delta(x))|}\int_{\partial B(x,t\delta(x))}|b(x)-b(y)||f(y)|\,d\mathcal{H}^{n-1}(y)\\ &\leq (n-\alpha)\mathfrak{M}_{b,\alpha-1,\Omega}f(x) + \sqrt{n}\|b\|_{Lip(\Omega)}M_{\alpha,\Omega}f(x) + 2n\|b\|_{\infty,\Omega}\mathcal{S}_{\alpha-1,\Omega}f(x)\end{aligned}$$

for almost every $x \in \Omega$.

The rest of the proof follows an approximation argument. Assume that $f \in L^p(\Omega)$ for some $p \in (1,n)$. There exists a sequence of functions $\{\varphi_j\}_{j=1}^\infty$ in $L^p(\Omega)\cap\mathcal{C}^\infty(\Omega)$ such that $\varphi_j \to f$ in $L^p(\Omega)$ as $j \to \infty$. It was known that

$$\lim_{j\to\infty} A_{t,b,\alpha}(\varphi_j) = A_{t,b,\alpha}(f)(x)$$

for all $x \in \Omega$. Moreover, it was proved that

$$\begin{aligned}|\nabla A_{t,b,\alpha}(\varphi_j)(x)| \leq (n-\alpha)\mathfrak{M}_{b,\alpha-1,\Omega}\varphi_j(x) + \sqrt{n}\|b\|_{Lip(\Omega)}M_{\alpha,\Omega}\varphi_j(x)\\ + 2n\|b\|_{\infty,\Omega}\mathcal{S}_{\alpha-1,\Omega}\varphi_j(x)\end{aligned} \tag{5.15}$$

for almost every $x \in \Omega$. Let $q_1 = np/(n - \alpha p)$. Clearly, $q < q_1$. By (1.10), (1.13), (5.15), Minkowski's inequality, Hölder's inequality and Lemma 2.7, we have

$$\begin{aligned}
&\|\nabla A_{t,b,\alpha}(\varphi_j)\|_{q,\Omega} \\
&\le (n-\alpha)\|\mathfrak{M}_{b,\alpha-1,\Omega}\varphi_j\|_{q,\Omega} + \sqrt{n}\|b\|_{Lip(\Omega)}\|M_{\alpha,\Omega}\varphi_j\|_{q,\Omega} + 2n\|b\|_{\infty,\Omega}\|\mathcal{S}_{\alpha-1,\Omega}\varphi_j\|_{q,\Omega} \\
&\le C_{\alpha,n,p}\|b\|_{\infty,\Omega}\|\varphi_j\|_{p,\Omega} + \sqrt{n}\|b\|_{Lip(\Omega)}|\Omega|^{1/q-1/q_1}\|M_{\alpha,\Omega}\varphi_j\|_{q_1,\Omega} \\
&\quad + C_{\alpha,n,p}\|b\|_{\infty,\Omega}\|\varphi_j\|_{p,\Omega} \\
&\le C_{\alpha,n,p,|\Omega|}\|b\|_{\mathrm{Lip}(\Omega)}\|\varphi_j\|_{p,\Omega}.
\end{aligned} \tag{5.16}$$

On the other hand, by Remark 1.11, we have that $\mathfrak{M}_{b,\alpha-1,\Omega}\varphi_j \to \mathfrak{M}_{b,\alpha-1,\Omega}f$ in $L^q(\Omega)$ as $j \to \infty$. Moreover, $M_{\alpha,\Omega}\varphi_j \to M_{\alpha,\Omega}f$ in $L^{q_1}(\Omega)$ as $j \to \infty$. This together with Hölder's inequality implies that $M_{\alpha,\Omega}\varphi_j \to M_{\alpha,\Omega}f$ in $L^q(\Omega)$ as $j \to \infty$. By the sublinearity and Lemma 2.7, one sees that $\mathcal{S}_{\alpha-1,\Omega}\varphi_j \to \mathcal{S}_{\alpha-1,\Omega}f$ in $L^q(\Omega)$ as $j \to \infty$. The rest of the proof follows from the arguments similar to those used in the proof of Lemma 5.1. We omit the details. □

5.2. Proof of Theorem 1.17

We adopt the method of [17] to prove Theorem 1.17. Let t_j, $j = 1, 2, \ldots$, be an enumeration of the rationals between 0 and 1. For any $k \ge 1$, $0 \le \alpha < n$ and two suitable functions f, b defined on Ω, we define the operator $u_{k,b,\alpha}$ by

$$u_{k,b,\alpha}(f)(x) = \max_{1\le j\le k} A_{t_j,b,\alpha}(f)(x).$$

For $\alpha = 0$, we denote $u_{k,b,\alpha} = u_{k,b}$.

