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Normal Forms for Rigid $\mathfrak{C}_{2,1}$ Hypersurfaces $M^5 \subset \mathbb{C}^3$

Dedicated to the memory of Alexander Isaev

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Abstract. Consider a 2-nondegenerate constant Levi rank 1 rigid $\mathscr{C}^\omega$ hypersurface $M^5 \subset \mathbb{C}^3$ in coordinates $(z, \zeta, w = u + iv)$:

$$u = F(z, \zeta, \overline{z}, \overline{\zeta}).$$

The Gaussier-Merker model $u = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1 - \zeta\overline{\zeta}}$ was shown by Fels-Kaup 2007 to be locally CR-equivalent to the light cone $\{x_1^2 + x_2^2 - x_3^2 = 0\}$. Another representation is the tube $u = \frac{(\operatorname{Re} z)^2}{1 - \operatorname{Re}\zeta}$. The Gaussier-Merker model has 7-dimensional rigid automorphisms group.

Inspired by Alexander Isaev, we study *rigid* biholomorphisms:

$$(z, \zeta, w) \longmapsto (f(z, \zeta), g(z, \zeta), \rho w + h(z, \zeta)) =: (z', \zeta', w').$$

The goal is to establish the Poincaré-Moser complete normal form:

$$u = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1 - \zeta\overline{\zeta}} + \sum_{\substack{a,b,c,d \in \mathbb{N} \\ a+c \geq 3}} G_{a,b,c,d}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d$$

with $0 = G_{a,b,0,0} = G_{a,b,1,0} = G_{a,b,2,0}$ and $0 = G_{3,0,0,1} = \operatorname{Im} G_{3,0,1,1}$.

1. Introduction

The problem of equivalence for CR manifolds was begun by Poincaré [24] in 1907, who, by a plain counting argument, pointed out that real hypersurfaces $M^3 \subset \mathbb{C}^2$ must *a priori* possess infinitely many *invariants* under biholomorphic transformations. This created a local classification problem, not even terminated nowadays for hypersurfaces in $\mathbb{C}^3$. Our goal is to bring a contribution to this problem, by treating a certain already remarkably rich class of special hypersurfaces $M^5 \subset \mathbb{C}^3$.

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Throughout this article, all CR manifolds will be assumed real analytic ($\mathscr{C}^\omega$). An elementary complex Frobenius theorem proved, e.g., by Paulette Libermann in [15], guarantees embeddability into some $\mathbb{C}^{\mathrm{N}}$. We will restrict ourselves to the definite class of $\mathscr{C}^\omega$ *hypersurfaces* $M^5 \subset \mathbb{C}^3$, which are automatically CR.

The interest of studying *rigidly equivalent*—in Alexander Isaev's terminology—*rigid* hypersurfaces was pointed out to us during his February 2019 stay in Orsay. A local hypersurface $M^5 \subset \mathbb{C}^3$ with coordinates $\mathrm{Z} = (\mathrm{Z}_1, \mathrm{Z}_2, \mathrm{Z}_3)$ is said to be *rigid* if there exists an infinitesimal CR automorphism, namely a vector field T tangent to M of the form $T = X + \overline{X}$ with a nonzero holomorphic vector field $X = \sum_{i=1}^3 a_i(\mathrm{Z})\partial_{\mathrm{Z}_i}$, which is *transversal* to the complex tangent space T^cM in the sense that $TM = T^cM \oplus \mathbb{R}T$. After a local biholomorphic straightening, one makes $X = i\frac{\partial}{\partial w}$ with $w := \mathrm{Z}_3$, and tangency of $X + \overline{X} = \frac{\partial}{\partial v}$ to M shows that, writing coordinates $\mathbb{C}^3 \ni (z, \zeta, w)$, the right-hand side $\mathscr{C}^\omega$ graphing function

$$M^5: \quad u = F(z, \zeta, \overline{z}, \overline{\zeta})$$

is independent of v, where $w = u + iv$.

Alexander Isaev's concept of *rigid biholomorphic transformation* is less popular or widespread. In $\mathbb{C}^3$, such are biholomorphisms of the shape:

$$(z, \zeta, w) \longmapsto \big(f(z, \zeta), g(z, \zeta), \rho w + h(z, \zeta)\big),$$

where f, g, h are holomorphic in their arguments, *independently of* w, and where $\rho \in \mathbb{R}^*$. The interest is that rigid biholomorphisms trivially send rigid hypersurfaces to rigid hypersurfaces: they respect the pre-given CR symmetry $2\,\mathrm{Re}\, i\partial_w = \partial_v$.

The study of biholomorphic equivalence classes of *general* (not necessarily rigid) hypersurfaces $M^5 \subset \mathbb{C}^3$ has raised remarkable attention recently, especially about the class denoted $\mathfrak{C}_{2,1}$ of constant Levi rank 1 and 2-nondegenerate hypersurfaces $M^5 \subset \mathbb{C}^3$, see [3–13, 16–19, 21–23].

In the rigid context, this class $\mathfrak{C}_{2,1}^{\mathrm{rigid}}$ consists of local hypersurfaces $\{u = F(z, \zeta, \overline{z}, \overline{\zeta})\}$ passing through the origin which satisfy

$$\begin{vmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{\zeta\overline{z}} & F_{\zeta\overline{\zeta}} \end{vmatrix} \equiv 0 \neq \begin{vmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{zz\overline{z}} & F_{zz\overline{\zeta}} \end{vmatrix}.$$

Propositions 3.1 and 3.2 will show below that both conditions are *invariant* under rigid biholomorphisms. Without loss of generality, we may also assume $0 \neq F_{z\overline{z}}$. Then the first condition means constant Levi rank 1, while the second condition means 2-nondegeneracy.

In Section 2, we will present a central example, the so-called *Gaussier-Merker model*:

$$M_{\mathrm{GM}}: \quad u = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1 - \zeta\overline{\zeta}} =: m(z, \zeta, \overline{z}, \overline{\zeta}),$$

which is known to be *maximally homogeneous*, as follows from an application of Cartan's equivalence method performed in [6]. More precisely, if one defines the Lie algebra of *rigid infinitesimal holomorphic automorphisms* of any $M^5 \in \mathfrak{C}_{2,1}^{\text{rigid}}$ as

$$\mathfrak{hol}^{\text{rigid}}(M^5) := \big\{X = a(z,\zeta)\partial_z + b(z,\zeta)\partial_\zeta + (\sigma w + c(z,\zeta))\partial_w : X + \overline{X} \text{ is tangent to } M\big\}$$

with $\sigma \in \mathbb{R}$ and a, b, c three holomorphic functions *independent of* w, then from [6, Theorem 1.1] it follows that

$$\dim \mathfrak{hol}^{\text{rigid}}(M^5) \leq \dim \mathfrak{hol}^{\text{rigid}}(M_{\text{GM}}) = 7$$

with equality holding if and only if $M^5 \cong M_{\text{GM}}$ is *rigidly* biholomorphically equivalent to the model. Furthermore, $\mathfrak{hol}^{\text{rigid}}(M_{\text{GM}})$ is spanned by

$$X_1 := i\partial_w, \quad X_2 := (\zeta - 1)\partial_z - 2z\partial_w, \quad X_3 := (i + i\zeta)\partial_z - 2iz\partial_w,$$
$$X_4 := z\zeta\partial_z + (\zeta^2 - 1)\partial_\zeta - z^2\partial_w, \quad X_5 := iz\zeta\partial_z + (i + i\zeta^2)\partial_\zeta - iz^2\partial_w,$$
$$X_6 := z\partial_z + 2w\partial_w, \quad X_7 := iz\partial_z + 2i\zeta\partial_\zeta$$

with $\exp(tX_6)(\cdot)$ and $\exp(tX_7)(\cdot)$ generating the 2-dimensional isotropy subgroup of automorphisms of M_{GM} fixing the origin $0 \in M_{\text{GM}}$.

After that an $\{e\}$-structure and a canonical Cartan connection have been constructed in [6], our main objective in this article is to produce a Moser-like normal form for any $M^5 \in \mathfrak{C}_{2,1}^{\text{rigid}}$. We may assume that M passes through the origin and has power series expansion

$$u = \sum_{a+b+c+d \geq 1} F_{a,b,c,d}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d.$$

Since M has Levi form of rank 1 and is 2-nondegenerate at the origin, it is not difficult (see Section 4) to bring its cubic approximation to

$$u = z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta + \sum_{\substack{a+b+c+d \geq 4 \\ a+b \geq 1 \\ c+d \geq 1}} F_{a,b,c,d}\, z^a \zeta^b \overline{z}^c \overline{\zeta}^d.$$

Notice that this general cubic approximation coincides with that of M_{GM}.

And now, an idea of *absorption* by factorization appears. Writing initial monomials as $\overline{z}(z)$ and $\overline{z}^2\big(\frac{1}{2}\zeta\big)$, we may *capture* all holomorphic monomials behind $\overline{z}(\cdots)$ and behind $\overline{z}^2(\cdots)$, by making the rigid biholomorphism

$$z' := z + \sum_{a+b \geq 1} F_{a,b,1,0}\, z^a \zeta^b, \quad \zeta' := \zeta + 2 \sum_{a+b \geq 2} F_{a,b,2,0}\, z^a \zeta^b$$

with unchanged $w' := w$. After this is done, dropping primes, we obtain a graph $u = F(z,\zeta,\overline{z},\overline{\zeta})$ which is *prenormalized* in the sense that

$$0 = F_{a,b,0,0} = F_{0,0,c,d}, \quad 0 = F_{a,b,1,0} = F_{1,0,c,d}, \quad 0 = F_{a,b,2,0} = F_{2,0,c,d},$$

except of course $F_{1,0,1,0} = 1$ and $F_{2,0,0,1} = 1/2 = F_{0,1,2,0}$. The true story is a little more subtle, requires more care, and will be told with rigorous details in Section 4. The next task is to normalize F beyond prenormalization.

Because in $\mathbb{C}^2$ a general rigid hypersurface $u = F(z,\overline{z}) = z\overline{z} + \mathrm{O}_{z,\overline{z}}(3)$ is naturally represented as a perturbation of the (flat) model $u = z\overline{z}$, we must represent a general rigid $M \in \mathfrak{C}_{2,1}^{\text{rigid}}$ as a perturbation of the Gaussier-Merker model

$$u = F(z,\zeta,\overline{z},\overline{\zeta}) = m(z,\zeta,\overline{z},\overline{\zeta}) + G(z,\zeta,\overline{z},\overline{\zeta}).$$

Here, the remainder function G cannot be arbitrary, it must be so that the Levi form is indeed degenerate

$$0 \equiv \begin{vmatrix} m_{z\overline{z}} + G_{z\overline{z}} & m_{z\overline{\zeta}} + G_{z\overline{\zeta}} \\ m_{\zeta\overline{z}} + G_{\zeta\overline{z}} & m_{\zeta\overline{\zeta}} + G_{\zeta\overline{\zeta}} \end{vmatrix}.$$

Using this zero determinant, in our key Proposition 4.4, we show that in prenormalized coordinates, one necessarily has

$$G = \mathrm{O}_{z,\overline{z}}(3) = z^3(\cdots) + z^2\overline{z}(\cdots) + z\overline{z}^2(\cdots) + \overline{z}^3(\cdots).$$

Next, since the Gaussier-Merker function

$$m(z,\zeta,\overline{z},\overline{\zeta}) = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \overline{z}^2\zeta}{1 - \zeta\overline{\zeta}}$$

is homogeneous of degree 2 in $(z,\overline{z})$, we are conducted to assign the following weights to the coordinate variables

$$[z] := 1 =: [\overline{z}], \quad [\zeta] := 0 =: \left[\overline{\zeta}\right], \quad [w] := 2 =: [\overline{w}].$$

We then expand G in weighted homogeneous parts

$$G = \sum_{\nu \geq 3} G_\nu, \quad G_\nu = \sum_{a+c=\nu} z^a \overline{z}^c G_{a,c}(\zeta,\overline{\zeta}),$$

and we normalize progressively the G_ν, in Sections 5 and 6. This conducts us to our main

Theorem 1.1. *Every hypersurface $M^5 \in \mathfrak{C}_{2,1}^{\text{rigid}}$ is equivalent, through a local rigid biholomorphism, to a rigid $\mathscr{C}^\omega$ hypersurface $M'^5 \subset \mathbb{C}'^3$ which, dropping primes for target coordinates, is a perturbation of the Gaussier-Merker model*

$$u = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1 - \zeta\overline{\zeta}} + \sum_{\substack{a,b,c,d \in \mathbb{N} \\ a+c \geq 3}} G_{a,b,c,d} z^a \zeta^b \overline{z}^c \overline{\zeta}^d$$

with a simplified remainder G which

(1) *is normalized to be an* $\mathrm{O}_{z,\overline{z}}(3)$*;*

(2) *satisfies the prenormalization conditions* $G = \mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1) = \mathrm{O}_z(3) + \mathrm{O}_\zeta(1)$*:*

$$G_{a,b,0,0} = 0 = G_{0,0,c,d}, \quad G_{a,b,1,0} = 0 = G_{1,0,c,d}, \quad G_{a,b,2,0} = 0 = G_{2,0,c,d};$$

(3) *satisfies in addition the sporadic normalization conditions*

$$G_{3,0,0,1} = 0 = G_{0,1,3,0}, \quad \Im G_{3,0,1,1} = 0 = \Im G_{1,1,3,0}.$$

We would like to stress that, as a by-product, this result can be used to easily produce an extremely large class of new examples of 2-nondegenerate constant Levi rank 1 hypersurfaces, none of them CR equivalent to the other. We thank the referee for pointing out this consequence to us.

A standard consequence of a reduction to a CR normal form (cf. [14]), is the finite-dimensionality (here 2D) of the remaining ambiguity, as stated by

Theorem 1.2. *Furthermore, two such rigid* $\mathscr{C}^\omega$ *hypersurfaces* $M^5 \subset \mathbb{C}^3$ *and* $M'^5 \subset \mathbb{C}'^3$*, both brought into such a normal form, are rigidly biholomorphically equivalent if and only if there exist two constants* $\rho \in \mathbb{R}_+^*$*,* $\varphi \in \mathbb{R}$*, such that for all* a*,* b*,* c*,* d*,*

$$G_{a,b,c,d} = G'_{a,b,c,d} \rho^{(a+c-2)/2} e^{i\varphi(a+2b-c-2d)}.$$

A longer memoir prepublished as in [1] exposes some other aspects not conserved (plainly for length reasons) in this article:

- an introduction to the differences between two of the classical ways of studying the geometry of real submanifolds of $\mathbb{C}^n$, namely Cartan's equivalence method, and Moser's normal forms method;

- some hints on how to construct a '*theoretical bridge*' between these two methods, bringing new light on the concerned algebras of differential invariants;

- a detailed exposition of the so-called '*power series method*', developed e.g. in [2], for determining explicit expressions of all (relative) differential invariants.

These aspects are currently being reorganized to be submitted elsewhere, and hopefully, will appear in print.

2. The Gaussier-Merker model

What is the appropriate local graphed model for 2-nondegenerate constant Levi rank 1 hypersurfaces $M^5 \subset \mathbb{C}^3$ in the class $\mathfrak{C}_{2,1}$? It is known from [13, 16, 21] that the local model is any neighborhood of any smooth point of the tube in $\mathbb{C}^3$ over the light cone in $\mathbb{R}^3$ having equation $x_2^2 - x_3^2 = x_1^2$ with $x_1 > 0$. But it is not graphed!

