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Solutions of Perturbed Fractional Schrödinger–Poisson System with Critical Nonlinearity in $\mathbb{R}^3$

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Abstract. In this paper, we study the existence of semiclassical solutions of the fractional Schrödinger–Poisson system with critical exponent in $\mathbb{R}^3$. Under suitable assumptions, we show that the problem has at least one nontrivial solution provided that $\epsilon \le \varepsilon_\delta$ where ε_δ is sufficiently small positive number.

1. Introduction and main result

Let us consider the fractional Schrödinger–Poisson system of the form

$$\begin{cases} \epsilon^{2s}(-\Delta)^s u + V(x)u + \Phi(x)u = Q(x)|u|^{2^*(s)-2}u + f(x,u), & x \in \mathbb{R}^3, \\ (-\Delta)^t \Phi = u^2, & x \in \mathbb{R}^3, \end{cases} \tag{1.1}$$

where $\epsilon > 0$, $2^*(s) = 6/(3-2s)$, $0 < t < 1$, $3/4 < s < 1$. $V(x)$, $Q(x)$ and $f(x,u)$ satisfy the following assumptions:

(V_1) $V \in C(\mathbb{R}^3,\mathbb{R})$ and there is $b > 0$ such that the set $V^b := \{x \in \mathbb{R}^3 : V(x) < b\}$ has finite Lebesgue measure.

(V_2) $0 = V(0) = \min_{x\in\mathbb{R}^3} V(x) \le V(x) < M$.

(Q) $Q \in C(\mathbb{R}^3,\mathbb{R})$, $0 < Q_1 := \inf_{x\in\mathbb{R}^3} Q(x) \le Q_2 := \sup_{x\in\mathbb{R}^3} Q(x) < \infty$.

(f_1) $f \in C(\mathbb{R}^3 \times \mathbb{R},\mathbb{R})$, $f(x,u) = o(|u|)$ uniformly in x as $u \to 0$.

(f_2) There are $c_0 > 0$ and $q \in (2, 2^*(s))$ such that

$$|f(x,u)| \le c_0(1 + |u|^{q-1}) \quad \text{for all } (x,u).$$

(f_3) There are $a_0 > 0$ and $4 < \mu, l < 2^*(s)$ such that $F(x,u) \ge a_0|u|^l$ and $\mu F(x,u) \le f(x,u)u$ for all (x,u), where $F(x,u) = \int_0^u f(x,s)\,ds$.

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The fractional Schrödinger–Poisson system appears in an interesting physical context. In (1.1), the first equation is a nonlinear fractional Schrödinger equation in which the potential Φ satisfies a nonlinear fractional Poisson equation. For this reason, system (1.1) is called the fractional Schrödinger–Poisson system. Recently, a great attention has been focused on the study of problems involving the fractional Laplacian, since this operator arises in several areas such as probability, physics and finance. Also, the fractional Laplacian can be understood as the infinitesimal generator of a stable Lévy process.

When $s = t = 1$, the system reduces to the classical Schrödinger–Poisson system. The Schrödinger–Poisson system has been investigated in the literature, see [1,4,6,10,13–16,32] and the references therein for existence results. In recent years, there have been a lot of works dealing with the fractional Schrödinger–Poisson equations. Zhang, Do Ó and Squassina in [34] obtained the existence of radial ground state solution to the following fractional Schrödinger–Poisson system with a general subcritical or critical nonlinearity

$$\begin{cases} (-\Delta)^s u + \lambda\Phi(y)u = g(u), & y \in \mathbb{R}^3, \\ (-\Delta)^t \Phi = \lambda u^2, & y \in \mathbb{R}^3. \end{cases}$$

Without assuming the Ambrosetti–Rabinowitz condition by using the fountain theorem, Zhang in [35] considered the existence of infinitely many large energy solutions to the following system

$$\begin{cases} (-\Delta)^s u + V(y)u + \Phi(y)u = f(y,u), & y \in \mathbb{R}^3, \\ (-\Delta)^t \Phi = \lambda u^2, & y \in \mathbb{R}^3. \end{cases} \tag{1.2}$$

Using the Lyapunov–Schmidt reduction method, Liu [17] obtained the existence of multi-bump solutions for the following fractional Schrödinger–Poisson system

$$\begin{cases} (-\Delta)^s u + u + \epsilon V(y)\Phi(y)u = |u|^{p-1}u, & y \in \mathbb{R}^3, \\ (-\Delta)^t \Phi = V(y)u^2, & y \in \mathbb{R}^3. \end{cases}$$

In [25], Shen and Yao obtained the existence of positive ground state solutions and concentration results for (1.2) via some new analytical skills and Nehari–Pohožaev identity. In [24], Shen concerned with the nontrivial solutions for fractional Schrödinger–Poisson system with the Bessel operator. Under certain assumptions on the nonlinearity, a nontrivial nonnegative solution is obtained by perturbation method. In [8], using the Non-Nehari manifold approach, the authors established the existence of the Nehari-type ground state solutions for fractional Schrödinger–Poisson system (1.2). By introducing some new tricks, in [9], Chen and Tang proved that the single problem admits a ground state solution of Pohoźaev type and a least energy solution. For more related results, one can refer to [3,5,7,18–22,27,28,31,33] and the references therein.

The purpose of this paper is to study the problem (1.1) and obtain the existence and multiplicity of semiclassical solutions. To the best of our knowledge, we cannot find any result in the literature that can be directly applied to obtain the existence and multiplicity of solutions to problem (1.1). In this work, we employ the methods of [2, 12, 23, 29, 30] to obtain our main result. Especially, in [12], Ding and Lin considered the single Schrödinger equation. Here, although the idea was used in [12], the adaptation to the procedure to our problem is not trivial at all, since the appearance of non-local term, we must consider our problem for suitable space and so we need more delicate estimates.

The main result of this paper is stated below.