We first prove (i). Let $f \in W^{1,p}(\Omega)$ with $p \in (1, \infty)$ and $b \in \mathrm{Lip}(\Omega)$. Invoking Lemma 5.2, one has that $A_{t_j,b}(f) \in W^{1,p}(\Omega)$ and

$$|\nabla A_{t_j,b}(f)(x)| \le 3\sqrt{n}\|b\|_{Lip(\Omega)}M_\Omega f(x) + 2\mathfrak{M}_{b,\Omega}|\nabla f|(x) \tag{5.17}$$

for all $j = 1, 2, \ldots$ and almost every $x \in \Omega$. On the other hand, it is easy to see that

$$\mathfrak{M}_{b,\Omega}f(x) = \sup_{j\ge1} A_{t_j,b}(f)(x)$$

for all $x \in \Omega$. Moreover, the sequence $\{u_{k,b}\}_{k=1}^{\infty}$ is an increasing sequence of functions converging pointwise to $\mathfrak{M}_{b,\Omega}f$. Using (5.17) and the fact that the maximum of two Sobolev functions belongs to the Sobolev space (see [11, Lemma 7.6]), one finds that

$$\begin{aligned}
|\nabla u_{k,b}(x)| &= \left|\nabla \max_{1\le j\le k} A_{t_j,b}(f)(x)\right| \le \max_{1\le j\le k} |\nabla A_{t_j,b}(f)(x)| \\
&\le 3\sqrt{n}\|b\|_{Lip(\Omega)}M_\Omega f(x) + 2\mathfrak{M}_{b,\Omega}|\nabla f|(x)
\end{aligned} \tag{5.18}$$

for all $k = 1, 2, \ldots$ and almost every $x \in \Omega$. By (5.18), (1.10), (1.13) and Minkowski's inequality, we have

$$\begin{aligned} \|\nabla u_{k,b}\|_{p,\Omega} &\leq 3\sqrt{n}\|b\|_{Lip(\Omega)}\|M_\Omega f\|_{p,\Omega} + 2\|\mathfrak{M}_{b,\Omega}|\nabla f|\|_{p,\Omega} \\ &\leq C_{n,p}\big(\|b\|_{Lip(\Omega)}\|f\|_{p,\Omega} + \|b\|_{\infty,\Omega}\|\nabla f\|_{p,\Omega}\big) \leq C_{n,p}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega}, \end{aligned} \tag{5.19}$$

which gives that $\{|\nabla u_{k,b}|\}_{k=1}^\infty$ is a bounded sequence in $L^p(\Omega)$. Since $u_{k,b}$ converges pointwise to $\mathfrak{M}_{b,\Omega}f$ as $k \to \infty$, then the weak gradient $\nabla\mathfrak{M}_{b,\Omega}f$ exists and there exists a subsequence $\{u_{k_\ell,b}\}_{\ell=1}^\infty$ of $\{u_{k,b}\}_{k=1}^\infty$ such that $|\nabla u_{k_\ell,b}| \to |\nabla\mathfrak{M}_{b,\Omega}f|$ weakly in $L^p(\Omega)$ as $\ell \to \infty$. The estimate (1.24) follows from the same argument as in the end of the proof of Lemma 5.1.

By (1.24) and the arguments similar to those used to derive (5.19), we have

$$\|\nabla\mathfrak{M}_{b,\Omega}f\|_{p,\Omega} \leq C_{n,p}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega},$$

which combining with (1.14) implies that

$$\|\mathfrak{M}_{b,\Omega}f\|_{1,p,\Omega} = \|\mathfrak{M}_{b,\Omega}f\|_{p,\Omega} + \|\nabla\mathfrak{M}_{b,\Omega}f\|_{p,\Omega} \leq C_{n,p}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega},$$

which proves (1.25).

Using Lemma 5.1 and the arguments similar to those used in deriving (1.24), we can prove (1.26). By (1.26) and the arguments similar to those used to derive (5.7),

$$\|\nabla\mathfrak{M}_{b,\alpha,\Omega}f\|_{q,\Omega} \leq C_{\alpha,n,p,|\Omega|}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega},$$

which together with (1.13) leads to (1.27).

