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Quasi-periodic Solutions for Nonlinear Schrödinger Equations with Legendre Potential

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Abstract. In this paper, the nonlinear Schrödinger equations with Legendre potential $\mathbf{i}u_t - u_{xx} + V_L(x)u + mu + \sec x \cdot |u|^2 u = 0$ subject to certain boundary conditions is considered, where $V_L(x) = -\frac{1}{2} - \frac{1}{4}\tan^2 x$, $x \in (-\pi/2, \pi/2)$. It is proved that for each given positive constant $m > 0$, the above equation admits lots of quasi-periodic solutions with two frequencies. The proof is based on a partial Birkhoff normal form technique and an infinite-dimensional Kolmogorov-Arnold-Moser theory.

1. Introduction and main results

Among various techniques for finding the quasi-periodic solutions for partial differential equations (PDEs) carrying a Hamtiltonian structure, Kolmogorov-Arnold-Moser (KAM, for short) theory has been proven to be one of the most powerful approaches. With the aid of KAM theory, one can obtain not only the existence of invariant tori along with small-amplitude quasi-periodic solutions, but also the linear stability of the invariant tori. Let us briefly recall the existing literatures in this line, Kuksin [13–15] and Wayne [25] were the first ones to extend the KAM theory in finite-dimensional case to infinite-dimensional ones, and construct the quasi-periodic solutions for some Hamiltonian PDEs. Among those PDEs, the nonlinear Schrödinger equation ($\mathbf{i}u_t - u_{xx} + V(x)u + f(|u|^2)u = 0$) and the nonlinear wave equation ($u_{tt} - u_{xx} + V(x)u + f(u) = 0$) in various situations have been studied by many authors, see, e.g., [4–6,8,11,18,24,26,27] and references therein. For those kind of PDEs with nonlinearity containing spatial derivative, like KdV equations, Benjamin-Ono equations, derivative nonlinear Schrödinger equations and derivative nonlinear wave equations, the corresponding unbounded KAM theorems were developed to establish the existence of quasi-periodic solutions for these PDEs, see, for instance, [2,3,12,16,17,19–22,28] for references. Moreover, we mention that the existence of the KAM tori and quasi-periodic solutions for some quasi-linear equations was studied in [1,9,10].

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It is well known that parameters need to be introduced so as to adjust frequencies to deal with small divisors problem in each KAM iteration step, an usual and efficient way to introduce parameters is to consider the nonlinear Hamiltonian PDEs with given potentials, where the Hamiltonian PDEs can be reformulated into a parameter-dependent integrable normal form plus a perturbation, and the normal form part possesses integrable directions whose amplitudes become natural parameters. In the aforementioned papers, the given potential V is always regular. If the potential possesses singularities, Cao and Yuan [7] prove that the corresponding nonlinear wave equation admits plenty of quasi-periodic solutions. As far as we know, this is the only result about finding the quasi-periodic solutions for PDEs with singular potentials via KAM theory. In [7], the authors introduce the following Legendre potential,

$$V_L(x) = -\frac{1}{2} - \frac{1}{4}\tan^2 x, \quad x \in (-\pi/2, \pi/2).$$

Obviously, the endpoints $x = \pm\pi/2$ are actually singular.

In this paper, we attempt to consider the following Schrödinger equation

$$\mathbf{i}u_t - u_{xx} + V_L(x)u + \sec x \cdot |u|^2 u = 0, \quad x \in (-\pi/2, \pi/2) \tag{1.1}$$

subject to boundary conditions

$$\lim_{x\to\pm\pi/2} u(x,t) = 0. \tag{1.2}$$

After the change of variables

$$y = \sin x, \ \ x \in (-\pi/2, \pi/2), \quad z = \frac{u}{\sqrt{\cos x}},$$

the equation (1.1) under boundary conditions (1.2) can be rewritten as

$$\mathbf{i}z_t - \big((1-y^2)z_y\big)_y + |z|^2 z = 0, \ \ y \in (-1,1), \quad \lim_{y\to\pm1} z(y)(1-y^2)^{1/4} = 0.$$

