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An Application of Liaison Theory to Zero-dimensional Schemes

Martin Kreuzer, Tran N. K. Linh*, Le Ngoc Long and Tu Chanh Nguyen

Abstract. Given a 0-dimensional scheme $\mathbb{X}$ in an n-dimensional projective space $\mathbb{P}^n_K$ over an arbitrary field K, we use liaison theory to characterize the Cayley-Bacharach property of $\mathbb{X}$. Our result extends the result for sets of K-rational points given in [8]. In addition, we examine and bound the Hilbert function and regularity index of the Dedekind different of $\mathbb{X}$ when $\mathbb{X}$ has the Cayley-Bacharach property.

1. Introduction

The theory of liaison has been used very extensively in the literature as a tool to study projective varieties in the n-dimensional projective space $\mathbb{P}^n_K$. The initial idea was to start with a projective variety, and look at its residual variety in a complete intersection. Since complete intersections are well understood in some sense, one can get information about the variety from its residual variety or vice versa, and so it would be easier to pass to a "simpler" variety instead of considering a complicated one. This idea has been also generalized by allowing links by arithmetically Gorenstein schemes (see, e.g., [24]). Currently, liaison theory is an area of active research [2,4–6,8,13,23,25,26], and has many useful applications, for instance, constructing interesting projective varieties [2,23,26], or computing invariants and establishing properties of projective varieties [4–6,10].

In this paper we are interested in applying the theory of liaison to investigate the geometrical structure of 0-dimensional subschemes of the n-dimensional projective space $\mathbb{P}^n_K$ over an arbitrary field K. This approach was introduced by Geramita et al. [8] in their study of finite sets of K-rational points with the Cayley-Bacharach property. Classically, a finite set of K-rational points $\mathbb{X}$ in $\mathbb{P}^n_K$ is called a *Cayley-Bachrach scheme* if any hypersurface of degree less than the regularity index of the coordinate ring of $\mathbb{X}$ which contains all points of $\mathbb{X}$ but one automatically contains the last point. One of main results of [8] is stated as follows:

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*Corresponding author.

Theorem 1.1. *Let $\mathbb{W}$ be a set of points in $\mathbb{P}^n_K$ which is a complete intersection, let $\mathbb{X} \subseteq \mathbb{W}$, let $\mathbb{Y} = \mathbb{W} \setminus \mathbb{X}$, and let $I_\mathbb{W}$, $I_\mathbb{X}$ and $I_\mathbb{Y}$ denote the homogeneous vanishing ideals of $\mathbb{W}$, $\mathbb{X}$ and $\mathbb{Y}$ in $P = K[X_0, \dots, X_n]$, respectively. Set $\alpha_{\mathbb{Y}/\mathbb{W}} = \min\{i \in \mathbb{N} \mid (I_\mathbb{Y}/I_\mathbb{W})_i \neq \langle 0 \rangle\}$. Then the following conditions are equivalent.*

(a) *$\mathbb{X}$ is a Cayley-Bachrach scheme.*

(b) *A generic element of $(I_\mathbb{Y})_{\alpha_{\mathbb{Y}/\mathbb{W}}}$ does not vanish at any point of $\mathbb{X}$.*

(c) *We have $I_\mathbb{W} : (I_\mathbb{Y})_{\alpha_{\mathbb{Y}/\mathbb{W}}} = I_\mathbb{X}$.*

This result nicely leads to an efficient algorithm for checking whether a given set $\mathbb{X}$ is a Cayley-Bacharach scheme. Later investigations of the Cayley-Bacharach property have included the work of Fouli, Polini, and Ulrich [7], Robbiano [18], Gold, Little, and Schenck [9], and Guardo [12]. Moreover, this property has also been extended for 0-dimensional schemes in $\mathbb{P}^n_K$ (see [14, 15, 17, 22]). When $\mathbb{X} \subseteq \mathbb{P}^n_K$ is a 0-dimensional scheme over an algebraically closed field K, Robbiano and the first author [18] considered subschemes of $\mathbb{X}$ of degree $\deg(\mathbb{X}) - 1$ to show that the conditions (a) and (c) of Theorem 1.1 are still equivalent. However, we get no further information for a generalization of condition (b) in this case. It is worth noting here that if K is not algebraically closed then the scheme $\mathbb{X}$ may have no subschemes of degree $\deg(\mathbb{X}) - 1$. For example, the 0-dimensional scheme $\mathbb{X} = \mathcal{Z}(2X_0^4 + X_0^2X_1^2 - X_1^4) \subseteq \mathbb{P}^1_\mathbb{Q}$ is of degree 4, but it has no subscheme of degree 3.

Our focus in this paper is to look at an extension of the Cayley-Bacharach property and to generalize the above theorem for 0-dimensional schemes $\mathbb{X}$ in $\mathbb{P}^n_K$ over an arbitrary field K. In particular, we will look closely at the natural question whether conditions (a) and (b) of the above theorem are equivalent for our more general setting. Our approach is to use the notion of maximal p_j-subschemes of $\mathbb{X}$ which are introduced and studied in the papers [15, 16]. Also, we discuss a characterization of the Cayley-Bacharach property of degree d with $d \in \mathbb{N}$ in terms of the canonical module of the coordinate ring of $\mathbb{X}$ and apply this result to bound the Hilbert function of the Dedekind different of $\mathbb{X}$ and determine its regularity index in some special cases. As explained above, liaison is inherently a technique from projective algebraic geometry. Therefore we found it convenient to use the usual notation for 0-dimensional subschemes of projective space throughout this paper. Frequently, however, we need to choose a coordinate system, and in particular a hyperplane $\mathcal{Z}(X_0)$ at infinity. Thus, by paying the price of being a little bit less "canonical", the paper could have been written in the language of 0-dimensional subschemes of affine spaces, as well. To reach a wide audience, we chose to stay as close as possible to the usual notation of projective algebraic geometry, and we leave it to the interested reader to change the notation to the affine setting, if desired.

This paper is structured as follows. In Section 2, we introduce the relevant information about Hilbert functions, maximal p_j-subschemes, standard sets of separators, and liaison techniques. Especially, we give an explicit description of the residual scheme in a 0-dimensional arithmetically Gorenstein scheme of a maximal p_j-subscheme of $\mathbb{X}$. In Section 3, we prove the generalization of the results mentioned above (see Theorems 3.5 and 3.8). We also give Example 3.7 to show that the condition (b) in Theorem 1.1 is, in general, only a sufficient condition, not a necessary condition, for $\mathbb{X}$ being a Cayley-Bacharach scheme. In the final section, we characterize the Cayley-Bacharach property of degree d using the canonical module of the coordinate ring of $\mathbb{X}$, and then look at the Hilbert function of the Dedekind different of $\mathbb{X}$ and its regularity index when $\mathbb{X}$ has the Cayley-Bacharach property of degree d. In particular, we obtain a new characterization of 0-dimensional arithmetically Gorenstein schemes via the Hilbert function of their Dedekind different.

All examples in this paper were calculated by using the computer algebraic system ApCoCoA (see [1]).

2. Basic facts and notations

Throughout the paper, we work over an arbitrary field K. The n-dimensional projective space over K is denoted by $\mathbb{P}^n_K$ and its homogeneous coordinate ring is the polynomial ring $P = K[X_0, \dots, X_n]$ equipped with the standard grading. Our object of interest is a 0-dimensional subscheme $\mathbb{X}$ of $\mathbb{P}^n_K$. Its homogeneous vanishing ideal in P is denoted by $I_\mathbb{X}$ and its homogeneous coordinate ring is given by $R_\mathbb{X} = P/I_\mathbb{X}$. The set of closed points of $\mathbb{X}$ is called the *support* of $\mathbb{X}$ and is denoted by $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$. We always assume that $\mathrm{Supp}(\mathbb{X}) \cap \mathcal{Z}(X_0) = \emptyset$. Under this assumption, the image x_0 of X_0 in $R_\mathbb{X}$ is a non-zerodivisor, and $R_\mathbb{X}$ is a 1-dimensional Cohen-Macaulay ring. To each point $p_j \in \mathrm{Supp}(\mathbb{X})$ we have the associated local ring $\mathcal{O}_{\mathbb{X},p_j}$. Its maximal ideal is denoted by $\mathfrak{m}_{\mathbb{X},p_j}$, and the residue field of $\mathbb{X}$ at p_j is denoted by $\kappa(p_j)$. The *degree* of $\mathbb{X}$ is defined as $\deg(\mathbb{X}) = \sum_{j=1}^{s} \dim_K(\mathcal{O}_{\mathbb{X},p_j})$.

Given any finitely generated graded $R_\mathbb{X}$-module M, the *Hilbert function* of M is a map $\mathrm{HF}_M \colon \mathbb{Z} \to \mathbb{N}$ given by $\mathrm{HF}_M(i) = \dim_K(M_i)$. The unique polynomial $\mathrm{HP}_M(z) \in \mathbb{Q}[z]$ for which $\mathrm{HF}_M(i) = \mathrm{HP}_M(i)$ for all $i \gg 0$ is called the *Hilbert polynomial* of M. The number

$$\mathrm{ri}(M) = \min\{i \in \mathbb{Z} \mid \mathrm{HF}_M(j) = \mathrm{HP}_M(j) \text{ for all } j \geq i\}$$

is called the *regularity index* of M (or of HF_M). Whenever $\mathrm{HF}_M(i) = \mathrm{HP}_M(i)$ for all $i \in \mathbb{Z}$, we let $\mathrm{ri}(M) = -\infty$. Instead of $\mathrm{HF}_{R_\mathbb{X}}$ we also write $\mathrm{HF}_\mathbb{X}$ and call it the Hilbert function of $\mathbb{X}$. Its regularity index is denoted by $r_\mathbb{X}$. Note that $\mathrm{HF}_\mathbb{X}(i) = 0$ for $i < 0$ and

$$1 = \mathrm{HF}_\mathbb{X}(0) < \mathrm{HF}_\mathbb{X}(1) < \cdots < \mathrm{HF}_\mathbb{X}(r_\mathbb{X} - 1) < \deg(\mathbb{X})$$

and $\mathrm{HF}_{\mathbb{X}}(i) = \deg(\mathbb{X})$ for $i \geq r_{\mathbb{X}}$.

Definition 2.1. Let $1 \leq j \leq s$. A subscheme $\mathbb{X}' \subsetneq \mathbb{X}$ is called a *p_j-subscheme* if we have $\mathcal{O}_{\mathbb{X},p_j} \neq \mathcal{O}_{\mathbb{X}',p_j}$ and $\mathcal{O}_{\mathbb{X}',p_k} = \mathcal{O}_{\mathbb{X},p_k}$ for $k \neq j$. A p_j-subscheme $\mathbb{X}' \subseteq \mathbb{X}$ is called *maximal* if $\deg(\mathbb{X}') = \deg(\mathbb{X}) - \dim_K \kappa(p_j)$.

In case $\mathbb{X}$ has K-rational support (i.e., all points $p_1, \ldots, p_s$ are K-rational), a maximal p_j-subscheme of $\mathbb{X}$ is nothing but a subscheme $\mathbb{X}' \subseteq \mathbb{X}$ of degree $\deg(\mathbb{X}') = \deg(\mathbb{X}) - 1$ with $\mathcal{O}_{\mathbb{X}',p_j} \neq \mathcal{O}_{\mathbb{X},p_j}$. According to [16, Proposition 3.2], there is a one-to-one correspondence between a maximal p_j-subscheme $\mathbb{X}'$ and an ideal $\langle s_j \rangle$ in $\mathcal{O}_{\mathbb{X},p_j}$, where s_j is an element in the socle $\mathrm{Ann}_{\mathcal{O}_{\mathbb{X},p_j}}(\mathfrak{m}_{\mathbb{X},p_j})$ of $\mathcal{O}_{\mathbb{X},p_j}$. The vanishing ideal of the scheme $\mathbb{X}'$ in $R_{\mathbb{X}}$ is denoted by $I_{\mathbb{X}'/\mathbb{X}}$ and its initial degree is given by $\alpha_{\mathbb{X}'/\mathbb{X}} = \min\{i \in \mathbb{N} \mid (I_{\mathbb{X}'/\mathbb{X}})_i \neq \langle 0 \rangle\}$. We find a non-zero element $f'_{\mathbb{X}} \in (I_{\mathbb{X}'/\mathbb{X}})_i$, $i \geq \alpha_{\mathbb{X}'/\mathbb{X}}$, such that $\widetilde{\imath}(f'_{\mathbb{X}}) = (0, \ldots, 0, s_j T_j^i, 0, \ldots, 0)$, where the map

$$\widetilde{\imath}\colon R_{\mathbb{X}} \to Q^h(R_{\mathbb{X}}) \cong \prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]$$

is the injection given by $\widetilde{\imath}(f) = (f_{p_1} T_1^i, \ldots, f_{p_s} T_s^i)$, for $f \in (R_{\mathbb{X}})_i$ with $i \geq 0$, where f_{p_j} is the germ of f at p_j. Here the ring $Q^h(R_{\mathbb{X}})$ is the *homogeneous ring of quotients* of $R_{\mathbb{X}}$ defined as the localization of $R_{\mathbb{X}}$ with respect to the set of all homogeneous non-zerodivisors of $R_{\mathbb{X}}$ (cf. [16, Section 3]).