We claim that in different notations, this cone has local graphed equation

$$u = \frac{x^2}{1-y}$$

with x, y, u being the real parts of three complex coordinates on $\mathbb{C}^3 \ni (z, \zeta, w)$. As we agreed orally with Alexander Isaev, this is the best, most compact existing graphed equation. It happens to also be the central model of parabolic surface $S^2 \subset \mathbb{R}^3$ occurring in [2].

The claim is easy. By CR-homogeneity, one can recenter at any smooth point, e.g. at $(0, 1, 1)$, write $(1+x_2)^2 - (1+x_3)^2 = x_1^2$, factor, divide, get $x_2 - x_3 = \frac{x_1^2}{2+x_2+x_3}$, and linearly change coordinates.

However, this tube graphed equation contains many pluriharmonic terms

$$\frac{w+\overline{w}}{2} = \frac{(z+\overline{z})^2}{4-2\zeta-2\overline{\zeta}} = \frac{1}{8}z^2\zeta + \frac{1}{8}\overline{z}^2\overline{\zeta} + \cdots,$$

that Moser's normal forms method would compulsorily kill at the very beginning. Thus, $u = \frac{x^2}{1-y}$ is not the right start. Similarly, $u = x^2 = \frac{1}{2}z^2 + \frac{1}{2}\overline{z}^2 + \cdots$ in $\mathbb{C}^2$ is not the right start from Moser's point of view.

The right graphed equation for the model light cone $M_{\rm GM} \subset \mathbb{C}^3$ in $\mathfrak{C}_{2,1}$ was discovered by Gaussier-Merker in [8]:

$$M_{\rm GM}\colon \qquad u = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1-\zeta\overline{\zeta}} =: m(z, \zeta, \overline{z}, \overline{\zeta}).$$

Here, the letter m is from model. By luck, $M_{\rm GM}$ is rigid!

Now, let us review the reasoning which conducted to $M_{\rm GM}$. Start with $M^5 \subset \mathbb{C}^3$ with $0 \in M$, rigid, graphed as

$$u = F(z, \zeta, \overline{z}, \overline{\zeta}).$$

Constant Levi rank 1 means, possibly after a linear transformation in $\mathbb{C}^2_{z,\zeta}$, that

$$F_{z\overline{z}} \neq 0 \equiv \begin{vmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{\zeta\overline{z}} & F_{\zeta\overline{\zeta}} \end{vmatrix} =: \mathrm{Levi}(F), \tag{2.1}$$

while 2-nondegeneracy means that

$$0 \neq \begin{vmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{zz\overline{z}} & F_{zz\overline{\zeta}} \end{vmatrix}. \tag{2.2}$$

By direct symbolic computations, Propositions 3.1 and 3.2 will establish *invariancy* of these vanishing/nonvanishing properties under rigid changes of holomorphic coordinates.

At the origin, $M_{\rm GM}$ of equation

$$u = z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta + {\rm O}_{z,\zeta,\overline{z},\overline{\zeta}}(4)$$

is obviously 2-nondegenerate, thanks to the cubic monomial $\frac{1}{2}z^2\overline{\zeta}$ which gives that (2.2) at $(z,\zeta) = (0,0)$ becomes $\left|\begin{smallmatrix} 1 & 0 \\ * & 1 \end{smallmatrix}\right| = 1$. As for constant Levi rank 1, order two terms $u = z\overline{z} + \cdots$ show that this condition is true at the origin, and simple computations show that (2.1) is identically zero:

$$\begin{vmatrix} m_{z\overline{z}} & m_{z\overline{\zeta}} \\ m_{\zeta\overline{z}} & m_{\zeta\overline{\zeta}} \end{vmatrix} = \begin{vmatrix} \frac{1}{1-\zeta\overline{\zeta}} & \frac{\overline{z}+z\overline{\zeta}}{(1-\zeta\overline{\zeta})^2} \\ \frac{z+\overline{z}\zeta}{(1-\zeta\overline{\zeta})^2} & \frac{(\overline{z}+z\overline{\zeta})(z+\overline{z}\zeta)}{(1-\zeta\overline{\zeta})^3} \end{vmatrix} \equiv 0.$$

So how to easily produce one simple example? How $M_{\rm GM}$ was born?

Normalizing the Levi form at the origin, one can assume $F = z\overline{z} + \cdots$. Hence the 2-nondegeneracy determinant (2.2) becomes at the origin $\begin{vmatrix} 1 & 0 \\ * & F_{zz\overline{\zeta}}(0) \end{vmatrix} = 1$. Thus, a monomial like $\frac{1}{2}z^2\overline{\zeta}$ must be present. Since F is real, its conjugate $\frac{1}{2}\overline{z}^2\zeta$ also comes

$$u = F = z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta + \sum_{k\geq 4} F^k(z,\zeta,\overline{z},\overline{\zeta});$$

here of course, the F^k are homogeneous polynomials of degree k. Without remainders, i.e., with all $F^k = 0$, the cubic equation is *not* of constant Levi rank 1 (exercise).

The idea of Gaussier-Merker was to take the simplest possible successive $F^4, F^5, F^6, \dots$ in order to guarantee ${\rm Levi}(F) \equiv 0$. Thus, plug all this in

$$0 \overset{?}{\equiv} \begin{vmatrix} 1 + F^4_{z\overline{z}} + F^5_{z\overline{z}} + F^5_{z\overline{z}} + \cdots & \overline{z} + F^4_{\zeta\overline{z}} + F^5_{\zeta\overline{z}} + F^6_{\zeta\overline{z}} + \cdots \\ z + F^4_{z\overline{\zeta}} + F^5_{z\overline{\zeta}} + F^6_{z\overline{\zeta}} + \cdots & F^4_{\zeta\overline{\zeta}} + F^5_{\zeta\overline{\zeta}} + F^6_{\zeta\overline{\zeta}} + \cdots \end{vmatrix}.$$

At first, look at terms of order 2, get $0 = F^4_{\zeta\overline{\zeta}} - z\overline{z}$, integrate as the simplest possible $F^4 := z\overline{z}\zeta\overline{\zeta}$. Next, plug this F^4 in, chase only homogeneous terms of degree 3, get $F^5_{\zeta\overline{\zeta}} = z^2\overline{\zeta} + \overline{z}^2\zeta$, and integrate most simply as $F^5 := \frac{1}{2}z^2\overline{\zeta}(\zeta\overline{\zeta}) + \frac{1}{2}\overline{z}^2\zeta(\zeta\overline{\zeta})$. Next, plug this F^5 in, get $F^6_{\zeta\overline{\zeta}} = 4z\overline{z}\zeta\overline{\zeta}$, integrate $F^6 := z\overline{z}(\zeta\overline{\zeta})^2$, and so on.

An easy induction then shows that powers $(\zeta\overline{\zeta})^k$ appear, and a geometric summation reconstitutes the denominator $\frac{1}{1-\zeta\overline{\zeta}}$ in the Gaussier-Merker model.

We can now pass to general $M \in \mathfrak{C}_{2,1}^{\rm rigid}$.

3. Two invariant determinants for hypersurfaces $M^5 \subset \mathbb{C}^3$

Consider a rigid biholomorphism

$$H\colon (z,\zeta,w) \longmapsto (f(z,\zeta), g(z,\zeta), \rho w + h(z,\zeta)) =: (z',\zeta',w'), \quad \rho \in \mathbb{R}^*,$$

hence with Jacobian $f_z g_\zeta - f_\zeta g_z \neq 0$, between two rigid $\mathscr{C}^\omega$ hypersurfaces

$$w = -\overline{w} + 2F(z, \zeta, \overline{z}, \overline{\zeta}) =: Q \quad \text{and} \quad w' = -\overline{w}' + 2F'(z', \zeta', \overline{z}', \overline{\zeta}') =: Q'.$$

Plugging the three components of H in the target equation

$$\rho w + h(z,\zeta) + \rho\overline{w} + \overline{h}(\overline{z},\overline{\zeta}) = 2F'\big(f(z,\zeta), g(z,\zeta), \overline{f}(\overline{z},\overline{\zeta}), \overline{g}(\overline{z},\overline{\zeta})\big),$$

and replacing $w+\overline{w} = 2F$, one receives the *fundamental equation* expressing $H(M) \subset M'$:

$$2\rho F(z,\zeta,\overline{z},\overline{\zeta}) + h(z,\zeta) + \overline{h}(\overline{z},\overline{\zeta}) \equiv 2F'\big(f(z,\zeta), g(z,\zeta), \overline{f}(\overline{z},\overline{\zeta}), \overline{g}(\overline{z},\overline{\zeta})\big).$$

By differentiating it (exercise! use a computer!), one expresses as follows the invariancy of the Levi determinant defined for general biholomorphisms [20] as

$$\begin{vmatrix} Q_{\overline{z}} & Q_{\overline{\zeta}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{\zeta}} & Q_{z\overline{w}} \\ Q_{\zeta\overline{z}} & Q_{\zeta\overline{\zeta}} & Q_{\zeta\overline{w}} \end{vmatrix} = 2^2 \begin{vmatrix} F_{\overline{z}} & F_{\overline{\zeta}} & -1 \\ F_{z\overline{z}} & F_{z\overline{\zeta}} & 0 \\ F_{\zeta\overline{z}} & F_{\zeta\overline{\zeta}} & 0 \end{vmatrix}.$$

Proposition 3.1. *Through any rigid biholomorphism*

$$\begin{vmatrix} F'_{z'\overline{z}'} & F'_{z'\overline{\zeta}'} \\ F'_{\zeta'\overline{z}'} & F'_{\zeta'\overline{\zeta}'} \end{vmatrix} = \frac{\rho^2}{\begin{vmatrix} f_z & f_\zeta \\ g_z & g_\zeta \end{vmatrix} \begin{vmatrix} \overline{f}_{\overline{z}} & \overline{f}_{\overline{\zeta}} \\ \overline{g}_{\overline{z}} & \overline{g}_{\overline{\zeta}} \end{vmatrix}} \begin{vmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{\zeta\overline{z}} & F_{\zeta\overline{\zeta}} \end{vmatrix}.$$

Consequently, the property that the Levi form is of constant rank 1 is biholomorphically invariant. The 2-nondegeneracy property [20] then expresses as the nonvanishing of

$$\begin{vmatrix} Q_{\overline{z}} & Q_{\overline{\zeta}} & Q_{\overline{w}} \\ Q_{z\overline{z}} & Q_{z\overline{\zeta}} & Q_{z\overline{w}} \\ Q_{zz\overline{z}} & Q_{zz\overline{\zeta}} & Q_{zz\overline{w}} \end{vmatrix} = 2^2 \begin{vmatrix} F_{\overline{z}} & F_{\overline{\zeta}} & -1 \\ F_{z\overline{z}} & F_{z\overline{\zeta}} & 0 \\ F_{zz\overline{z}} & F_{zz\overline{\zeta}} & 0 \end{vmatrix}.$$

Proposition 3.2. *When the Levi form is of constant rank* 1*, through any rigid biholomorphism,*

$$\begin{vmatrix} F'_{z'\overline{z}'} & F'_{z'\overline{\zeta}'} \\ F'_{z'z'\overline{z}'} & F'_{z'z'\overline{\zeta}'} \end{vmatrix} = \frac{\rho^2 \big(g_\zeta F_{z\overline{z}} - g_z F_{\zeta\overline{z}}\big)^3}{\begin{vmatrix} f_z & f_\zeta \\ g_z & g_\zeta \end{vmatrix}^3 \begin{vmatrix} \overline{f}_{\overline{z}} & \overline{f}_{\overline{\zeta}} \\ \overline{g}_{\overline{z}} & \overline{g}_{\overline{\zeta}} \end{vmatrix}} \begin{vmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{zz\overline{z}} & F_{zz\overline{\zeta}} \end{vmatrix}.$$

4. Prenormalization

In coordinates $(z, \zeta, w) \in \mathbb{C}^3$ with $w = u + iv$, consider a local $\mathscr{C}^\omega$ rigid hypersurface $M^5 \subset \mathbb{C}^3$ graphed as $u = F(z, \zeta, \overline{z}, \overline{\zeta})$ passing through the origin. Expand $\sum_{a+b+c+d\geq 1} F_{a,b,c,d}$ $z^a \zeta^b \overline{z}^c \overline{\zeta}^d$, and define by conjugating only coefficients

$$\overline{F}(z, \zeta, \overline{z}, \overline{\zeta}) := \sum_{a+b+c+d\geq 1} \overline{F}_{a,b,c,d} z^a \zeta^b \overline{z}^c \overline{\zeta}^d.$$

The reality $\overline{u} = u$ forces $\overline{F(z, \zeta, \overline{z}, \overline{\zeta})} = F(z, \zeta, \overline{z}, \overline{\zeta})$ which becomes

$$\overline{F}(\overline{z}, \overline{\zeta}, z, \zeta) \equiv F(z, \zeta, \overline{z}, \overline{\zeta}).$$

The 4 independent derivations ∂_z, ∂_ζ, $\partial_{\overline{z}}$, $\partial_{\overline{\zeta}}$ commute. Applying $\frac{1}{a!}\partial_z^a \frac{1}{b!}\partial_\zeta^b \frac{1}{c!}\partial_{\overline{z}}^c \frac{1}{d!}\partial_{\overline{\zeta}}^d$ at the origin $(0,0,0,0)$, it comes

$$\overline{F}_{c,d,a,b} = F_{a,b,c,d}.$$

With $\chi(z, \zeta) := F(z, \zeta, 0, 0)$ which is holomorphic, setting $w' := w - 2\chi(z, \zeta)$, we get

$$\frac{w' + \overline{w}'}{2} = u' = F(z, \zeta, \overline{z}, \overline{\zeta}) - \chi(z, \zeta) - \overline{\chi}(\overline{z}, \overline{\zeta}) =: F'(z, \zeta, \overline{z}, \overline{\zeta})$$

with now $0 \equiv F'(z, \zeta, 0, 0) \equiv F'(0, 0, \overline{z}, \overline{\zeta})$.

By $\mathrm{O}_x(3)$, we mean a (remainder) function equal to $x^3(\cdots)$, where $(\cdots)$ is any function of one or several variables. By $\mathrm{O}_{x,y}(2)$, we mean $x^2(\cdots) + xy(\cdots) + y^2(\cdots)$, and so on.

