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Cofiniteness with Respect to the Class of Modules in Dimension less than a Fixed Integer

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Abstract. Let R be a commutative Noetherian ring with non-zero identity, n a non-negative integer, $\mathfrak{a}$ an ideal of R with $\dim(R/\mathfrak{a}) \leq n+1$, and X an arbitrary R-module. In this paper, we prove the following results:

(i) If X is an $\mathfrak{a}$-torsion R-module such that $\operatorname{Hom}_R(R/\mathfrak{a}, X)$ and $\operatorname{Ext}^1_R(R/\mathfrak{a}, X)$ are $\mathrm{FD}_{<n}$ R-modules, then X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module;

(ii) The category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules is an Abelian category;

(iii) $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module and $\{\mathfrak{p} \in \operatorname{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X)) : \dim(R/\mathfrak{p}) \geq n\}$ is a finite set for all i when $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i.

We observe that, among other things, $\operatorname{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ is a finite set for all i whenever R is a semi-local ring with $\dim(R/\mathfrak{a}) \leq 2$ and $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<1}$ R-module for all i.

1. Introduction

Throughout this paper R is a commutative Noetherian ring with non-zero identity, $\mathfrak{a}$ is an ideal of R, M is a finite (i.e., finitely generated) R-module, X is an arbitrary R-module which is not necessarily finite, and n is a non-negative integer. For basic results, notations, and terminology not given in this paper, readers are referred to [12, 13, 34].

In [20], Hartshorne defined an $\mathfrak{a}$-torsion R-module X to be $\mathfrak{a}$-cofinite if $\operatorname{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all i and asked the following questions.

Question 1.1. Is the category of $\mathfrak{a}$-cofinite R-modules an Abelian category?

Question 1.2. Is $\mathrm{H}^i_{\mathfrak{a}}(M)$ an $\mathfrak{a}$-cofinite R-module for all i?

The followings are also important problems in local cohomology (see [19, Expose XIII, Conjecture 1.1] and [22, Problem 4]).

Question 1.3. Is $\operatorname{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$ a finite R-module for all i?

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Question 1.4. Is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))$ a finite set for all i?

In [20, Proposition 7.6 and Corollary 7.7], [23, Theorem 4.1], [15, Theorem 3], [16, Theorems 1 and 2], [36, Theorem 1.1], [14, Theorem 1.4], [7, Theorem 2.6], [24, Theorems 1 and 8], [30, Theorems 2.6 and 2.10], [9, Theorem 2.7], [5, Theorem 4.3], [2, Theorems 3.4 and 3.7], and [31, Theorem 3.3], the authors studied these questions and prepared affirmative answers to them for the case that $\dim(R/\mathfrak{a}) = 1$.

Recall that X is said to be an $\mathrm{FD}_{<n}$ (or in dimension $< n$) R-module if there exists a finite submodule X' of X such that $\dim_R(X/X') < n$ [2, 4]. Also, we say that X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module if X is an $\mathfrak{a}$-torsion R-module and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i [3, Definition 4.1]. Note that, by [37, Theorem 2.3], the class of $\mathrm{FD}_{<n}$ R-modules forms a Serre subcategory of the category of R-modules (i.e., the class of R-modules which is closed under taking submodules, quotients, and extensions). Also, X is a finite R-module if and only if X is an $\mathrm{FD}_{<0}$ R-module, and so X is an $\mathfrak{a}$-cofinite R-module if and only if X is an $(\mathrm{FD}_{<0}, \mathfrak{a})$-cofinite R-module. Thus, it is natural to raise the following questions as generalizations of Questions 1.1–1.4 (see [1, Question]). Here, we denote the set $\{\mathfrak{p} \in \mathrm{Ass}_R(X) : \dim(R/\mathfrak{p}) \geq n\}$ by $\mathrm{Ass}_R(X)_{\geq n}$.

Question 1.5. Is the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules an Abelian category?

Question 1.6. Is $\mathrm{H}^i_{\mathfrak{a}}(M)$ an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i?

Question 1.7. Is $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^i_{\mathfrak{a}}(M))$ an $\mathrm{FD}_{<n}$ R-module for all i?

Question 1.8. Is $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(M))_{\geq n}$ a finite set for all i?

By Abazari-Bahmanpour's results [1, Theorems 2.5 and 2.10], the answer to Questions 1.6–1.8 is yes if R is a complete local ring with $\dim(R/\mathfrak{a}) \leq n + 1$. In this paper, we remove complete local assumption on R. It follows that, among other things, the answer to Question 1.4 is yes if R is a semi-local ring with $\dim(R/\mathfrak{a}) \leq 2$. We also study Question 1.5 and prepare an affirmative answer to it for the case that $\dim(R/\mathfrak{a}) \leq n + 1$.

In the main result of Section 2, we prove that if X is an $\mathfrak{a}$-torsion $\mathrm{FD}_{<n+2}$ R-module (e.g., X is $\mathfrak{a}$-torsion and $\dim(R/\mathfrak{a}) \leq n + 1$) such that $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, X)$ are $\mathrm{FD}_{<n}$ R-modules, then X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module. This result plays an important role in the study of the above questions.

In Section 3, with respect to Question 1.5, we show that the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite $\mathrm{FD}_{<n+2}$ R-modules is an Abelian category. In particular, the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules is an Abelian category if $\dim(R/\mathfrak{a}) \leq n + 1$.

Section 4 is devoted to the study of Questions 1.6–1.8. Let t be a non-negative integer. We prove that if X is an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq t$ (e.g., X is an $\mathrm{FD}_{<n}$ R-module) and $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<n+2}$ R-module

for all $i < t$ (e.g., $\dim(R/\mathfrak{a}) \leq n+1$), then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i < t$, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X))$ is an $\mathrm{FD}_{<n}$ R-module, and $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))_{\geq n}$ is a finite set for all $i \leq t$.