Let $\varkappa_j := \dim_K \kappa(p_j)$, and let $\{e_{j1}, \ldots, e_{j\varkappa_j}\} \subseteq \mathcal{O}_{\mathbb{X},p_j}$ be elements whose residue classes form a K-basis of $\kappa(p_j)$. For $a \in \mathcal{O}_{\mathbb{X},p_j}$, we set

$$\mu(a) := \min\{i \in \mathbb{N} \mid (0, \ldots, 0, a T_j^i, 0, \ldots, 0) \in \widetilde{\imath}(R_{\mathbb{X}})\}.$$

Since the restriction map $\widetilde{\imath}|_{(R_{\mathbb{X}})_{r_{\mathbb{X}}}}\colon (R_{\mathbb{X}})_{r_{\mathbb{X}}} \to \left(\prod_{j=1}^{s} \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]\right)_{r_{\mathbb{X}}}$ is an isomorphism of K-vector spaces, we have $\mu(a) \leq r_{\mathbb{X}}$ for all $a \in \mathcal{O}_{\mathbb{X},p_j}$. Using this notation, we recall from [15, Section 1] the following notion of separators.

Definition 2.2. Let $\mathbb{X}'$ be a maximal p_j-subscheme as above, and let

$$f^*_{jk_j} := \widetilde{\imath}^{-1}((0, \ldots, 0, e_{jk_j} s_j T_j^{\mu(e_{jk_j} s_j)}, 0, \ldots, 0))$$

and $f_{jk_j} = x_0^{r_{\mathbb{X}} - \mu(e_{jk_j} s_j)} f^*_{jk_j}$ for $k_j = 1, \ldots, \varkappa_j$.

(a) The set $\{f^*_{j1}, \ldots, f^*_{j\varkappa_j}\}$ is called the *set of minimal separators of* $\mathbb{X}'$ *in* $\mathbb{X}$ with respect to s_j and $\{e_{j1}, \ldots, e_{j\varkappa_j}\}$.

(b) The set $\{f_{j1}, \ldots, f_{j\varkappa_j}\}$ is called the *standard set of separators of* $\mathbb{X}'$ *in* $\mathbb{X}$ with respect to s_j and $\{e_{j1}, \ldots, e_{j\varkappa_j}\}$.

(c) The number
$$\mu_{\mathbb{X}'/\mathbb{X}} := \max\{\deg(f^*_{jk_j}) \mid k_j = 1, \dots, \varkappa_j\}$$
is called the *maximal degree of a minimal separator of* $\mathbb{X}'$ *in* $\mathbb{X}$.

Remark 2.3. Let $\mathbb{X}'$ be a maximal p_j-subscheme of $\mathbb{X}$.

(a) The maximal degree of a minimal separator of $\mathbb{X}'$ in $\mathbb{X}$ depends neither on the choice of the socle element s_j nor on the specific choice of $\{e_{j1}, \dots, e_{j\varkappa_j}\}$ (see [16, Lemma 4.4]). Moreover, we have $\mu_{\mathbb{X}'/\mathbb{X}} \le r_{\mathbb{X}}$.

(b) For $k_j = 1, \dots, \varkappa_j$, let F_{jk_j} (respectively, $F^*_{jk_j}$) be a representative of f_{jk_j} (respectively, $f^*_{jk_j}$) in P. We also say that the set $\{F_{j1}, \dots, F_{j\varkappa_j}\}$ is a standard set of separators of $\mathbb{X}'$ in $\mathbb{X}$ and the set $\{F^*_{j1}, \dots, F^*_{j\varkappa_j}\}$ is a set of minimal separators of $\mathbb{X}'$ in $\mathbb{X}$.

(c) According to [15, Proposition 2.5(c)], one may choose a set of minimal separators $\{f^*_{j1}, \dots, f^*_{j\varkappa_j}\}$ of $\mathbb{X}'$ in $\mathbb{X}$ such that
$$(I_{\mathbb{X}'/\mathbb{X}})_i = \left\langle x_0^{\,i-\deg(f^*_{jk_j})} f^*_{jk_j} \;\middle|\; \deg(f^*_{jk_j}) \le i \right\rangle_K$$
for all $i \ge 0$.

Recall that a 0-dimensional scheme $\mathbb{X}$ is called a *complete intersection* if $I_{\mathbb{X}}$ can be generated by n homogeneous polynomials in P, and it is called an *arithmetically Gorenstein scheme* if $R_{\mathbb{X}}$ is a Gorenstein ring. Note that every complete intersections are arithmetically Gorenstein, however, except for the case $n = 2$, an arithmetically Gorenstein scheme is not a complete intersection in general (see [16, Example 2.12]).

The graded $R_{\mathbb{X}}$-module $\omega_{R_{\mathbb{X}}} = \mathrm{Hom}_{K[x_0]}(R_{\mathbb{X}}, K[x_0])(-1)$ is called the *canonical module* of $R_{\mathbb{X}}$. Its $R_{\mathbb{X}}$-module structure is defined by $(f \cdot \varphi)(g) = \varphi(fg)$ for all $f, g \in R_{\mathbb{X}}$ and $\varphi \in \omega_{R_{\mathbb{X}}}$. It is also a finitely generated graded $R_{\mathbb{X}}$-module and
$$\mathrm{HF}_{\omega_{R_{\mathbb{X}}}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-i)$$
for all $i \in \mathbb{Z}$ (see e.g., [3, Theorem 4.4.5]). Moreover, by [11, Proposition 2.1.3], the scheme $\mathbb{X}$ is arithmetically Gorenstein if and only if $\omega_{R_{\mathbb{X}}} \cong R_{\mathbb{X}}(r_{\mathbb{X}} - 1)$.

In what follows, we let $\mathbb{W} \subseteq \mathbb{P}^n_K$ be a 0-dimensional arithmetically Gorenstein scheme, let $\mathbb{X}$ be a subscheme of $\mathbb{W}$, and let $I_{\mathbb{X}/\mathbb{W}}$ be the ideal of $\mathbb{X}$ in $R_{\mathbb{W}}$. Then the homogeneous ideal $\mathrm{Ann}_{R_{\mathbb{W}}}(I_{\mathbb{X}/\mathbb{W}}) \subseteq R_{\mathbb{W}}$ is saturated and defines a 0-dimensional subscheme $\mathbb{Y}$ of $\mathbb{W}$.

Definition 2.4. (a) The subscheme $\mathbb{Y} \subseteq \mathbb{W}$ which is defined by the saturated homogeneous ideal $I_{\mathbb{Y}/\mathbb{W}} = \mathrm{Ann}_{R_{\mathbb{W}}}(I_{\mathbb{X}/\mathbb{W}})$ is said to be the *residual scheme* of $\mathbb{X}$ in $\mathbb{W}$. We also say that $\mathbb{X}$ and $\mathbb{Y}$ are *(algebraically) G-linked* by $\mathbb{W}$.

(b) Two G-linked schemes $\mathbb{X}$ and $\mathbb{Y}$ by $\mathbb{W}$ are said to be *geometrically G-linked* by $\mathbb{W}$ if they have no common irreducible component.

Remark 2.5. From the point of view of the saturated ideals, the schemes $\mathbb{X}$ and $\mathbb{Y}$ are geometrically G-linked by $\mathbb{W}$ if and only if $I_{\mathbb{W}} = I_{\mathbb{X}} \cap I_{\mathbb{Y}}$ and neither $I_{\mathbb{X}}$ nor $I_{\mathbb{Y}}$ is contained in any associated prime of the other (see [24, Section 5.2]). In this case, if we write $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$ and $\mathrm{Supp}(\mathbb{Y}) = \{p'_1, \dots, p'_t\}$, then we have $\mathrm{Supp}(\mathbb{W}) = \{p_1, \dots, p_s, p'_1, \dots, p'_t\}$ and $\mathrm{Supp}(\mathbb{X}) \cap \mathrm{Supp}(\mathbb{Y}) = \emptyset$. In particular, we have $\mathcal{O}_{\mathbb{W},p_j} = \mathcal{O}_{\mathbb{X},p_j}$ for $j = 1, \dots, s$ and $\mathcal{O}_{\mathbb{W},p'_j} = \mathcal{O}_{\mathbb{Y},p'_j}$ for $j = 1, \dots, t$.

First we collect some useful results about the G-linked schemes $\mathbb{X}$ and $\mathbb{Y}$ by the arithmetically Gorenstein scheme $\mathbb{W}$.

Proposition 2.6. (a) *We have* $I_{\mathbb{X}/\mathbb{W}} = \mathrm{Ann}_{R_{\mathbb{W}}}(I_{\mathbb{Y}/\mathbb{W}})$.

(b) *We have* $\deg(\mathbb{W}) = \deg(\mathbb{X}) + \deg(\mathbb{Y})$ *and* $r_{\mathbb{W}} = r_{\mathbb{X}} + \alpha_{\mathbb{Y}/\mathbb{W}} = r_{\mathbb{Y}} + \alpha_{\mathbb{X}/\mathbb{W}}$.

(c) *The Hilbert function of* $I_{\mathbb{Y}/\mathbb{W}}$ *satisfies*

$$\mathrm{HF}_{I_{\mathbb{Y}/\mathbb{W}}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{W}} - i - 1) \quad \textit{for all } i \in \mathbb{Z}.$$

Proof. Claims (a) and (b) follow from [5]. To prove (c), we use (a) and [11, Proposition 2.2.9] to get the following sequence of isomorphism of graded $R_{\mathbb{W}}$-modules

$$\begin{aligned} I_{\mathbb{Y}/\mathbb{W}} &= \mathrm{Ann}_{R_{\mathbb{W}}}(I_{\mathbb{X}/\mathbb{W}}) \cong \mathrm{Hom}_{R_{\mathbb{W}}}(R_{\mathbb{W}}/I_{\mathbb{X}/\mathbb{W}}, R_{\mathbb{W}}) \\ &\cong \mathrm{Hom}_{R_{\mathbb{W}}}(R_{\mathbb{X}}, R_{\mathbb{W}}(r_{\mathbb{W}} - 1))(-r_{\mathbb{W}} + 1) \cong \mathrm{Hom}_{R_{\mathbb{W}}}(R_{\mathbb{X}}, \omega_{R_{\mathbb{W}}})(-r_{\mathbb{W}} + 1) \\ &\cong \omega_{R_{\mathbb{X}}}(-r_{\mathbb{W}} + 1). \end{aligned}$$

Since $\mathrm{HF}_{\omega_{R_{\mathbb{X}}}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-i)$ for all $i \in \mathbb{Z}$, we get the desired formula for the Hilbert function of $I_{\mathbb{Y}/\mathbb{W}}$ and claim (c) follows. □

In the following we shall use "$\overline{\cdot}$" to denote residue classes modulo X_0.

Lemma 2.7. *For every* $d \in \{1, \dots, r_{\mathbb{X}}\}$, *we have* $(\overline{I}_{\mathbb{W}})_{r_{\mathbb{W}}} : (\overline{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d)} = (\overline{I}_{\mathbb{X}})_d$.