Theorem 1.1. *Let* (V_1), (V_2), (Q), (f_1), (f_2) *and* (f_3) *hold,* $0 < t < 1$, $3/4 < s < 1$. *Then for any* $\delta > 0$, *there is* $\varepsilon_\delta > 0$ *such that if* $\epsilon \le \varepsilon_\delta$, *problem* (1.1) *has at least one nontrivial solution* u_ϵ *satisfying*

(i) $\displaystyle \left(\frac{1}{2} - \frac{1}{\mu}\right) \int_{\mathbb{R}^3} \left(|(-\Delta)^{s/2} u_\epsilon|^2 + V(x)|u_\epsilon|^2\right) \le \delta \lambda^{1-3/(2s)}$ *and*

(ii) $\displaystyle \left(\frac{1}{4} - \frac{1}{2^*(s)}\right) \int_{\mathbb{R}^3} Q(x)|u_\epsilon|^{2^*(s)} + \frac{\mu - 4}{4} \int_{\mathbb{R}^3} F(x, u_\epsilon) \le \delta \lambda^{-3/(2s)}.$

The paper is organized as below. In Section 2, we present some preliminary results. Section 3 is devoted to the behavior of Palais–Smale sequences. In Sections 4 and 5, we accomplish the proof of the main result.

2. Preliminaries

In this section, we recall some preliminary results which will be useful along the paper. First, we will give some useful facts of the fractional order Sobolev spaces. Then, we recall some useful facts of the homogeneous Sobolev space and some properties of the Poisson potential Φ.

For any $s \in (0, 1)$, the fractional Sobolev space $H^s(\mathbb{R}^3)$ is defined by

$$\begin{aligned} H^s(\mathbb{R}^3) &= \left\{ u \in L^2(\mathbb{R}^3) : \frac{|u(x) - u(y)|}{|x - y|^{3/2+s}} \in L^2(\mathbb{R}^3 \times \mathbb{R}^3) \right\} \\ &= \left\{ u \in L^2(\mathbb{R}^3) : \int_{\mathbb{R}^3} (1 + |\xi|^{2s}) |\widehat{u}(\xi)|^2 \, d\xi < \infty \right\} \end{aligned}$$

endowed with the natural norm

$$\|u\|_{H^s(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} \int_{\mathbb{R}^3} \frac{|u(x) - u(y)|^2}{|x - y|^{3+2s}} \, dxdy + \int_{\mathbb{R}^3} |u|^2 \, dx \right)^{1/2},$$

which is induced by the inner product

$$\begin{aligned}\langle u, v\rangle_{H^s(\mathbb{R}^3)} &= \langle u, v\rangle_s + \langle u, v\rangle_{L^2(\mathbb{R}^3)}\\ &= \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{(u(x)-u(y))(v(x)-v(y))}{|x-y|^{3+2s}}\,dxdy + \int_{\mathbb{R}^3} u(x)v(x)\,dx.\end{aligned}$$

Here the term

$$[u]_{H^s(\mathbb{R}^3)} := \left(\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u(x)-u(y)|^2}{|x-y|^{3+2s}}\,dxdy\right)^{1/2}$$

is the so-called Gagliardo (semi-)norm of u and $\widehat{u}$ means the Fourier transform of u. The following identity yields the relation between the fractional operator $(-\Delta)^s$ and the fractional Sobolev space $H^s(\mathbb{R}^3)$,

$$[u]_{H^s(\mathbb{R}^3)} = C\left(\int_{\mathbb{R}^3} |\xi|^{2s}|\widehat{u}(\xi)|^2\,d\xi\right)^{1/2} = C\|(-\Delta)^{s/2}u\|_{L^2(\mathbb{R}^3)}$$

for a suitable positive constant C depending only on s.

For the reader's convenience, we review the main embedding result for this class of fractional Sobolev spaces.

Lemma 2.1. [11] *$H^s(\mathbb{R}^3)$ is continuously embedded into $L^q(\mathbb{R}^3)$ for $q \in [2, 6/(3-2s)]$, and the embedding is compact whenever $q \in [2, 6/(3-2s))$ on bounded domains.*

We introduce the space

$$E = \left\{u \in H^s(\mathbb{R}^3) : \int_{\mathbb{R}^3} V(x)u^2 < \infty\right\},$$

which is a Hilbert space under the scalar product

$$\langle v_1, v_2\rangle_E = \int_{\mathbb{R}^3} (-\Delta)^{s/2}v_1(-\Delta)^{s/2}v_2 + \int_{\mathbb{R}^3} V(x)v_1v_2.$$

The norm induced by the product $\langle\,\cdot\,,\cdot\,\rangle_E$ is

$$\|u\|_E = \sqrt{\langle u, u\rangle_E}, \quad u \in E.$$

The homogeneous Sobolev space $D^{t,2}(\mathbb{R}^3)$ is defined by

$$D^{t,2}(\mathbb{R}^3) = \left\{u \in L^{2^*(t)}(\mathbb{R}^3) : \int_{\mathbb{R}^3} |\xi|^{2s}|\widehat{u}(\xi)|^2\,d\xi < \infty\right\},$$

which is the completion of $C_0^\infty(\mathbb{R}^3)$ under the norm

$$\|u\|_{D^{t,2}} = \left(\int_{\mathbb{R}^3} |\xi|^{2t}|\widehat{u}(\xi)|^2\,d\xi\right)^{1/2} = \|(-\Delta)^{t/2}u\|_{L^2(\mathbb{R}^3)}$$

and the inner product

$$(u,v)_{D^{t,2}} = \int_{\mathbb{R}^3} (-\Delta)^{t/2} u (-\Delta)^{t/2} v \, dx, \quad u,v \in D^{t,2}(\mathbb{R}^3).$$

By the assumption (V_1), we know that the embedding $E \hookrightarrow H^s(\mathbb{R}^3)$ is continuous. Note that the norm $\|\cdot\|_E$ is equivalent to the one $\|\cdot\|_\lambda$ defined by

$$\|u\|_\lambda = \left(\int_{\mathbb{R}^3} |(-\Delta)^{s/2} u|^2 + \lambda \int_{\mathbb{R}^3} V(x) u^2 \right)^{1/2}$$

for each $\lambda > 0$. It is obvious that for each $p \in [2, 2^*(s)]$, there is $c_p > 0$ such that if $\lambda \geq 1$,

$$|u|_p \leq c_p \|u\|_E \leq c_p \|u\|_\lambda. \tag{2.1}$$

To prove our results, the following Sobolev embedding result is necessary.