Section 5 consists of some applications on ordinary cofiniteness and weakly cofiniteness of local cohomology modules. Recall that X is said to be a weakly Laskerian R-module if the set of associated prime ideals of any quotient module of X is finite [17, Definition 2.1]. Also, we say that X is an $\mathfrak{a}$-weakly cofinite R-module if X is an $\mathfrak{a}$-torsion R-module and $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a weakly Laskerian R-module for all i [18, Definition 2.4]. Note that, if X is a weakly Laskerian R-module (resp. an $\mathrm{FD}_{<1}$ R-module and R is a semi-local ring), then X is an $\mathrm{FD}_{<2}$ R-module (resp. a weakly Laskerian R-module) by [5, Theorem 3.3].

It is perhaps worth noting that the results of this paper generalize all of the previous results concerning Questions 1.1–1.8 (see e.g., [1, 2, 5–9, 11, 14–16, 20, 23–25, 27, 30–33, 36]). Note also that, some results of the finiteness of associated prime ideals in this paper follow from [10, Theorem 1.2] when X is finite.

2. Cofinite modules

The following lemma is needed in this paper.

Lemma 2.1. *Suppose that X is an $\mathfrak{a}$-torsion $\mathrm{FD}_{<n+1}$ R-module such that $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module. Then X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module.*

Proof. We can, and do, assume that $\dim_R(X) = n$. Since $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module, there exists a short exact sequence

$$0 \longrightarrow X' \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, X) \longrightarrow X'' \longrightarrow 0$$

such that X' is a finite submodule of $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ and $\dim_R(X'') < n$. Also, by [13, Exercise 1.2.28], $\mathrm{Ass}_R(\mathrm{Hom}_R(R/\mathfrak{a}, X)) = \mathrm{Ass}_R(X)$ because X is an $\mathfrak{a}$-torsion R-module. Let j be a positive integer such that $\dim_R(\mathrm{Ext}^j_R(R/\mathfrak{a}, X)) = n$ and set $A = \{\mathfrak{p} \in \mathrm{Supp}_R(\mathrm{Ext}^j_R(R/\mathfrak{a}, X)) : \dim(R/\mathfrak{p}) = n\}$. Then A is a non-empty and finite set because $A \subseteq \mathrm{Ass}_R(X')$. Let $A = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$ and $S = R \setminus \bigcup_{k=1}^{l} \mathfrak{p}_k$. Then $\dim_{S^{-1}R}(S^{-1}X) \leq 0$, $S^{-1}X$ is an $S^{-1}\mathfrak{a}$-torsion $S^{-1}R$-module, and the $S^{-1}R$-module $\mathrm{Hom}_{S^{-1}R}(S^{-1}R/S^{-1}\mathfrak{a}, S^{-1}X)$ is finite and so has finite length. Thus, by [29, Proposition 4.1], $S^{-1}X$ is an $S^{-1}\mathfrak{a}$-cofinite $S^{-1}R$-module. Hence $S^{-1}\mathrm{Ext}^j_R(R/\mathfrak{a}, X) \cong \mathrm{Ext}^j_{S^{-1}R}(S^{-1}R/S^{-1}\mathfrak{a}, S^{-1}X)$ is a finite $S^{-1}R$-module. Therefore there is a finite submodule Y of $\mathrm{Ext}^j_R(R/\mathfrak{a}, X)$ such that $S^{-1}Y = S^{-1}\mathrm{Ext}^j_R(R/\mathfrak{a}, X)$. Now, since $S^{-1}(\mathrm{Ext}^j_R(R/\mathfrak{a}, X)/Y) = 0$, it is easy to see that $\dim_R(\mathrm{Ext}^j_R(R/\mathfrak{a}, X)/Y) < n$. Thus $\mathrm{Ext}^j_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module. □

We are now ready to state and prove the main result of this section.

Theorem 2.2. *Suppose that X is an $\mathfrak{a}$-torsion $\mathrm{FD}_{<n+2}$ R-module such that $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, X)$ are $\mathrm{FD}_{<n}$ R-modules. Then X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module.*

Proof. By Lemma 2.1, we can, and do, assume that $\dim_R(X) = n + 1$. Suppose, on the contrary, that X is not an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module and seek a contradiction. Let A be the set of ideals $(0 :_R Y)$, where Y is an $\mathfrak{a}$-torsion R-module, $\dim_R(Y) = n + 1$, $\mathrm{Hom}_R(R/\mathfrak{a}, Y)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, Y)$ are $\mathrm{FD}_{<n}$ R-modules, and Y is not an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module. Then A is a non-empty set of ideals of Noetherian ring R. Let $(0 :_R Y)$ be a maximal member of A. Since Y is an $\mathfrak{a}$-torsion R-module and $\mathrm{Hom}_R(R/\mathfrak{a}, Y)$ is an $\mathrm{FD}_{<n}$ R-module, the set $B = \{\mathfrak{p} \in \mathrm{Supp}_R(Y) : \dim(R/\mathfrak{p}) = n+1\}$ is finite. Let $B = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_l\}$ and $S = R \setminus \bigcup_{k=1}^{l} \mathfrak{p}_k$. Then $\dim_{S^{-1}R}(S^{-1}Y) \leq 0$, $S^{-1}Y$ is an $S^{-1}\mathfrak{a}$-torsion $S^{-1}R$-module, and the $S^{-1}R$-module $\mathrm{Hom}_{S^{-1}R}(S^{-1}R/S^{-1}\mathfrak{a}, S^{-1}Y)$ is finite and so has finite length. Thus, by [29, Proposition 4.1], $S^{-1}Y$ is an Artinian $S^{-1}\mathfrak{a}$-cofinite $S^{-1}R$-module. Therefore $S^{-1}(Y/\mathfrak{a}Y) \cong S^{-1}Y/S^{-1}\mathfrak{a}S^{-1}Y$ is a finite $S^{-1}R$-module from [29, Theorem 2.1]. Hence there is a finite submodule Y' of Y such that $S^{-1}((\mathfrak{a}Y + Y')/\mathfrak{a}Y) = S^{-1}(Y/\mathfrak{a}Y)$. Let $Z = Y/Y'$. Since $S^{-1}Z$ is an Artinian $S^{-1}R$-module and $S^{-1}Z = S^{-1}\mathfrak{a}S^{-1}Z$, there is an element a of $\mathfrak{a}$ such that $S^{-1}Z = \frac{a}{1}S^{-1}Z$ from [26, 2.8]. Therefore $S^{-1}(Z/aZ) = 0$ and so it is easy to see that $\dim_R(Z/aZ) \leq n$. By the short exact sequence