Proof. Clearly, we have $I_{\mathbb{X}} \cdot I_{\mathbb{Y}} \subseteq I_{\mathbb{W}}$. This implies $(\overline{I}_{\mathbb{X}})_d \subseteq (\overline{I}_{\mathbb{W}})_{r_{\mathbb{W}}} : (\overline{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d)}$. For the other inclusion, let $f \in (\overline{I}_{\mathbb{W}})_{r_{\mathbb{W}}} : (\overline{I}_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d)}$. In $\overline{R}_{\mathbb{W}} = R_{\mathbb{W}}/\langle x_0 \rangle$, we have $\overline{f} \in (\mathrm{Ann}_{\overline{R}_{\mathbb{W}}}((\overline{I}_{\mathbb{Y}/\mathbb{W}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d)}))_d$. Since $\mathbb{W}$ is arithmetically Gorenstein, the ring $\overline{R}_{\mathbb{W}}$ is a 0-dimensional local Gorenstein ring with socle $(\overline{R}_{\mathbb{W}})_{r_{\mathbb{W}}} \cong K$. Thus we can argue in the same way as Lemma 4.1 and Proposition 4.3.a of [8] to get

$$\begin{aligned} (\mathrm{Ann}_{\overline{R}_{\mathbb{W}}}((\overline{I}_{\mathbb{Y}/\mathbb{W}})_{\alpha_{\mathbb{Y}/\mathbb{W}} + (r_{\mathbb{X}} - d)}))_d &= (\mathrm{Ann}_{\overline{R}_{\mathbb{W}}}((\overline{I}_{\mathbb{Y}/\mathbb{W}})_{r_{\mathbb{W}} - d}))_d \\ &= (\mathrm{Ann}_{\overline{R}_{\mathbb{W}}}(\overline{I}_{\mathbb{Y}/\mathbb{W}}))_d = (\overline{I}_{\mathbb{X}/\mathbb{W}})_d. \end{aligned}$$

Consequently, we have $\overline{f} \in (\overline{I}_{\mathbb{X}/\mathbb{W}})_d$, and hence $f \in (\overline{I}_{\mathbb{X}})_d$, as desired. □

The next lemma follows for instance from [21, 3.15 and 16.38-40].

Lemma 2.8. *Let A/K be a finite Gorenstein algebra.*

(a) *There is a non-degenerate K-bilinear form $\Phi\colon A \times A \to K$ with the property that $\Phi(xy, z) = \Phi(x, yz)$ for all $x, y, z \in A$.*

(b) *Let I be a non-zero ideal of A, and let $I^0 = \{x \in A \mid \Phi(I, x) = 0\}$. Then we have $\operatorname{Ann}_A(I) = I^0$ and $\dim_K I + \dim_K \operatorname{Ann}_A(I) = \dim_K A$.*

A concrete description of the residual scheme in $\mathbb{W}$ of a maximal p_j-subscheme of $\mathbb{X}$ is given by the following proposition.

Proposition 2.9. *Let $\mathbb{W} \subseteq \mathbb{P}^n_K$ be a 0-dimensional arithmetically Gorenstein scheme, let $\mathbb{X}$ and $\mathbb{X}'$ be subschemes of $\mathbb{W}$, let $\mathbb{Y}$ and $\mathbb{Y}'$ be the residual schemes of $\mathbb{X}$ and $\mathbb{X}'$ in $\mathbb{W}$ respectively, and let $p_j \in \operatorname{Supp}(\mathbb{X})$. Then $\mathbb{X}'$ is a (maximal) p_j-subscheme of $\mathbb{X}$ if and only if $\mathbb{Y}'$ contains $\mathbb{Y}$ as a (maximal) p_j-subscheme.*

Proof. As sets, we have $\operatorname{Supp}(\mathbb{W}) = \operatorname{Supp}(\mathbb{X}) \cup \operatorname{Supp}(\mathbb{Y})$ by [24, Proposition 5.2.2]. Let us write $\operatorname{Supp}(\mathbb{W}) = \{p_1, \dots, p_u\}$ and consider the ring epimorphism $\theta\colon R_{\mathbb{W}} \to \Gamma_{\mathbb{W}} := \prod_{j=1}^u \mathcal{O}_{\mathbb{W},p_j}$ given by $f \mapsto (f_{p_1}, \dots, f_{p_u})$. According to [14, Lemma 1.1], the restriction $\theta|_{(R_{\mathbb{W}})_i}$ is an injection for $0 \le i < r_{\mathbb{W}}$ and is an isomorphism for all $i \ge r_{\mathbb{W}}$. By $I^a_{\mathbb{X}/\mathbb{W}}$ (respectively, $I^a_{\mathbb{X}'/\mathbb{W}}$, $I^a_{\mathbb{Y}/\mathbb{W}}$, $I^a_{\mathbb{Y}'/\mathbb{W}}$) we denote the image of $I_{\mathbb{X}/\mathbb{W}}$ (respectively, $I_{\mathbb{X}'/\mathbb{W}}$, $I_{\mathbb{Y}/\mathbb{W}}$, $I_{\mathbb{Y}'/\mathbb{W}}$) under θ. We verify that $I^a_{\mathbb{Y}/\mathbb{W}} = \operatorname{Ann}_{\Gamma_{\mathbb{W}}}(I^a_{\mathbb{X}/\mathbb{W}})$. Clearly, the equality $I_{\mathbb{Y}/\mathbb{W}} = \operatorname{Ann}_{R_{\mathbb{W}}}(I_{\mathbb{X}/\mathbb{W}})$ implies $I^a_{\mathbb{Y}/\mathbb{W}} \subseteq \operatorname{Ann}_{\Gamma_{\mathbb{W}}}(I^a_{\mathbb{X}/\mathbb{W}})$. Also, Proposition 2.6(b) and Lemma 2.8 yields

$$\begin{aligned}\dim_K I^a_{\mathbb{Y}/\mathbb{W}} &= \dim_K (I_{\mathbb{Y}/\mathbb{W}})_{r_{\mathbb{W}}} = \deg(\mathbb{W}) - \deg(\mathbb{Y}) = \deg(\mathbb{X})\\ &= \deg(\mathbb{W}) - \dim_K (I_{\mathbb{X}/\mathbb{W}})_{r_{\mathbb{W}}} = \deg(\mathbb{W}) - \dim_K I^a_{\mathbb{X}/\mathbb{W}}\\ &= \dim_K \operatorname{Ann}_{\Gamma_{\mathbb{W}}}(I^a_{\mathbb{X}/\mathbb{W}}).\end{aligned}$$

So, the equality $I^a_{\mathbb{Y}/\mathbb{W}} = \operatorname{Ann}_{\Gamma_{\mathbb{W}}}(I^a_{\mathbb{X}/\mathbb{W}})$ holds true. In $\Gamma_{\mathbb{W}}$, we have $I^a_{\mathbb{X}/\mathbb{W}} \subseteq I^a_{\mathbb{X}'/\mathbb{W}}$ and the quotient $I^a_{\mathbb{X}'/\mathbb{W}}/I^a_{\mathbb{X}/\mathbb{W}}$ is non-zero and its support has exactly one minimal prime ideal, which is also a minimal prime ideal of $\Gamma_{\mathbb{W}}$ corresponding to the point p_j. Hence we get $I^a_{\mathbb{Y}/\mathbb{W}} = \operatorname{Ann}_{\Gamma_{\mathbb{W}}}(I^a_{\mathbb{X}/\mathbb{W}}) \supseteq \operatorname{Ann}_{\Gamma_{\mathbb{W}}}(I^a_{\mathbb{X}'/\mathbb{W}}) = I^a_{\mathbb{Y}'/\mathbb{W}}$ with a non-zero quotient whose support has exactly the same minimal prime ideal. This proves the asserted correspondence, without the adjective "maximal". Now suppose that $\mathbb{X}'$ is a maximal p_j-subscheme of $\mathbb{X}$ and let $\mathfrak{q}$, $\mathfrak{q}'$ be the kernels of $\mathcal{O}_{\mathbb{W},p_j} \to \mathcal{O}_{\mathbb{X},p_j}$, $\mathcal{O}_{\mathbb{W},p_j} \to \mathcal{O}_{\mathbb{X}',p_j}$ respectively. Note that $\mathcal{O}_{\mathbb{Y},p_j} = \mathcal{O}_{\mathbb{W},p_j}/\operatorname{Ann}_{\mathcal{O}_{\mathbb{W},p_j}}(\mathfrak{q})$ and $\mathcal{O}_{\mathbb{Y}',p_j} = \mathcal{O}_{\mathbb{W},p_j}/\operatorname{Ann}_{\mathcal{O}_{\mathbb{W},p_j}}(\mathfrak{q}')$. Then [16, Proposition 3.2] yields

$$0 \longrightarrow \kappa(p_j) \longrightarrow \mathcal{O}_{\mathbb{X},p_j} \longrightarrow \mathcal{O}_{\mathbb{X}',p_j} \longrightarrow 0$$

which, dually, gives

$$0 \longrightarrow \omega_{\mathcal{O}_{\mathbb{X}',p_j}} \longrightarrow \omega_{\mathcal{O}_{\mathbb{X},p_j}} \longrightarrow \kappa(p_j) \longrightarrow 0.$$

But $\omega_{\mathcal{O}_{\mathbb{X}',p_j}} = \mathrm{Ann}_{\mathcal{O}_{\mathbb{W},p_j}}(\mathfrak{q}')$ and $\omega_{\mathcal{O}_{\mathbb{X},p_j}} = \mathrm{Ann}_{\mathcal{O}_{\mathbb{W},p_j}}(\mathfrak{q})$, so we get

$$0 \longrightarrow \kappa(p_j) \longrightarrow \mathcal{O}_{\mathbb{Y}',p_j} \longrightarrow \mathcal{O}_{\mathbb{Y},p_j} \longrightarrow 0.$$

Therefore $\mathbb{Y}$ is a maximal p_j-subscheme of $\mathbb{Y}'$. $\square$

3. The Cayley-Bacharach property and liaison

In this section we use liaison techniques to characterize the Cayley-Bacharach property of a 0-dimensional scheme $\mathbb{X}$ in $\mathbb{P}^n_K$. First we recall the notions of the degree of a point in $\mathbb{X}$ and the Cayley-Bacharach property (see [15, Section 4]).

Definition 3.1. Let $d \geq 0$, let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme, and let $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$.

(a) For $1 \leq j \leq s$, the *degree of p_j in $\mathbb{X}$* is defined as

$$\deg_{\mathbb{X}}(p_j) := \min\big\{\mu_{\mathbb{X}'/\mathbb{X}} \mid \mathbb{X}' \text{ is a maximal } p_j\text{-subscheme of } \mathbb{X}\big\},$$

where $\mu_{\mathbb{X}'/\mathbb{X}}$ is the maximal degree of a minimal separator of $\mathbb{X}'$ in $\mathbb{X}$.

(b) We say that $\mathbb{X}$ has the *Cayley-Bacharach property of degree d* (in short, $\mathbb{X}$ has $\mathrm{CBP}(d)$) if $\deg_{\mathbb{X}}(p_j) \geq d+1$ for every $j \in \{1, \dots, s\}$. In the case that $\mathbb{X}$ has $\mathrm{CBP}(r_{\mathbb{X}}-1)$ we also say that $\mathbb{X}$ is a *Cayley-Bacharach scheme.*

According to Remark 2.3(a), we have $0 \leq \deg_{\mathbb{X}}(p_j) \leq r_{\mathbb{X}}$. So, the number $r_{\mathbb{X}} - 1$ is the largest degree $d \geq 0$ such that $\mathbb{X}$ can have $\mathrm{CBP}(d)$. Hence it suffices to consider the Cayley-Bacharach property in degree $d \in \{0, \dots, r_{\mathbb{X}} - 1\}$. Using standard sets of separators of $\mathbb{X}$, we can characterize the Cayley-Bacharach property as follows (see [15, Proposition 4.3]).

Proposition 3.2. *Let $0 \leq d < r_{\mathbb{X}}$, let $\mathrm{Supp}(\mathbb{X}) = \{p_1, \dots, p_s\}$, and let $\varkappa_j = \dim \kappa(p_j)$. Then the following statements are equivalent.*

(a) *The scheme $\mathbb{X}$ has $\mathrm{CBP}(d)$.*

(b) *If $\mathbb{X}' \subseteq \mathbb{X}$ is a maximal p_j-subscheme and $\{f_{j1}, \dots, f_{j\varkappa_j}\} \subseteq R_{\mathbb{X}}$ is a standard set of separators of $\mathbb{X}'$ in $\mathbb{X}$, then there exists $k_j \in \{1, \dots, \varkappa_j\}$ such that $x_0^{r_{\mathbb{X}}-d} \nmid f_{jk_j}$.*

(c) *If $\mathbb{X}' \subseteq \mathbb{X}$ is a maximal p_j-subscheme and $\{F_{j1}, \dots, F_{j\varkappa_j}\} \subseteq P$ is a standard set of separators of $\mathbb{X}'$ in $\mathbb{X}$, then there exists $k_j \in \{1, \dots, \varkappa_j\}$ such that $F_{jk_j} \notin \langle X_0^{r_{\mathbb{X}}-d}, (I_{\mathbb{X}})_{r_{\mathbb{X}}}\rangle_P$.*

(d) *For all $p_j \in \operatorname{Supp}(\mathbb{X})$, every maximal p_j-subscheme $\mathbb{X}' \subseteq \mathbb{X}$ satisfies*

$$\dim_K(I_{\mathbb{X}'/\mathbb{X}})_d < \varkappa_j.$$

Now we give two useful lemmas that will be used in the proof of Theorem 3.5.