Proposition 4.1. *After a rigid biholomorphism, an $M \in \mathfrak{C}_{2,1}$ satisfies*

$$F(z, \zeta, \overline{z}, 0) = z\overline{z} + \frac{1}{2}\zeta\overline{z}^2 + \mathrm{O}_{\overline{z}}(3).$$

Employing the letter $\mathscr{R}$ for unspecified functions, this amounts to

$$F(z, \zeta, \overline{z}, \overline{\zeta}) = z\overline{z} + \frac{1}{2}\zeta\overline{z}^2 + \overline{z}^3\mathscr{R}(z, \zeta, \overline{z}) + \overline{\zeta}\mathscr{R}(z, \zeta, \overline{z}, \overline{\zeta}). \tag{4.1}$$

We will use without mention

$$\mathscr{R}(z, \zeta, \overline{z}, \overline{\zeta}) = \mathscr{R}(z, \zeta, \overline{z}) + \overline{\zeta}\mathscr{R}(z, \zeta, \overline{z}, \overline{\zeta}).$$

Proof of Proposition 4.1. We will perform rigid biholomorphisms of the form $z' = z'(z, \zeta)$, $\zeta' = \zeta'(z, \zeta)$, $w' = w$ fixing 0. They transform $u = F(z, \zeta, \overline{z}, \overline{\zeta})$ into $u' = F'(z', \zeta', \overline{z}', \overline{\zeta}')$ with

$$F'(z', \zeta', \overline{z}', \overline{\zeta}') := F\big(z(z', \zeta'), \zeta(z', \zeta'), \overline{z}(\overline{z}', \overline{\zeta}'), \overline{\zeta}(\overline{z}', \overline{\zeta}')\big),$$

hence they conserves $F'(z', \zeta', 0, 0) \equiv 0$.

The Levi form being of rank 1 at 0, we may assume

$$u = z\overline{z} + \mathrm{O}_3(z, \zeta, \overline{z}, \overline{\zeta}).$$

Assertion 4.2. *After a rigid biholomorphism fixing* 0*,*

$$F = z\overline{z} + \overline{z}^2\mathscr{R} + \overline{\zeta}\mathscr{R}.$$

Proof. We can decompose

$$F(z,\zeta,\overline{z},\overline{\zeta}) = F(z,\zeta,\overline{z},0) + \overline{\zeta}\mathscr{R} = \overline{z}(z + \chi(z,\zeta)) + \overline{z}^2\mathscr{R} + \overline{\zeta}\mathscr{R}$$

with $\chi = \mathrm{O}(2)$. Then

$$F = (z+\chi)(\overline{z}+\overline{\chi}) - z\overline{\chi} - \chi\overline{\chi} + \overline{z}^2\mathscr{R} + \overline{\zeta}\mathscr{R}.$$

But $\overline{\chi} = \overline{z}^2\mathscr{R}(\overline{z}) + \overline{\zeta}\mathscr{R}(\overline{z},\overline{\zeta})$ is absorbable, hence

$$F = (z+\chi)(\overline{z}+\overline{\chi}) + \overline{z}^2\mathscr{R} + \overline{\zeta}\mathscr{R}.$$

Thus, we perform the rigid biholomorphism $z' := z + \chi(z,\zeta)$, $\zeta' := \zeta$ with inverse

$$z = z' + \mathrm{O}_{z',\zeta'}(2) = z' + {z'}^2\mathscr{R}' + \zeta'\mathscr{R}'.$$

Hence $\overline{z}^2 = \overline{z}'^2\mathscr{R}' + \overline{\zeta}'\mathscr{R}'$, and lastly

$$F'(z',\zeta',\overline{z}',\overline{\zeta}') = z'\overline{z}' + \overline{z}'^2\mathscr{R}' + \overline{\zeta}'\mathscr{R}'. \qquad \square$$

Next, dropping primes, specifying 3rd order (real) terms $P = P_3$ in $F = z\overline{z} + P_3 + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4)$, let us inspect the Levi determinant

$$0 \equiv \begin{vmatrix} 1 + P_{z\overline{z}} + \mathrm{O}_2 & P_{\zeta\overline{z}} + \mathrm{O}_2 \\ P_{z\overline{\zeta}} + \mathrm{O}_2 & P_{\zeta\overline{\zeta}} + \mathrm{O}_2 \end{vmatrix}, \quad \text{whence } 0 \equiv P_{\zeta\overline{\zeta}},$$

i.e., P is harmonic with respect to ζ when z, $\overline{z}$ are seen as constants. Thus taking account of $0 \equiv P(z,\zeta,0,0)$,

$$P = az^2\overline{z} + \overline{a}z\overline{z}^2 + \zeta(bz\overline{z} + c\overline{z}^2) + \overline{\zeta}(\overline{b}z\overline{z} + \overline{c}z^2) + \zeta^2(d\overline{z}) + \overline{\zeta}^2(\overline{d}z).$$

But Assertion 4.2 forces $a = 0$, $b = 0$, $d = 0$, whence

$$u = z\overline{z} + c\zeta\overline{z}^2 + \overline{c}\overline{\zeta}z^2 + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4).$$

From Proposition 3.2, we know that $c \neq 0$, hence $c\zeta =: \frac{1}{2}\zeta'$ conducts to

$$u = z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4) = z\overline{z} + \overline{z}^2\mathscr{R} + \overline{\zeta}\mathscr{R}. \tag{4.2}$$

Next, let us look at 4th order terms which depend only on $(z,\overline{z})$, especially at the monomial $ez^2\overline{z}^2$ with $e := F_{2,0,2,0} \in \mathbb{R}$. We can make $e = 0$ thanks to $\zeta' := \zeta + ez^2$,

$$u = z\overline{z} + \frac{1}{2}(\zeta + ez^2)\overline{z}^2 + \frac{1}{2}(\overline{\zeta} + e\overline{z}^2)z^2 + \overline{z}^2\mathscr{R} + \overline{\zeta}\mathscr{R}.$$

So we can assume $F_{2,0,2,0} = 0$. We then write

$$u = z\overline{z} + \frac{1}{2}\overline{z}^2 S(z,\zeta,\overline{z}) + \overline{\zeta}\mathscr{R}(z,\zeta,\overline{z},\overline{\zeta})$$

with $S = \zeta + \mathrm{O}_{z,\zeta,\overline{z}}(2)$ and with *no* z^2 monomial in the remainder. Hence with some function $\tau(z)$ which *is* an $\mathrm{O}_z(3)$, and with some function $\omega(z,\zeta) = \mathrm{O}_{z,\zeta}(1)$, we devise which biholomorphism to perform

$$\begin{aligned} u &= z\overline{z} + \frac{1}{2}\overline{z}^2(\zeta + \tau(z) + \zeta\omega(z,\zeta) + \overline{z}\theta(z,\zeta,\overline{z})) + \overline{\zeta}\mathscr{R} \\ &= z\overline{z} + \frac{1}{2}\overline{z}^2\big(\underbrace{\zeta + \tau(z) + \zeta\omega(z,\zeta)}_{=:\zeta',\text{ while } z=:z'}\big) + \overline{z}^3\mathscr{R} + \overline{\zeta}\mathscr{R}. \end{aligned}$$

Assertion 4.3. *The inverse* $\zeta = \zeta' + \mathrm{O}(2) = \tau'(z') + \zeta'[1 + \omega'(z',\zeta')]$ *also satisfies* $\tau'(z') = \mathrm{O}_{z'}(3)$.

Proof. Indeed, by definition,

$$\zeta \equiv \tau'(z) + [\tau(z) + \zeta(1 + \omega(z,\zeta))][1 + \omega'(z, \tau(z) + \zeta(1 + \omega(z,\zeta)))],$$

and it suffices to put $\zeta := 0$ to get a concluding relation which even shows that $\mathrm{ord}_0\,\tau = \mathrm{ord}_0\,\tau'$:

$$0 \equiv \tau'(z) + \tau(z)[1 + \omega'(z,\tau(z))]. \qquad \square$$

All this enables to reach the goal (4.1) since $\overline{\tau}'(\overline{z}')$ is absorbable in $\overline{z}'^3\mathscr{R}'$:

$$u = z'\overline{z}' + \frac{1}{2}\overline{z}'^2\zeta' + \overline{z}'^3\mathscr{R}' + \big(\overline{\zeta}' + \overline{\tau}'(\overline{z}') + \overline{\zeta}'\overline{\omega}'(\overline{z}',\overline{\zeta}')\big)\mathscr{R}'. \qquad \square$$

Coordinates like in Proposition 4.1 will be called *prenormalized.* Equivalently (exercise),

$$0 = F_{a,b,0,0} = F_{0,0,c,d}, \quad 0 = F_{a,b,1,0} = F_{1,0,c,d}, \quad 0 = F_{a,b,2,0} = F_{2,0,c,d}$$

with only three exceptions $F_{1,0,1,0} = 1$ and $F_{2,0,0,1} = 1/2 = F_{0,1,2,0}$. During the proof, in (4.2), we obtained simultaneously

$$u = F = z\overline{z} + \frac{1}{2}\overline{z}^2\zeta + \mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1) = z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4). \tag{4.3}$$

Now, recall that the Gaussier-Merker model is homogeneous of degree 2 in z, $\overline{z}$, when ζ, $\overline{\zeta}$ are treated as constants:

$$u = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1 - \zeta\overline{\zeta}} =: m(z,\zeta,\overline{z},\overline{\zeta}).$$

A general $M \in \mathfrak{C}_{2,1}$ is just a perturbation of it:

$$u = F = m + G \quad \text{with } G := F - m = \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4).$$

Proposition 4.4. *In prenormalized coordinates, one has* $G = \mathrm{O}_{z,\overline{z}}(3)$.

Proof. Expand

$$m = z\overline{z}\sum_{i\geq 0}\zeta^i\overline{\zeta}^i + \frac{1}{2}z^2\sum_{i\geq 0}\zeta^i\overline{\zeta}^{i+1} + \frac{1}{2}\overline{z}^2\sum_{i\geq 0}\zeta^{i+1}\overline{\zeta}^i = z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(4),$$

$$G = \sum_{k\geq 4}\sum_{a+b+c+d=k} G_{a,b,c,d}\, z^a\zeta^b\overline{z}^c\overline{\zeta}^d =: \sum_{k\geq 4} G^k.$$

Of course, $F^k = m^k + G^k$ with $G^2 = G^3 = 0$.

Assertion 4.5. *For every* $k \geq 2$, *one has* $G^k = \mathrm{O}_{z,\overline{z}}(3)$.

Proof. For some $k \geq 4$, assume by induction that $G^2, G^3, \dots, G^{k-1}$ are $\mathrm{O}_{z,\overline{z}}(3)$, whence

$$G^\ell_{z\overline{z}} = \mathrm{O}_{z,\overline{z}}(1), \quad G^\ell_{\zeta\overline{z}} = \mathrm{O}_{z,\overline{z}}(2) = G^\ell_{z\overline{\zeta}}, \quad G^\ell_{\zeta\overline{\zeta}} = \mathrm{O}_{z,\overline{z}}(3), \quad 1 \leq \ell \leq k-1.$$

Next, insert $F = \sum_{i\geq 2} F^i$ in the Levi determinant:

$$0 \equiv \begin{vmatrix} \sum_i F^i_{z\overline{z}} & \sum_j F^j_{\zeta\overline{z}} \\ \sum_i F^i_{z\overline{\zeta}} & \sum_j F^j_{\zeta\overline{\zeta}} \end{vmatrix} = \sum_{\ell\geq 4}\left(\sum_{\substack{i+j=\ell\\ i,j\geq 2}} \left(F^i_{z\overline{z}}F^j_{\zeta\overline{\zeta}} - F^i_{z\overline{\zeta}}F^j_{\zeta\overline{z}}\right)\right).$$

Behind $\sum_\ell$, all terms are of constant homogeneous order $i-2+j-2 = \ell-4$, hence $0 \equiv \sum_{i+j=\ell}(\text{above})$ for each $\ell \geq 4$. Take $\ell := k+2$ and expand

$$\begin{aligned} 0 \equiv F^2_{z\overline{z}}F^k_{\zeta\overline{\zeta}} &+ \sum_{3\leq i\leq k-1} F^i_{z\overline{z}}F^{k+2-i}_{\zeta\overline{\zeta}} + F^k_{z\overline{z}}\underline{F^2_{\zeta\overline{\zeta}}}_\circ \\ - \underline{F^2_{z\overline{\zeta}}}_\circ F^k_{\zeta\overline{z}} &- \sum_{3\leq i\leq k-1} F^i_{z\overline{\zeta}}F^{k+2-i}_{\zeta\overline{z}} - F^k_{z\overline{\zeta}}\underline{F^2_{\zeta\overline{z}}}_\circ. \end{aligned}$$

Observe from (4.3) that $1 \equiv F^2_{z\overline{z}}$ while $0 \equiv F^2_{\zeta\overline{\zeta}} \equiv F^2_{z\overline{\zeta}} \equiv F^2_{\zeta\overline{z}}$. Of course, Levi determinant vanishing holds for $F := m$,

$$\begin{aligned} 0 \equiv m^2_{z\overline{z}}m^k_{\zeta\overline{\zeta}} &+ \sum_{3\leq i\leq k-1} m^i_{z\overline{z}}m^{k+2-i}_{\zeta\overline{\zeta}} + m^k_{z\overline{z}}\underline{m^2_{\zeta\overline{\zeta}}}_\circ \\ - \underline{m^2_{z\overline{\zeta}}}_\circ m^k_{\zeta\overline{z}} &- \sum_{3\leq i\leq k-1} m^i_{z\overline{\zeta}}m^{k+2-i}_{\zeta\overline{z}} - m^k_{z\overline{\zeta}}\underline{m^2_{\zeta\overline{z}}}_\circ. \end{aligned}$$

Substituting the boxed term $F^k_{\zeta\overline{\zeta}}$ with $m^k_{\zeta\overline{\zeta}} + G^k_{\zeta\overline{\zeta}}$, solving for $G^k_{\zeta\overline{\zeta}}$, substituting as well the other $F^\ell_{\cdot\cdot} = m^\ell_{\cdot\cdot} + G^\ell_{\cdot\cdot}$, and subtracting, we obtain

$$\begin{aligned} -G^k_{\zeta\overline{\zeta}} \equiv &\sum_{3\leq i\leq k-1}\left(m^i_{z\overline{z}}G^{k+2-i}_{\zeta\overline{\zeta}} + G^i_{z\overline{z}}m^{k+2-i}_{\zeta\overline{\zeta}} + G^i_{z\overline{z}}G^{k+2-i}_{\zeta\overline{\zeta}}\right) \\ &- \sum_{3\leq i\leq k-1}\left(m^i_{z\overline{\zeta}}G^{k+2-i}_{\zeta\overline{z}} + G^i_{z\overline{\zeta}}m^{k+2-i}_{\zeta\overline{z}} + G^i_{z\overline{\zeta}}G^{k+2-i}_{\zeta\overline{z}}\right). \end{aligned}$$

Since we also have $3 \leq k+2-i \leq k-1$, induction applies to all six products to get $G^k_{\zeta\overline{\zeta}} = \mathrm{O}_{z,\overline{z}}(3)$.

By integration, $G^k = \lambda^k(z,\zeta,\overline{z}) + \overline{\lambda}^k(\overline{z},\overline{\zeta},z) + \mathrm{O}_{z,\overline{z}}(3)$. After absorption in $\mathrm{O}_{z,\overline{z}}(3)$, we can assume that λ^k is of degree ≤ 2 in $(z,\overline{z})$, hence contains only monomials $z^a\zeta^b\overline{z}^c$ with $a+c \leq 2$ and $a+b+c=k$. So $b \geq k-2$.