$$0 \longrightarrow Y' \longrightarrow Y \longrightarrow Z \longrightarrow 0,$$

we have $\mathrm{Hom}_R(R/\mathfrak{a}, Z)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, Z)$ are $\mathrm{FD}_{<n}$ R-modules. If $\dim_R(Z) < n+1$, then Z is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from Lemma 2.1. Therefore Y is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by the above short exact sequence, a contradiction. Thus $\dim_R(Z) = n + 1$ and Z is not an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module, and hence $(0 :_R Z) \in A$. Thus $(0 :_R Y) = (0 :_R Z)$ because $(0 :_R Y) \subseteq (0 :_R Z)$ and $(0 :_R Y)$ is a maximal member of A. Let $Z' = (0 :_Z a)$. Since $aZ \neq 0$, $(0 :_R Z) \subsetneqq (0 :_R Z')$ and so $(0 :_R Z') \notin A$. $\mathrm{Hom}_R(R/\mathfrak{a}, Z')$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, Z')$ are $\mathrm{FD}_{<n}$ R-modules from the short exact sequences

$$0 \longrightarrow Z' \longrightarrow Z \longrightarrow aZ \longrightarrow 0 \quad \text{and} \quad 0 \longrightarrow aZ \longrightarrow Z \longrightarrow Z/aZ \longrightarrow 0.$$

If $\dim_R(Z') < n + 1$, then Z' is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from Lemma 2.1. Otherwise, $\dim_R(Z') = n + 1$ and so Z' is again an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module as $(0 :_R Z') \notin A$. Thus $\mathrm{Hom}_R(R/\mathfrak{a}, Z/aZ)$ is an $\mathrm{FD}_{<n}$ R-module by the above short exact sequences. Therefore Z/aZ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by Lemma 2.1. Hence Z is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from [29, Corollary 3.2]. This contradiction shows that X is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module as desired. □

As immediate applications of the above theorem, we have the following corollaries.

Corollary 2.3. *Suppose that* $\dim(R/\mathfrak{a}) \leq n+1$ *and that* X *is an* $\mathfrak{a}$*-torsion* R*-module such that* $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ *and* $\mathrm{Ext}^1_R(R/\mathfrak{a}, X)$ *are* $\mathrm{FD}_{<n}$ R*-modules. Then* X *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* R*-module.*

Corollary 2.4. (see [9, Proposition 2.6], [5, Corollary 4.2], and [2, Theorem 3.1]) *Suppose that* X *is an* $\mathfrak{a}$*-torsion* $\mathrm{FD}_{<2}$ R*-module such that* $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ *and* $\mathrm{Ext}^1_R(R/\mathfrak{a}, X)$ *are finite* R*-modules. Then* X *is an* $\mathfrak{a}$*-cofinite* R*-module.*

Corollary 2.5. (see [30, Theorem 2.3]) *Suppose that* $\dim(R/\mathfrak{a}) \leq 1$ *and that* X *is an* $\mathfrak{a}$*-torsion* R*-module such that* $\mathrm{Hom}_R(R/\mathfrak{a}, X)$ *and* $\mathrm{Ext}^1_R(R/\mathfrak{a}, X)$ *are finite* R*-modules. Then* X *is an* $\mathfrak{a}$*-cofinite* R*-module.*

3. Abelianness of the category of cofinite modules

With respect to Question 1.1, Hartshorne in [20, Proposition 7.6] showed that the category of $\mathfrak{a}$-cofinite R-modules is an Abelian category if R is a complete regular local ring and $\mathfrak{a}$ is a prime ideal of R with $\dim(R/\mathfrak{a}) = 1$. In [16, Theorem 2], Delfino and Marley generalized Hartshorne's result to arbitrary complete local rings. Kawasaki in [24, Theorem 1] extended this result for an arbitrary ideal $\mathfrak{a}$ with $\dim(R/\mathfrak{a}) = 1$ in a local ring R. In [30, Theorem 2.6], Melkersson removed local assumption on R. Recently, Aghapournahr and Bahmanpour in [2, Theorem 3.7] (see also [9, Theorem 2.7] and [5, Theorem 4.3]) generalized Melkersson's result and proved that the category of $\mathfrak{a}$-cofinite $\mathrm{FD}_{<2}$ R-modules is an Abelian category.

In this section, we extend Aghapournahr-Bahmanpour's result [2, Theorem 3.7] and show that the category of $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite $\mathrm{FD}_{<n+2}$ R-modules is an Abelian category. In particular, we prepare an affirmative answer to Question 1.5 for the case that $\dim(R/\mathfrak{a}) \leq n+1$.

Theorem 3.1. *The category of* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* $\mathrm{FD}_{<n+2}$ R*-modules is an Abelian category.*

Proof. Let X and Y be $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite $\mathrm{FD}_{<n+2}$ R-modules and let $f\colon X \longrightarrow Y$ be an R-homomorphism. It is enough to show that $\ker f$ and $\operatorname{coker} f$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite $\mathrm{FD}_{<n+2}$ R-modules. From the short exact sequences

$$0 \longrightarrow \ker f \longrightarrow X \longrightarrow \operatorname{im} f \longrightarrow 0$$

and

$$0 \longrightarrow \operatorname{im} f \longrightarrow Y \longrightarrow \operatorname{coker} f \longrightarrow 0,$$

$\ker f$ and $\operatorname{coker} f$ are $\mathrm{FD}_{<n+2}$ R-modules and $\mathrm{Hom}_R(R/\mathfrak{a}, \ker f)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \ker f)$ are $\mathrm{FD}_{<n}$ R-modules. Thus $\ker f$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by Theorem 2.2. Hence $\operatorname{coker} f$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from the above short exact sequences. □