Lemma 3.3. *Let $\mathbb{W} \subseteq \mathbb{P}^n_K$ be a 0-dimensional arithmetically Gorenstein scheme, let $\mathbb{X}$ be a subscheme of $\mathbb{W}$ with its residual scheme $\mathbb{Y}$, and let $0 \le d < r_{\mathbb{X}}$. Furthermore, let $\mathbb{X}' \subseteq \mathbb{X}$ be a maximal p_j-subscheme, and let $\{F_{j1}, \dots, F_{j\varkappa_j}\} \subseteq P_{r_{\mathbb{X}}}$ be a standard set of separators of $\mathbb{X}'$ in $\mathbb{X}$. Suppose that $\langle F_{j1}, \dots, F_{j\varkappa_j}\rangle_K \nsubseteq \langle X_0^{r_{\mathbb{X}}-d}, (I_{\mathbb{X}})_{r_{\mathbb{X}}}\rangle_P$ and $\langle F_{j1}, \dots, F_{j\varkappa_j}\rangle_K \subseteq \langle X_0^{r_{\mathbb{X}}-d-1}, (I_{\mathbb{X}})_{r_{\mathbb{X}}}\rangle_P$, and write $F_{jk_j} = F'_{jk_j} + X_0^{r_{\mathbb{X}}-d-1}G_{jk_j}$ with $F'_{jk_j} \in (I_{\mathbb{X}})_{r_{\mathbb{X}}}$ and $G_{jk_j} \in P_{d+1}$.*

Then there is $k_j \in \{1, \dots, \varkappa_j\}$ such that $G_{jk_j} \notin (I_{\mathbb{W}})_{r_{\mathbb{W}}} : (I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1}$.

Proof. Suppose that $G_{jk_j} \in (I_{\mathbb{W}})_{r_{\mathbb{W}}} : (I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1}$ for all $k_j = 1, \dots, \varkappa_j$. By modulo X_0 we have $\overline{G}_{jk_j}(\overline{I}_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1} \subseteq (\overline{I}_{\mathbb{W}})_{r_{\mathbb{W}}}$. Note that $r_{\mathbb{W}} = \alpha_{\mathbb{Y}/\mathbb{W}} + r_{\mathbb{X}}$ by Proposition 2.6(b). Thus Lemma 2.7 yields that $\overline{G}_{jk_j} \in (\overline{I}_{\mathbb{X}})_{d+1}$. This allows us to write $G_{jk_j} = G'_{jk_j} + X_0 H_{jk_j}$ with $G'_{jk_j} \in (I_{\mathbb{X}})_{d+1}$ and $H_{jk_j} \in P_d$. It is clear that $H_{jk_j} \in (I_{\mathbb{X}'})_d$. From this we get $F_{jk_j} = (F'_{jk_j} + X_0^{r_{\mathbb{X}}-d-1}G'_{jk_j}) + X_0^{r_{\mathbb{X}}-d}H_{jk_j}$ for all $k_j = 1, \dots, \varkappa_j$. It follows that $F_{jk_j} \in \langle X_0^{r_{\mathbb{X}}-d}, (I_{\mathbb{X}})_{r_{\mathbb{X}}}\rangle_P$ for all $k_j = 1, \dots, \varkappa_j$. This is a contradiction to our hypothesis, and hence the claim is completely proved. □

Lemma 3.4. *Let A be a 0-dimensional local affine K-algebra with maximal ideal $\mathfrak{m}$, let $\mathfrak{q}$ be a $\mathfrak{m}$-primary ideal, let $R = A/\mathfrak{q}$, and let $\pi\colon A \to R$ be the canonical epimorphism. Let $g \in A$ be an element such that $\pi(g) \in \operatorname{Ann}_R(\pi(\mathfrak{m}))$ is a non-zero socle element of R, and suppose $h \in \operatorname{Ann}_A(\mathfrak{q})$ and $gh \neq 0$.*

(a) *We have $gh \in \operatorname{Ann}_A(\mathfrak{m})$ and $\langle 0\rangle :_{\langle g\rangle} \langle h\rangle \subseteq \mathfrak{q}$.*

(b) *Every element $f \in A$ with $\pi(f) \in \langle \pi(g)\rangle_R \setminus \{0\}$ satisfies $fh \neq 0$.*

(c) *Let $g_1, \dots, g_r \in A \setminus \{0\}$. If the set $\{\pi(g_1), \dots, \pi(g_r)\} \subseteq \langle \pi(g)\rangle_R$ is K-linearly independent, then the set $\{g_1h, \dots, g_rh\}$ is K-linearly independent.*

Proof. For (a), let $a \in \mathfrak{m}$ be a non-zero element. In R we have $\pi(a) \in \pi(\mathfrak{m})$, and so we get $\pi(ag) = \pi(a)\pi(g) = 0$ or $ag \in \mathfrak{q}$. It follows that $agh = 0$. Hence $gh \in \operatorname{Ann}_A(\mathfrak{m})$. Moreover, for $f \in \langle 0\rangle :_{\langle g\rangle} \langle h\rangle$ we have $f = gf'$ for some $f' \in A$ and $gf'h = fh = 0$. Since gh is a socle element of A and $bgh \neq 0$ for $b \in A \setminus \mathfrak{m}$, we have $\operatorname{Ann}_A(gh) = \mathfrak{m}$. This implies $f' \in \mathfrak{m}$. Thus $f = gf' \in \mathfrak{q}$.

To prove (b), we consider an element $f \in A$ with $\pi(f) \in \langle \pi(g)\rangle_R \setminus \{0\}$. Writing $\pi(f) = \pi(g)\pi(f')$ for some $f' \in A \setminus \{0\}$, we see that $f' \notin \mathfrak{m}$ is a unit and $f = gf' + f''$ with $f'' \in \mathfrak{q}$. So, we obtain $fh = f'gh + f''h = f'gh \neq 0$.

Next, we prove (c). Suppose that there are $a_1, \dots, a_r \in K$ such that $a_1 g_1 h + \cdots + a_r g_r h = (a_1 g_1 + \cdots + a_r g_r)h = 0$. Since $\pi(a_1 g_1 + \cdots + a_r g_r) \in \langle \pi(g) \rangle_R$, it follows from (b) that $\pi(a_1 g_1 + \cdots + a_r g_r) = a_1 \pi(g_1) + \cdots + a_r \pi(g_r) = 0$. By assumption, we get $a_1 = \cdots = a_r = 0$. □

The first main result of this section is the following characterization of the Cayley-Bacharach property, which is a generalization of results for finite sets of K-rational points or for the case that K is an algebraically closed field found in [8, Theorem 4.6] and [18, Theorem 4.1]. For $i \geq 0$ we write F_p for the image in $\mathcal{O}_{\mathbb{W},p}$ of $F \in P_i$ under the composition map $P_i \to (R_{\mathbb{W}})_i \to \prod_{p \in \operatorname{Supp}(\mathbb{W})} \mathcal{O}_{\mathbb{W},p} \to \mathcal{O}_{\mathbb{W},p}$. Notice that $F \in I_{\mathbb{W}}$ if and only if $F_p = 0$ for all $p \in \operatorname{Supp}(\mathbb{W})$ (cf. [14, Lemma 1.1]).

Theorem 3.5. *Let $\mathbb{W} \subseteq \mathbb{P}^n_K$ be a 0-dimensional arithmetically Gorenstein scheme, let $\mathbb{X}$ be a subscheme of $\mathbb{W}$, let $\mathbb{Y}$ be the residual scheme of $\mathbb{X}$ in $\mathbb{W}$, and let $0 \leq d \leq r_{\mathbb{X}} - 1$. Then the following statements are equivalent.*

(a) *The scheme $\mathbb{X}$ has* CBP(d).

(b) *Every subscheme $\mathbb{Y}' \subseteq \mathbb{W}$ containing $\mathbb{Y}$ as a maximal p_j-subscheme, where $p_j \in \operatorname{Supp}(\mathbb{X})$, satisfies $\operatorname{HF}_{I_{\mathbb{Y}/\mathbb{Y}'}}(r_{\mathbb{W}} - d - 1) > 0$.*

(c) *We have $I_{\mathbb{W}} : (I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1} = I_{\mathbb{X}}$.*

(d) *We have $(I_{\mathbb{W}})_{r_{\mathbb{W}}-1} : (I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1} = (I_{\mathbb{X}})_d$.*

(e) *For all $p_j \in \operatorname{Supp}(\mathbb{X})$ and for every maximal p_j-subscheme $\mathbb{X}' \subseteq \mathbb{X}$ with standard set of separators $\{F_{j1}, \dots, F_{j\varkappa_j}\}$ there exists a homogeneous element $H_j \in (I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1}$ such that $H_j \cdot \langle F_{j1}, \dots, F_{j\varkappa_j} \rangle_K \nsubseteq I_{\mathbb{W}}$.*

Proof. First we prove the implication (a) ⇒ (b). Let $p_j \in \operatorname{Supp}(\mathbb{X})$, let $\varkappa_j = \dim_K \kappa(p_j)$, let $\mathbb{Y}' \subseteq \mathbb{W}$ be a subscheme containing $\mathbb{Y}$ as a maximal p_j-subscheme, and let $\mathbb{X}'$ be the residual scheme of $\mathbb{Y}'$ in $\mathbb{W}$. Proposition 2.9 shows that $\mathbb{X}'$ is exactly a maximal p_j-subscheme of $\mathbb{X}$ of degree $\deg(\mathbb{X}') = \deg(\mathbb{X}) - \varkappa_j$. By Proposition 2.6, we observe that $r_{\mathbb{X}'} + \alpha_{\mathbb{Y}'/\mathbb{W}} = r_{\mathbb{W}} = r_{\mathbb{X}} + \alpha_{\mathbb{Y}/\mathbb{W}}$, and $\operatorname{HF}_{I_{\mathbb{Y}/\mathbb{W}}}(i) = \deg(\mathbb{X}) - \operatorname{HF}_{\mathbb{X}}(r_{\mathbb{W}} - i - 1)$ and $\operatorname{HF}_{I_{\mathbb{Y}'/\mathbb{W}}}(i) = \deg(\mathbb{X}') - \operatorname{HF}_{\mathbb{X}'}(r_{\mathbb{W}} - i - 1)$ for all $i \in \mathbb{Z}$. So, for all $i \in \mathbb{Z}$, we have $\operatorname{HF}_{I_{\mathbb{Y}/\mathbb{Y}'}}(i) = \varkappa_j - \operatorname{HF}_{I_{\mathbb{X}'/\mathbb{X}}}(r_{\mathbb{W}} - i - 1)$. According to Proposition 3.2, the Hilbert function of $I_{\mathbb{X}'/\mathbb{X}}$ satisfies $\operatorname{HF}_{I_{\mathbb{X}'/\mathbb{X}}}(d) < \varkappa_j$. Consequently, we get $\operatorname{HF}_{I_{\mathbb{Y}/\mathbb{Y}'}}(r_{\mathbb{W}} - d - 1) = \varkappa_j - \operatorname{HF}_{I_{\mathbb{X}'/\mathbb{X}}}(d) > 0$, as wanted.