Further, $G^k(z,\zeta,0,0) \equiv 0$ imposes $\lambda^k(z,\zeta,0) \equiv 0$. So $1 \leq c \leq 2$. Consequently, λ^k can contain only three monomials

$$\lambda^k(z,\zeta,\overline{z}) = a\overline{z}\zeta^{k-1} + bz\overline{z}\zeta^{k-2} + c\overline{z}^2\zeta^{k-2}.$$

Since $k \geq 4$, we see that the conjugate $\overline{\lambda}^k(\overline{z},\overline{\zeta},z)$ is multiple of $\overline{\zeta}^{k-2\geq 2}$, hence

$$G^k(z,\zeta,\overline{z},0) = \lambda^k(z,\zeta,\overline{z}) + \underline{\overline{\lambda}^k(\overline{z},0,z)}_{\circ} + \mathrm{O}_{z,\overline{z}}(3).$$

Finally, because the prenormalized coordinates of Proposition 4.1 require $G^k(z,\zeta,\overline{z},0) = \mathrm{O}_{\overline{z}}(3)$, we reach $\lambda^k(z,\zeta,\overline{z}) = \mathrm{O}_{z,\overline{z}}(3)$, which forces $a=b=c=0=\lambda^k$, so as asserted $G^k = \mathrm{O}_{z,\overline{z}}(3)$. □

In conclusion, $G = \sum G^k = \mathrm{O}_{z,\overline{z}}(3)$. □

According to [6] the Lie group G of rigid holomorphic automorphisms of the Gaussier-Merker model $\{u=m\}$ has Lie algebra of dimension 7, generated by the vector fields $X_1,\dots,X_7$ shown in Section 1. The 2-dimensional isotropy subgroup $G_0 \subset G$ of the origin $0 \in \mathbb{C}^3$ has Lie algebra generated by

$$X_6 := z\partial_z + 2w\partial_w, \quad X_7 := iz\partial_z + 2i\zeta\partial_\zeta.$$

By computing the flows $\exp(tX_\sigma)(z,\zeta,w)$ for $t \in \mathbb{R}$ and $\sigma = 6,7$, one verifies that G_0 consists of scalings coupled with 'rotations':

$$z' = \rho^{1/2}e^{i\varphi}z, \quad \zeta' = e^{2i\varphi}\zeta, \quad w' = \rho w, \quad \rho \in \mathbb{R}^*_+,\ \varphi \in \mathbb{R}.$$

Next, any holomorphic function $e = e(z,\zeta)$ decomposes in weighted homogeneous terms as

$$e(z,\zeta) = \sum_{a,b} e_{a,b}z^a\zeta^b = \sum_{k\geq 0}\left(\sum_b e_{k,b}\zeta^b\right)z^k =: \sum_{k\geq 0} e_k.$$

Mind notation: for weights, indices e_k are lower case, while for orders, as e.g. in G^k before, they were upper case. Similarly,

$$E(z,\zeta,\overline{z},\overline{\zeta}) = \sum_{k\geq 0}\left(\sum_{a+c=k}\left(\sum_{b,d} E_{a,b,c,d}\zeta^b\overline{\zeta}^d\right)z^a\overline{z}^c\right) =: \sum_{k\geq 0} E_k.$$

According to what precedes, we can assume that both the source M and the target M' rigid hypersurfaces are prenormalized. Assume therefore that a rigid biholomorphism

$$H\colon (z,\zeta,w) \longmapsto (f(z,\zeta), g(z,\zeta), \rho w + h(z,\zeta)) =: (z',\zeta',w'),$$

fixing the origin is given between

$$\begin{aligned} u = F = z\overline{z} + \frac{1}{2}\overline{z}^2\zeta + \mathrm{O}_{\overline{z}}(3) = m + G = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1-\zeta\overline{\zeta}} + \mathrm{O}_{z,\overline{z}}(3), \\ u' = F' = z'\overline{z}' + \frac{1}{2}\overline{z}'^2\zeta' + \mathrm{O}_{\overline{z}'}(3) = m' + G' = \frac{z'\overline{z}' + \frac{1}{2}z'^2\overline{\zeta}' + \frac{1}{2}\overline{z}'^2\zeta'}{1-\zeta'\overline{\zeta}'} + \mathrm{O}_{z',\overline{z}'}(3). \end{aligned}$$

Observation 4.6. Scalings and rotations $(z',\zeta',w') \longmapsto \big(\rho^{1/2}e^{i\varphi}z', e^{2i\varphi}\zeta', \rho w'\big)$ preserve prenormalizations.

Since $T_0^cM = \{w = 0\}$ and $T_0^cM' = \{w' = 0\}$, and since $H_*T_0^cM = T_0^cM'$, we necessarily have $h = \mathrm{O}_{z,\zeta}(2)$. After the scaling $w' \longmapsto \frac{1}{\rho}w'$, we may therefore assume that the last component of H is $w + \mathrm{O}_{z,\zeta}(2)$.

Let us decompose the components of H in weighted homogeneous parts

$$f = f_0 + f_1 + f_2 + f_3 + \cdots, \quad g = g_0 + g_1 + g_2 + \cdots, \quad h = h_0 + h_1 + h_2 + h_3 + h_4 + \cdots.$$

Plug in the components of H in the target rigid equation $\frac{w'+\overline{w}'}{2} = F'(z',\zeta',\overline{z}',\overline{\zeta}')$:

$$w + h(z,\zeta) + \overline{w} + \overline{h}(\overline{z},\overline{\zeta}) = 2F'\big(f(z,\zeta), g(z,\zeta), \overline{f}(\overline{z},\overline{\zeta}), \overline{g}(\overline{z},\overline{\zeta})\big),$$

and then, substitute $w + \overline{w} = 2F$ to get a *fundamental equation*, holding identically:

$$2F(z,\zeta,\overline{z},\overline{\zeta}) + h(z,\zeta) + \overline{h}(\overline{z},\overline{\zeta}) \equiv 2F'\big(f(z,\zeta), g(z,\zeta), \overline{f}(\overline{z},\overline{\zeta}), \overline{g}(\overline{z},\overline{\zeta})\big). \tag{4.4}$$

Proposition 4.7. *Possibly after a rotation* $(z',\zeta',w') \longmapsto (e^{i\varphi}z', e^{2i\varphi}\zeta', w')$*, one has*

$$f = z + f_2 + f_3 + \cdots, \quad g = \zeta + g_1 + g_2 + \cdots, \quad h = w + h_3 + h_4 + \cdots$$

or equivalently: $f_0 = 0$*,* $f_1 = z$*;* $g_0 = \zeta$*;* $h_0 = 0$*,* $h_1 = 0$*,* $h_2 = w$*.*

Proof. Recall that $F = m + G$, that $m = m_2$ and that $G = G_3 + G_4 + \cdots$ with the same about $F' = m' + G'$. So F and F' have no terms of weights 0 or 1. Of course $f_0 = f_0(\zeta)$, $g_0 = g_0(\zeta)$, $h_0 = h_0(\zeta)$ depend on ζ only.

In (4.4), pick terms of weight zero:

$$0 + h_0(\zeta) + \overline{h}_0(\overline{\zeta}) \equiv 2F'\big(f_0(\zeta), g_0(\zeta), \overline{f}_0(\overline{\zeta}), \overline{g}_0(\overline{\zeta})\big),$$

put $\overline{\zeta} := 0$, use $F'(z',\zeta',0,0) \equiv 0$, and get $h_0 = 0$.

Once again, pick in (4.4) terms of weight zero using $F' = m' + \mathrm{O}_{z',\overline{z}'}(3)$:

$$0 \equiv \frac{f_0(\zeta)\overline{f}_0(\overline{\zeta}) + \frac{1}{2} f_0(\zeta)^2 \overline{g}_0(\overline{\zeta}) + \frac{1}{2} \overline{f}_0(\overline{\zeta}) g_0(\zeta)}{1 - g_0(\zeta)\overline{g}_0(\overline{\zeta})} + \mathrm{O}_{f_0(\zeta), \overline{f}_0(\overline{\zeta})}(3).$$

We claim that $f_0(\zeta) \equiv 0$. Otherwise, $f_0 = c\zeta^\nu + \mathrm{O}_\zeta(\nu + 1)$ with $c \neq 0$, but on the right, the monomial $c\overline{c}\zeta^\nu\overline{\zeta}^\nu$ cannot be killed—contradiction. This finishes examination of weight zero, for it remains only $0 \equiv 0$.

Hence, pass to weight 1. We claim that $h_1 = 0$. Of course, $f_1 = zf_1(\zeta)$ and $h_1 = zh_1(\zeta)$. Since m' is weighted homogeneous of degree 2, we have $F' = \mathrm{O}_{z',\overline{z}'}(2)$, and we get from (4.4) what forces $h_1 = 0$:

$$\mathrm{O}_{z,\overline{z}}(2) + zh_1(\zeta) + \overline{z}\overline{h}_1(\overline{\zeta}) \equiv \mathrm{O}_{zf_1(\zeta), \overline{z}\overline{f}_1(\zeta)}(2) \equiv \mathrm{O}_{z,\overline{z}}(2).$$

Before passing to weight 2, since $f = zf_1(\zeta) + \mathrm{O}_z(2)$ and $g = g_0(\zeta) + zg_1(\zeta) + \mathrm{O}_z(2)$, the nonzero Jacobian $\begin{vmatrix} f_z & f_\zeta \\ g_z & g_\zeta \end{vmatrix}$ has value at the origin $\begin{vmatrix} f_1(0) & 0 \\ g_1(0) & g_0'(0) \end{vmatrix}$, hence $f_1(0) \neq 0 \neq g_0'(0)$.

Lastly, picking weighted degree 2 terms in (4.4), we get

$$2m(z, \zeta, \overline{z}, \overline{\zeta}) + z^2 h_2(\zeta) + \overline{z}^2 \overline{h}_2(\overline{\zeta}) \equiv 2m\big(zf_1(\zeta), g_0(\zeta), \overline{z}\overline{f}_1(\overline{\zeta}), \overline{g}_0(\overline{\zeta})\big).$$

This identity means that the map $(z, \zeta, w) \longmapsto (zf_1(\zeta), g_0(\zeta), w + z^2 h_2(\zeta))$ is an automorphism of the Gaussier-Merker model fixing the origin, hence is a rotation, so that $f_1(\zeta) = e^{i\varphi}$, $g_0(\zeta) = e^{2i\varphi}\zeta$, $h_2(z, \zeta) \equiv 0$. Post-composing with the inverse rotation, we attain the conclusion. □

Question 4.8. Suppose given two rigid hypersurfaces prenormalized as before,

$$u = F = z\overline{z} + \frac{1}{2}\overline{z}^2\zeta + \mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1) = m + G = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1 - \zeta\overline{\zeta}} + \mathrm{O}_{z,\overline{z}}(3),$$

$$u' = F' = z'\overline{z}' + \frac{1}{2}\overline{z}'^2\zeta' + \mathrm{O}_{\overline{z}'}(3) + \mathrm{O}_{\overline{\zeta}'}(1) = m' + G' = \frac{z'\overline{z}' + \frac{1}{2}z'^2\overline{\zeta}' + \frac{1}{2}\overline{z}'^2\zeta'}{1 - \zeta'\overline{\zeta}'} + \mathrm{O}_{z',\overline{z}'}(3).$$

Is it true that the group of rigid biholomorphisms at the origin between them:

$$(z, \zeta, w) \longmapsto \big(z + f(z, \zeta), \zeta + g(z, \zeta), w + h(z, \zeta)\big) =: (z', \zeta', w'),$$

where $f = f_2 + f_3 + \cdots$, $g = g_1 + g_2 + \cdots$, $h = h_3 + h_4 + \cdots$, is finite-dimensional?

Here, the two appearing remainders $\mathrm{O}_{z,\overline{z}}(3)$ and $\mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1)$ are different. By expanding $1/(1 - \zeta\overline{\zeta})$ we see that

$$m = z\overline{z} + \frac{1}{2}\overline{z}^2\zeta + \frac{1}{2}z^2\overline{\zeta} + \zeta\overline{\zeta}(\cdots) = z\overline{z} + \frac{1}{2}\overline{z}^2\zeta + \mathrm{O}_{\overline{\zeta}}(1),$$

hence by subtraction, we get that G is more than just an $\mathrm{O}_{z,\overline{z}}(3)$.

Observation 4.9. The remainder function satisfies $G = \mathrm{O}_{z,\overline{z}}(3) = \mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1)$.

The synthesis between these two conditions will be made in Section 6.

5. Weighted homogeneous normalizing biholomorphisms

Now, inspired by Jacobowitz's presentation [14] of Moser's normal form in $\mathbb{C}^2$, Propositions 4.4 and 4.7 justify to introduce the spaces

$$\begin{aligned}\mathscr{G} &:= \big\{G = G(z,\zeta,\overline{z},\overline{\zeta}) : G = G_3 + G_4 + \cdots\big\},\\ \mathscr{D} &:= \big\{(z+f(z,\zeta), \zeta+g(z,\zeta), w+h(z,\zeta)) : f = f_2+f_3+\cdots, g = g_1+g_2+\cdots,\\ &\qquad\qquad h = h_3+h_4+\cdots\big\},\end{aligned}$$

where lower indices denote homogeneous components with respect to the weighting $[z]=1$, $[\zeta]=0$, $[w]=2$ of Section 1, leading to

$$\big[z^a\zeta^b\overline{z}^c\overline{\zeta}^d\big] = a+c.$$

The goal is to use the '*freedom*' space $\mathscr{D}$ of rigid biholomorphisms in order to '*normalize*' as much as possible the remainder G in the graphed equation $\{u = m+G\}$ of any given hypersurface. Here, $m = \frac{z\overline{z}+\frac{1}{2}z^2\overline{\zeta}+\frac{1}{2}\overline{z}^2\zeta}{1-\zeta\overline{\zeta}}$ is homogeneous of weight 2.

Both $\mathscr{G}$ and $\mathscr{D}$ decompose as direct sums graded by increasing weights

$$\begin{aligned}\mathscr{G} &= \bigcup_{\nu\geq 3}\mathscr{G}_\nu, \qquad \mathscr{G}_\nu := \{G_\nu\},\\ \mathscr{D} &= \bigcup_{\nu\geq 3}\mathscr{D}_\nu, \qquad \mathscr{D}_\nu := \{(f_{\nu-1}, g_{\nu-2}, h_\nu)\},\end{aligned}$$

and the (upcoming) justification for the shifts in $\mathscr{D}_\nu$ will be due to two multipliers

$$m_z = \frac{\overline{z}+z\overline{\zeta}}{1-\zeta\overline{\zeta}} \text{ of weight 1} \quad\text{and}\quad m_\zeta = \frac{(\overline{z}+z\overline{\zeta})^2}{2(1-\zeta\overline{\zeta})^2} \text{ of weight 2.}$$

One can figure out that $G_2 := m$ and $G_2' := m'$ are already finalized/normalized. With increasing weights $\nu = 3,4,5,\ldots$, we shall perform successive holomorphic rigid transformations of the shape

$$z' := z + f_{\nu-1}, \quad \zeta' := \zeta + g_{\nu-2}, \quad w' := w + h_\nu. \tag{5.1}$$

When $\nu \gg 1$ is high, it is intuitively clear that such transformations close to the identity will preserve previously achieved low order normalizations; to make this claim precise, let us follow and adapt [14, Chapter 3].