Lemma 2.2. [11] *For any $t \in (0,1)$, $D^{t,2}(\mathbb{R}^3)$ is continuously embedded into $L^{2^*(t)}(\mathbb{R}^3)$, i.e., there exists $S_t > 0$ such that*

$$\left(\int_{\mathbb{R}^3} |u|^{2^*(t)} \, dx \right)^{2/2^*(t)} \leq S_t \int_{\mathbb{R}^3} |(-\Delta)^{t/2} u|^2 \, dx, \quad u \in D^{t,2}(\mathbb{R}^3).$$

By Lemma 2.1, if $0 < t < 1$, $3/4 < s < 1$, $H^s(\mathbb{R}^3) \hookrightarrow L^{12/(3+2t)}(\mathbb{R}^3)$. Then, for $u \in H^s(\mathbb{R}^3)$,

$$\int_{\mathbb{R}^3} u^2 v \, dx \leq \|u\|^2_{12/(3+2t)} \|v\|_{2^*(t)} \leq C \|u\|^2_{H^s(\mathbb{R}^3)} \|v\|_{D^{t,2}}.$$

Hence there exists a unique Φ_u^t such that $(-\Delta)^t \Phi_u^t = u^2$ and the t-Riesz potential

$$\Phi_u^t(x) = \frac{\Gamma(3/2 - 2t)}{\pi^{3/2} 2^{2t} \Gamma(t)} \int_{\mathbb{R}^3} \frac{u^2(y)}{|x-y|^{3-2t}} \, dy.$$

Furthermore, $\Phi_u^t \geq 0$ and we have

$$\|\Phi_u^t\|_{D^{t,2}} \leq C \|u\|^2_{H^s(\mathbb{R}^3)} \quad \text{if } 0 < t < 1, \ \frac{3}{4} < s < 1.$$

Substituting Φ_u^t into (1.1), we are led to the equation

$$\epsilon^{2s} (-\Delta)^s u + V(x) u + \Phi_u^t(x) u = Q(x) |u|^{2^*(s)-2} u + f(x,u). \tag{2.2}$$

Let $\lambda = \epsilon^{-2s}$, then (2.2) becomes

$$(-\Delta)^s u + \lambda V(x) u + \lambda \Phi_u^t u = \lambda Q(x) |u|^{2^*(s)-2} u + \lambda f(x,u). \tag{2.3}$$

Let

$$g(x,u) = Q(x) |u|^{2^*(s)-2} u + f(x,u)$$

and

$$G(x,u)=\int_0^u g(x,s)\,ds=\frac{1}{2^*(s)}Q(x)|u|^{2^*(s)}+F(x,u).$$

Then, after the change of variables, we obtain the following functional

$$I_\lambda(u)=\frac{1}{2}\int_{\mathbb{R}^3}|(-\Delta)^{s/2}u|^2+\frac{\lambda}{2}\int_{\mathbb{R}^3}V(x)|u|^2+\frac{\lambda}{4}\int_{\mathbb{R}^3}\Phi_u^t|u|^2-\lambda\int_{\mathbb{R}^3}G(x,u).$$

Now we can restate Theorem 1.1 as follows:

Theorem 2.3. *Let* (V_1), (V_2), (Q), (f_1), (f_2) *and* (f_3) *hold,* $0<t<1$, $3/4<s<1$. *Then for any* $\delta>0$, *there is* $\Lambda_\delta>0$ *such that if* $\lambda\ge\Lambda_\delta$, *problem* (2.3) *has at least one nontrivial solution* v_λ *satisfying*

(i) $\displaystyle\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\left(|(-\Delta)^{s/2}u_\epsilon|^2+V(x)|u_\epsilon|^2\right)\le\delta\lambda^{1-3/(2s)}$ *and*

(ii) $\displaystyle\left(\frac{1}{4}-\frac{1}{2^*(s)}\right)\int_{\mathbb{R}^3}Q(x)|u_\epsilon|^{2^*(s)}+\frac{\mu-4}{4}\int_{\mathbb{R}^3}F(x,u_\epsilon)\le\delta\lambda^{-3/(2s)}$.

3. Behaviors of $(\mathrm{PS})_c$ sequences

Let E be a real Banach space and $I_\lambda\colon E\to\mathbb{R}$ be a function of class C^1. We say that a sequence $\{u_n\}\subset E$ is a $(\mathrm{PS})_c$ sequence at level c if $I_\lambda(u_n)\to c$ and $I'_\lambda(u_n)\to 0$. I_λ is said to satisfy the $(\mathrm{PS})_c$ condition if any $(\mathrm{PS})_c$ sequence contains a convergent subsequence. The main result of this section is the following compactness result.

Lemma 3.1. *Let* (V_1), (V_2), (Q), (f_1), (f_2) *and* (f_3) *hold. If* $\{u_n\}$ *is a* $(\mathrm{PS})_c$ *sequence of* I_λ, *then* $\{u_n\}$ *is bounded in* E *and* $c\ge 0$.

Proof. If $\{u_n\}\subset E$ is a $(\mathrm{PS})_c$ sequence of I_λ, one has

$$I_\lambda(u_n)-\frac{1}{\mu}I'_\lambda(u_n)u_n=c+o(1)\|u_n\|_\lambda \tag{3.1}$$

as $n\to\infty$.