Corollary 3.2. *Let N be a finite R-module and let X be an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* $\mathrm{FD}_{<n+2}$ *R-module. Then* $\mathrm{Ext}^j_R(N, X)$ *and* $\mathrm{Tor}^R_j(N, X)$ *are* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite* $\mathrm{FD}_{<n+2}$ *R-modules for all j.*

Proof. Assume that

$$F_{\bullet N} = \cdots \longrightarrow F_{j+1} \longrightarrow F_j \longrightarrow F_{j-1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

is a deleted free resolution of N such that F_j is finite for all j. By applying $\mathrm{Hom}_R(-, X)$ and $- \otimes_R X$ to $F_{\bullet N}$, the assertion follows from Theorem 3.1. □

Corollary 3.3. *If* $\dim(R/\mathfrak{a}) \leq n + 1$*, then the category of* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-modules is an Abelian category.*

Corollary 3.4. *Let* $\dim(R/\mathfrak{a}) \leq n + 1$*, let N be a finite R-module, and let X be an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-module. Then* $\mathrm{Ext}^j_R(N, X)$ *and* $\mathrm{Tor}^R_j(N, X)$ *are* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-modules for all j.*

Corollary 3.5. *The category of* $\mathfrak{a}$*-cofinite* $\mathrm{FD}_{<2}$ *R-modules is an Abelian category.*

Corollary 3.6. *Let N be a finite R-module and let X be an* $\mathfrak{a}$*-cofinite* $\mathrm{FD}_{<2}$ *R-module. Then* $\mathrm{Ext}^j_R(N, X)$ *and* $\mathrm{Tor}^R_j(N, X)$ *are* $\mathfrak{a}$*-cofinite* $\mathrm{FD}_{<2}$ *R-modules for all j.*

Corollary 3.7. *If* $\dim(R/\mathfrak{a}) \leq 1$*, then the category of* $\mathfrak{a}$*-cofinite R-modules is an Abelian category.*

Corollary 3.8. *Let* $\dim(R/\mathfrak{a}) \leq 1$*, let N be a finite R-module, and let X be an* $\mathfrak{a}$*-cofinite R-module. Then* $\mathrm{Ext}^j_R(N, X)$ *and* $\mathrm{Tor}^R_j(N, X)$ *are* $\mathfrak{a}$*-cofinite R-modules for all j.*

4. Cofiniteness of local cohomology modules

Abazari and Bahmanpour in [1, Theorems 2.5 and 2.10] prepared affirmative answers to Questions 1.6–1.8 for the case that R is a complete local ring with $\dim(R/\mathfrak{a}) \leq n + 1$. In this section, we remove complete local assumption on R. They showed that if R is a complete local ring, X is a finite R-module, and t is a non-negative integer such that $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<n+2}$ R-module for all $i < t$, then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i < t$, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X))$ are $\mathrm{FD}_{<n}$ R-modules, and $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))_{\geq n}$ is a finite set for all $i \leq t$. In the main result of this section, we prove it without assuming that R is a complete local ring and X is a finite R-module. As applications of this result, in Section 5, we generalize all of the previous results concerning Questions 1.2–1.4 (see e.g., [2, 6–8, 11, 14–16, 20, 23–25, 27, 30–33, 36]).

Lemma 4.1. *Let X be an arbitrary R-module and let s, t be non-negative integers such that* $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ *is an* $\mathrm{FD}_{<n}$ *R-module for all* $s \leq i \leq s+t+1$, $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-module for all $i < s$, and* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $\mathrm{FD}_{<n+2}$ *R-module for all* $s \leq i \leq s+t$. *Then* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-module for all* $i \leq s+t$.

Proof. We prove by using induction on t. Let $t = 0$. From [3, Theorem 2.3], $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^s_{\mathfrak{a}}(X))$ are $\mathrm{FD}_{<n}$ R-modules. Thus $\mathrm{H}^s_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by Theorem 2.2. Suppose that $t > 0$ and that $t-1$ is settled. It is enough to show that $\mathrm{H}^{s+t}_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module because $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i \leq s+t-1$ from the induction hypothesis on $t-1$. By [3, Theorem 2.3], $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{s+t}_{\mathfrak{a}}(X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^{s+t}_{\mathfrak{a}}(X))$ are $\mathrm{FD}_{<n}$ R-modules. Therefore $\mathrm{H}^{s+t}_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from Theorem 2.2. □

Theorem 4.2. *Let X be an arbitrary R-module and let t be a non-negative integer such that* $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ *is an* $\mathrm{FD}_{<n}$ *R-module for all $i \leq t$ and* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $\mathrm{FD}_{<n+2}$ *R-module for all $i < t$. Then the following statements hold true:*

(i) *Y and* $\mathrm{H}^i_{\mathfrak{a}}(X)/Y$ *are* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-modules for all $i < t$ and every* $\mathrm{FD}_{<n+1}$ *R-submodule Y of* $\mathrm{H}^i_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{H}^i_{\mathfrak{a}}(X)$ *is an* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-module for all $i < t$;*

(ii) *Let N be a finite R-module. Then* $\mathrm{Ext}^j_R(N, Y)$, $\mathrm{Tor}^R_j(N, Y)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$, *and* $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$ *are* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-modules for all $i < t$, all j, and every* $\mathrm{FD}_{<n+1}$ *R-submodule Y of* $\mathrm{H}^i_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ *and* $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ *are* $(\mathrm{FD}_{<n}, \mathfrak{a})$*-cofinite R-modules for all $i < t$ and all j;*

(iii) $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ *is an* $\mathrm{FD}_{<n}$ *R-module for every* $\mathrm{FD}_{<n+1}$ *R-submodule Y of* $\mathrm{H}^t_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X))$ *is an* $\mathrm{FD}_{<n}$ *R-module;*

(iv) $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X)/Y)_{\geq n}$ *is a finite set for all $i \leq t$ and every* $\mathrm{FD}_{<n+1}$ *R-submodule Y of* $\mathrm{H}^i_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))_{\geq n}$ *is a finite set for all $i \leq t$;*