For $\mu \geq 0$, denote by $\mathrm{O}(\mu)$ power series whose monomials $z^a\zeta^b\overline{z}^c\overline{\zeta}^d$ are all of weight $a+c\geq\mu$, and introduce the projection operators

$$\pi_\mu\Bigg(\sum_{a,b,c,d\geq 0} T_{a,b,c,d}\, z^a\zeta^b\overline{z}^c\overline{\zeta}^d\Bigg) := \sum_{a+c\leq\mu}\sum_{b,d\geq 0} T_{a,b,c,d}\, z^a\zeta^b\overline{z}^c\overline{\zeta}^d.$$

Proposition 5.1. *Through any biholomorphism* (5.1) *which transforms*

$u = m + G_3 + \cdots + G_{\nu-1} + G_\nu + \mathrm{O}(\nu+1)$ *into* $u' = m + G_3' + \cdots + G_{\nu-1}' + G_\nu' + \mathrm{O}'(\nu+1)$,

homogeneous terms are kept untouched up to order $\leq \nu - 1$,

$$G_\mu'(z, \zeta, \overline{z}, \overline{\zeta}) = G_\mu(z, \zeta, \overline{z}, \overline{\zeta}), \quad 3 \leq \mu \leq \nu - 1,$$

while

$$\begin{aligned} &G_\nu'(z, \zeta, \overline{z}, \overline{\zeta}) \\ &= G_\nu(z, \zeta, \overline{z}, \overline{\zeta}) - 2\,\mathrm{Re}\left\{ \frac{\overline{z} + z\overline{\zeta}}{1 - \zeta\overline{\zeta}} f_{\nu-1}(z, \zeta) + \frac{(\overline{z} + z\overline{\zeta})^2}{2(1 - \zeta\overline{\zeta})^2} g_{\nu-2}(z, \zeta) - \frac{1}{2} h_\nu(z, \zeta) \right\}. \end{aligned}$$

Thus, by appropriately choosing $(f_{\nu-1}, g_{\nu-2}, h_\nu)$, we will be able to 'kill' many monomials in G_ν, hence make G_ν' simpler, or *normalized*. We leave to the reader to verify that in fact $h_\nu \equiv 0$ necessarily, when F and F' are assumed to be prenormalized.

Proof. As already seen, the fundamental equation, holding identically, is

$$\mathrm{Re}(w + h_\nu) = F(z, \zeta, \overline{z}, \overline{\zeta}) + \mathrm{Re}\, h_\nu \equiv F'(z + f_{\nu-1}(z, \zeta), \zeta + g_{\nu-2}(z, \zeta), w + h_\nu(z, \zeta)).$$

Decomposing $F = m + G$, $F' = m' + G'$ and reorganizing, it becomes

$$\begin{aligned} &\frac{(z + f_{\nu-1})(\overline{z} + \overline{f}_{\nu-1}) + \frac{1}{2}(z + f_{\nu-1})^2(\overline{\zeta} + \overline{g}_{\nu-2}) + \frac{1}{2}(\overline{z} + \overline{f}_{\nu-1})^2(\zeta + g_{\nu-2})}{1 - (\zeta + g_{\nu-2})(\overline{\zeta} + \overline{g}_{\nu-2})} \\ &- \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1 - \zeta\overline{\zeta}} - \mathrm{Re}\, h_\nu \\ &= G - G'. \end{aligned}$$

A reduction of the left hand side to the same denominator shows after algebraic simplifications:

$$\begin{aligned} &\frac{(1 - \zeta\overline{\zeta})\big[z\overline{f}_{\nu-1} + \overline{z}f_{\nu-1} + \frac{1}{2}(2zf_{\nu-1}\overline{\zeta} + z^2\overline{g}_{\nu-2}) + \frac{1}{2}(2\overline{z}\overline{f}_{\nu-1}\zeta + \overline{z}^2 g_{\nu-2})\big]}{(1 - \zeta\overline{\zeta})(1 - \zeta\overline{\zeta} - \zeta\overline{g}_{\nu-2} - \overline{\zeta}g_{\nu-2} - g_{\nu-2}\overline{g}_{\nu-2})} \\ &+ \frac{(\zeta\overline{g}_{\nu-2} + \overline{\zeta}g_{\nu-2})\big(z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta\big)}{(1 - \zeta\overline{\zeta})(1 - \zeta\overline{\zeta} - \zeta\overline{g}_{\nu-2} - \overline{\zeta}g_{\nu-2} - g_{\nu-2}\overline{g}_{\nu-2})} - \mathrm{Re}\, h_\nu \end{aligned}$$

that this left-hand side is $\mathrm{O}(\nu)$, hence has zero $\pi_{\nu-1}(\bullet) = 0$. Moreover, its homogeneous degree ν part is obtained by taking only weighted degree zero terms in the denominator, namely $\frac{\text{numerator}}{(1-\zeta\overline{\zeta})^2} - \mathrm{Re}\, h_\nu$, and one recognizes/reconstitutes m_z, m_ζ as homogeneous multipliers of weights 1, 2:

$$\pi_\nu(m' - m - \mathrm{Re}\, h_\nu) = 2\,\mathrm{Re}\left\{ \frac{\overline{z} + z\overline{\zeta}}{1 - \zeta\overline{\zeta}} f_{\nu-1}(z, \zeta) + \frac{(\overline{z} + z\overline{\zeta})^2}{2(1 - \zeta\overline{\zeta})^2} g_{\nu-2}(z, \zeta) - \frac{1}{2} h_\nu(z, \zeta) \right\}.$$

It remains to treat $\pi_\nu(\bullet)$ of the right-hand side:

$$\sum_{3\leq\mu\leq\nu} G_\mu(z,\zeta,\overline{z},\overline{\zeta}) - \pi_\nu\left(\sum_{3\leq\mu\leq\nu} G'_\mu(z+f_{\nu-1},\zeta+g_{\nu-2},\overline{z}+\overline{f}_{\nu-1},\overline{\zeta}+\overline{g}_{\nu-2})\right).$$

Assertion 5.2. *For each $3\leq\mu\leq\nu$,*

$$\pi_\nu\big(G'_\mu(z+f_{\nu-1},\zeta+g_{\nu-2},\overline{z}+\overline{f}_{\nu-1},\overline{\zeta}+\overline{g}_{\nu-2})\big) = G'_\mu(z,\zeta,\overline{z},\overline{\zeta}).$$

Proof. All possible monomials in G'_μ with $a+c=\mu\geq 3$ after binomial expansion

$$\begin{aligned}
&(z+f_{\nu-1})^a(\zeta+g_{\nu-2})^b(\overline{z}+\overline{f}_{\nu-1})^c(\overline{\zeta}+\overline{g}_{\nu-2})^d\\
&=(z^a+\mathrm{O}(a-1+\nu-1))(\zeta^b+\mathrm{O}(\nu-2))(\overline{z}^c+\mathrm{O}(c-1+\nu-1))(\overline{\zeta}^d+\mathrm{O}(\nu-2))\\
&=z^a\zeta^b\overline{z}^c\overline{\zeta}^d+\mathrm{O}(a+c-2+\nu)
\end{aligned}$$

have the simple projection $\pi_\nu(\bullet)=z^a\zeta^b\overline{z}^c\overline{\zeta}^d$ since $a+c-2+\nu\geq 1+\nu$. □

We therefore obtain an identity in which all arguments are $(z,\zeta,\overline{z},\overline{\zeta})$:

$$2\,\mathrm{Re}\left\{\frac{\overline{z}+z\overline{\zeta}}{1-\zeta\overline{\zeta}}f_{\nu-1}+\frac{(\overline{z}+z\overline{\zeta})^2}{2(1-\zeta\overline{\zeta})^2}g_{\nu-2}-\frac{1}{2}h_\nu\right\}\equiv\sum_{3\leq\mu\leq\nu-1}\underline{\big(G_\mu-G'_\mu\big)}_{\circ}+G_\nu-G'_\nu.$$

Applying $\pi_{\nu-1}$ annihilates both the left-hand side and $G_\nu-G'_\nu$, whence $G_\mu=G'_\mu$ for $3\leq\mu\leq\nu-1$, which concludes. □

6. Normal form

The assumption that the Levi form is of constant rank 1:

$$F_{z\overline{z}}\neq 0\equiv F_{z\overline{z}}F_{\zeta\overline{\zeta}}-F_{\zeta\overline{z}}F_{z\overline{\zeta}},$$

enables to solve identically as functions of $(z,\zeta,\overline{z},\overline{\zeta})$:

$$F_{\zeta\overline{\zeta}}\equiv\frac{F_{\zeta\overline{z}}F_{z\overline{\zeta}}}{F_{z\overline{z}}}.$$

By successively differentiating this identity and performing replacements, we get formulas.

Lemma 6.1. *For every jet multiindex $(a,b,c,d)\in\mathbb{N}^4$ with $b\geq 1$ and $d\geq 1$, abbreviating $n:=a+b+c+d$, there exists a polynomial $P_{a,b,c,d}$ in its arguments and an integer $\mathrm{N}_{a,b,c,d}\geq 1$ such that*

$$\begin{aligned}
&F_{z^a\zeta^b\overline{z}^c\overline{\zeta}^d}\\
&\equiv\frac{1}{(F_{z\overline{z}})^{\mathrm{N}_{a,b,c,d}}}P_{a,b,c,d}\left(\big\{F_{z^{a'}\overline{z}^{c'}}\big\}_{a'+c'\leq n},\big\{F_{z^{a'}\zeta^{b'}\overline{z}^{c'}}\big\}^{b'\geq 1}_{a'+b'+c'\leq n},\big\{F_{z^{a'}\overline{z}^{c'}\overline{\zeta}^{d'}}\big\}^{d'\geq 1}_{a'+c'+d'\leq n}\right).
\end{aligned}$$

In other words, the Levi rank 1 assumption implies that all Taylor coefficients at the origin of $\sum_{a,b,c,d} F_{a,b,c,d} z^a \zeta^b \overline{z}^c \overline{\zeta}^d$ for which $b \geq 1$ and $d \geq 1$ are determined by the free Taylor coefficients

$$\{F_{a,0,c,0}\}_{a\geq 0,\, c\geq 0} \cup \{F_{a,b,c,0}\}_{a\geq 0,\, b\geq 1,\, c\geq 0} \cup \{F_{a,0,c,d}\}_{a\geq 0,\, c\geq 0,\, d\geq 1}.$$

In subsequent computations, we will therefore normalize only these free (independent) Taylor coefficients at the origin, while those (dependent) attached to monomials that are multiple of $\zeta\overline{\zeta}$ will then be automatically determined by the formulas of Lemma 6.1.

As promised, we can now explore Observation 4.9 further. What precedes shows that it is best appropriate to expand G with respect to $(\zeta, \overline{\zeta})$:

$$\begin{aligned} G = \sum_{a,c\geq 0} G_{a,0,c,0} z^a \overline{z}^c + \sum_{b\geq 1} \zeta^b \left(\sum_{a,c\geq 0} G_{a,b,c,0} z^a \overline{z}^c \right) \\ + \sum_{d\geq 1} \overline{\zeta}^d \left(\sum_{a,c\geq 0} G_{a,0,c,d} z^a \overline{z}^c \right) + \sum_{b,d\geq 1} \sum_{a,c\geq 0} G_{a,b,c,d} z^a \zeta^b \overline{z}^c \overline{\zeta}^d. \end{aligned}$$

The last quadruple sum gathers all dependent jets. We will abbreviate this remainder as $\zeta\overline{\zeta}(\cdots)$. With different notations, we can therefore write

$$G = a(z,\overline{z}) + \sum_{k\geq 0} \zeta^{k+1} \Pi_k(z,\overline{z}) + \sum_{k\geq 0} \overline{\zeta}^{k+1} \overline{\Pi}_k(\overline{z}, z) + \zeta\overline{\zeta}(\cdots)$$

with $a(z,\overline{z}) \equiv \overline{a}(\overline{z}, z)$ real, but no reality constraint on the $\Pi_k(z,\overline{z})$.

Recall that $G = \mathrm{O}_{z,\overline{z}}(3)$. In view of Proposition 5.1, we must, for every weight $\nu \geq 3$, extract G_ν, while writing $\zeta^{k+1} = \zeta\zeta^k$,

$$\begin{aligned} G_\nu = {} & a_{\nu,0} z^\nu + a_{\nu-1,1} z^{\nu-1}\overline{z} + \cdots + a_{1,\nu-1} z\overline{z}^{\nu-1} + a_{0,\nu}\overline{z}^\nu \\ & + \sum_{k\geq 0} \zeta\zeta^k \big(z^\nu \Pi_{k,\nu,0} + z^{\nu-1}\overline{z}\Pi_{k,\nu-1,1} + \cdots + z\overline{z}^{\nu-1}\Pi_{k,1,\nu-1} + \overline{z}^\nu \Pi_{k,0,\nu} \big) \\ & + \sum_{k\geq 0} \overline{\zeta\zeta}^k \big(\overline{z}^\nu \overline{\Pi}_{k,\nu,0} + \overline{z}^{\nu-1} z \overline{\Pi}_{k,\nu-1,1} + \cdots + \overline{z} z^{\nu-1} \overline{\Pi}_{k,1,\nu-1} + z^\nu \overline{\Pi}_{k,0,\nu} \big) + \zeta\overline{\zeta}(\cdots). \end{aligned}$$

To reorganize all this in powers of $(z,\overline{z})$, let us introduce the two collections for all $0 \leq \mu \leq \nu$ of (anti)holomorphic functions (mind the inversion $\nu - \mu \longleftrightarrow \mu$ at the end):

$$B_{\nu-\mu,\mu}(\zeta) := \sum_{k\geq 0} \zeta^k \Pi_{k,\nu-\mu,\mu} \quad \text{and} \quad \overline{C}_{\nu-\mu,\mu}(\overline{\zeta}) := \sum_{k\geq 0} \overline{\zeta}^k \overline{\Pi}_{k,\mu,\nu-\mu}.$$

The definition of these $B_{\bullet,\bullet}$ and $\overline{C}_{\bullet,\bullet}$ enables us to emphasize that the obtained functions $\zeta B_{\bullet,\bullet}(\zeta)$ and $\overline{\zeta C}_{\bullet,\bullet}(\overline{\zeta})$ vanish when either $\zeta := 0$ or $\overline{\zeta} := 0$, and we therefore obtain, taking also account of the fact that G_ν is real:

$$\begin{aligned} G_\nu = z^\nu\big(a_{\nu,0} + \zeta B_{\nu,0}(\zeta) + \overline{\zeta C}_{\nu,0}(\overline{\zeta})\big) + z^{\nu-1}\overline{z}\big(a_{\nu-1,1} + \zeta B_{\nu-1,1}(\zeta) + \overline{\zeta C}_{\nu-1,1}(\overline{\zeta})\big) \\ + \cdots + z\overline{z}^{\nu-1}\big(\overline{a}_{\nu-1,1} + \overline{\zeta B}_{\nu-1,1}(\overline{\zeta}) + \zeta C_{\nu-1,1}(\zeta)\big) \\ + \overline{z}^\nu\big(\overline{a}_{\nu,0} + \overline{\zeta B}_{\nu,0}(\overline{\zeta}) + \zeta C_{\nu,0}(\zeta)\big) + \zeta\overline{\zeta}(\cdots). \end{aligned}$$

Of course, all these weighted homogeneous functions G_ν automatically satisfy $G_\nu = \mathrm{O}_{z,\overline{z}}(3)$, since $\nu \geq 3$ thanks to Proposition 4.4. Now, Observation 4.9 also requires that they satisfy, since they are real:

$$G_\nu = \mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1) = \mathrm{O}_z(3) + \mathrm{O}_\zeta(1). \tag{6.1}$$

Lemma 6.2. *For each weight $\nu \geq 5$, the function G_ν satisfies* (6.1) *if and only if it is of the form*

$$\begin{aligned} G_\nu = z^\nu\big(0 + 0 + \overline{\zeta C}_{\nu,0}(\overline{\zeta})\big) + z^{\nu-1}\overline{z}\big(0 + 0 + \overline{\zeta C}_{\nu-1,1}(\overline{\zeta})\big) + z^{\nu-2}\overline{z}^2\big(0 + 0 + \overline{\zeta C}_{\nu-2,2}(\overline{\zeta})\big) \\ + z^{\nu-3}\overline{z}^3\big(a_{\nu-3,3} + \zeta B_{\nu-3,3}(\zeta) + \overline{\zeta C}_{\nu-3,3}(\overline{\zeta})\big) + \cdots \\ + z^3\overline{z}^{\nu-3}\big(\overline{a}_{\nu-3,3} + \zeta C_{\nu-3,3}(\zeta) + \overline{\zeta B}_{\nu-3,3}(\overline{\zeta})\big) + z^2\overline{z}^{\nu-2}(0 + \zeta C_{\nu-2,2}(\zeta) + 0) \\ + z^1\overline{z}^{\nu-1}(0 + \zeta C_{\nu-1,1}(\zeta) + 0) + \overline{z}^\nu(0 + \zeta C_{\nu,0}(\zeta) + 0) + \zeta\overline{\zeta}(\cdots). \end{aligned}$$

Just after, we will treat the two weights $\nu = 3, 4$ separately.