Gathering (Q) and (f_3), we obtain

$$\begin{aligned}
&I_\lambda(u_n)-\frac{1}{\mu}I'_\lambda(u_n)u_n\\
&=\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\left(|(-\Delta)^{s/2}u_n|^2+\lambda V(x)|u_n|^2\right)+\left(\frac{1}{\mu}-\frac{1}{2^*(s)}\right)\int_{\mathbb{R}^3}\lambda Q(x)|u_n|^{2^*(s)}\\
&\quad+\frac{\lambda}{\mu}\int_{\mathbb{R}^3}f(x,u_n)u_n-\lambda\int_{\mathbb{R}^N}F(x,u_n)+\lambda\left(\frac{1}{4}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\Phi_{u_n}^t|u_n|^2\\
&\ge\left(\frac{1}{2}-\frac{1}{\mu}\right)\int_{\mathbb{R}^3}\left(|(-\Delta)^{s/2}u_n|^2+\lambda V(x)|u_n|^2\right).
\end{aligned}\tag{3.2}$$

It follows from (3.1) and (3.2), we have

$$\left(\frac{1}{2}-\frac{1}{\mu}\right)\|u_n\|_\lambda^2 \le c+o(1)\|u_n\|_\lambda.$$

This estimate implies that $\|u_n\|_\lambda$ is bounded as $n\to\infty$. Taking the limit in (3.2) shows that $c\ge 0$ and the lemma is proved. □

Note that u_n is a $(\mathrm{PS})_c$ sequence. By Lemma 3.1, we may assume, without loss of generality, that $u_n \rightharpoonup u$ in E, $u_n\to u$ in $L^t_{\mathrm{loc}}(\mathbb{R}^3)$ for $2\le t<2^*(s)$ and $u_n\to u$ a.e. for $x\in\mathbb{R}^3$. Obviously, u is a critical point of I_λ.

Lemma 3.2. *Let $t\in[2,2^*(s))$ and $\{u_n\}$ be a bounded $(\mathrm{PS})_c$ sequence. Then there is a subsequence $\{u_{n_j}\}$ such that, for each $\epsilon>0$, there exists $r_\epsilon>0$ such that*

$$\limsup_{j\to\infty}\int_{B_j\setminus B_r}|u_{n_j}|^t\le\epsilon$$

for all $r\ge r_\epsilon$, where $B_k=\{x\in\mathbb{R}^N:|x|\le k\}$.

Proof. Noting that $u_n\to u$ in $L^t_{\mathrm{loc}}(\mathbb{R}^N)$ for $2\le t<2^*(s)$ as $n\to\infty$, we have, for each $j\in\mathbb{N}$,

$$\int_{B_j}|u_n|^t\to\int_{B_j}|u|^t\quad\text{as } n\to\infty,$$

and there exists $\overline{n}_j\in\mathbb{N}$ such that

$$\int_{B_j}(|u_n|^t-|u|^t)<\frac{1}{j}\quad\text{for all } n=\overline{n}_j+j,\ j=1,2,\ldots.$$

Without loss of generality, we can assume $\overline{n}_{j+1}\ge\overline{n}_j$. For $n_j=\overline{n}_j+j$, we deduce

$$\int_{B_j}(|u_{n_j}|^t-|u|^t)<\frac{1}{j}.$$

Notice that there exists an r_ϵ such that $r\ge r_\epsilon$, and the following relation is satisfied:

$$\int_{B_r^c}|u|^t<\frac{1}{3}\epsilon.$$

We have

$$\begin{aligned}\int_{B_j\setminus B_r}|u_{n_j}|^t&=\int_{B_j}(|u_{n_j}|^t-|u|^t)+\int_{B_j\setminus B_r}|u|^t+\int_{B_r}(|u|^t-|u_{n_j}|^t)\\&\le\frac{1}{j}+\int_{B_r^c}|u|^t+\int_{B_r}(|u|^t-|u_{n_j}|^t)\\&\le\epsilon\quad\text{as } j\to\infty.\end{aligned}$$

□

Remark 3.3. From the proof of Lemma 3.2, we can find the same subsequence $\{u_{n_j}\}$ such that the result of Lemma 3.2 holds for both $s = 2$ and $s = q$.

Let $\eta\colon [0,\infty) \to [0,1]$ be a smooth function satisfying $\eta(t) = 1$ if $t \leq 1$, $\eta(t) = 0$ if $t \geq 2$. Define

$$\widetilde{u}_j(x) = \eta\left(\frac{2|x|}{j}\right) u(x).$$

Using the same argument in [26], we obtain

$$\|u - \widetilde{u}_j\|_\lambda \to 0 \quad \text{as } j \to \infty. \tag{3.3}$$

Then we have the following lemma which was proved in [29].

Lemma 3.4. [12, Lemma 3.4] *Let $\{u_{n_j}\}$ be defined in Lemma 3.2. Then we have*

$$\lim_{j\to\infty} \left| \int_{\mathbb{R}^3} \big(f(x,u_{n_j}) - f(x,u_{n_j} - \widetilde{u}_j) - f(x,\widetilde{u}_j)\big)\phi \right| = 0$$

uniformly in $\phi \in E$ with $\|\phi\|_\lambda \leq 1$.

Lemma 3.5. *Let $\{u_n\}$ be defined in Lemma 3.2. Then*

(i) $I_\lambda(u_n - \widetilde{u}_n) \to c - I_\lambda(u)$;

(ii) $I'_\lambda(u_n - \widetilde{u}_n) \to 0$.

Proof. Since $V(x)$ and $Q(x)$ are bounded, by (3.3) and along the lines in the Brezis–Lieb lemma, we have

$$\begin{gathered}
\int_{\mathbb{R}^3} V(x)\big(|u_n|^2 - |u_n - \widetilde{u}_n|^2 - |\widetilde{u}_n|^2\big) \to 0, \\
\int_{\mathbb{R}^3} Q(x)\big(|u_n|^{2^*(s)} - |u_n - \widetilde{u}_n|^{2^*(s)} - |\widetilde{u}_n|^{2^*(s)}\big) \to 0, \\
\begin{aligned}
\int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x) - u_n(y)|^2}{|x-y|^{3+2s}}\,dxdy = {}& \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|u_n(x) - \widetilde{u}_n(x) - u_n(y) + \widetilde{u}_n(y)|^2}{|x-y|^{3+2s}}\,dxdy \\
&+ \int_{\mathbb{R}^3}\int_{\mathbb{R}^3} \frac{|\widetilde{u}_n(x) - \widetilde{u}_n(y)|^2}{|x-y|^{3+2s}}\,dxdy + o(1)
\end{aligned}
\end{gathered} \tag{3.4}$$

and

$$\int_{\mathbb{R}^3} \Phi_{u_n - \widetilde{u}_n}|u_n - \widetilde{u}_n|^2 + \int_{\mathbb{R}^3} \Phi_{\widetilde{u}_n}|\widetilde{u}_n|^2 - \int_{\mathbb{R}^3} \Phi_{u_n}|u_n|^2 \to 0.$$

Similar to the proof of Lemma 3.4, it is not difficult to check that

$$\int_{\mathbb{R}^3} \big(F(x,u_n) - F(x,u_n - \widetilde{u}_n) - F(x,\widetilde{u}_n)\big) \to 0. \tag{3.5}$$

Gathering (3.4) to (3.5) and the facts that $I_\lambda(u_n) \to c$ and $I_\lambda(\widetilde{u}_n) \to I_\lambda(u)$ as $n \to \infty$, we get Lemma 3.5(i).