(v) *Assume that* $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, X)$ *is an* $\mathrm{FD}_{<n}$ *R-module. Then* $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ *is an* $\mathrm{FD}_{<n}$ *R-module for every* $\mathrm{FD}_{<n+1}$ *R-submodule Y of* $\mathrm{H}^t_{\mathfrak{a}}(X)$. *In particular,* $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X))$ *is an* $\mathrm{FD}_{<n}$ *R-module.*

Proof. (i) Since $\mathrm{Hom}_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(X))$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \Gamma_{\mathfrak{a}}(X))$ are $\mathrm{FD}_{<n}$ R-modules by [3, Theorem 2.3], $\Gamma_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from Theorem 2.2, and so $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i < t$ by Lemma 4.1. Let $i < t$ and let Y be an $\mathrm{FD}_{<n+1}$ R-submodule of $\mathrm{H}^i_{\mathfrak{a}}(X)$. Then $\mathrm{Hom}_R(R/\mathfrak{a}, Y)$ is an $\mathrm{FD}_{<n}$ R-module and so Y is

an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module from Lemma 2.1. Thus $\mathrm{H}^i_{\mathfrak{a}}(X)/Y$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by the short exact sequence

$$0 \longrightarrow Y \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(X) \longrightarrow \mathrm{H}^i_{\mathfrak{a}}(X)/Y \longrightarrow 0.$$

(ii) This follows from the first part and Corollary 3.2.

(iii) Let Y be an $\mathrm{FD}_{<n+1}$ R-submodule of $\mathrm{H}^t_{\mathfrak{a}}(X)$. From the first part and [3, Theorem 2.3], $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X))$ is an $\mathrm{FD}_{<n}$ R-module. Thus $\mathrm{Hom}_R(R/\mathfrak{a}, Y)$ is an $\mathrm{FD}_{<n}$ R-module and so Y is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module by Lemma 2.1. Hence, from the exact sequence

$$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X)) \longrightarrow \mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X)/Y) \longrightarrow \mathrm{Ext}^1_R(R/\mathfrak{a}, Y),$$

$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ is an $\mathrm{FD}_{<n}$ R-module.

(iv) It follows by the first part, the third part, and [13, Exercise 1.2.28].

(v) This is similar to the proof of the third part. □

Remark 4.3. (see [1, Theorems 2.6 and 2.10]) Let N be an $\mathfrak{a}$-torsion finite R-module, let X be an arbitrary R-module, and let t be a non-negative integer such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \le t$ and $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<n+2}$ R-module for all $i < t$. Then, by Theorem 4.2 and [21, Proposition 3.4(i)], the following statements hold true:

(i) $\mathrm{Ext}^j_R(N, Y)$ and $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$ are $\mathrm{FD}_{<n}$ R-modules and $\mathrm{Ass}_R(\mathrm{Ext}^j_R(N, Y))_{\ge n}$ and $\mathrm{Ass}_R(\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y))_{\ge n}$ are finite sets for all $i < t$, all j, and every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ is an $\mathrm{FD}_{<n}$ R-module and $\mathrm{Ass}_R(\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)))_{\ge n}$ is a finite set for all $i < t$ and all j;

(ii) $\mathrm{Hom}_R(N, \mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ is an $\mathrm{FD}_{<n}$ R-module and $\mathrm{Ass}_R(\mathrm{Hom}_R(N, \mathrm{H}^t_{\mathfrak{a}}(X)/Y))_{\ge n}$ is a finite set for every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^t_{\mathfrak{a}}(X)$. In particular, $\mathrm{Hom}_R(N, \mathrm{H}^t_{\mathfrak{a}}(X))$ is an $\mathrm{FD}_{<n}$ R-module and $\mathrm{Ass}_R(\mathrm{Hom}_R(N, \mathrm{H}^t_{\mathfrak{a}}(X)))_{\ge n}$ is a finite set;

(iii) Assume that $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module. Then $\mathrm{Ext}^1_R(N, \mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ is an $\mathrm{FD}_{<n}$ R-module and $\mathrm{Ass}_R(\mathrm{Ext}^1_R(N, \mathrm{H}^t_{\mathfrak{a}}(X)/Y))_{\ge n}$ is a finite set for every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^t_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^1_R(N, \mathrm{H}^t_{\mathfrak{a}}(X))$ is an $\mathrm{FD}_{<n}$ R-module and $\mathrm{Ass}_R(\mathrm{Ext}^1_R(N, \mathrm{H}^t_{\mathfrak{a}}(X)))_{\ge n}$ is a finite set.

Let X be an arbitrary R-module which is not necessarily finite and let n be a non-negative integer. We set $f_{\mathfrak{a}}(X) = \inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X) \text{ is not a finite } R\text{-module}\}$ and $f^n_{\mathfrak{a}}(X) = \inf\{f_{\mathfrak{a}R_{\mathfrak{p}}}(X_{\mathfrak{p}}) : \mathfrak{p} \in \mathrm{Spec}(R) \text{ and } \dim(R/\mathfrak{p}) \ge n\}$ which are called finiteness dimension and nth finiteness dimension of X with respect to $\mathfrak{a}$, respectively [8,35]. In [35, Corollary 2.3], it is shown that if X is an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i (in fact, for all $i \le f^n_{\mathfrak{a}}(X)$), then the equality $f^n_{\mathfrak{a}}(X) =$

$\inf\{i \in \mathbb{N}_0 : \mathrm{H}^i_{\mathfrak{a}}(X)$ is not an $\mathrm{FD}_{<n}$ R-module$\}$ holds (see also [4, Theorem 2.5] and [28, Theorem 2.10]).