Proof of Lemma 6.2. Putting $\overline{\zeta} := 0$ above, it must hold that

$$\begin{aligned} \mathrm{O}_{\overline{z}}(3) + 0 = G_\nu\big|_{\overline{\zeta}=0} = z^\nu(a_{\nu,0} + \zeta B_{\nu,0}(\zeta) + 0) + z^{\nu-1}\overline{z}(a_{\nu-1,1} + \zeta B_{\nu-1,1}(\zeta) + 0) \\ + z^{\nu-2}\overline{z}^2(a_{\nu-2,2} + \zeta B_{\nu-2,2}(\zeta) + 0) + \mathrm{O}_{\overline{z}}(3) + 0. \end{aligned}$$

Thus, all the appearing $a_{\bullet,\bullet}$ and $B_{\bullet,\bullet}$ should vanish, as stated, and the converse is clear. □

Proceeding similarly, the reader will find for $\nu = 3$ that G_3 satisfies (6.1) if and only if

$$G_3 = z^3\big(0+0+\overline{\zeta C}_{3,0}(\overline{\zeta})\big) + z^2\overline{z}(0+0+0) + z\overline{z}^2(0+0+0) + \overline{z}^3(0+\zeta C_{3,0}(\zeta)+0) + \zeta\overline{\zeta}(\cdots),$$

as well as

$$\begin{aligned} G_4 = z^4\big(0 + 0 + \overline{\zeta C}_{4,0}(\overline{\zeta})\big) + z^3\overline{z}\big(0 + 0 + \overline{\zeta C}_{3,1}(\overline{\zeta})\big) + z^2\overline{z}^2(0 + 0 + 0) \\ + z\overline{z}^3(0 + \zeta C_{1,3}(\zeta) + 0) + \overline{z}^4(0 + \zeta C_{4,0}(\zeta) + 0) + \zeta\overline{\zeta}(\cdots). \end{aligned}$$

Now, consider a rigid biholomorphism $z' = f(z,\zeta)$, $\zeta' = g(z,\zeta)$, $w' = \rho w + h(z,\zeta)$ between two rigid hypersurfaces M and M'. Of course, as in Question 4.8, we may assume that both M and M' have already been prenormalized, and thanks to Proposition 4.7 also that $f = f_2 + f_3 + \cdots$, $g = g_1 + g_2 + \cdots$, $\rho = 1$, $h = h_3 + h_4 + \cdots$.

The goal is to *normalize* M' even further, by means of appropriate choices of f, g, h.

We saw that it is natural to decompose $G = G_3 + G_4 + G_5 + \cdots$ and $G' = G_3' + G_4' + G_5' + \cdots$ in weighted homogeneous parts, and we just finished to express what prenormalization means about these G_ν and G_ν'. Proceeding with increasing weights $\nu = 3, 4, 5, \ldots$, we therefore consider biholomorphisms of the shape $z' = z + f_{\nu-1}$, $\zeta' = \zeta + g_{\nu-2}$, $w' = w + h_\nu$, and we recall that Proposition 5.1 showed that

$$\begin{aligned} &G_\nu'(z, \zeta, \overline{z}, \overline{\zeta}) \\ &= G_\nu(z, \zeta, \overline{z}, \overline{\zeta}) - 2\,\mathrm{Re}\left\{ \frac{\overline{z} + z\overline{\zeta}}{1 - \zeta\overline{\zeta}} f_{\nu-1}(z, \zeta) + \frac{(\overline{z} + z\overline{\zeta})^2}{2(1 - \zeta\overline{\zeta})^2} g_{\nu-2}(z, \zeta) - \frac{1}{2} h_\nu(z, \zeta) \right\}. \end{aligned}$$

The freedom to '*normalize*' G_ν' even more that G_ν, namely the term $-2\,\mathrm{Re}\{\cdots\}$, is parametrized by the completely free choice for the triple of holomorphic functions $(f_{\nu-1}, g_{\nu-2}, h_\nu)$. However, prenormalizations should be left untouched.

Lemma 6.3. *At every weight level $\nu \geq 5$, only the identity biholomorphic transformation $z' = z$, $\zeta' = \zeta$, $w' = w$ stabilizes prenormalization in source and target spaces*

$$G_\nu(z, \zeta, \overline{z}, \overline{\zeta}) = \mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1) = G_\nu'(z, \zeta, \overline{z}, \overline{\zeta}),$$

or equivalently, the 'freedom function' respects prenormalization

$$\begin{aligned} \mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1) &= 2\,\mathrm{Re}\left\{ \frac{\overline{z} + z\overline{\zeta}}{1 - \zeta\overline{\zeta}} f_{\nu-1}(z, \zeta) + \frac{(\overline{z} + z\overline{\zeta})^2}{2(1 - \zeta\overline{\zeta})^2} g_{\nu-2}(z, \zeta) - \frac{1}{2} h_\nu(z, \zeta) \right\} \\ &=: \Phi(z, \zeta, \overline{z}, \overline{\zeta}) \end{aligned}$$

if and only if $0 = f_{\nu-1} = g_{\nu-2} = h_\nu$.

Proof. It is easy to verify that the vanishings $G_\nu(z, \zeta, 0, 0) \equiv 0 \equiv G_\nu'(z, \zeta, 0, 0)$, which hold from the very beginning (of Proposition 4.1) already suffice to force $h_\nu(z, \zeta) \equiv 0$.

Next, write

$$\begin{aligned} f_{\nu-1}(z, \zeta) &= z^{\nu-1} f(\zeta) = z^{\nu-1}(f_0 + f_1\zeta + f_2\zeta^2 + \cdots), \\ g_{\nu-2}(z, \zeta) &= z^{\nu-2} g(\zeta) = z^{\nu-2}(g_0 + g_1\zeta + g_2\zeta^2 + \cdots). \end{aligned}$$

The goal is to show that $f(\zeta) \equiv 0$ and $g(\zeta) \equiv 0$.

Prenormalization being expressed modulo $\zeta\overline{\zeta}(\cdots)$, when we expand the two denominators of Φ, we have by luck $\frac{1}{1-\zeta\overline{\zeta}} \equiv 1$ and $\frac{1}{2(1-\zeta\overline{\zeta}^2)} \equiv \frac{1}{2}$, and hence it suffices to require that

$$\mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1) \stackrel{?}{=} 2\,\mathrm{Re}\left\{ (\overline{z} + z\overline{\zeta}) z^{\nu-1} \sum_{k \geq 0} f_k \zeta^k + \frac{1}{2} (\overline{z} + z\overline{\zeta})^2 z^{\nu-2} \sum_{k \geq 0} g_k \zeta^k \right\}.$$

Using $\nu \geq 5$ to guarantee that there is no interference when extracting the first three powers z^ν, $z^{\nu-1}\overline{z}$, $z^{\nu-2}\overline{z}^2$, let us compute the three relevant terms of the freedom function:

$$\begin{aligned}
\Phi(z,\zeta,\overline{z},\overline{\zeta}) &= (\overline{z}+z\overline{\zeta})z^{\nu-1}(f_0+f_1\zeta+f_2\zeta^2+\cdots)\\
&\quad+\left(\frac{1}{2}\overline{z}^2+z\overline{z}\overline{\zeta}+\frac{1}{2}z^2\overline{\zeta}^2\right)z^{\nu-2}(g_0+g_1\zeta+g_2\zeta^2+\cdots)\\
&\quad+(z+\overline{z}\zeta)\overline{z}^{\nu-1}(\overline{f}_0+\overline{f}_1\overline{\zeta}+\overline{f}_2\overline{\zeta}^2+\cdots)\\
&\quad+\left(\frac{1}{2}z^2+\overline{z}z\zeta+\frac{1}{2}\overline{z}^2\zeta^2\right)\overline{z}^{\nu-2}(\overline{g}_0+\overline{g}_1\overline{\zeta}+\overline{g}_2\overline{\zeta}^2+\cdots)\\
&= z^\nu\left(f_0\overline{\zeta}+\underline{f_1\zeta\overline{\zeta}+f_2\zeta^2\overline{\zeta}+\cdots}_{\circ}+\frac{1}{2}g_0\overline{\zeta}^2+\underline{\frac{1}{2}g_1\zeta\overline{\zeta}^2+\frac{1}{2}g_2\zeta^2\overline{\zeta}^2+\cdots}_{\circ}\right)\\
&\quad+z^{\nu-1}\overline{z}\big(f_0+f_1\zeta+f_2\zeta^2+\cdots+g_0\overline{\zeta}+\underline{g_1\zeta\overline{\zeta}+g_2\zeta^2\overline{\zeta}+\cdots}_{\circ}\big)\\
&\quad+z^{\nu-2}\overline{z}^2\left(\frac{1}{2}g_0+\frac{1}{2}g_1\zeta+\frac{1}{2}g_2\zeta^2+\cdots\right)+\overline{z}^3(\cdots)+\zeta\overline{\zeta}(\cdots).
\end{aligned}$$

Since the underlined terms can be absorbed into the remainder $\zeta\overline{\zeta}(\cdots)$, it remains only

$$\begin{aligned}
\Phi(z,\zeta,\overline{z},\overline{\zeta}) &= \frac{1}{2}z^\nu(2f_0\overline{\zeta}+g_0\overline{\zeta}^2)+z^{\nu-1}\overline{z}(f_0+f_1\zeta+f_2\zeta^2+\cdots+g_0\overline{\zeta})\\
&\quad+\frac{1}{2}z^{\nu-2}\overline{z}^2(g_0+g_1\zeta+g_2\zeta^2+\cdots)+\overline{z}^3(\cdots)+\zeta\overline{\zeta}(\cdots).
\end{aligned}$$

Putting $\overline{\zeta} := 0$, the result should be an $\mathrm{O}_{\overline{z}}(3)$, hence the first three lines should vanish, and lines 2 and 3 conclude that $f(\zeta)\equiv 0\equiv g(\zeta)$, as aimed at. □

Next, inspect the two remaining weights $\nu = 3, 4$. For $\nu = 3$, again modulo $\zeta\overline{\zeta}(\cdots)$, the freedom function is

$$\begin{aligned}
\Phi_3 \equiv 2\,\mathrm{Re}\Bigg\{&(\overline{z}+z\overline{\zeta})z^2(f_0+f_1\zeta+f_2\zeta^2+\cdots)\\
&+\left(\frac{1}{2}\overline{z}^2+z\overline{z}\overline{\zeta}+\frac{1}{2}z^2\overline{\zeta}^2\right)z^1(g_0+g_1\zeta+g_2\zeta^2+\cdots)\Bigg\}.
\end{aligned}$$

Assertion 6.4. *Prenormalization* $\Phi_3 = \mathrm{O}_{\overline{z}}(3)+\mathrm{O}_{\overline{\zeta}}(1)$ *is preserved if and only if*

$$0 = f_0+\frac{1}{2}\overline{g}_0,\quad 0 = f_1,\quad 0 = f_2,\quad 0 = \overline{g}_0+\frac{1}{2}g_1,\quad 0 = g_2,\quad \ldots.$$

Consequently, only 1 complex constant is free, f_0, in terms of which

$$g_0 = -2\overline{f}_0,\qquad g_1 = -4f_0.$$

With this, how can one normalize $G_3' = G_3 - \Phi_3$ further? Still modulo $\zeta\overline{\zeta}(\cdots)$:

$$\Phi_3 \equiv z^3(f_0\overline{\zeta}-\overline{f}_0\overline{\zeta}^2)+z^2\overline{z}(0)+z\overline{z}^2(0)+\overline{z}^3(\overline{f}_0\zeta-f_0\zeta^2),$$

hence

$$G'_{3,0,0,1} = G_{3,0,0,1} - f_0, \quad G'_{3,0,0,2} = G_{3,0,0,2} + \overline{f}_0.$$

It is natural to normalize the lowest jet order $4 = 3 + 0 + 0 + 1$ coefficient here.

Assertion 6.5. *One can normalize* $G'_{3,0,0,1} := 0$ *by choosing* $f_0 := G_{3,0,0,1}$.

Once this is done, it is easy to see that preserving/maintaining the normalization

$$G'_{3,0,0,1} = G_{3,0,0,1} = 0,$$

forces $f_0 = 0$ above.

Assertion 6.6. *In prenormalized coordinates which satisfy in addition* $G_{3,0,0,1} = 0$*, the coefficient*

$$G'_{3,0,0,2} = G_{3,0,0,2}$$

is an invariant (at the origin).

After such a normalization, we get

$$u = z\overline{z} + \frac{1}{2}\overline{z}^2\zeta + \frac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + az^2\overline{z}^2 + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(5)$$

with, possible, a nonzero real constant a, and possibly, a remainder that is *not* prenormalized.

Fortunately, we can apply the process of Proposition 4.1 to prenormalize again the coordinates, making in particular $a = 0$, without perturbing the normalizations obtained up to order 4 included.