To prove (ii), note that, for any $\phi \in E$,

$$\begin{aligned}
&I'_\lambda(u_n - \widetilde{u}_n)\phi \\
&= I'_\lambda(u_n)\phi - I'_\lambda(\widetilde{u}_n)\phi \\
&\quad + \lambda \int_{\mathbb{R}^3} Q(x)\big(|u_n|^{2^*(s)-2}u_n - |u_n - \widetilde{u}_n|^{2^*(s)-2}(u_n - \widetilde{u}_n) + |\widetilde{u}_n|^{2^*(s)-2}\widetilde{u}_n\big)\phi \\
&\quad + \lambda \int_{\mathbb{R}^3} \big(f(x,u_n) - f(x,u_n - \widetilde{u}_n) - f(x,\widetilde{u}_n)\big)\phi \\
&\quad + \lambda \int_{\mathbb{R}^3} \big(\Phi_{u_n - \widetilde{u}_n}(u_n - \widetilde{u}_n) + \Phi_{\widetilde{u}_n}\widetilde{u}_n - \Phi_{u_n}u_n\big)\phi.
\end{aligned}$$

By standard argument, we can check that

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} Q(x)\big(|u_n|^{2^*(s)-2}u_n - |u_n - \widetilde{u}_n|^{2^*(s)-2}(u_n - \widetilde{u}_n) + |\widetilde{u}_n|^{2^*(s)-2}\widetilde{u}_n\big)\phi = 0$$

uniformly in $\|\phi\|_\lambda \le 1$. By Lemma 3.4, we have

$$\lim_{n\to\infty} \int_{\mathbb{R}^3} \big(f(x,u_n) - f(x,u_n - \widetilde{u}_n) - f(x,\widetilde{u}_n)\big)\phi = 0$$

uniformly in $\|\phi\|_\lambda \le 1$. Hence, we have

$$I'_\lambda(u_n - \widetilde{u}_n) \to 0. \qquad \square$$

Proposition 3.6. *Assume that* (V_1), (V_2), (Q), (f_1), (f_2) *and* (f_3) *hold,* $0 < t < 1$, $3/4 < s < 1$. *If* $\{u_n\}$ *is a* $(\mathrm{PS})_c$ *sequence of* I_λ, *then there exists a constant* $\alpha_0 > 0$ *independent of* λ, *either* $u_n \to u$ *in* E, *or*

$$c - I_\lambda(u) \ge \alpha_0 \lambda^{1-3/(2s)}.$$

Proof. Taking $v_n := u_n - \widetilde{u}_n$. We have $u_n - u = v_n + (\widetilde{u}_n - u)$, and by (3.3), $u_n \to u$ if and only if $v_n \to 0$. By Lemma 3.5, one has $I_\lambda(v_n) \to c - I_\lambda(u)$ and $I'_\lambda(v_n) \to 0$. Using (f_3), we get

$$\begin{aligned}
&I_\lambda(v_n) - \frac{1}{4}I'_\lambda(v_n)v_n \\
&= \frac{1}{4}\int_{\mathbb{R}^3} \big(|(-\Delta)^{s/2}v_n|^2 + \lambda V(x)|v_n|^2\big) \\
&\quad + \left(\frac{1}{4} - \frac{1}{2^*(s)}\right)\int_{\mathbb{R}^3} \lambda Q(x)|v_n|^{2^*(s)} + \frac{\lambda}{4}\int_{\mathbb{R}^3} f(x,v_n)v_n - \lambda\int_{\mathbb{R}^3} F(x,v_n) \\
&\ge \left(\frac{1}{4} - \frac{1}{2^*(s)}\right)\int_{\mathbb{R}^3} \lambda Q_1|v_n|^{2^*(s)}.
\end{aligned}$$

Therefore,

$$|v_n|_{2^*(s)}^{2^*(s)} \le \frac{12(c - I_\lambda(u))}{(4s-3)Q_1\lambda} + o(1). \tag{3.6}$$

Set

$$V_b := \max\{V(x), b\},$$

where the positive constant b is mentioned in (V_1). Since the V^b has a finite measure and $v_n \to 0$ in $L^2_{\mathrm{loc}}(\mathbb{R}^3)$, we see that

$$\int_{\mathbb{R}^3} V(x)|v_n|^2 = \int_{\mathbb{R}^3} V_b|v_n|^2 + o(1). \tag{3.7}$$

From (f_1) to (f_2), we deduce for any fixed $\epsilon > 0$ that there exists C_ϵ such that

$$f(x,u)u \le \epsilon|u|^2 + C_\epsilon|u|^{2^*(s)},$$

thus by (Q), we can find a constant C_b such that

$$Q(x)|u|^{2^*(s)} + f(x,u)u \le b|u|^2 + C_b|u|^{2^*(s)}. \tag{3.8}$$

Let S be the best fractional Sobolev constant

$$S|u|^2_{2^*(s)} \le \int_{\mathbb{R}^3} |(-\Delta)^{s/2}u|^2 \quad \text{for all } u \in H^s(\mathbb{R}^3).$$