Corollary 4.4. *Let X be an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all $i \leq f^{n+2}_{\mathfrak{a}}(X)$. Then the following statements hold true:*

(i) *Y and $\mathrm{H}^i_{\mathfrak{a}}(X)/Y$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules for all $i < f^{n+2}_{\mathfrak{a}}(X)$ and every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all $i < f^{n+2}_{\mathfrak{a}}(X)$;*

(ii) *Let N be a finite R-module. Then $\mathrm{Ext}^j_R(N, Y)$, $\mathrm{Tor}^R_j(N, Y)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$, and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules for all $i < f^{n+2}_{\mathfrak{a}}(X)$, all j, and every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules for all $i < f^{n+2}_{\mathfrak{a}}(X)$ and all j;*

(iii) *$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{f^{n+2}_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)/Y)$ is an $\mathrm{FD}_{<n}$ R-module for every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^{f^{n+2}_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)$. In particular, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{f^{n+2}_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X))$ is an $\mathrm{FD}_{<n}$ R-module;*

(iv) *$\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X)/Y)_{\geq n}$ is a finite set for all $i \leq f^{n+2}_{\mathfrak{a}}(X)$ and every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))_{\geq n}$ is a finite set for all $i \leq f^{n+2}_{\mathfrak{a}}(X)$;*

(v) *Assume that $\mathrm{Ext}^{f^{n+2}_{\mathfrak{a}}(X)+1}_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module. Then $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^{f^{n+2}_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)/Y)$ is an $\mathrm{FD}_{<n}$ R-module for every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^{f^{n+2}_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^{f^{n+2}_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X))$ is an $\mathrm{FD}_{<n}$ R-module.*

Corollary 4.5. *Suppose that one of the following conditions holds:*

(a) *$\dim(R/\mathfrak{a}) \leq n+1$ and X is an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<n}$ R-module for all i;*

(b) *X is a finite R-module with $\dim_R(X/\mathfrak{a}X) \leq n+1$.*

Then the following statements are true:

(i) *Y and $\mathrm{H}^i_{\mathfrak{a}}(X)/Y$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules for all i and every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-module for all i;*

(ii) *Let N be a finite R-module. Then $\mathrm{Ext}^j_R(N, Y)$, $\mathrm{Tor}^R_j(N, Y)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$, and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules for all i, all j, and every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ are $(\mathrm{FD}_{<n}, \mathfrak{a})$-cofinite R-modules for all i and all j;*

(iii) *$\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X)/Y)_{\geq n}$ is a finite set for all i and every $\mathrm{FD}_{<n+1}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))_{\geq n}$ is a finite set for all i.*

5. More applications

5.1. Ordinary cofiniteness of local cohomology modules

By putting $n = 0$ in Theorem 4.2 and Corollaries 4.4–4.5, we have the following results which generalize [20, Corollary 7.7], [23, Theorem 4.1], [15, Theorem 3], [16, Theorem 1], [36, Theorem 1.1], [25, Theorem B], [11, Theorem 2.2], [14, Theorem 1.4], [32, Theorem 5.6], [6, Theorems 2.3 and 2.5], [7, Theorem 2.6], [33, Theorem 3.2], [24, Theorem 8], [30, Theorem 2.10], [8, Theorems 2.3 and 3.2], [2, Theorem 3.4], and [31, Theorem 3.3].

Corollary 5.1. *Let X be an arbitrary R-module and let t be a non-negative integer such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \le t$ and $\mathrm{H}^i_\mathfrak{a}(X)$ is an $\mathrm{FD}_{<2}$ R-module for all $i < t$. Then the following statements hold true:*

(i) *Y and $\mathrm{H}^i_\mathfrak{a}(X)/Y$ are $\mathfrak{a}$-cofinite R-modules for all $i < t$ and every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^i_\mathfrak{a}(X)$. In particular, $\mathrm{H}^i_\mathfrak{a}(X)$ is an $\mathfrak{a}$-cofinite R-module for all $i < t$;*

(ii) *Let N be a finite R-module. Then $\mathrm{Ext}^j_R(N, Y)$, $\mathrm{Tor}^R_j(N, Y)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_\mathfrak{a}(X)/Y)$, and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_\mathfrak{a}(X)/Y)$ are $\mathfrak{a}$-cofinite R-modules for all $i < t$, all j, and every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^i_\mathfrak{a}(X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_\mathfrak{a}(X))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_\mathfrak{a}(X))$ are $\mathfrak{a}$-cofinite R-modules for all $i < t$ and all j;*

(iii) *$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(X)/Y)$ is a finite R-module for every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^t_\mathfrak{a}(X)$. In particular, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(X))$ is a finite R-module;*

(iv) *$\mathrm{Ass}_R(\mathrm{H}^i_\mathfrak{a}(X)/Y)$ is a finite set for all $i \le t$ and every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^i_\mathfrak{a}(X)$. In particular, $\mathrm{Ass}_R(\mathrm{H}^i_\mathfrak{a}(X))$ is a finite set for all $i \le t$;*

(v) *Assume that $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, X)$ is a finite R-module. Then $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(X)/Y)$ is a finite R-module for every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^t_\mathfrak{a}(X)$. In particular, $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_\mathfrak{a}(X))$ is a finite R-module.*

Corollary 5.2. *Let X be an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is a finite R-module for all $i \le f^2_\mathfrak{a}(X)$. Then the following statements hold true:*

(i) *Y and $\mathrm{H}^i_\mathfrak{a}(X)/Y$ are $\mathfrak{a}$-cofinite R-modules for all $i < f^2_\mathfrak{a}(X)$ and every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^i_\mathfrak{a}(X)$. In particular, $\mathrm{H}^i_\mathfrak{a}(X)$ is an $\mathfrak{a}$-cofinite R-module for all $i < f^2_\mathfrak{a}(X)$;*

(ii) *Let N be a finite R-module. Then $\mathrm{Ext}^j_R(N, Y)$, $\mathrm{Tor}^R_j(N, Y)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_\mathfrak{a}(X)/Y)$, and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_\mathfrak{a}(X)/Y)$ are $\mathfrak{a}$-cofinite R-modules for all $i < f^2_\mathfrak{a}(X)$, all j, and every $\mathrm{FD}_{<1}$ R-submodule Y of $\mathrm{H}^i_\mathfrak{a}(X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_\mathfrak{a}(X))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_\mathfrak{a}(X))$ are $\mathfrak{a}$-cofinite R-modules for all $i < f^2_\mathfrak{a}(X)$ and all j;*

(iii) $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(X)}(X)/Y)$ *is a finite R-module for every* $\mathrm{FD}_{<1}$ *R-submodule Y of* $\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(X)}(X)$. *In particular,* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(X)}(X))$ *is a finite R-module;*