In convention, we still use the notations $u(x) := z(y)$, $x := y$. Denote

$$\mathscr{B} = -\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x},$$

it is easy to draw a conclusion that the operator $\mathscr{B}$ possesses the only pure point spectrum, i.e., $\sigma(\mathscr{B}) = \sigma_p(\mathscr{B})$. Besides,

$$(\mathscr{B}u, u) = \int_{-1}^{1} -\frac{\mathrm{d}}{\mathrm{d}x}\left[(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right]\overline{u(x)}\,\mathrm{d}x = \int_{-1}^{1}(1-x^2)\left|\frac{\mathrm{d}}{\mathrm{d}x}u(x)\right|^2 \mathrm{d}x \geq 0,$$

which implies that

$$\sigma(\mathscr{B}) \subset [0,\infty).$$

To ensure the strict positive definiteness of the singular differential operator $\mathscr{B}$, we modify it by adding a positive constant m. We denote the new operator by B in the following manner

$$B = \mathscr{B} + m, \quad m > 0.$$

Thus, in the rest of the paper, we focus on the modified Schrödinger equation

$$\mathbf{i}u_t - \big((1-x^2)u_x\big)_x + mu + |u|^2u = 0, \ \ x \in (-1,1), \quad \lim_{x\to\pm 1} u(x)(1-x^2)^{1/4} = 0. \tag{1.3}$$

Let λ_j and ϕ_j ($j = 1, 2, \dots$) be the eigenvalues and eigenfunctions of B respectively. For simplicity, we choose them in the following form

$$\lambda_j = 2j(2j-1) + m, \quad \phi_j = \sqrt{2j - \frac{1}{2}}P_{2j-1}(x), \quad j = 1, 2, \dots, \tag{1.4}$$

where $P_j(x)$ are Legendre polynomials. It is well known that the sequence $\{\phi_j\}_j^\infty$ forms a complete orthogonal basis in the space defined by all the odd functions in L^2. Note that the eigenfunctions are not triangle functions $\sqrt{2/\pi}\sin jx$ as usual. So the calculation of the integral $\int_{-1}^1 \phi_i\phi_j\phi_k\phi_l \,\mathrm{d}x$ is not an easy task. This is the reason why we shall obtain only 2-dimensional KAM tori. Let E be a complex invariant linear space with dimension 2 which is completely foliated into rotational tori, that is,

$$E = \{u = q_1\phi_1 + q_2\phi_2 : q \in \mathbb{C}^2\} = \bigcup_{I\in\overline{\mathbb{P}^2}} \mathscr{T}(I),$$

where $\mathbb{P}^2 = \{I \in \mathbb{R}^2 : I_j > 0 \text{ for } j = 1, 2\}$ is the positive quadrant in $\mathbb{R}^2$ and

$$\mathscr{T}(I) = \{u = q_1\phi_1 + q_2\phi_2 : |q|_j^2 = 2I_j \text{ for } j = 1, 2\}.$$

This is the linear situation. Upon restoration of the nonlinearity $|u|^2u$, the invariant manifold E will not persist in their entirety due to the resonances. We show, however, that in sufficiently small neighbourhood of the origin, a large Cantor subfamily of rotational 2-tori persists.

Theorem 1.1 (Main Theorem)**.** *For each $m > 0$, there exist a set $\mathscr{C}$ in $\mathbb{P}^2$ with positive Lebesgue measure, a family of 2-tori*

$$\mathscr{T}[\mathscr{C}] = \bigcup_{I\in\mathscr{C}} \mathscr{T}(I) \subset E$$

over $\mathscr{C}$, as well as a Lipschitz continuous embedding into phase space $\mathscr{P}$

$$\Phi\colon \mathscr{T}[\mathscr{C}] \hookrightarrow \mathscr{P},$$

which is a higher order perturbation of the inclusion map $\Phi_0\colon E \hookrightarrow \mathscr{P}$ restricted to $\mathscr{T}[\mathscr{C}]$, such that the restriction of Φ to each $\mathscr{T}(I)$ in the family is an embedding of a rotational invariant 2-torus for the nonlinear Schrödinger equation (1.3)*.*