Now we prove the implication (b) ⇒ (c). Clearly, $I_{\mathbb{X}} \subseteq I_{\mathbb{W}} : (I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1}$. Suppose for a contradiction that $F \in I_{\mathbb{W}} : (I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1}$ and $F \notin I_{\mathbb{X}}$. There is a point $p_j \in \operatorname{Supp}(\mathbb{X})$ such that $F_{p_j} \neq 0$. By [20, Lemma 4.5.9(a)] there is $a_j \in \mathcal{O}_{\mathbb{X},p_j}$ such that $a_j \cdot F_{p_j}$ is a socle element of $\mathcal{O}_{\mathbb{X},p_j}$. This socle element defines a maximal p_j-subscheme $\mathbb{X}'$ of $\mathbb{X}$ by [16, Proposition 4.2]. Then the residual scheme $\mathbb{Y}'$ of $\mathbb{X}'$ in $\mathbb{W}$ satisfies $\mathrm{HF}_{I_{\mathbb{Y}/\mathbb{Y}'}}(r_{\mathbb{W}}-d-1)>0$ by Proposition 2.9 and (b). On the other hand, letting $G\in(I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1}$, then $FG\in I_{\mathbb{W}}$ and $G\cdot I_{\mathbb{X}}\subseteq I_{\mathbb{W}}$. Since $I_{\mathbb{W}}$ is saturated, we have $G\cdot\langle F,I_{\mathbb{X}}\rangle^{\mathrm{sat}}\subseteq I_{\mathbb{W}}$. So, $G\cdot I_{\mathbb{X}'}\subseteq I_{\mathbb{W}}$ or $G\in(I_{\mathbb{Y}'})_{r_{\mathbb{W}}-d-1}$, as $I_{\mathbb{X}'}\subseteq\langle F,I_{\mathbb{X}}\rangle^{\mathrm{sat}}$. Hence we get $\mathrm{HF}_{I_{\mathbb{Y}/\mathbb{Y}'}}(r_{\mathbb{W}}-d-1)=0$, a contradiction.

Moreover, the implication (c) $\Rightarrow$ (d) is clear. Next, we prove the implication (d) $\Rightarrow$ (e). Let $\mathbb{X}'\subseteq\mathbb{X}$ be a maximal p_j-subscheme with set of minimal separators $\{F^*_{j1},\dots,F^*_{j\varkappa_j}\}$. If there exists some index $k_j\in\{1,\dots,\varkappa_j\}$ such that $\deg(F^*_{jk_j})\le d$, then $G_{jk_j}=X_0^{d-\deg(F^*_{jk_j})}F^*_{jk_j}\notin(I_{\mathbb{X}})_d$, and so claim (d) implies $G_{jk_j}\notin(I_{\mathbb{W}})_{r_{\mathbb{W}}-1}:(I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1}$. Let $H_j\in(I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1}\setminus\{0\}$ be such that $G_{jk_j}H_j\notin(I_{\mathbb{W}})_{r_{\mathbb{W}}-1}$. Since X_0 is a non-zerodivisor for $R_{\mathbb{W}}$, we have $F_{jk_j}H_j=X_0^{r_{\mathbb{X}}-d}G_{jk_j}H_j\notin I_{\mathbb{W}}$. In case $\deg(F^*_{jk_j})>d$ for all $k_j=1,\dots,\varkappa_j$, we see that $F_{jk_j}\notin\langle X_0^{r_{\mathbb{X}}-d},(I_{\mathbb{X}})_{r_{\mathbb{X}}}\rangle_P$. Let $1\le\delta\le r_{\mathbb{X}}-d$ be the smallest number such that $\langle F_{j1},\dots,F_{jk_j}\rangle_K\nsubseteq\langle X_0^{\delta},(I_{\mathbb{X}})_{r_{\mathbb{X}}}\rangle_P$. Write $F_{jk_j}=F'_{jk_j}+X_0^{\delta-1}G_{jk_j}$ with $F'_{jk_j}\in(I_{\mathbb{X}})_{r_{\mathbb{X}}}$ and $G_{jk_j}\in P_{r_{\mathbb{X}}-\delta+1}$. Then Lemma 3.3 yields that $G_{jk_j}\notin(I_{\mathbb{W}})_{r_{\mathbb{W}}}:(I_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+\delta-1}$ for some $k_j\in\{1,\dots,\varkappa_j\}$. So, there is an element $\widetilde{H}_j\in(I_{\mathbb{Y}})_{\alpha_{\mathbb{Y}/\mathbb{W}}+\delta-1}$ such that $G_{jk_j}\widetilde{H}_j\notin(I_{\mathbb{W}})_{r_{\mathbb{W}}}$. Set $H_j=X_0^{r_{\mathbb{X}}-d-\delta}\widetilde{H}_j\in(I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1}$. Since $F'_{jk_j}\widetilde{H}_j\in I_{\mathbb{W}}$, we get $F_{jk_j}H_j\notin I_{\mathbb{W}}$.

Finally, we prove the implication (e) $\Rightarrow$ (a). For a contradiction, assume that $\mathbb{X}$ does not have CBP(d), and let $\mathbb{X}'\subseteq\mathbb{X}$ be a maximal p_j-subscheme such that its minimal separators satisfies $\deg(F^*_{jk_j})\le d$ for all $k_j=1,\dots,\varkappa_j$. Set $G_{jk_j}=X_0^{d-\deg(F^*_{jk_j})}F^*_{jk_j}$ for $k_j=1,\dots,\varkappa_j$. By (e) there exists $H_j\in(I_{\mathbb{Y}})_{r_{\mathbb{W}}-d-1}$ and some $k_j\in\{1,\dots,\varkappa_j\}$ such that $G_{jk_j}H_j\notin(I_{\mathbb{W}})_{r_{\mathbb{W}}-1}$. Without loss of generality, we assume that $G_{j1}H_j\notin(I_{\mathbb{W}})_{r_{\mathbb{W}}-1}$. Notice that, as sets, $\mathrm{Supp}(\mathbb{W})=\mathrm{Supp}(\mathbb{X})\cup\mathrm{Supp}(\mathbb{Y})$. In $\mathcal{O}_{\mathbb{W},p_j}$, we have $(G_{j1}H_j)_{p_j}\ne0$ and $(G_{j1}H_j)_p=0$ for any $p\in\mathrm{Supp}(\mathbb{W})\setminus\{p_j\}$. Also, by writing $\mathcal{O}_{\mathbb{X},p_j}=\mathcal{O}_{\mathbb{W},p_j}/\mathfrak{q}_j$ for some ideal $\mathfrak{q}_j$ of $\mathcal{O}_{\mathbb{W},p_j}$, we have $\mathfrak{q}_j\cdot(H_j)_{p_j}=\langle0\rangle$ in $\mathcal{O}_{\mathbb{W},p}$ and $(G_{j1})_{p_j}\in\mathcal{O}_{\mathbb{X},p_j}$ is a socle element with $(G_{jk_j})_{p_j}\in\langle(G_{j1})_{p_j}\rangle_{\mathcal{O}_{\mathbb{X},p_j}}\setminus\{0\}$ for all $k_j=1,\dots,\varkappa_j$. In particular, by the definition of minimal separators, the set $\{(G_{j1})_{p_j},\dots,(G_{j\varkappa_j})_{p_j}\}$ is K-linearly independent. Thus Lemma 3.4 yields that $(G_{j1}H_j)_{p_j}$ is a socle element of $\mathcal{O}_{\mathbb{W},p_j}$ and $\{(G_{j1}H_j)_{p_j},\dots,(G_{j\varkappa_j}H_j)_{p_j}\}\subseteq\mathcal{O}_{\mathbb{W},p_j}$ is K-linearly independent. Set $J:=\langle G_{jk}H_j+I_{\mathbb{W}}\mid 1\le k\le\varkappa_j\rangle_{R_{\mathbb{W}}}$. Obviously, we have

$$\dim_K J_{r_{\mathbb{W}}-1+i}\ge\dim_K\langle(G_{j1}H_j)_{p_j},\dots,(G_{j\varkappa_j}H_j)_{p_j}\rangle_K=\varkappa_j$$

for all $i\ge0$. Furthermore, using [15, Lemma 2.8] we write

$$X_iG_{jl}+I_{\mathbb{X}}=\sum_{k_j=1}^{\varkappa_j}c_{jk_jl}X_0G_{jk_j}+I_{\mathbb{X}}$$

for some $c_{j1l},\dots,c_{j\varkappa_jl}\in K$, where $0\le i\le n$ and $1\le l\le\varkappa_j$. Then we get

$$X_iG_{jl}H_j + I_{\mathbb{W}} = \sum_{k_j=1}^{\varkappa_j} c_{jk_jl}X_0G_{jk_j}H_j + I_{\mathbb{W}},$$

and subsequently $\dim_K J_{r_{\mathbb{W}}-1+i} = \varkappa_j$ for all $i \geq 0$. Consequently, the homogeneous ideal J defines a maximal p_j-subscheme $\mathbb{W}' \subseteq \mathbb{W}$ such that $\dim_K(I_{\mathbb{W}'/\mathbb{W}})_{r_{\mathbb{W}}-1} = \varkappa_j$. Therefore Proposition 3.2 implies that $\mathbb{W}$ is not a Cayley-Bacharach scheme. But $\mathbb{W}$ is arithmetically Gorenstein, and so it is a Cayley-Bacharach scheme by [16, Proposition 4.8], and this is a contradiction. $\square$

Let us apply Theorem 3.5 to a concrete case.

Example 3.6. Let K be a field with $\mathrm{char}(K) \neq 2, 3$, and let $\mathbb{W} \subseteq \mathbb{P}^2_K$ be the 0-dimensional complete intersection defined by $I_{\mathbb{W}} = \langle F, G\rangle$, where $F = X_1(X_1 - 2X_0)(X_1 + 2X_0)$ and $G = (X_2 - X_0)(X_1^2 + X_2^2 - 4X_0^2)$. Then $\mathbb{W}$ has degree 9 and the support of $\mathbb{W}$ is $\mathrm{Supp}(\mathbb{W}) = \{p_1, \dots, p_7\}$ with $p_1 = (1 : 0 : 1)$, $p_2 = (1 : 0 : 2)$, $p_3 = (1 : 0 : -2)$, $p_4 = (1 : 2 : 1)$, $p_5 = (1 : 2 : 0)$, $p_6 = (1 : -2 : 1)$, and $p_7 = (1 : -2 : 0)$. A homogeneous primary decomposition of $I_{\mathbb{W}}$ is $I_{\mathbb{W}} = I_1 \cap \cdots \cap I_7$, where I_i is the homogeneous prime ideal corresponding to p_i for $i \neq 5, 7$, $I_5 = \langle X_1 - 2X_0, X_2^2\rangle$, and $I_7 = \langle X_1 + 2X_0, X_2^2\rangle$. So, the scheme $\mathbb{W}$ is arithmetically Gorenstein, but not reduced at p_5 and p_7.

Now we consider the 0-dimensional subscheme $\mathbb{X}$ of $\mathbb{W}$ defined by the ideal $I_{\mathbb{X}} = I_1 \cap I_3 \cap I_4 \cap I_5 \subseteq P$. Then $\deg(\mathbb{X}) = 5$ and $\mathbb{X}$ is not reduced. The residual scheme of $\mathbb{X}$ in $\mathbb{W}$ is denoted by $\mathbb{Y}$. It is easy to see that $\mathbb{X}$ and $\mathbb{Y}$ are geometrically G-linked. We have $r_{\mathbb{W}} = 4$ and $r_{\mathbb{X}} = \alpha_{\mathbb{X}/\mathbb{W}} = r_{\mathbb{Y}} = \alpha_{\mathbb{Y}/\mathbb{W}} = 2$. In this case there is a homogeneous polynomial $H \in (I_{\mathbb{Y}})_2$ such that its image in $R_{\mathbb{X}}$ is a non-zerodivisor, for instance, $H = X_0^2 + X_0X_1 + \frac{1}{4}X_1^2 - \frac{1}{2}X_0X_2 - \frac{1}{4}X_1X_2$. This polynomial satisfies the condition (e) in Theorem 3.5. Therefore $\mathbb{X}$ is a Cayley-Bacharach scheme.

Example 3.6 shows that, setting $I_{\mathbb{Y},\mathbb{X}} := (I_{\mathbb{Y}}+I_{\mathbb{X}})/I_{\mathbb{X}}$, the condition $\mathrm{Ann}_{R_{\mathbb{X}}}((I_{\mathbb{Y},\mathbb{X}})_{r_{\mathbb{W}}-d-1}) = \langle 0\rangle$ is a sufficient condition for $\mathbb{X}$ having $\mathrm{CBP}(d)$ in this case. In general case, this is also true. Indeed, if $\mathrm{Ann}_{R_{\mathbb{X}}}((I_{\mathbb{Y},\mathbb{X}})_{r_{\mathbb{W}}-d-1}) = \langle 0\rangle$ then for each maximal p_j-subscheme $\mathbb{X}' \subseteq \mathbb{X}$ with standard set of separators $\{F_{j1}, \dots, F_{j\varkappa_j}\}$ there is a non-zero homogeneous element $H_j \in (I_{\mathbb{Y},\mathbb{X}})_{r_{\mathbb{W}}-d-1}$ such that $(H_j)_{p_j} \in \mathcal{O}_{\mathbb{X},p_j} \setminus \mathfrak{m}_{\mathbb{X},p_j}$, and so $(H_j)_{p_j} \notin \mathfrak{m}_{\mathbb{W},p_j}$ and $(H_jF_{jk_j})_{p_j} \neq 0$ in $\mathcal{O}_{\mathbb{W},p_j}$. This means that $H_jF_{jk_j} \notin I_{\mathbb{W}}$. Subsequently, the condition (e) of Theorem 3.5 is satisfied, and hence $\mathbb{X}$ has $\mathrm{CBP}(d)$.