Lastly, treat weight $\nu = 4$. The freedom function modulo $\zeta\overline{\zeta}(\cdots)$, is

$$\Phi_4 \equiv 2\,\mathrm{Re}\bigg\{(\overline{z} + z\overline{\zeta})z^3(f_0 + f_1\zeta + f_2\zeta^2 + \cdots) + \left(\frac{1}{2}\overline{z}^2 + z\overline{z}\overline{\zeta} + \frac{1}{2}z^2\overline{\zeta}^2\right)z^2(g_0 + g_1\zeta + g_2\zeta^2 + \cdots)\bigg\}.$$

Assertion 6.7. *Prenormalization* $\Phi_4 = \mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1)$ *is preserved if and only if*

$$0 = f_0 = f_1 = f_2 = \cdots, \quad 0 = g_0 + \overline{g}_0 = g_1 = g_2 = \cdots.$$

Thus now, only 1 *real* degree of freedom is left:

$$g_0 = i\tau, \quad \tau \in \mathbb{R}.$$

With this, how can one normalize $G'_4 = G_4 - \Phi_4$ further? Still modulo $\zeta\overline{\zeta}(\cdots)$:

$$\Phi_4 \equiv z^4\left(\frac{i}{2}\tau\overline{\zeta}^2\right) + z^3\overline{z}(i\tau\overline{\zeta}) + z^2\overline{z}^2(0) + z\overline{z}^3(-i\tau\zeta) + z^4\left(-\frac{i}{2}\tau\zeta^2\right),$$

hence

$$G'_{4,0,0,2} = G_{4,0,0,2} - \frac{i}{2}\tau, \quad G'_{3,0,1,1} = G_{3,0,1,1} - i\tau, \quad G'_{2,0,2,0} = G_{2,0,0,2}.$$

The third line shows an invariant. Notice also that $G'_{4,0,0,1} = G_{4,0,0,1}$ is an invariant. We choose to normalize the lowest jet order $3+0+1+1=5$ coefficient here.

Assertion 6.8. *One can normalize* $\Im G'_{3,0,1,1} := 0$ *by choosing* $\tau := \Im G_{3,0,1,1}$.

Once this is done, $G'_{3,0,1,1} = G_{3,0,1,1} \in \mathbb{R}$ is an invariant.

Again, we can re-apply the process of Proposition 4.1 to prenormalize the coordinates without touching the lower order normalizations.

We already saw in Lemma 6.3 that for any weight $\nu \geq 5$, no degree of freedom exists. Since only $2+1=3$ real degrees of freedom have been encountered, namely $f_0 \in \mathbb{C}$ in weight $\nu = 3$ and $\Im g_0 \in \mathbb{R}$ in weight $\nu = 4$, we conclude that the answer to Question 4.8 is positive.

All this enables us to conclude the present section by stating results which come from our analysis.

Theorem 6.9. *Every local rigid* $\mathscr{C}^\omega$ *graphed hypersurface* $M^5 \subset \mathbb{C}^3 \ni (z, \zeta, w = u + iv)$ *passing through the origin of equation*

$$u = \sum_{a+b+c+d \geq 1} F_{a,b,c,d} z^a \zeta^b \overline{z}^c \overline{\zeta}^d,$$

whose Levi form is of constant rank 1 *and which is* 2*-nondegenerate:*

$$F_{z\overline{z}} \neq 0 \equiv \begin{vmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{\zeta\overline{z}} & F_{\zeta\overline{\zeta}} \end{vmatrix} \quad \text{and} \quad 0 \neq \begin{vmatrix} F_{z\overline{z}} & F_{z\overline{\zeta}} \\ F_{zz\overline{z}} & F_{zz\overline{\zeta}} \end{vmatrix}$$

is equivalent, through a local rigid biholomorphism

$$(z, \zeta, w) \longmapsto (f(z,\zeta), g(z,\zeta), \rho w + h(z,\zeta)) =: (z', \zeta', w'), \quad \rho \in \mathbb{R}^*$$

to a rigid $\mathscr{C}^\omega$ *hypersurface* $M'^5 \subset \mathbb{C}'^3$ *which, dropping primes for target coordinates, is a perturbation of the Gaussier-Merker model—homogeneous of order* 2 *in* $(z, \overline{z})$*—*

$$u = \frac{z\overline{z} + \frac{1}{2}z^2\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta}{1 - \zeta\overline{\zeta}} + \sum_{\substack{a,b,c,d \in \mathbb{N} \\ a+c \geq 3}} G_{a,b,c,d} z^a \zeta^b \overline{z}^c \overline{\zeta}^d$$

with a simplified remainder G *which*

(1) *is normalized to be an* $\mathrm{O}_{z,\overline{z}}(3)$*;*

(2) *satisfies the prenormalization conditions* $G = \mathrm{O}_{\overline{z}}(3) + \mathrm{O}_{\overline{\zeta}}(1) = \mathrm{O}_z(3) + \mathrm{O}_\zeta(1)$, *or equivalently,*

$$G_{a,b,0,0} = 0 = G_{0,0,c,d}, \quad G_{a,b,1,0} = 0 = G_{1,0,c,d}, \quad G_{a,b,2,0} = 0 = G_{2,0,c,d};$$

(3) *satisfies in addition the sporadic normalization conditions*

$$G_{3,0,0,1} = 0 = G_{0,1,3,0}, \quad \Im G_{3,0,1,1} = 0 = \Im G_{1,1,3,0}.$$

There is of course *no* uniqueness of a rigid biholomorphic map which transfers M to an M' satisfying all these normalization conditions (1), (2), (3), just because any post-composition with a dilation-rotation map

$$(z', \zeta', w') \longmapsto (\rho^{1/2} e^{i\varphi} z', e^{2i\varphi} \zeta', \rho w') = (z'', \zeta'', w''), \quad \rho \in \mathbb{R}_+^*, \ \varphi \in \mathbb{R}$$

will transfer M' into an $M'' = \{u'' = m'' + G''\}$ which enjoys again the normalization conditions (1), (2), (3), since one obviously has

$$G''_{a,b,c,d} \rho^{\frac{a+c-2}{2}} e^{i\varphi(a+2b-c-2d)} = G'_{a,b,c,d}.$$

Remind that such dilation-rotation maps parametrize the 2-dimensional isotropy group of the origin for the Gaussier-Merker model $\{u' = m(z', \zeta', \overline{z}', \overline{\zeta}')\}$. Fortunately, an examination of our analysis above can show that these two parameters ρ, φ are the only ambiguity, since once one assumes that $f = z + f_2 + f_3 + \cdots$ with no $\rho^{1/2} e^{i\varphi}$ in front of z, that $g = \zeta + g_1 + g_2 + \cdots$, and that $h = w + h_3 + h_4 + \cdots$, with no $\rho^{1/2} e^{i\varphi}$, our reasonings showed *uniqueness* (exercise) of the map to normal form.

To finish, let us abbreviate the space of power series $G = G(z, \zeta, \overline{z}, \overline{\zeta})$ satisfying the normalization conditions (1), (2), (3) as

$$\mathfrak{N}_{2,1}.$$

Corollary 6.10. *Two rigid* $\mathscr{C}^\omega$ *hypersurfaces* $M^5 \subset \mathbb{C}^3$ *and* ${M'}^5 \subset {\mathbb{C}'}^3$ *belonging to* $\mathfrak{C}_{2,1}$, *both brought into normal form*

$$u = m + G, \qquad G \in \mathfrak{N}_{2,1},$$
$$u' = m' + G', \qquad G' \in \mathfrak{N}'_{2,1}$$

are rigidly biholomorphically equivalent if and only if there exist two constants $\rho \in \mathbb{R}_+^*$, $\varphi \in \mathbb{R}$, *such that for all* a, b, c, d,

$$G_{a,b,c,d} = G'_{a,b,c,d} \rho^{\frac{a+c-2}{2}} e^{i\varphi(a+2b-c-2d)}.$$

Granted that hypersurfaces can be put into such a normal form, this criterion is quite effective to determine whether two $M, M' \in \mathfrak{C}_{2,1}$ are rigidly equivalent.

7. A summary of further results

As an epilog, we now briefly describe some results which were detailed in the longer memoir prepublished as in [1], and which will appear elsewhere.

Adding factorials for technical reasons, consider a rigid $\mathfrak{C}_{2,1}$ hypersurface $M^5 \subset \mathbb{C}^3$ with $0 \in M$,

$$u = F = \sum_{a+b+c+d \geq 1} \frac{F_{a,b,c,d}}{a!b!c!d!} z^a \zeta^b \overline{z}^c \overline{\zeta}^d.$$

By Theorem 1.1, there exists a rigid biholomorphism which transforms M into normal form

$$u = m(z, \zeta, \overline{z}, \overline{\zeta}) + \sum_{a+c \geq 3} \frac{G_{a,b,c,d}}{a!b!c!d!} z^a \zeta^b \overline{z}^c \overline{\zeta}^d$$

with the $G_{a,b,c,d}$ satisfying the normalizing conditions stated there.

Question 7.1. How do the final coefficients $G_{\bullet,\bullet,\bullet,\bullet}$ express in terms of the initial coefficients $F_{\bullet,\bullet,\bullet,\bullet}$?

In Section 9 of [1], we present a general method inspired from [2] which proceeds with truncated group actions on jet spaces of increasing orders in order to keep track of how the $G_{\bullet,\bullet,\bullet,\bullet}$ express in terms of the $F_{\bullet,\bullet,\bullet,\bullet}$. Without proofs, we would like to show what the outcome is, up to order 5 included.

With the standard weighting $[z^a \zeta^b \overline{z}^c \overline{\zeta}^d] := a + b + c + d$, looking at the terms G_4 and G_5 after Lemma 6.2, we see that, in normal form, the remainder G has no order 4 term, and just the following 3 couples of order 5 monomials remain

$$\begin{aligned} u = z\overline{z} + \frac{1}{2}\overline{z}^2\zeta + \frac{1}{2}z^2\overline{\zeta} + z\overline{z}\zeta\overline{\zeta} + \frac{1}{2}\overline{z}^2\zeta\zeta\overline{\zeta} + \frac{1}{2}z^2\overline{\zeta}\zeta\overline{\zeta} \\ + \frac{1}{24}\overline{G_{0,1,4,0}}z^4\overline{\zeta} + \frac{1}{24}G_{0,1,4,0}\zeta\overline{z}^4 + \frac{1}{12}\overline{G_{0,2,3,0}}z^3\overline{\zeta}^2 + \frac{1}{12}G_{0,2,3,0}\zeta^2\overline{z}^3 \\ + \frac{1}{6}G_{1,1,3,0}z^3\overline{z}\overline{\zeta} + \frac{1}{6}G_{1,1,3,0}z\zeta\overline{z}^3 + \mathrm{O}_{z,\zeta,\overline{z},\overline{\zeta}}(6). \end{aligned}$$

Question 7.2. How $G_{0,1,4,0} \in \mathbb{C}$, how $G_{0,2,3,0} \in \mathbb{C}$, how $G_{1,1,3,0} \in \mathbb{R}$ express in terms of $\{F_{a,b,c,d}\}_{a+b+c+d \leq 5}$?

In [1], we show with details that the three quantities

$$\begin{aligned} V_0 &:= G_{0,1,4,0}\big(\{F_{a,b,c,d}\}_{a+b+c+d \leq 5}\big), \\ I_0 &:= G_{0,2,3,0}\big(\{F_{a,b,c,d}\}_{a+b+c+d \leq 5}\big), \\ Q_0 &:= G_{1,1,3,0}\big(\{F_{a,b,c,d}\}_{a+b+c+d \leq 5}\big) \end{aligned}$$

are relative differential invariants under rigid biholomorphisms, in accordance with Theorem 1.2. Furthermore,

$$\begin{aligned}
V_0 &= \frac{11 \text{ terms in degree } 4}{3F_{1,0,1,0}(F_{0,1,1,0}F_{1,0,2,0} - F_{0,1,2,0}F_{1,0,1,0})^2},\\
I_0 &= \frac{52 \text{ terms in degree } 9}{F_{1,0,1,0}^{3/2}(F_{0,1,1,0}F_{1,0,2,0} - F_{0,1,2,0}F_{1,0,1,0})^3(F_{1,0,0,1}F_{2,0,1,0} - F_{1,0,1,0}F_{2,0,0,1})},\\
Q_0 &= \frac{824 \text{ terms in degree } 18}{6F_{1,0,1,0}^3(F_{0,1,1,0}F_{1,0,2,0} - F_{0,1,2,0}F_{1,0,1,0})^4(F_{1,0,0,1}F_{2,0,1,0} - F_{1,0,1,0}F_{2,0,0,1})^4},
\end{aligned}$$

where the numerator of V_0 is

$$\begin{aligned}
&3F_{0,1,1,0}^2F_{1,0,2,0}F_{1,0,4,0} - 5F_{0,1,1,0}^2F_{1,0,3,0}^2 - 3F_{0,1,1,0}F_{0,1,2,0}F_{1,0,1,0}F_{1,0,4,0}\\
&+ 12F_{0,1,1,0}F_{0,1,2,0}F_{1,0,2,0}F_{1,0,3,0} + 10F_{0,1,1,0}F_{0,1,3,0}F_{1,0,1,0}F_{1,0,3,0} - 12F_{0,1,1,0}F_{0,1,3,0}F_{1,0,2,0}^2\\
&- 3F_{0,1,1,0}F_{0,1,4,0}F_{1,0,1,0}F_{1,0,2,0} - 12F_{0,1,2,0}^2F_{1,0,1,0}F_{1,0,3,0} + 12F_{0,1,2,0}F_{0,1,3,0}F_{1,0,1,0}F_{1,0,2,0}\\
&+ 3F_{0,1,2,0}F_{0,1,4,0}F_{1,0,1,0}^2 - 5F_{0,1,3,0}^2F_{1,0,1,0}^2,
\end{aligned}$$