By Lemma 5.3 and the arguments similar to those used in deriving (1.24), we can prove (1.28). By (1.28) and the arguments similar to those used to derive (5.14),

$$\|\nabla\mathfrak{M}_{b,\alpha,\Omega}f\|_{q,\Omega} \leq C_{\alpha,n,p}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{1,p,\Omega},$$

which together with (1.13) leads to (1.29).

Finally, using Lemma 5.4 and the arguments similar to those used in deriving (1.24), we can prove (1.30). By (1.30) and the arguments similar to those used to derive (5.16),

$$\|\nabla\mathfrak{M}_{b,\alpha,\Omega}f\|_{q,\Omega} \leq C_{\alpha,n,p,|\Omega|}\|b\|_{\mathrm{Lip}(\Omega)}\|f\|_{p,\Omega},$$

which together with (1.13) leads to (1.31). This completes the proof of Theorem 1.17.

6. Proof of Corollary 1.18

To prove Corollary 1.18, we need the following property of the Sobolev space with zero boundary values.

Lemma 6.1. [18] *Let $\Omega \subset \mathbb{R}^n$, $\Omega \neq \mathbb{R}^n$, be an open set. Let $f \in W^{1,p}(\Omega)$ for $1 < p < \infty$ and $\int_\Omega \big(\frac{f(x)}{\operatorname{dist}(x,\Omega^c)}\big)^p\,dx < \infty$. Then $f \in W_0^{1,p}(\Omega)$.*

Proof of Corollary 1.18. The proof of Corollary 1.18 will be divided into four steps:

(i) Let $f \in W_0^{1,p}(\Omega)$ with $p \in (1,\infty)$ and $b \in \mathrm{Lip}(\Omega)$. There exists a sequence of functions $\{\varphi_k\}_{k=1}^\infty$ in $\mathcal{C}_0^\infty(\Omega)$ such that $\varphi_k \to f$ in $W^{1,p}(\Omega)$ as $k \to \infty$. It follows from Theorem 1.16(i) that $[b, M_\Omega](\varphi_k) \in W^{1,p}(\Omega)$, $k = 1, 2, \ldots$. Note that $[b, M_\Omega](\varphi_k)(x) = 0$ whenever $\operatorname{dist}(x, \partial\Omega) < 1/(2\operatorname{dist}(\operatorname{supp}\varphi_k, \partial\Omega))$. Thus we have $[b, M_\Omega](\varphi_k) \in W_0^{1,p}(\Omega)$. By Remark 1.11(ii) we see that $[b, M_\Omega](\varphi_k) \to [b, M_\Omega](f)$ in $L^p(\Omega)$ as $k \to \infty$. On the other hand, by Theorem 1.16(i),

$$|\nabla[b, M_\Omega](\varphi_k)(x)| \leq 4\|b\|_{\infty,\Omega} M_\Omega|\nabla\varphi_k|(x) + 3\sqrt{n}\|b\|_{Lip(\Omega)} M_\Omega\varphi_k(x)$$

for almost every $x \in \Omega$. This together with the arguments similar to those used to derive (1.17) implies

$$\|[b, M_\Omega](\varphi_k)\|_{1,p,\Omega} \leq C_{n,p}\|b\|_{\mathrm{Lip}(\Omega)}\|\varphi_k\|_{1,p,\Omega}.$$

This yields that $\{[b, M_\Omega](\varphi_k)\}$ is a bounded sequence in $W_0^{1,p}(\Omega)$ converging to $[b, M_\Omega](f)$ in $L^p(\Omega)$. A weak compactness implies $[b, M_\Omega](f) \in W_0^{1,p}(\Omega)$. Similarly, we can prove $\mathfrak{M}_{b,\Omega} f \in W_0^{1,p}(\Omega)$.

(ii) Let $f \in W^{1,p}(\Omega)$ with $p \in (1, n)$ and α, q, Ω be given as in Corollary 1.18(ii). It is easy to see that $M_{\alpha,\Omega} f(x) \leq \operatorname{dist}(x, \Omega^c) M_{\alpha-1,\Omega} f(x)$ for any $x \in \Omega$. It follows that

$$\begin{aligned} |[b, M_{\alpha,\Omega}](f)(x)| &\leq \operatorname{dist}(x, \Omega^c)\big(|b(x)| M_{\alpha-1,\Omega} f(x) + M_{\alpha-1,\Omega}(bf)(x)\big) \\ &\leq 2\|b\|_{\infty,\Omega} \operatorname{dist}(x, \Omega^c) M_{\alpha-1,\Omega} f(x) \end{aligned} \tag{6.1}$$