(iv) $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X)/Y)$ *is a finite set for all $i \leq f_{\mathfrak{a}}^2(X)$ and every* $\mathrm{FD}_{<1}$ *R-submodule Y of* $\mathrm{H}_{\mathfrak{a}}^i(X)$. *In particular,* $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X))$ *is a finite set for all $i \leq f_{\mathfrak{a}}^2(X)$;*

(v) *Assume* $\mathrm{Ext}_R^{f_{\mathfrak{a}}^2(X)+1}(R/\mathfrak{a}, X)$ *is a finite R-module. Then* $\mathrm{Ext}_R^1(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(X)}(X)/Y)$ *is a finite R-module for every* $\mathrm{FD}_{<1}$ *R-submodule Y of* $\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(X)}(X)$. *In particular,* $\mathrm{Ext}_R^1(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^2(X)}(X))$ *is a finite R-module.*

Corollary 5.3. *Suppose that one of the following conditions holds:*

(a) $\dim(R/\mathfrak{a}) \leq 1$ *and X is an arbitrary R-module such that* $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ *is a finite R-module for all i;*

(b) *X is a finite R-module with* $\dim_R(X/\mathfrak{a}X) \leq 1$.

Then the following statements are true:

(i) *Y and* $\mathrm{H}_{\mathfrak{a}}^i(X)/Y$ *are $\mathfrak{a}$-cofinite R-modules for all i and every* $\mathrm{FD}_{<1}$ *R-submodule Y of* $\mathrm{H}_{\mathfrak{a}}^i(X)$. *In particular,* $\mathrm{H}_{\mathfrak{a}}^i(X)$ *is an $\mathfrak{a}$-cofinite R-module for all i;*

(ii) *Let N be a finite R-module. Then* $\mathrm{Ext}_R^j(N, Y)$, $\mathrm{Tor}_j^R(N, Y)$, $\mathrm{Ext}_R^j(N, \mathrm{H}_{\mathfrak{a}}^i(X)/Y)$, *and* $\mathrm{Tor}_j^R(N, \mathrm{H}_{\mathfrak{a}}^i(X)/Y)$ *are $\mathfrak{a}$-cofinite R-modules for all i, all j, and every* $\mathrm{FD}_{<1}$ *R-submodule Y of* $\mathrm{H}_{\mathfrak{a}}^i(X)$. *In particular,* $\mathrm{Ext}_R^j(N, \mathrm{H}_{\mathfrak{a}}^i(X))$ *and* $\mathrm{Tor}_j^R(N, \mathrm{H}_{\mathfrak{a}}^i(X))$ *are $\mathfrak{a}$-cofinite R-modules for all i and all j;*

(iii) $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X)/Y)$ *is a finite set for all i and every* $\mathrm{FD}_{<1}$ *R-submodule Y of* $\mathrm{H}_{\mathfrak{a}}^i(X)$. *In particular,* $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X))$ *is a finite set for all i.*

5.2. Weakly cofiniteness of local cohomology modules

Bahmanpour and Naghipour in [7, Theorem 3.1] prepared an affirmative answer to Question 1.4 for the case that R is a local ring with $\dim(R/\mathfrak{a}) \leq 2$ (see also [27, Theorem 3.3(c)]). We generalize this result to arbitrary semi-local rings. They showed that if R is a local ring, X is a finite R-module, and t is a non-negative integer such that $\dim_R(\mathrm{H}_{\mathfrak{a}}^i(X)) \leq 2$ for all $i < t$, then $\mathrm{H}_{\mathfrak{a}}^i(X)$ is an $\mathfrak{a}$-weakly cofinite R-module for all $i < t$, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^t(X))$ is a weakly Laskerian R-module, and $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X))$ is a finite set for all $i \leq t$. Here, by taking $n = 1$ in Theorem 4.2 and considering [5, Theorem 3.3], we prove it with weaker assumptions that R is a semi-local ring and X is an arbitrary R-module such that $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<1}$ R-module for all $i \leq t$ and $\mathrm{H}_{\mathfrak{a}}^i(X)$ is an $\mathrm{FD}_{<3}$ R-module for all $i < t$.

Corollary 5.4. *Let R be a semi-local ring, let X be an arbitrary R-module, and let t be a non-negative integer such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<1}$ R-module for all $i \leq t$ and $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathrm{FD}_{<3}$ R-module for all $i < t$. Then the following statements hold true:*

(i) *Y and $\mathrm{H}^i_{\mathfrak{a}}(X)/Y$ are $\mathfrak{a}$-weakly cofinite R-modules for all $i < t$ and every $\mathrm{FD}_{<2}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-weakly cofinite R-module for all $i < t$;*

(ii) *Let N be a finite R-module. Then $\mathrm{Ext}^j_R(N, Y)$, $\mathrm{Tor}^R_j(N, Y)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$, and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$ are $\mathfrak{a}$-weakly cofinite R-modules for all $i < t$, all j, and every $\mathrm{FD}_{<2}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ are $\mathfrak{a}$-weakly cofinite R-modules for all $i < t$ and all j;*

(iii) *$\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ is a weakly Laskerian R-module for every $\mathrm{FD}_{<2}$ R-submodule Y of $\mathrm{H}^t_{\mathfrak{a}}(X)$. In particular, $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X))$ is a weakly Laskerian R-module;*

(iv) *$\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X)/Y)$ is a finite set for all $i \leq t$ and every $\mathrm{FD}_{<2}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ass}_R(\mathrm{H}^i_{\mathfrak{a}}(X))$ is a finite set for all $i \leq t$;*

(v) *Assume that $\mathrm{Ext}^{t+1}_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<1}$ R-module. Then $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X)/Y)$ is a weakly Laskerian R-module for every $\mathrm{FD}_{<2}$ R-submodule Y of $\mathrm{H}^t_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^t_{\mathfrak{a}}(X))$ is a weakly Laskerian R-module.*

In [8, Theorem 3.9], the authors showed that if R is a complete local ring, X is a finite R-module, and Y is a weakly Laskerian R-submodule of $\mathrm{H}^{f^3_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)$, then $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-weakly cofinite R-module for all $i < f^3_{\mathfrak{a}}(X)$ and the R-modules $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}^{f^3_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)/Y)$ and $\mathrm{Ext}^1_R(R/\mathfrak{a}, \mathrm{H}^{f^3_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)/Y)$ are weakly Laskerian. Here, we prove this result with weaker assumptions that R is an arbitrary semi-local ring, X is an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<1}$ R-module for all $i \leq f^3_{\mathfrak{a}}(X) + 1$, and Y is an $\mathrm{FD}_{<2}$ R-submodule of $\mathrm{H}^{f^3_{\mathfrak{a}}(X)}_{\mathfrak{a}}(X)$.