Assume that $\{v_n\}$ has no convergent subsequence. Then $\liminf_{n\to\infty} \|v_n\|_\lambda > 0$ and $c - I_\lambda(u) > 0$. From (3.7) and (3.8), we obtain

$$\begin{aligned}
&S|v_n|^2_{2^*(s)} \\
&\le \int_{\mathbb{R}^3} |(-\Delta)^{s/2}v_n|^2 + \lambda\int_{\mathbb{R}^3} V(x)|v_n|^2 - \lambda\int_{\mathbb{R}^3} V(x)|v_n|^2 \\
&= \lambda\int_{\mathbb{R}^3} Q(x)|v_n|^{2^*(s)} + \lambda\int_{\mathbb{R}^3} f(x,v_n)v_n - \lambda\int_{\mathbb{R}^3} V_b|v_n|^2 + o(1) - \lambda\int_{\mathbb{R}^3} \Phi^t_{v_n}|v_n|^2 \\
&\le \lambda\int_{\mathbb{R}^3} Q(x)|v_n|^{2^*(s)} + \lambda\int_{\mathbb{R}^3} f(x,v_n)v_n - \lambda\int_{\mathbb{R}^3} b|v_n|^2 + o(1) \\
&\le \lambda C_b|v_n|^{2^*(s)}_{2^*(s)} + o(1).
\end{aligned} \tag{3.9}$$

From (3.6) and (3.9), we have

$$S \le \lambda C_b|v_n|^{2^*(s)-2}_{2^*(s)} + o(1) \le \lambda C_b\left(\frac{12(c - I_\lambda(u))}{(4s-3)Q_1\lambda}\right)^{2s/3} + o(1).$$

Therefore, we get

$$\alpha_0\lambda^{1-3/(2s)} \le c - I_\lambda(u) + o(1),$$

where $\alpha_0 > 0$ is a constant. □

From above lemmas, we have the following conclusions.

Corollary 3.7. *Assume that* (V_1), (V_2), (Q), (f_1), (f_2) *and* (f_3) *hold. Then* I_λ *satisfies the* $(\mathrm{PS})_c$ *condition for all* $c < \alpha_0\lambda^{1-3/(2s)}$.

4. The mountain-pass structure

Lemma 4.1. *Let* (V_1), (V_2), (Q), (f_1), (f_2) *and* (f_3) *hold. There exist* $\alpha_\lambda, \rho_\lambda > 0$ *such that* $I_\lambda(u) > 0$ *if* $u \in B_{\rho_\lambda} \setminus \{0\}$ *and* $I_\lambda(u) \geq \alpha_\lambda$ *if* $u \in \partial B_{\rho_\lambda}$, *where* $B_{\rho_\lambda} = \{u \in E : \|u\|_\lambda \leq \rho_\lambda\}$.

Proof. By (f_1), (f_2) and (f_3), there is $C_\epsilon > 0$ such that

$$F(x,u) \leq \epsilon |u|^2 + C_\epsilon |u|^{2^*(s)}.$$

Thus, for $\epsilon \leq (4\lambda c_2^2)^{-1}$,

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|(-\Delta)^{s/2}u|^2 + \lambda V(x)|u|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_u^t |u|^2 - \lambda \int_{\mathbb{R}^3} F(x,u) \\ &\geq \frac{1}{2}\|u\|_\lambda^2 - \lambda\epsilon |u|_2^2 - \lambda C_\epsilon |u|_{2^*(s)}^{2^*(s)} \\ &\geq \left(\frac{1}{2} - \lambda\epsilon c_2^2\right)\|u\|_\lambda^2 - \lambda C_\epsilon |u|_{2^*(s)}^{2^*(s)} \\ &\geq \frac{1}{4}\|u\|_\lambda^2 - \lambda C_\epsilon |u|_{2^*(s)}^{2^*(s)}, \end{aligned}$$

where c_2 is the embedding constant of (2.1). Since $2 < 2^*(s)$, we know that the conclusion of Lemma 4.1 holds. □

Lemma 4.2. *Under the assumptions of Lemma* 4.1, *for any finite-dimensional subspace* $E_1 \subset E$,

$$I_\lambda(u) \to -\infty \quad \text{as } u \in E_1,\ \|u\|_\lambda \to \infty.$$

Proof. By (f_3), we have

$$\begin{aligned} I_\lambda(u) &= \frac{1}{2}\int_{\mathbb{R}^3} \left(|(-\Delta)^{s/2}u|^2 + \lambda V(x)|u|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi_u^t |u|^2 - \lambda \int_{\mathbb{R}^3} F(x,u) \\ &\leq \frac{1}{2}\int_{\mathbb{R}^3} \left(|(-\Delta)^{s/2}u|^2 + \lambda V(x)|u|^2\right) + C\frac{\lambda}{4}\|u\|_\lambda^4 - \lambda \int_{\mathbb{R}^3} a_0 |u|^l \end{aligned}$$

for all $u \in E_1$. Since all norms in a finite-dimensional space are equivalent and $l > 4$, one easily obtains the desired conclusion. □

However, in general we do not know if I_λ satisfies $(\mathrm{PS})_c$ condition. Then, we will find a special finite-dimensional subspaces by which we construct sufficiently small minimax levels.

Note that

$$\inf\left\{\int_{\mathbb{R}^3} |(-\Delta)^{s/2}\phi|^2 : \phi \in C_0^\infty(\mathbb{R}^3), |\phi|_r = 1\right\} = 0.$$

For any $\sigma > 0$, we can choose $\phi_\sigma \in C_0^\infty(\mathbb{R}^3)$ with $|\phi_\sigma|_r = 1$ and $\operatorname{supp}\phi_\sigma \subset B_{r_\sigma}(0)$ so that $|(-\Delta)^{s/2}\phi_\sigma|_2^2 < \sigma$. Denote

$$e_\lambda(x) = \phi_\sigma(\lambda^{1/(2s)}x). \tag{4.1}$$

Then $\operatorname{supp} e_\lambda \subset B_{\lambda^{-1/(2s)}r_\sigma}(0)$.