However, the above condition is not a necessary condition for $\mathbb{X}$ having $\mathrm{CBP}(d)$, as our next example shows.

Example 3.7. Let K be a field with $\mathrm{char}(K) \neq 2, 3$, let $\mathbb{W} \subseteq \mathbb{P}^2_K$ be the 0-dimensional complete intersection given in Example 3.6, and let $\mathbb{X}'$ be the set of points in $\mathbb{W}$ with its homogeneous vanishing ideal $I_{\mathbb{X}'} = I_1 \cap I_3 \cap I_4 \cap I_5'$, where $I_5' = \langle X_1 - 2X_0, X_2\rangle$

is the homogeneous prime ideal corresponding to p_5. Then the residual scheme $\mathbb{Y}'$ of $\mathbb{X}'$ in $\mathbb{W}$ has the homogeneous vanishing ideal $I_{\mathbb{Y}'} = I_2 \cap I_5' \cap I_6 \cap I_7$. It is clear that $r_{\mathbb{X}'} = \alpha_{\mathbb{X}'/\mathbb{W}} = r_{\mathbb{Y}'} = \alpha_{\mathbb{Y}'/\mathbb{W}} = 2$ and

$$I_{\mathbb{Y}'} = \langle X_0^2 - \tfrac{1}{4}X_1^2 - \tfrac{1}{2}X_0X_2 - \tfrac{1}{4}X_1X_2, X_0X_1X_2 + \tfrac{1}{2}X_1^2X_2, X_0X_2^2 + \tfrac{1}{4}X_1X_2^2 - \tfrac{1}{2}X_2^3\rangle.$$

In this case it is not difficult to verify that the scheme $\mathbb{X}'$ is a complete intersection, and hence it is a Cayley-Bacharach scheme. However, there is no element H in $(I_{\mathbb{Y}'})_2$ such that $H_{p_5} \neq 0$ in $\mathcal{O}_{\mathbb{X}',p_5}$. Hence the condition $\mathrm{Ann}_{R_{\mathbb{X}'}}((I_{\mathbb{Y}',\mathbb{X}'})_{r_\mathbb{W}-r_{\mathbb{X}'}}) = \langle 0\rangle$ is not satisfied, even when $\mathbb{X}'$ is a Cayley-Bacharach scheme. Moreover, we see that the element $F_5 = X_1^2 - 2X_1X_2$ is a minimal separator of $\mathbb{X}' \setminus \{p_5\}$ in $\mathbb{X}'$ and $(F_5H_5)_{p_5}$ is a socle element of $\mathcal{O}_{\mathbb{W},p_5}$, where $H_5 = X_0^2 - \frac{1}{4}X_1^2 - \frac{1}{2}X_0X_2 - \frac{1}{4}X_1X_2 \in (I_{\mathbb{Y}'})_2$.

It is interesting to examine the natural question whether the condition that $\mathbb{X}$ has $\mathrm{CBP}(d)$ is equivalent to $\mathrm{Ann}_{R_\mathbb{X}}((I_{\mathbb{Y},\mathbb{X}})_{r_\mathbb{W}-d-1}) = \langle 0\rangle$. When the schemes $\mathbb{W}$, $\mathbb{X}$ and $\mathbb{Y}$ are finite sets of K-rational points in $\mathbb{P}^n_K$ and $\mathbb{W}$ is a complete intersection, this question has an affirmative answer as was shown in [8, Theorem 4.6]. In our more general setting, this result can be generalized as follows.

Theorem 3.8. *Let $\mathbb{X}$ and $\mathbb{Y}$ be geometrically G-linked by a 0-dimensional arithmetically Gorenstein scheme $\mathbb{W}$, and let $I_{\mathbb{Y},\mathbb{X}} = (I_\mathbb{Y} + I_\mathbb{X})/I_\mathbb{X}$. Then the scheme $\mathbb{X}$ has* $\mathrm{CBP}(d)$ *if and only if we have* $\mathrm{Ann}_{R_\mathbb{X}}((I_{\mathbb{Y},\mathbb{X}})_{r_\mathbb{W}-d-1}) = \langle 0\rangle$.

Proof. From the argument before Example 3.7, it suffices to show that $\mathrm{Ann}_{R_\mathbb{X}}((I_{\mathbb{Y},\mathbb{X}})_{r_\mathbb{W}-d-1}) = \langle 0\rangle$ if $\mathbb{X}$ has $\mathrm{CBP}(d)$. To this end, let $\mathbb{X}' \subseteq \mathbb{X}$ be a maximal p_j-subscheme with standard set of separators $\{F_{j1},\dots,F_{j\varkappa_j}\} \subseteq P_{r_\mathbb{X}}$. Since $\mathbb{X}$ has $\mathrm{CBP}(d)$, Theorem 3.5 yields that there is an element $H_j \in (I_\mathbb{Y})_{r_\mathbb{W}-d-1}$ such that $H_j \cdot \langle F_{j1},\dots,F_{j\varkappa_j}\rangle_K \nsubseteq I_\mathbb{W}$. Without loss of generality, we assume that $H_jF_{j1} \notin I_\mathbb{W}$. Since $\mathbb{X}$ and $\mathbb{Y}$ are geometrically G-linked, $(F_{j1})_{p_j}$ is a socle element in $\mathcal{O}_{\mathbb{W},p_j} = \mathcal{O}_{\mathbb{X},p_j}$. Since $(H_jF_{j1})_p = 0$ in $\mathcal{O}_{\mathbb{W},p}$ for every $p \in \mathrm{Supp}(\mathbb{W}) \setminus \{p_j\}$ and $(H_jF_{j1})_{p_j} \neq 0$, we get $(H_j)_{p_j} \notin \mathfrak{m}_{\mathbb{X},p_j}$. Consequently, for each point p_j of $\mathrm{Supp}(\mathbb{X})$, we can find an element $H_j \in (I_\mathbb{Y})_{r_\mathbb{W}-d-1}$ such that $(H_j)_{p_j}$ is a unit of $\mathcal{O}_{\mathbb{X},p_j}$. By [14, Lemma 1.1], this condition is exactly the right condition to have $\mathrm{Ann}_{R_\mathbb{X}}((I_{\mathbb{Y},\mathbb{X}})_{r_\mathbb{W}-d-1}) = \langle 0\rangle$. □

Remark 3.9. Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme.

(a) If $\mathbb{X}$ is reduced and has K-rational support, then there is a complete intersection consisting of distinct K-rational points $\mathbb{W}$ containing $\mathbb{X}$ such that $\mathbb{X}$ and its residue scheme by $\mathbb{W}$ are geometrically G-linked (see, e.g., [8, Remark 4.11]).

(b) If $\mathcal{O}_{\mathbb{X},p_j}$ is not a Gorenstein local ring for some point $p_j \in \mathrm{Supp}(\mathbb{X})$, then there is no 0-dimensional arithmetically Gorenstein scheme $\mathbb{W} \subseteq \mathbb{P}^n_K$ containing $\mathbb{X}$ such that $\mathbb{X}$ and its residual scheme in $\mathbb{W}$ are geometrically G-linked.

We end this section with the following immediate consequence of Theorem 3.5. This result allows us to check whether $\mathbb{X}$ has CBP(d) by using a truncated Gröbner basis calculation (cf. [19, Section 4.5]). For the case of sets of distinct K-rational points and $d = r_{\mathbb{X}} - 1$ see also [8, Corollary 4.10].

Corollary 3.10. *In the setting of Theorem 3.5, the scheme $\mathbb{X}$ has CBP(d) if and only if $\mathrm{HF}_{P/(I_{\mathbb{W}}:(I_{\mathbb{W}}:I_{\mathbb{X}})_{r_{\mathbb{W}}-d-1})}(d) = \mathrm{HF}_{\mathbb{X}}(d)$.*

4. A bound for the Hilbert function of the Dedekind different

In this section, we let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme and we let $0 \le d < r_{\mathbb{X}}$. The aim of this section is to characterize the Cayley-Bacharach property using the canonical module of the homogeneous coordinate ring $R_{\mathbb{X}}$, and apply these results to bound the Hilbert function and determine the regularity index of the Dedekind different of $\mathbb{X}$ under some additional hypotheses. As an application, we get a new characterization of 0-dimensional arithmetically Gorenstein schemes in terms of their Dedekind differents.

The following two lemmas give us some more information about the canonical module $\omega_{R_{\mathbb{X}}}$.

Lemma 4.1. *For every homogeneous element $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ its restriction $\overline{\varphi} = \varphi|_{(R_{\mathbb{X}})_{d+1}}\colon (R_{\mathbb{X}})_{d+1} \to K$ is a K-linear map such that $\overline{\varphi}(x_0(R_{\mathbb{X}})_d) = \langle 0\rangle$. Conversely, if $\overline{\varphi}\colon (R_{\mathbb{X}})_{d+1} \to K$ is a K-linear map such that $\overline{\varphi}(x_0(R_{\mathbb{X}})_d) = \langle 0\rangle$, then there exists a homogeneous element $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ such that $\varphi|_{(R_{\mathbb{X}})_{d+1}} = \overline{\varphi}$.*

Proof. Clearly, for every homogeneous element $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ its restriction $\overline{\varphi} = \varphi|_{(R_{\mathbb{X}})_{d+1}}$ is a K-linear map. Also, we have

$$\overline{\varphi}(x_0(R_{\mathbb{X}})_d) = \varphi(x_0(R_{\mathbb{X}})_d) = x_0\varphi((R_{\mathbb{X}})_d) \subseteq x_0(K[x_0])_{-1} = \langle 0\rangle.$$

Now let $\overline{\varphi}\colon (R_{\mathbb{X}})_{d+1} \to K$ is a K-linear map such that $\overline{\varphi}(x_0(R_{\mathbb{X}})_d) = \langle 0\rangle$. Let $h_i = \mathrm{HF}_{\mathbb{X}}(i) - \mathrm{HF}_{\mathbb{X}}(i-1)$ for $i \in \mathbb{N}$. Note that $(R_{\mathbb{X}})_i = x_0^{i-r_{\mathbb{X}}}(R_{\mathbb{X}})_{r_{\mathbb{X}}}$ and $h_i = 0$ for all $i > r_{\mathbb{X}}$. To define an element $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ with the desired properties, we start taking a K-basis $g_1, \dots, g_{\sum_{0\le k\le d+1} h_k}$ of $(R_{\mathbb{X}})_{d+1}$. For $i = d+2, \dots, r_{\mathbb{X}}$, we choose $g_{\sum_{0\le k<i} h_k+1}, \dots, g_{\sum_{0\le k\le i} h_k}$ such that the set

$$\Big\{x_0^{i-d-1}g_1, \dots, x_0^{i-d-1}g_{\sum_{0\le k\le d+1} h_k}, \dots, g_{\sum_{0\le k<i} h_k+1}, \dots, g_{\sum_{0\le k\le i} h_k}\Big\}$$

forms a K-basis of $(R_{\mathbb{X}})_i$. Then we get

$$\begin{aligned}&(R_{\mathbb{X}})_i\\ &= \Big\langle x_0^{i-d-1}g_1, \dots, x_0^{i-d-1}g_{\sum_{0\le k\le d+1} h_k}, \dots, x_0^{i-r_{\mathbb{X}}}g_{\sum_{0\le k<r_{\mathbb{X}}} h_k+1}, \dots, x_0^{i-r_{\mathbb{X}}}g_{\sum_{0\le k\le r_{\mathbb{X}}} h_k}\Big\rangle_K\end{aligned}$$

for all $i \geq r_{\mathbb{X}}$. Let $\varphi\colon R_{\mathbb{X}} \to K[x_0]$ be the homogeneous K-linear map of degree $-d$ defined as: for $f \in R_i$ with $i \leq d$ we let $\varphi(f) = 0$, and for $f \in R_i$ with $i \geq d+1$ we write

$$f = \sum_{1 \leq j \leq \sum_{0 \leq k \leq d+1} h_k} a_j x_0^{i-d-1} g_j + \cdots + \sum_{\sum_{0 \leq k < r_{\mathbb{X}}} h_k + 1 \leq j \leq \sum_{0 \leq k \leq r_{\mathbb{X}}} h_k} a_j x_0^{i-r_{\mathbb{X}}} g_j$$

and let $\varphi(f) = \sum_{1 \leq j \leq \sum_{0 \leq k \leq d+1} h_k} a_j x_0^{i-d-1} \overline{\varphi}(g_j)$. The condition $\overline{\varphi}(x_0(R_{\mathbb{X}})_d) = \langle 0 \rangle$ implies that the map φ is $K[x_0]$-linear. Hence $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ is the desired element that we wanted to construct. $\square$