for all $x \in \Omega$. In light of (6.1) and (1.10) we would have

$$\begin{aligned} \left(\int_\Omega \left(\frac{[b, M_{\alpha,\Omega}](f)(x)}{\operatorname{dist}(x, \Omega^c)}\right)^q dx\right)^{1/q} &\leq 2\|b\|_{\infty,\Omega}\|M_{\alpha-1,\Omega} f\|_{q,\Omega} \\ &\leq C_{\alpha,n,p}\|b\|_{\infty,\Omega}\|f\|_{p,\Omega} < \infty. \end{aligned} \tag{6.2}$$

On the other hand, we get from Theorem 1.16(ii) that $[b, M_{\alpha,\Omega}](f) \in W^{1,q}(\Omega)$. This together with (6.2) and Lemma 6.1 yields $[b, M_{\alpha,\Omega}](f) \in W_0^{1,q}(\Omega)$.

One can easily check that

$$\mathfrak{M}_{b,\alpha,\Omega} f(x) \leq 2\|b\|_{\infty,\Omega} M_{\alpha,\Omega} f(x) \leq 2\|b\|_{\infty,\Omega} \operatorname{dist}(x, \Omega^c) M_{\alpha-1,\Omega} f(x), \tag{6.3}$$

which together with Theorem 1.17(ii) and the arguments similar to those used in deriving $[b, M_{\alpha,\Omega}](f) \in W_0^{1,q}(\Omega)$ implies $\mathfrak{M}_{b,\alpha,\Omega} f \in W_0^{1,q}(\Omega)$.

(iii) Let $f \in W^{1,p}(\Omega)$ and α, p, q, Ω be given as in Corollary 1.18(iii). Let $1/\widetilde{p} = 1/p - 1/n$. Clearly, $1/q = 1/\widetilde{p} - (\alpha - 1)/n$. By (6.1), (1.10) and the p-Sobolev embedding property of Ω, we have

$$\begin{aligned} \left(\int_\Omega \left(\frac{[b, M_{\alpha,\Omega}](f)(x)}{\operatorname{dist}(x, \Omega^c)}\right)^q dx\right)^{1/q} &\le 2\|b\|_{\infty,\Omega}\|M_{\alpha-1,\Omega}f\|_{q,\Omega} \le C_{\alpha,n,p}\|b\|_{\infty,\Omega}\|f\|_{\widetilde{p},\Omega} \\ &\le C_{\alpha,n,p}\|b\|_{\infty,\Omega}\|f\|_{1,p,\Omega} < \infty. \end{aligned} \tag{6.4}$$

On the other hand, we get from Theorem 1.16(iii) that $[b, M_{\alpha,\Omega}](f) \in W^{1,q}(\Omega)$. This together with (6.4) and Lemma 6.1 yields $[b, M_{\alpha,\Omega}](f) \in W_0^{1,q}(\Omega)$. Similarly, we get by (6.3) and Theorem 1.17(iii) that $\mathfrak{M}_{b,\alpha,\Omega}f \in W_0^{1,q}(\Omega)$.

(iv) Let $f \in L^p(\Omega)$ and p, α, q be given as in Corollary 1.18(iv). By (6.2), we have

$$\left(\int_\Omega \left(\frac{[b, M_{\alpha,\Omega}](f)(x)}{\operatorname{dist}(x, \Omega^c)}\right)^q dx\right)^{1/q} \le C_{\alpha,n,p}\|f\|_{p,\Omega} < \infty. \tag{6.5}$$

By Theorem 1.16(iv) we have $[b, M_{\alpha,\Omega}](f) \in W^{1,q}(\Omega)$. This together with (6.5) and Lemma 6.1 yields $[b, M_{\alpha,\Omega}](f) \in W_0^{1,q}(\Omega)$.

On the other hand, by (6.3) and the arguments similar to those used to derive (6.2),

$$\left(\int_\Omega \left(\frac{\mathfrak{M}_{b,\alpha,\Omega}f(x)}{\operatorname{dist}(x, \Omega^c)}\right)^q dx\right)^{1/q} \le C_{\alpha,n,p}\|f\|_{p,\Omega} < \infty.$$

This together with Theorem 1.17(iv) and Lemma 6.1 implies $\mathfrak{M}_{b,\alpha,\Omega}f \in W_0^{1,q}(\Omega)$. □

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