Remark that for $k \geq 0$,

$$\begin{aligned}
I_\lambda(ke_\lambda) &= \frac{1}{2}\int_{\mathbb{R}^3}\left(|(-\Delta)^{s/2}(ke_\lambda)|^2 + \lambda V(x)|ke_\lambda|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3}\Phi^t_{ke_\lambda}|ke_\lambda|^2 - \lambda\int_{\mathbb{R}^3}G(x,ke_\lambda)\\
&\leq \frac{k^2}{2}\int_{\mathbb{R}^3}\left(|(-\Delta)^{s/2}e_\lambda|^2 + \lambda V(x)|e_\lambda|^2\right) + \frac{\lambda}{4}k^4\int_{\mathbb{R}^3}\Phi^t_{e_\lambda}|e_\lambda|^2 - \lambda\int_{\mathbb{R}^3}F(x,ke_\lambda)\\
&\leq \frac{k^2}{2}\int_{\mathbb{R}^3}\left(|(-\Delta)^{s/2}e_\lambda|^2 + \lambda V(x)|e_\lambda|^2\right) + \frac{\lambda}{4}k^4\int_{\mathbb{R}^3}\Phi^t_{e_\lambda}|e_\lambda|^2 - \lambda a_0 k^l\int_{\mathbb{R}^3}|e_\lambda|^l\\
&= \lambda^{1-3/(2s)}\left(\frac{k^2}{2}\int_{\mathbb{R}^3}\left(|(-\Delta)^{s/2}\phi_\sigma|^2 + V(\lambda^{-1/(2s)}x)|\phi_\sigma|^2\right)\right.\\
&\qquad\left. + \frac{k^4}{4}\lambda^{-t/s}\int_{\mathbb{R}^3}\Phi^t_{\phi_\sigma}|\phi_\sigma|^2 - a_0k^l\int_{\mathbb{R}^3}|\phi_\sigma|^l\right)\\
&:= \lambda^{1-3/(2s)}\Psi_\lambda(k\phi_\sigma),
\end{aligned}$$

where $\Psi_\lambda \in C^1(E,\mathbb{R})$ is defined by

$$\Psi_\lambda(v) = \frac{1}{2}\int_{\mathbb{R}^3}\left(|(-\Delta)^{s/2}v|^2 + V(\lambda^{-1/(2s)}x)|v|^2\right) + \frac{1}{4}\lambda^{-t/s}\int_{\mathbb{R}^3}\Phi_v|v|^2 - a_0\int_{\mathbb{R}^3}|v|^l.$$

Since $l > 4$, thus there exists finite number $k_0 \in [0,+\infty)$ such that

$$\begin{aligned}
&\max_{k\geq 0}\Psi_\lambda(k\phi_\sigma)\\
&= \frac{k_0^2}{2}\int_{\mathbb{R}^3}\left(|(-\Delta)^{s/2}\phi_\sigma|^2 + V(\lambda^{-1/(2s)}x)|\phi_\sigma|^2\right) + \frac{k_0^4}{4}\lambda^{-t/s}\int_{\mathbb{R}^3}\Phi_{\phi_\sigma}|\phi_\sigma|^2 - a_0k_0^l\int_{\mathbb{R}^3}|\phi_\sigma|^l\\
&\leq \frac{k_0^2}{2}\int_{\mathbb{R}^3}\left(|(-\Delta)^{s/2}\phi_\sigma|^2 + V(\lambda^{-1/(2s)}x)|\phi_\sigma|^2\right) + \frac{k_0^4}{4}C\lambda^{-t/s}\left(\int_{\mathbb{R}^3}|\phi_\sigma|^{12/5}\right)^{5/3}.
\end{aligned}$$

On the one hand, since $V(0) = 0$ and $\operatorname{supp}\phi_\sigma \subset B_{r_\sigma}(0)$, there is Λ_σ such that

$$V(\lambda^{-1/(2s)}x) \leq \frac{\sigma}{|\phi_\sigma|_2^2} \quad \text{for all } |x| \leq r_\sigma \text{ and } \lambda \geq \Lambda_\sigma,$$

and $\lambda^{-t/s} < \sigma$. Then

$$\max_{k\geq 0}\Psi_\lambda(k\phi_\sigma) \leq \widetilde{C}\sigma.$$

Hence, for any $\lambda \geq \Lambda_\sigma$,

$$\max_{k\geq 0} I_\lambda(ke_\lambda) \leq \widetilde{C}\sigma\lambda^{1-3/(2s)}. \tag{4.2}$$

Now, we show the following lemma.

Lemma 4.3. *Under the assumptions of Lemma 4.1, for any $\delta > 0$ there exists $\overline{\Lambda}_\delta > 0$ such that, for each $\lambda \geq \overline{\Lambda}_\delta$, there is $\overline{e}_\lambda$ with $\|\overline{e}_\lambda\| > \rho_\lambda$, $I_\lambda(\overline{e}_\lambda) \leq 0$ and*

$$\max_{t\geq 0} I_\lambda(t\overline{e}_\lambda) \leq \delta\lambda^{1-3/(2s)},$$

where ρ_λ is given by Lemma 4.1.

Proof. Choose $\sigma > 0$ so small that $\widetilde{C}\sigma \leq \delta$, and the function $e_\lambda \in E$ is defined in (4.1). Take $\overline{\Lambda}_\delta = \Lambda_\sigma$. Let $\bar{t}_\lambda > 0$ be such that $\bar{t}_\lambda \|e_\lambda\|_\lambda > \rho_\lambda$ and $I_\lambda(te_\lambda) \leq 0$ for all $t \geq \bar{t}_\lambda$. Let $\bar{e}_\lambda := \bar{t}_\lambda e_\lambda$. As a consequence of (4.2), we know the conclusion of Lemma 4.3 holds. $\square$

For any $n^* \in \mathbb{N}$, we can choose n^* functions $\phi_\sigma^j \in C_0^\infty(\mathbb{R}^3)$ with $|\phi_\sigma^j|_r = 1$ and $\operatorname{supp} \phi_\sigma^j \cap \operatorname{supp} \phi_\sigma^k = \emptyset$, $j \neq k$ so that $|(-\Delta)^{s/2}\phi_\sigma^j|_2^2 < \sigma$. Let $r_\sigma^{n^*} > 0$ be such that $\operatorname{supp} \phi_\sigma^j \subset B_{r_\sigma^{n^*}}(0)$ for $j = 1, 2, \ldots, n^*$. Let

$$e_\lambda^j(x) = \phi_\sigma^j(\lambda^{1/(2s)}x) \text{ for } j = 1, 2, \ldots, n^* \quad \text{and} \quad H_{\lambda\sigma}^{n^*} = \operatorname{span}\{e_\lambda^1, \ldots, e_\lambda^{n^*}\}.$$