Let us compare our results with those of Cao-Yuan [7]. There are two main differences. The first difference lies in that in our proof of the main theorem, we get a bad Birkhoff normal form, which is caused by the fact that we just remove the fourth order terms with at most two normal variables instead of three ones in [7]. Actually, in the following Lemma 3.2, we shall show a conclusion that the combinations of the frequencies vanish in many situations when one attempts to cancel those fourth order terms with three normal variables, which is a worse situation that never happened in [7]. This is why we can not make use of the Cantor Manifold Theorem in [18] directly. Fortunately, Pöschel's Theorem in [23] is benefit for our proof. For the second difference, in [7] the authors need to restrict $m \in (0, 1/4) \cup (1/4, 41/4)$ when checking the non-degeneracy conditions, however, we adopt an effective technique to verify it without any restriction on m.

The rest of the paper is organized as follows. In Section 2 the Hamiltonian function is written in infinitely many coordinates and regularity is established. Section 3 is devoted to transforming the Hamiltonian into partial normal form. In Section 4 the assumptions for the infinite-dimensional KAM theory are checked and the Main theorem is proved.

2. The Hamiltonian

The Hamiltonian of the nonlinear Schrödinger equation (1.3) is

$$H = \frac{1}{2}\langle Bu, u\rangle + \frac{1}{4}\int_{-1}^{1} |u|^4 \,\mathrm{d}x,$$

where $B = -\frac{\mathrm{d}}{\mathrm{d}x}(1-x^2)\frac{\mathrm{d}}{\mathrm{d}x} + m$. We rewrite H as a Hamiltonian in infinitely many coordinates by making the ansatz

$$u = \mathcal{S}q = \sum_{j\geq 1} q_j \phi_j(x).$$

The coordinates are taken from the Hilbert space ℓ_s^2 of all complex-valued sequences $q = (q_1, q_2, \ldots)$ with

$$\|q\|_s^2 = \sum_{j\geq 1} j^{2s}|q_j|^2 < \infty.$$

We shall choose $s = 4$ later. One then gets the Hamiltonian

$$H = \Lambda + G = \frac{1}{2}\sum_{j\geq 1}\lambda_j|q_j|^2 + \frac{1}{4}\int_{-1}^{1}|\mathcal{S}q|^4\,\mathrm{d}x \tag{2.1}$$

on the phase space with symplectic structure $\frac{\mathbf{i}}{2}\sum_{j\geq 1} dq_j \wedge d\overline{q}_j$. Its equations of motion are

$$\dot{q}_j = 2\mathbf{i}\frac{\partial H}{\partial \overline{q}_j}, \quad j \geq 1.$$

They are the classical Hamiltonian equations of motions for the real and imaginary parts of $q_j = x_j + \mathbf{i}y_j$ written in complex notation.

Next, we shall establish the regularity of the gradient of G. To this end, we need to verify the algebraic property analogous to Theorem 2.1 in [7]. At first, we make some preparations. Fix $s > 1$. Let

$$L_s^2 = \Big\{ u = \sum_{j\geq 1} u_j\phi_j \in L^2[-1,1] \;\Big|\; \sum_{j\geq 1} j^{2s}|u_j|^2 < \infty \Big\},$$

and the norm of L_s^2 is defined by $\|u\|_s = \big(\sum_{j\geq 1} j^{2s}|u_j|^2\big)^{1/2}$. By the positive definiteness of operator B, we can also define another norm

$$\|B^{s/2}u\|_{L^2[-1,1]} = \langle B^{s/2}u, B^{s/2}u\rangle^{1/2} = \langle B^s u, u\rangle^{1/2},$$