Lemma 4.2. *The canonical module $\omega_{R_{\mathbb{X}}}$ satisfies $\operatorname{Ann}_{R_{\mathbb{X}}}((\omega_{R_{\mathbb{X}}})_{-d}) = \langle 0 \rangle$ if and only if for every $p_j \in \operatorname{Supp}(\mathbb{X})$ and for every maximal p_j-subscheme $\mathbb{X}' \subseteq \mathbb{X}$ there exists a homogeneous element $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ such that $I_{\mathbb{X}'/\mathbb{X}} \cdot \varphi \neq \langle 0 \rangle$.*

Proof. We need only to prove that if for every $p_j \in \operatorname{Supp}(\mathbb{X})$ and for every maximal p_j-subscheme $\mathbb{X}' \subseteq \mathbb{X}$ there exists a homogeneous element $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ such that $I_{\mathbb{X}'/\mathbb{X}} \cdot \varphi \neq \langle 0 \rangle$ then $\operatorname{Ann}_{R_{\mathbb{X}}}((\omega_{R_{\mathbb{X}}})_{-d}) = \langle 0 \rangle$. Suppose for a contradiction that $f \cdot (\omega_{R_{\mathbb{X}}})_{-d} = \langle 0 \rangle$ for some $f \in (R_{\mathbb{X}})_i \setminus \{0\}$ with $i \geq 0$. Since $f \neq 0$, we may assume the germ $f_{p_j} \neq 0$ for some $j \in \{1, \ldots, s\}$. In the local ring $\mathcal{O}_{\mathbb{X},p_j}$ we find an element $a \in \mathcal{O}_{\mathbb{X},p_j}$ such that $s_j = a f_{p_j}$ is a socle element of $\mathcal{O}_{\mathbb{X},p_j}$ (cf. [20, Lemma 4.5.9(a)]). Now let $g = \widetilde{\imath}^{-1}((0, \ldots, 0, s_j T_j^{r_{\mathbb{X}}}, 0, \ldots, 0))$ and $h = \widetilde{\imath}^{-1}((0, \ldots, 0, a T_j^{r_{\mathbb{X}}}, 0, \ldots, 0))$. Then $g, h \in (R_{\mathbb{X}})_{r_{\mathbb{X}}}$ satisfies $x_0^i g = fh$. Also, the ideal $\langle g \rangle$ defines a maximal p_j-subscheme $\mathbb{X}'$ of $\mathbb{X}$, that is, we have $I_{\mathbb{X}'/\mathbb{X}} = \langle g \rangle^{\mathrm{sat}}$. Thus there is $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ such that $\langle g \rangle^{\mathrm{sat}} \cdot \varphi \neq \langle 0 \rangle$, in particularly, $g \cdot \varphi \neq 0$. It follows that $g \cdot \varphi(\widetilde{g}) \neq 0$ for some non-zero homogeneous element $\widetilde{g} \in R_{\mathbb{X}}$. Hence we get $0 = (f \cdot \varphi)(h\widetilde{g}) = (fh \cdot \varphi)(\widetilde{g}) = (x_0^i g \cdot \varphi)(\widetilde{g}) = (g \cdot \varphi)(x_0^i \widetilde{g}) = x_0^i (g \cdot \varphi)(\widetilde{g}) \neq 0$, a contradiction. $\square$

Using the above properties we prove the following characterization of the Cayley-Bacharach property in terms of the canonical module.

Proposition 4.3. *Let $\mathbb{X} \subseteq \mathbb{P}^n_K$ be a 0-dimensional scheme, and let $0 \leq d < r_{\mathbb{X}}$. Then the following conditions are equivalent.*

(a) *The scheme $\mathbb{X}$ has* $\operatorname{CBP}(d)$.

(b) *We have $\operatorname{Ann}_{R_{\mathbb{X}}}((\omega_{R_{\mathbb{X}}})_{-d}) = \langle 0 \rangle$.*

Proof. Suppose that $\mathbb{X}$ has $\operatorname{CBP}(d)$. Let $\mathbb{X}' \subseteq \mathbb{X}$ be a maximal p_j-subscheme with set of minimal separators $\{f^*_{j1}, \ldots, f^*_{j\varkappa_j}\}$. By Proposition 3.2, there exists an index $k \in \{1, \ldots, \varkappa_j\}$ such that $\rho = \deg(f^*_{jk}) \geq d+1$ and $f^*_{jk} \notin x_0(R_{\mathbb{X}})_{\rho-1}$. Without loss of generality, we assume that $k = 1$. So, we can define a K-linear map $\overline{\varphi}_j\colon (R_{\mathbb{X}})_\rho \to K$ such that $\overline{\varphi}_j(x_0(R_{\mathbb{X}})_{\rho-1}) = \langle 0 \rangle$ and $\overline{\varphi}_j(f^*_{j1}) \neq 0$. Using Lemma 4.1 we lift this map to obtain a homogeneous element $\varphi_j \in (\omega_{R_{\mathbb{X}}})_{-\rho+1}$ such that $\varphi_j(f_{j1}^*) \neq 0$. Since x_0 is a non-zerodivisor of $R_{\mathbb{X}}$, it follows that $x_0^{\rho-d-1}\varphi_j(f_{j1}^*) \neq 0$. Especially, we have $x_0^{\rho-d-1} \cdot \varphi_j \in (\omega_{R_{\mathbb{X}}})_{-d}$ and $I_{\mathbb{X}'/\mathbb{X}} \cdot (x_0^{\rho-d-1} \cdot \varphi_j) \neq \langle 0 \rangle$. Hence Lemma 4.2 yields the condition $\operatorname{Ann}_{R_{\mathbb{X}}}((\omega_{R_{\mathbb{X}}})_{-d}) = \langle 0 \rangle$.

Conversely, assume for a contradiction that $\mathbb{X}$ does not have CBP(d). There is a maximal p_j-subscheme $\mathbb{X}' \subseteq \mathbb{X}$ such that its set of minimal separators $\{f_{j1}^*, \dots, f_{j\varkappa_j}^*\}$ satisfies $\deg(f_{jk}^*) \leq d$ for all $k = 1, \dots, \varkappa_j$. By Remark 2.3(a)–(c), we may assume that, for $i \geq 0$, the set $\{x_0^{i-\deg(f_{jk_j}^*)} f_{jk_j}^* \mid \deg(f_{jk_j}^*) \leq i\}$ is a K-basis of $(I_{\mathbb{X}'/\mathbb{X}})_i$. In this case for every $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ we have $\varphi(f_{jk}^*) = 0$ for all $k = 1, \dots, \varkappa_j$. We shall show that $f_{jk}^* \cdot \varphi = 0$ for all $k = 1, \dots, \varkappa_j$. Let $i \geq 0$ and let $h \in R_i \setminus \{0\}$ be a homogeneous element. If $hf_{jk}^* = 0$ then $(f_{jk}^* \cdot \varphi)(h) = \varphi(hf_{jk}^*) = 0$. Suppose that $hf_{jk}^* \neq 0$. Since $hf_{jk}^* \in I_{\mathbb{X}'/\mathbb{X}}$, this allows us to write $hf_{jk}^* = \sum_{l=1}^{\varkappa_j} c_{jl} x_0^{i+\deg(f_{jk}^*)-\deg(f_{jl}^*)} f_{jl}^*$ for some $c_{j1}, \dots, c_{j\varkappa_j} \in K$. This implies $(f_{jk}^* \cdot \varphi)(h) = \varphi(hf_{jk}^*) = \sum_{l=1}^{\varkappa_j} c_{jl} x_0^{i+\deg(f_{jk}^*)-\deg(f_{jl}^*)} \varphi(f_{jl}^*) = 0$. Hence we have shown $f_{jk}^* \cdot \varphi = 0$ for all $k = 1, \dots, \varkappa_j$. In addition, we have $I_{\mathbb{X}'/\mathbb{X}} = \langle f_{j1}^*, \dots, f_{j\varkappa_j}^* \rangle$. It follows that $I_{\mathbb{X}'/\mathbb{X}} \cdot \varphi = \langle 0 \rangle$ for any homogeneous element $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$. Therefore we get $\operatorname{Ann}_{R_{\mathbb{X}}}((\omega_{R_{\mathbb{X}}})_{-d}) \neq \langle 0 \rangle$, a contradiction. □

As a consequence of the proposition, we get the following property. Here we recall that a 0-dimensional scheme $\mathbb{X}$ is called *locally Gorenstein* if the local ring $\mathcal{O}_{\mathbb{X},p_j}$ is a Gorenstein ring for every point $p_j \in \operatorname{Supp}(\mathbb{X})$.

Corollary 4.4. *Let K be an infinite field, let $\mathbb{X} \subseteq \mathbb{P}_K^n$ be a 0-dimensional locally Gorenstein scheme, and let $0 \leq d < r_{\mathbb{X}}$. Then $\mathbb{X}$ has* CBP(d) *if and only if there exists an element $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ such that $\operatorname{Ann}_{R_{\mathbb{X}}}(\varphi) = \langle 0 \rangle$.*

Proof. Since $\mathbb{X}$ is locally Gorenstein, there is for each point $p_j \in \operatorname{Supp}(\mathbb{X})$ a uniquely maximal p_j-subscheme $\mathbb{X}_j'$ of $\mathbb{X}$. So, the condition (b) of Proposition 4.3 is equivalent to the condition that for each $j \in \{1, \dots, s\}$ there exists $\varphi_j \in (\omega_{R_{\mathbb{X}}})_{-d}$ such that $I_{\mathbb{X}_j'/\mathbb{X}} \cdot \varphi_j \neq \langle 0 \rangle$. This is in turn equivalent to that there exists $\varphi \in (\omega_{R_{\mathbb{X}}})_{-d}$ such that $I_{\mathbb{X}_j'/\mathbb{X}} \cdot \varphi \neq \langle 0 \rangle$ for $j = 1, \dots, s$, since the base field K is infinite, and this condition is exactly the right condition to make $\operatorname{Ann}_{R_{\mathbb{X}}}(\varphi) = \langle 0 \rangle$. □

Remark 4.5. This corollary is a generalization of a result for the case $d = r_{\mathbb{X}} - 1$ found in [16, Proposition 4.12]. Moreover, the hypothesis in the corollary that K is infinite is necessary (cf. [16, Example 4.14]).