and where the numerator of I_0 is

$$\begin{aligned}
&F_{0,1,1,0}^3F_{1,0,0,1}F_{1,0,1,0}^2F_{1,0,2,0}F_{2,0,1,0}F_{2,0,3,0} - F_{0,1,1,0}^3F_{1,0,0,1}F_{1,0,1,0}^2F_{1,0,3,0}F_{2,0,1,0}F_{2,0,2,0}\\
&+ 2F_{0,1,1,0}^3F_{1,0,0,1}F_{1,0,1,0}F_{1,0,2,0}^3F_{3,0,1,0} - 6F_{0,1,1,0}^3F_{1,0,0,1}F_{1,0,2,0}^3F_{2,0,1,0}^2\\
&- F_{0,1,1,0}^3F_{1,0,1,0}^3F_{1,0,2,0}F_{2,0,0,1}F_{2,0,3,0} + F_{0,1,1,0}^3F_{1,0,1,0}^3F_{1,0,3,0}F_{2,0,0,1}F_{2,0,2,0}\\
&- 2F_{0,1,1,0}^3F_{1,0,1,0}^2F_{1,0,2,0}^3F_{3,0,0,1} + 6F_{0,1,1,0}^3F_{1,0,1,0}F_{1,0,2,0}^3F_{2,0,0,1}F_{2,0,1,0}\\
&- F_{0,1,1,0}^2F_{0,1,2,0}F_{1,0,0,1}F_{1,0,1,0}^3F_{2,0,1,0}F_{2,0,3,0} - 6F_{0,1,1,0}^2F_{0,1,2,0}F_{1,0,0,1}F_{1,0,1,0}^2F_{1,0,2,0}^2F_{3,0,1,0}\\
&+ F_{0,1,1,0}^2F_{0,1,2,0}F_{1,0,0,1}F_{1,0,1,0}^2F_{1,0,3,0}F_{2,0,1,0}^2 + 18F_{0,1,1,0}^2F_{0,1,2,0}F_{1,0,0,1}F_{1,0,1,0}F_{1,0,2,0}^2F_{2,0,1,0}^2\\
&+ F_{0,1,1,0}^2F_{0,1,2,0}F_{1,0,1,0}^4F_{2,0,0,1}F_{2,0,3,0} + 6F_{0,1,1,0}^2F_{0,1,2,0}F_{1,0,1,0}^3F_{1,0,2,0}^2F_{3,0,0,1}\\
&- F_{0,1,1,0}^2F_{0,1,2,0}F_{1,0,1,0}^3F_{1,0,3,0}F_{2,0,0,1}F_{2,0,1,0} - 18F_{0,1,1,0}^2F_{0,1,2,0}F_{1,0,1,0}^2F_{1,0,2,0}^2F_{2,0,0,1}F_{2,0,1,0}\\
&+ F_{0,1,1,0}^2F_{0,1,3,0}F_{1,0,0,1}F_{1,0,1,0}^3F_{2,0,1,0}F_{2,0,2,0} - F_{0,1,1,0}^2F_{0,1,3,0}F_{1,0,0,1}F_{1,0,1,0}^2F_{1,0,2,0}F_{2,0,1,0}^2\\
&- F_{0,1,1,0}^2F_{0,1,3,0}F_{1,0,1,0}^4F_{2,0,0,1}F_{2,0,2,0} + F_{0,1,1,0}^2F_{0,1,3,0}F_{1,0,1,0}^3F_{1,0,2,0}F_{2,0,0,1}F_{2,0,1,0}\\
&- 2F_{0,1,1,0}^2F_{1,0,0,1}F_{1,0,1,0}^3F_{1,0,2,0}F_{1,1,3,0}F_{2,0,1,0} + 2F_{0,1,1,0}^2F_{1,0,0,1}F_{1,0,1,0}^3F_{1,0,3,0}F_{1,1,2,0}F_{2,0,1,0}\\
&+ 2F_{0,1,1,0}^2F_{1,0,1,0}^4F_{1,0,2,0}F_{1,1,3,0}F_{2,0,0,1} - 2F_{0,1,1,0}^2F_{1,0,1,0}^4F_{1,0,3,0}F_{1,1,2,0}F_{2,0,0,1}\\
&+ 6F_{0,1,1,0}F_{0,1,2,0}^2F_{1,0,0,1}F_{1,0,1,0}^3F_{1,0,2,0}F_{3,0,1,0} - 18F_{0,1,1,0}F_{0,1,2,0}^2F_{1,0,0,1}F_{1,0,1,0}^2F_{1,0,2,0}F_{2,0,1,0}^2\\
&- 6F_{0,1,1,0}F_{0,1,2,0}^2F_{1,0,1,0}^4F_{1,0,2,0}F_{3,0,0,1} + 18F_{0,1,1,0}F_{0,1,2,0}^2F_{1,0,1,0}^3F_{1,0,2,0}F_{2,0,0,1}F_{2,0,1,0}\\
&+ 2F_{0,1,1,0}F_{0,1,2,0}F_{1,0,0,1}F_{1,0,1,0}^4F_{1,1,3,0}F_{2,0,1,0} - 2F_{0,1,1,0}F_{0,1,2,0}F_{1,0,0,1}F_{1,0,1,0}^3F_{1,0,3,0}F_{1,1,1,0}F_{2,0,1,0}\\
&- 2F_{0,1,1,0}F_{0,1,2,0}F_{1,0,1,0}^5F_{1,1,3,0}F_{2,0,0,1} + 2F_{0,1,1,0}F_{0,1,2,0}F_{1,0,1,0}^4F_{1,0,3,0}F_{1,1,1,0}F_{2,0,0,1}\\
&- 2F_{0,1,1,0}F_{0,1,3,0}F_{1,0,0,1}F_{1,0,1,0}^4F_{1,1,2,0}F_{2,0,1,0} + 2F_{0,1,1,0}F_{0,1,3,0}F_{1,0,0,1}F_{1,0,1,0}^3F_{1,0,2,0}F_{1,1,1,0}F_{2,0,1,0}\\
&+ 2F_{0,1,1,0}F_{0,1,3,0}F_{1,0,1,0}^5F_{1,1,2,0}F_{2,0,0,1} - 2F_{0,1,1,0}F_{0,1,3,0}F_{1,0,1,0}^4F_{1,0,2,0}F_{1,1,1,0}F_{2,0,0,1}\\
&- F_{0,1,1,0}F_{0,2,2,0}F_{1,0,0,1}F_{1,0,1,0}^4F_{1,0,3,0}F_{2,0,1,0} + F_{0,1,1,0}F_{0,2,2,0}F_{1,0,1,0}^5F_{1,0,3,0}F_{2,0,0,1}\\
&+ F_{0,1,1,0}F_{0,2,3,0}F_{1,0,0,1}F_{1,0,1,0}^4F_{1,0,2,0}F_{2,0,1,0} - F_{0,1,1,0}F_{0,2,3,0}F_{1,0,1,0}^5F_{1,0,2,0}F_{2,0,0,1}
\end{aligned}$$

$$
\begin{aligned}
&- 2F_{0,1,2,0}^3F_{1,0,0,1}F_{1,0,1,0}^4F_{3,0,1,0} + 6F_{0,1,2,0}^3F_{1,0,0,1}F_{1,0,1,0}^3F_{2,0,1,0}^2\\
&+ 2F_{0,1,2,0}^3F_{1,0,1,0}^5F_{3,0,0,1} - 6F_{0,1,2,0}^3F_{1,0,1,0}^4F_{2,0,0,1}F_{2,0,1,0}\\
&+ F_{0,1,2,0}F_{0,2,1,0}F_{1,0,0,1}F_{1,0,1,0}^4F_{1,0,3,0}F_{2,0,1,0} - F_{0,1,2,0}F_{0,2,1,0}F_{1,0,1,0}^5F_{1,0,3,0}F_{2,0,0,1}\\
&- F_{0,1,2,0}F_{0,2,3,0}F_{1,0,0,1}F_{1,0,1,0}^5F_{2,0,1,0} + F_{0,1,2,0}F_{0,2,3,0}F_{1,0,1,0}^6F_{2,0,0,1}\\
&- F_{0,1,3,0}F_{0,2,1,0}F_{1,0,0,1}F_{1,0,1,0}^4F_{1,0,2,0}F_{2,0,1,0} + F_{0,1,3,0}F_{0,2,1,0}F_{1,0,1,0}^5F_{1,0,2,0}F_{2,0,0,1}\\
&+ F_{0,1,3,0}F_{0,2,2,0}F_{1,0,0,1}F_{1,0,1,0}^5F_{2,0,1,0} - F_{0,1,3,0}F_{0,2,2,0}F_{1,0,1,0}^6F_{2,0,0,1}.
\end{aligned}
$$

Question 7.3. Why chasing explicit expressions?

Before this article, in [6], we applied Cartan's equivalence method to rigid biholomorphic equivalences of rigid $\mathfrak{C}_{2,1}$ hypersurfaces $M^5 \subset \mathbb{C}^3$, and we found two primary relative differential invariants named V_0, I_0, plus a secondary one Q_0. Let us briefly describe the main result of [6], and argue that explicit expressions prove a perfect matching of the full expressions of V_0, I_0, Q_0 found by two completely different approaches.

Consider as before a rigid $M^5 \subset \mathbb{C}^3$ with $0 \in M$, which is 2-nondegenerate and has Levi form of constant rank 1, i.e., belongs to the class $\mathfrak{C}_{2,1}$, and which is graphed as

$$u = F(z_1, z_2, \overline{z}_1, \overline{z}_2).$$

Now, the letter ζ is protected, hence not used instead of z_2, since ζ will denote a 1-form. Two natural generators of $T^{1,0}M$ in the intrinsic coordinates $(z_1, z_2, \overline{z}_1, \overline{z}_2, v)$ on M are

$$\mathscr{L}_1 := \partial_{z_1} - iF_{z_1}\partial_v \quad \text{and} \quad \mathscr{L}_2 := \partial_{z_2} - iF_{z_2}\partial_v.$$

The Levi kernel bundle $K^{1,0}M \subset T^{1,0}M$ is generated by

$$\mathscr{K} := k\mathscr{L}_1 + \mathscr{L}_2, \quad \text{where } k := -\frac{F_{z_2\overline{z}_1}}{F_{z_1\overline{z}_1}}$$

is the slant function. The hypothesis of 2-nondegeneracy is equivalent to the nonvanishing

$$0 \neq \overline{\mathscr{L}}_1(k).$$

Also, the conjugate $\overline{\mathscr{K}}$ generates the conjugate Levi kernel bundle $K^{0,1} \subset T^{0,1}M$.

There is a second fundamental function, and no more

$$P := \frac{F_{z_1z_1\overline{z}_1}}{F_{z_1\overline{z}_1}}.$$

In the rigid case, it looks so simple, but in the *non*rigid case [5, 21], we would like to mention that P has a numerator involving 69 differential monomials (!).

In [6], we produced a reduction to an $\{e\}$-structure for the equivalence problem, under *rigid* (local) biholomorphic transformations, of such rigid $M^5 \in \mathfrak{C}_{2,1}$. We constructed an invariant 7-dimensional bundle $P^7 \longrightarrow M^5$ equipped with coordinates

$$(z_1, z_2, \overline{z}_1, \overline{z}_2, v, \mathrm{c}, \overline{\mathrm{c}})$$

with $\mathrm{c} \in \mathbb{C}$, together with a collection of seven complex-valued 1-forms which make a frame for T^*P^7, denoted

$$\{\rho, \kappa, \zeta, \overline{\kappa}, \overline{\zeta}, \alpha, \overline{\alpha}\}, \quad \overline{\rho} = \rho$$

which satisfy 7 *finalized* invariant exterior differential equations of the form

$$\begin{aligned}
d\rho &= (\alpha + \overline{\alpha}) \wedge \rho + i\kappa \wedge \overline{\kappa}, \\
d\kappa &= \alpha \wedge \kappa + \zeta \wedge \overline{\kappa}, \\
d\zeta &= (\alpha - \overline{\alpha}) \wedge \zeta + \frac{1}{\mathrm{c}} I_0 \kappa \wedge \zeta + \frac{1}{\mathrm{c}\overline{\mathrm{c}}} V_0 \kappa \wedge \overline{\kappa}, \\
d\alpha &= \zeta \wedge \overline{\zeta} - \frac{1}{\mathrm{c}} I_0 \zeta \wedge \overline{\kappa} + \frac{1}{\mathrm{c}\overline{\mathrm{c}}} Q_0 \kappa \wedge \overline{\kappa} + \frac{1}{\overline{\mathrm{c}}} \overline{I}_0 \overline{\zeta} \wedge \kappa
\end{aligned}$$

conjugate structure equations for $d\overline{\kappa}$, $d\overline{\zeta}$, $d\overline{\alpha}$ being easily deduced.

Here, there are exactly *two* primary Cartan-curvature invariants

$$\begin{aligned}
V_0 &:= -\frac{1}{3} \frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k)))}{\overline{\mathscr{L}}_1(k)} + \frac{5}{9} \left(\frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)} \right)^2 - \frac{1}{9} \frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))\overline{P}}{\overline{\mathscr{L}}_1(k)} + \frac{1}{3} \overline{\mathscr{L}}_1(\overline{P}) - \frac{1}{9} \overline{PP}, \\
I_0 &:= -\frac{1}{3} \frac{\mathscr{K}(\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k)))}{\overline{\mathscr{L}}_1(k)^2} + \frac{1}{3} \frac{\mathscr{K}(\overline{\mathscr{L}}_1(k))\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)^3} + \frac{2}{3} \frac{\mathscr{L}_1(\mathscr{L}_1(\overline{k}))}{\mathscr{L}_1(\overline{k})} + \frac{2}{3} \frac{\mathscr{L}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)}.
\end{aligned}$$

Furthermore, there is *one* secondary invariant whose unpolished expression is

$$Q_0 := \frac{1}{2} \overline{\mathscr{L}}_1(I_0) - \frac{1}{3} \left(P - \frac{\mathscr{L}_1(\mathscr{L}_1(\overline{k}))}{\mathscr{L}_1(\overline{k})} \right) \overline{I}_0 - \frac{1}{6} \left(\overline{P} - \frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)} \right) I_0 - \frac{1}{2} \frac{\mathscr{K}(V_0)}{\overline{\mathscr{L}}_1(k)}.$$

Visibly indeed, the vanishing of I_0 and V_0 implies the vanishing of Q_0. In fact, a consequence of Cartan's general theory is

$$0 \equiv V_0 \equiv I_0 \quad \Longleftrightarrow \quad M \text{ is rigidly equivalent to the Gaussier-Merker model.}$$

When one inserts the expressions of k, P in terms of F inside V_0, I_0, Q_0, and when one factorizes, simplifies, reorganizes, one obtains

Theorem 7.4 (On a computer)**.** *Up to multiplication by a complex number of modulus* 1*, the expressions of* V_0*,* I_0*,* Q_0 *obtained either by the normal forms method or by Cartan's equivalence method are exactly the same.*

However, the normal forms method showed by construction that $Q_0 = G_{1,1,3,0}(F_{\bullet,\bullet,\bullet,\bullet})$ is *real-valued*, whereas the expression of Q_0 found in [6] and copied just above does not look real-valued. Even a sub-part of Q_0 above which *seems* real-valued is *not*, because $-\frac{1}{3} \neq -\frac{1}{6}$! For some time, we thought there could be some errors somewhere, because computations in [6] were done manually.

Fortunately, there were no errors, and in Section 8 of the longer memoir prepublished as in [1], an equivalent clean finalized expression of Q_0, in terms of only the two fundamental functions k, P (and their conjugates), from which one immediately sees real-valuedness, has been obtained

$$\begin{aligned}
Q_0 = 2\,\mathrm{Re}\bigg\{ & \frac{1}{9}\frac{\mathscr{K}(\overline{\mathscr{L}}_1(k))\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))^2}{\overline{\mathscr{L}}_1(k)^4} - \frac{1}{9}\frac{\mathscr{K}(\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k)))\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)^3} \\
& - \frac{1}{9}\frac{\mathscr{K}(\overline{\mathscr{L}}_1(k))\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))\overline{P}}{\overline{\mathscr{L}}_1(k)^3} - \frac{1}{9}\frac{\mathscr{L}_1(\overline{\mathscr{L}}_1(k))\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)^2} \\
& + \frac{1}{9}\frac{\mathscr{K}(\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k)))\overline{P}}{\overline{\mathscr{L}}_1(k)^2} - \frac{2}{9}\frac{\mathscr{L}_1(\overline{\mathscr{L}}_1(k))\overline{P}}{\overline{\mathscr{L}}_1(k)} - \frac{1}{9}\frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))P}{\overline{\mathscr{L}}_1(k)} \\
& + \frac{1}{3}\frac{\mathscr{L}_1(\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k)))}{\overline{\mathscr{L}}_1(k)} + \frac{1}{6}\overline{\mathscr{L}}_1(P)\bigg\} \\
& - \frac{1}{9}|\overline{P}|^2 + \frac{1}{3}\left|\frac{\overline{\mathscr{L}}_1(\overline{\mathscr{L}}_1(k))}{\overline{\mathscr{L}}_1(k)}\right|^2.
\end{aligned}$$

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