Corollary 5.5. *Let R be a semi-local ring and let X be an arbitrary R-module such that $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is an $\mathrm{FD}_{<1}$ R-module for all $i \leq f^3_{\mathfrak{a}}(X)$. Then the following statements hold true:*

(i) *Y and $\mathrm{H}^i_{\mathfrak{a}}(X)/Y$ are $\mathfrak{a}$-weakly cofinite R-modules for all $i < f^3_{\mathfrak{a}}(X)$ and every $\mathrm{FD}_{<2}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{H}^i_{\mathfrak{a}}(X)$ is an $\mathfrak{a}$-weakly cofinite R-module for all $i < f^3_{\mathfrak{a}}(X)$;*

(ii) *Let N be a finite R-module. Then $\mathrm{Ext}^j_R(N, Y)$, $\mathrm{Tor}^R_j(N, Y)$, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$, and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X)/Y)$ are $\mathfrak{a}$-weakly cofinite R-modules for all $i < f^3_{\mathfrak{a}}(X)$, all j, and every $\mathrm{FD}_{<2}$ R-submodule Y of $\mathrm{H}^i_{\mathfrak{a}}(X)$. In particular, $\mathrm{Ext}^j_R(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ and $\mathrm{Tor}^R_j(N, \mathrm{H}^i_{\mathfrak{a}}(X))$ are $\mathfrak{a}$-weakly cofinite R-modules for all $i < f^3_{\mathfrak{a}}(X)$ and all j;*

(iii) $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^3(X)}(X)/Y)$ *is a weakly Laskerian* R*-module for every* $\mathrm{FD}_{<2}$ R*-submodule* Y *of* $\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^3(X)}(X)$*. In particular,* $\mathrm{Hom}_R(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^3(X)}(X))$ *is a weakly Laskerian* R*-module;*

(iv) $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X)/Y)$ *is a finite set for all* $i \le f_{\mathfrak{a}}^3(X)$ *and every* $\mathrm{FD}_{<2}$ R*-submodule* Y *of* $\mathrm{H}_{\mathfrak{a}}^i(X)$*. In particular,* $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X))$ *is a finite set for all* $i \le f_{\mathfrak{a}}^3(X)$*;*

(v) *Assume* $\mathrm{Ext}_R^{f_{\mathfrak{a}}^3(X)+1}(R/\mathfrak{a}, X)$ *is an* $\mathrm{FD}_{<1}$ R*-module. Then* $\mathrm{Ext}_R^1(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^3(X)}(X)/Y)$ *is a weakly Laskerian* R*-module for every* $\mathrm{FD}_{<2}$ R*-submodule* Y *of* $\mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^3(X)}(X)$*. In particular,* $\mathrm{Ext}_R^1(R/\mathfrak{a}, \mathrm{H}_{\mathfrak{a}}^{f_{\mathfrak{a}}^3(X)}(X))$ *is a weakly Laskerian* R*-module.*

Corollary 5.6. *Suppose that* R *is a semi-local ring and one of the following conditions holds:*

(a) $\dim(R/\mathfrak{a}) \le 2$ *and* X *is an arbitrary* R*-module such that* $\mathrm{Ext}_R^i(R/\mathfrak{a}, X)$ *is an* $\mathrm{FD}_{<1}$ R*-module for all* i*;*

(b) X *is a finite* R*-module with* $\dim_R(X/\mathfrak{a}X) \le 2$*.*

Then the following statements are true:

(i) Y *and* $\mathrm{H}_{\mathfrak{a}}^i(X)/Y$ *are* $\mathfrak{a}$*-weakly cofinite* R*-modules for all* i *and every* $\mathrm{FD}_{<2}$ R*-submodule* Y *of* $\mathrm{H}_{\mathfrak{a}}^i(X)$*. In particular,* $\mathrm{H}_{\mathfrak{a}}^i(X)$ *is an* $\mathfrak{a}$*-weakly cofinite* R*-module for all* i*;*

(ii) *Let* N *be a finite* R*-module. Then* $\mathrm{Ext}_R^j(N, Y)$*,* $\mathrm{Tor}_j^R(N, Y)$*,* $\mathrm{Ext}_R^j(N, \mathrm{H}_{\mathfrak{a}}^i(X)/Y)$*, and* $\mathrm{Tor}_j^R(N, \mathrm{H}_{\mathfrak{a}}^i(X)/Y)$ *are* $\mathfrak{a}$*-weakly cofinite* R*-modules for all* i*, all* j*, and every* $\mathrm{FD}_{<2}$ R*-submodule* Y *of* $\mathrm{H}_{\mathfrak{a}}^i(X)$*. In particular,* $\mathrm{Ext}_R^j(N, \mathrm{H}_{\mathfrak{a}}^i(X))$ *and* $\mathrm{Tor}_j^R(N, \mathrm{H}_{\mathfrak{a}}^i(X))$ *are* $\mathfrak{a}$*-weakly cofinite* R*-modules for all* i *and all* j*;*

(iii) $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X)/Y)$ *is a finite set for all* i *and every* $\mathrm{FD}_{<2}$ R*-submodule* Y *of* $\mathrm{H}_{\mathfrak{a}}^i(X)$*. In particular,* $\mathrm{Ass}_R(\mathrm{H}_{\mathfrak{a}}^i(X))$ *is a finite set for all* i*.*

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