Now let us apply the above results to look at the Hilbert function of the Dedekind different of $\mathbb{X}$. For this purpose, we assume, in what follows, that $\mathbb{X}$ is locally Gorenstein, and we let $L_0 = K[x_0, x_0^{-1}]$. The homogeneous ring of quotients of $R_{\mathbb{X}}$ is $Q^h(R_{\mathbb{X}}) \cong \prod_{j=1}^s \mathcal{O}_{\mathbb{X},p_j}[T_j, T_j^{-1}]$. According to [16, Proposition 3.3], the graded algebra $Q^h(R_{\mathbb{X}})/L_0$ has a homogeneous trace map σ of degree zero, i.e., $\sigma \in (\operatorname{Hom}_{L_0}(Q^h(R_{\mathbb{X}}), L_0))_0$ satisfies

$\operatorname{Hom}_{L_0}(Q^h(R_{\mathbb{X}}), L_0) = Q^h(R_{\mathbb{X}}) \cdot \sigma$. Thus there is an injective homomorphism of graded $R_{\mathbb{X}}$-modules

$$\begin{aligned} \Phi\colon \omega_{R_{\mathbb{X}}}(1) &\hookrightarrow \operatorname{Hom}_{L_0}(Q^h(R_{\mathbb{X}}), L_0) = Q^h(R_{\mathbb{X}}) \cdot \sigma \xrightarrow{\sim} Q^h(R_{\mathbb{X}}) \\ \varphi &\mapsto \varphi \otimes \mathrm{id}_{L_0} \end{aligned}$$

The image of Φ is a homogeneous fractional $R_{\mathbb{X}}$-ideal $\mathfrak{C}_{\mathbb{X}}^{\sigma}$ of $Q^h(R_{\mathbb{X}})$. It is also a finitely generated graded $R_{\mathbb{X}}$-module and

$$\mathrm{HF}_{\mathfrak{C}_{\mathbb{X}}^{\sigma}}(i) = \deg(\mathbb{X}) - \mathrm{HF}_{\mathbb{X}}(-i-1) \quad \text{for all } i \in \mathbb{Z}.$$

Definition 4.6. The R-module $\mathfrak{C}_{\mathbb{X}}^{\sigma}$ is called the *Dedekind complementary module* of $\mathbb{X}$ (or of $R_{\mathbb{X}}/K[x_0]$) with respect to σ. Its inverse,

$$\delta_{\mathbb{X}}^{\sigma} = (\mathfrak{C}_{\mathbb{X}}^{\sigma})^{-1} = \{f \in Q^h(R_{\mathbb{X}}) \mid f \cdot \mathfrak{C}_{\mathbb{X}}^{\sigma} \subseteq R_{\mathbb{X}}\},$$

is called the *Dedekind different* of $\mathbb{X}$ (or of $R_{\mathbb{X}}/K[x_0]$) with respect to σ.

The following basic properties of the Dedekind different of $\mathbb{X}$ are shown in [16, Proposition 3.7].

Proposition 4.7. *Let σ be a trace map of $Q^h(R_{\mathbb{X}})/L_0$.*

(a) *The Dedekind different $\delta_{\mathbb{X}}^{\sigma}$ is a homogeneous ideal of $R_{\mathbb{X}}$ and $x_0^{2r_{\mathbb{X}}} \in \delta_{\mathbb{X}}^{\sigma}$.*

(b) *The Hilbert function of $\delta_{\mathbb{X}}^{\sigma}$ satisfies $\mathrm{HF}_{\delta_{\mathbb{X}}^{\sigma}}(i) = 0$ for $i < 0$, $\mathrm{HF}_{\delta_{\mathbb{X}}^{\sigma}}(i) = \deg(\mathbb{X})$ for $i \geq 2r_{\mathbb{X}}$, and $0 \leq \mathrm{HF}_{\delta_{\mathbb{X}}^{\sigma}}(0) \leq \cdots \leq \mathrm{HF}_{\delta_{\mathbb{X}}^{\sigma}}(2r_{\mathbb{X}}) = \deg(\mathbb{X})$. In particular, the regularity index of $\delta_{\mathbb{X}}^{\sigma}$ satisfies $r_{\mathbb{X}} \leq \mathrm{ri}(\delta_{\mathbb{X}}^{\sigma}) \leq 2r_{\mathbb{X}}$.*

When $\mathbb{X}$ has $\mathrm{CBP}(d)$, the Hilbert function of the Dedekind different and its regularity index can be described as follows. We use the notation $\alpha_\delta = \min\{i \in \mathbb{N} \mid (\delta_{\mathbb{X}}^{\sigma})_i \neq \langle 0 \rangle\}$.

Proposition 4.8. *Let K be an infinite field, let σ be a trace map of $Q^h(R_{\mathbb{X}})/L_0$, and suppose that $\mathbb{X}$ has $\mathrm{CBP}(d)$ with $0 \leq d \leq r_{\mathbb{X}} - 1$.*

(a) *We have $d + 1 \leq \alpha_\delta \leq 2r_{\mathbb{X}}$ and $\mathrm{HF}_{\delta_{\mathbb{X}}^{\sigma}}(i) \leq \mathrm{HF}_{\mathbb{X}}(i - d - 1)$ for all $i \in \mathbb{Z}$.*

(b) *Let i_0 be the smallest number such that $\mathrm{HF}_{\delta_{\mathbb{X}}^{\sigma}}(i_0) = \mathrm{HF}_{\mathbb{X}}(i_0 - d - 1) > 0$. Then we have $\mathrm{HF}_{\delta_{\mathbb{X}}^{\sigma}}(i) = \mathrm{HF}_{\mathbb{X}}(i - d - 1)$ for all $i \geq i_0$ and*

$$\mathrm{ri}(\delta_{\mathbb{X}}^{\sigma}) = \max\{i_0, r_{\mathbb{X}} + d + 1\}.$$

Proof. Since $\mathfrak{C}_{\mathbb{X}}^{\sigma} \cong \omega_{R_{\mathbb{X}}}(1)$, Corollary 4.4 implies that there is $g \in (\mathfrak{C}_{\mathbb{X}}^{\sigma})_{-d-1}$ such that $\operatorname{Ann}_{R_{\mathbb{X}}}(g) = \langle 0 \rangle$. Notice that x_0 is a non-zerodivisor of $R_{\mathbb{X}}$. Then we find a non-zerodivisor

$\widetilde{g} \in (R_{\mathbb{X}})_{r_{\mathbb{X}}}$ such that $g = x_0^{-r_{\mathbb{X}}-d-1}\widetilde{g}$ by [16, Proposition 3.7]. Observe that $\widetilde{g} \cdot (\delta^{\sigma}_{\mathbb{X}})_i \subseteq x_0^{r_{\mathbb{X}}+d+1}(R_{\mathbb{X}})_{i-d-1}$. This implies $(\delta^{\sigma}_{\mathbb{X}})_i = \langle 0 \rangle$ for $i \leq d$, and so $d+1 \leq \alpha_{\delta}$. Moreover, for all $i \in \mathbb{Z}$, we have

$$\begin{aligned}\mathrm{HF}_{\delta^{\sigma}_{\mathbb{X}}}(i) &= \dim_K(\delta^{\sigma}_{\mathbb{X}})_i = \dim_K(\widetilde{g} \cdot \delta^{\sigma}_{\mathbb{X}})_i) \\ &\leq \dim_K(x_0^{r_{\mathbb{X}}+d+1}(R_{\mathbb{X}})_{i-d-1}) = \mathrm{HF}_{\mathbb{X}}(i-d-1).\end{aligned}$$

Thus claim (a) is completely proved.

Now we prove claim (b). Clearly, we have $d+1 \leq i_0 \leq 2r_{\mathbb{X}}$. By induction, we only need to show that $\mathrm{HF}_{\delta^{\sigma}_{\mathbb{X}}}(i_0+1) = \mathrm{HF}_{\mathbb{X}}(i_0-d) > 0$. Let $f \in (R_{\mathbb{X}})_{i_0-d} \setminus \{0\}$. There are $g_0, \dots, g_n \in (R_{\mathbb{X}})_{i_0-d-1}$ such that $f = x_0g_0 + x_1g_1 + \cdots + x_ng_n$. By assumption, we have $\widetilde{g} \cdot (\delta^{\sigma}_{\mathbb{X}})_{i_0} = x_0^{r_{\mathbb{X}}+d+1}(R_{\mathbb{X}})_{i_0-d-1}$. This enables us to write $x_0^{r_{\mathbb{X}}+d+1}g_j = \widetilde{g}h_j$ for some $h_j \in (\delta^{\sigma}_{\mathbb{X}})_{i_0}$, where $j \in \{0, \dots, n\}$. Thus we have

$$\begin{aligned}x_0^{r_{\mathbb{X}}+d+1}f &= x_0^{r_{\mathbb{X}}+d+1}(x_0g_0 + x_1g_1 + \cdots + x_ng_n) = x_0\widetilde{g}h_0 + x_1\widetilde{g}h_1 + \cdots + x_n\widetilde{g}h_n \\ &= \widetilde{g}(x_0h_0 + x_1h_1 + \cdots + x_nh_n)\end{aligned}$$

and so $x_0^{r_{\mathbb{X}}+d+1}f \in \widetilde{g} \cdot (\delta^{\sigma}_{\mathbb{X}})_{i_0+1}$. Hence $x_0^{r_{\mathbb{X}}+d+1}(R_{\mathbb{X}})_{i_0-d} = \widetilde{g} \cdot (\delta^{\sigma}_{\mathbb{X}})_{i_0+1}$. In other words, we get $\mathrm{HF}_{\delta^{\sigma}_{\mathbb{X}}}(i_0+1) = \mathrm{HF}_{\mathbb{X}}(i_0-d)$.

Let $k = \max\{i_0, r_{\mathbb{X}}+d+1\}$. In order to prove the equality $\mathrm{ri}(\delta^{\sigma}_{\mathbb{X}}) = k$, we consider the following two cases.

Case 1. Let $i_0 \geq r_{\mathbb{X}}+d+1$. Then we have $k = i_0$. Observe that

$$\deg(\mathbb{X}) \geq \mathrm{HF}_{\delta^{\sigma}_{\mathbb{X}}}(k) = \mathrm{HF}_{\mathbb{X}}(k-d-1) \geq \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}) = \deg(\mathbb{X}).$$

It follows that $\mathrm{HF}_{\delta^{\sigma}_{\mathbb{X}}}(k) = \deg(\mathbb{X})$, and hence $k \geq \mathrm{ri}(\delta^{\sigma}_{\mathbb{X}})$. Moreover, for $i < k = i_0$, we have $\mathrm{HF}_{\delta^{\sigma}_{\mathbb{X}}}(i) < \mathrm{HF}_{\mathbb{X}}(i-d-1) \leq \mathrm{HF}_{\mathbb{X}}(k-d-1) = \deg(\mathbb{X})$. Thus we get $\mathrm{ri}(\delta^{\sigma}_{\mathbb{X}}) = k$.

Case 2. Let $i_0 < r_{\mathbb{X}}+d+1$. Then we have $k = r_{\mathbb{X}}+d+1$ and $\mathrm{HF}_{\delta^{\sigma}_{\mathbb{X}}}(k) = \mathrm{HF}_{\mathbb{X}}(k-d-1) = \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}) = \deg(\mathbb{X})$. This implies $k \geq \mathrm{ri}(\delta^{\sigma}_{\mathbb{X}})$. For $i < k$, we have $\mathrm{HF}_{\delta^{\sigma}_{\mathbb{X}}}(i) \leq \mathrm{HF}_{\mathbb{X}}(i-d-1) \leq \mathrm{HF}_{\mathbb{X}}(r_{\mathbb{X}}-1) < \deg(\mathbb{X})$. Hence we obtain $\mathrm{ri}(\delta^{\sigma}_{\mathbb{X}}) = k$ again. □

In the special case that $\mathbb{X}$ is a locally Gorenstein Cayley-Bacharach scheme, the regularity index of the Dedekind different attains the maximal value. This also follows from [15, Proposition 4.8] with a different proof.

Corollary 4.9. *In the setting of Proposition* 4.8*, assume that* $\mathbb{X}$ *is a Cayley-Bacharach scheme.*

(a) *The regularity index of the Dedekind different* $\delta^{\sigma}_{\mathbb{X}}$ *is* $2r_{\mathbb{X}}$.

(b) *The scheme* $\mathbb{X}$ *is arithmetically Gorenstein if and only if the Hilbert function of* $\delta^{\sigma}_{\mathbb{X}}$ *satisfies* $\mathrm{HF}_{\delta^{\sigma}_{\mathbb{X}}}(i) = \mathrm{HF}_{\mathbb{X}}(i-r_{\mathbb{X}})$ *for all* $i \in \mathbb{Z}$.

Proof. Claim (a) follows directly from the proposition, and claim (b) follows by [16, Proposition 5.8]. □

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Martin Kreuzer
Fakultät für Informatik und Mathematik, Universität Passau, D-94030 Passau, Germany
E-mail address: `martin.kreuzer@uni-passau.de`

Tran N. K. Linh
Department of Mathematics, College of Education, Hue University, 34 Le Loi, Hue, Vietnam
E-mail address: `tnkhanhlinh@hueuni.edu.vn`

Le Ngoc Long
Fakultät für Informatik und Mathematik, Universität Passau, D-94030 Passau, Germany
and
Department of Mathematics, College of Education, Hue University, 34 Le Loi, Hue, Vietnam
E-mail address: `lelong@hueuni.edu.vn`

Tu Chanh Nguyen
Faculty of Advanced Science and Technology (FAST), University of Science and Technology, The University of Danang, 54 Nguyen Luong Bang, Danang, Vietnam
E-mail address: `nctu@dut.udn.vn`