Observe that for each

$$v = \sum_{j=1}^{n^*} c_j e_\lambda^j \in H_{\lambda\sigma}^{n^*},$$

we have

$$\int_{\mathbb{R}^N} |(-\Delta)^{s/2} v|^2 \leq C \sum_{j=1}^{n^*} |c_j|^2 \int_{\mathbb{R}^N} |(-\Delta)^{s/2} e_\lambda^j|^2$$

for some constant $C > 0$. Hence, we have

$$I_\lambda(v) \leq C \sum_{j=1}^{n^*} I_\lambda(c_j e_\lambda^j)$$

for some constant $C > 0$. It is the similar way to show that

$$I_\lambda(c_j e_\lambda^j) \leq \lambda^{1-3/(2s)} \Psi_\lambda(|c_j| e_\lambda^j).$$

Set

$$\beta_\sigma := \max\{|\phi_\sigma^j|_2^2, j = 1, 2, \ldots, n^*\}$$

and choose $\Lambda_{n^*\sigma}$ such that

$$V(\lambda^{-1/(2s)}x) \leq \frac{\sigma}{\beta_\sigma} \quad \text{for all } |x| \leq r_\sigma^{n^*} \text{ and } \lambda \geq \Lambda_{n^*\sigma}.$$

As before, we can obtain the following

$$\max_{u \in H_{\lambda\sigma}^{n^*}} I_\lambda(v) \leq \widetilde{C}\sigma\lambda^{1-3/(2s)} \tag{4.3}$$

for all $\lambda \geq \Lambda_{n^*\sigma}$.

As a consequence of this estimate, we obtain the following conclusion.

Lemma 4.4. *Assume that* (V_1), (V_2), (Q), (f_1), (f_2) *and* (f_3) *are satisfied. For any $n^* \in \mathbb{N}$ and $\delta > 0$ there exists $\overline{\Lambda}_{n^*\delta} > 0$ such that, for each $\lambda \geq \overline{\Lambda}_{n^*\delta}$, there exists an n^*-dimensional subspace $F_{\lambda n^*}$ satisfying*

$$\max_{u \in F_{\lambda n^*}} I_\lambda(v) \leq \delta\lambda^{1-3/(2s)}.$$

5. Proof of the main result

In section, we prove the main result.

Proof of Theorem 2.3. For any $0 < \delta < \alpha_0$, by Lemma 4.3 we choose Λ_δ and define for $\lambda \geq \Lambda_\delta$ the minimax value

$$c_\lambda := \inf_{\gamma \in \Gamma_\lambda} \max_{t \in [0,1]} I_\lambda(t\overline{e}_\lambda),$$

where

$$\Gamma_\lambda := \{\gamma \in C([0,1], E) : \gamma(0) = 0 \text{ and } \gamma(1) = \overline{e}_\lambda\}.$$

It follows from Lemma 4.1 and (4.3) that

$$\alpha_\lambda \leq c_\lambda \leq \delta\lambda^{1-3/(2s)}.$$

From Corollary 3.7, we know that I_λ satisfies the $(\mathrm{PS})_c$ condition. Using the Mountain-pass theorem, we get that there is $u_\lambda \in E$ such that $I'_\lambda(u_\lambda) = 0$ and $I_\lambda(u_\lambda) = c_\lambda$.

Because u_λ is a critical point of I_λ, for $\nu \in [4, 2^*(s)]$,

$$\begin{aligned}
&\delta\lambda^{1-3/(2s)} \geq I_\lambda(u_\lambda) = I_\lambda(u_\lambda) - \frac{1}{\nu} I'_\lambda(u_\lambda)u_\lambda \\
&= \frac{1}{2}\int_{\mathbb{R}^3} \left(|(-\Delta)^{s/2}u_\lambda|^2 + \lambda V(x)|u_\lambda|^2\right) + \frac{\lambda}{4}\int_{\mathbb{R}^3} \Phi^t_{u_\lambda}|u_\lambda|^2 - \lambda\int_{\mathbb{R}^3} G(x,u_\lambda) \\
&\quad - \frac{1}{\nu}\int_{\mathbb{R}^3} \left(|(-\Delta)^{s/2}u_\lambda|^2 + \lambda V(x)|u_\lambda|^2\right) - \frac{\lambda}{\nu}\int_{\mathbb{R}^3} \Phi^t_{u_\lambda}|u_\lambda|^2 + \frac{\lambda}{\nu}\int_{\mathbb{R}^3} g(x,u_\lambda)u_\lambda \\
&\geq \left(\frac{1}{2} - \frac{1}{\nu}\right)\int_{\mathbb{R}^3} \left(|(-\Delta)^{s/2}u_\lambda|^2 + \lambda V(x)|u_\lambda|^2\right) + \lambda\left(\frac{1}{4} - \frac{1}{\nu}\right)\int_{\mathbb{R}^3} \Phi^t_{u_\lambda}|u_\lambda|^2 \\
&\quad + \lambda\left(\frac{1}{\nu} - \frac{1}{2^*(s)}\right)\int_{\mathbb{R}^3} Q(x)|u_\lambda|^{2^*(s)} + \lambda\left(\frac{\mu}{\nu} - 1\right)\int_{\mathbb{R}^3} F(x,u_\lambda),
\end{aligned}$$

where the constant μ is defined in (f_3). Choosing $\nu = \mu$, we have

$$\left(\frac{1}{2} - \frac{1}{\mu}\right)\int_{\mathbb{R}^3} \left(|(-\Delta)^{s/2}u_\lambda|^2 + V(x)|u_\lambda|^2\right) \leq \delta\lambda^{1-3/(2s)}$$

and taking $\nu = 4$ we obtain

$$\left(\frac{1}{4} - \frac{1}{2^*(s)}\right)\int_{\mathbb{R}^3} Q(x)|u_\lambda|^{2^*(s)} + \frac{\mu - 4}{4}\int_{\mathbb{R}^3} F(x,u_\lambda) \leq \delta\lambda^{-3/(2s)}. \qquad \square$$

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