where $\langle\,\cdot\,,\cdot\,\rangle$ denotes the usual scalar product in $L^2[-1,1]$. Thus the two norms above are equivalent, i.e.,

$$\|B^{s/2}u\|_{L^2[-1,1]} \sim \|u\|_s.$$

In fact, on one hand,

$$\Big\| B^{s/2}\sum_{j=1}^n u_j\phi_j \Big\|_{L^2[-1,1]}^2 \leq \Big\| \sum_{j=1}^n u_j\lambda_j^s\phi_j \Big\|_{L^2[-1,1]}^2 \lesssim \sum_{j=1}^n |u_j|^2 j^{2s} \lesssim \|u\|_s^2, \quad n\to\infty.$$

On the other hand,

$$\sum_{j=1}^{+\infty} |u_j|^2 j^{2s} \lesssim \sum_{j=1}^{+\infty} |u_j|^2\lambda_j^{2s} = \Big\langle B^s\sum_{j\geq 1} u_j\phi_j, \sum_{j\geq 1} u_j\phi_j \Big\rangle = \|B^{s/2}u\|_{L^2[-1,1]}^2.$$

Then our lemma is as follows.

Lemma 2.1. *If $u(-x) = -u(x)$, $u \in L_4^2$, then*

$$\|B^2(u^2\overline{u})\|_{L^2[-1,1]} \lesssim \|B^2 u\|_{L^2[-1,1]}^3.$$

Proof. Due to the fact that $(u\pm\overline{u})^3 = u^3 \pm 3u^2\overline{u} + 3u\overline{u}^2 \pm \overline{u}^3$, it is easy to get

$$u^2\overline{u} = \frac{1}{6}\big[(u+\overline{u})^3 - (u-\overline{u})^3 - 2\overline{u}^3\big].$$

Then, using Theorem 2.1 in [7], we conclude the lemma. □

Lemma 2.2. *For $u\,(=\sum_{j\geq 1} q_j\phi_j) \in L_4^2$ (or $q = (q_j)_{j\geq 1} \in \ell_4^2$), the gradient $X_G = \big(\frac{\partial G}{\partial \overline{q}_j}\big)_{j\geq 1}$ is real analytic as a map from some neighbourhood of the origin in ℓ_4^2 into ℓ_4^2, with*

$$\|X_G\|_4 = O(\|q\|_4^3).$$

Proof. From the fact that

$$\frac{\partial G}{\partial \overline{q}_j} = \langle u^2\overline{u}, \phi_j\rangle,$$

we have

$$\|X_G\|_4 = \|(\langle u^2\overline{u}, \phi_j\rangle)_{j\geq 1}\|_4 \sim \|B^2(u^2\overline{u})\|_{L^2[-1,1]} \lesssim \|B^2 u\|^3_{L^2[-1,1]} \sim \|q\|_4^3. \qquad \square$$

For the nonlinearity $u^2\overline{u}$, one finds

$$G = \frac{1}{4}\int_{-1}^{1} |u(x)|^4 \,\mathrm{d}x = \frac{1}{4}\sum_{i,j,k,l} G_{ijkl} q_i q_j \overline{q}_k \overline{q}_l \tag{2.2}$$

with

$$G_{ijkl} = \int_{-1}^{1} \phi_i\phi_j\phi_k\phi_l \,\mathrm{d}x.$$

If ϕ_j is trigonometric function $\sqrt{2/\pi}\sin x$, it is easy to calculate the coefficient G_{ijkl}. However, in this paper, ϕ_j is Legendre polynomial, which leads to the complicated calculation of the integral $\int_{-1}^{1}\phi_i\phi_j\phi_k\phi_l \,\mathrm{d}x$. Fortunately, the paper [7] provides enough information about the coefficient G_{ijkl}. We list some useful facts.

Lemma 2.3. *The coefficients $G_{ijkl} \geq 0$. Moreover, there exists a constant $C > 0$ such that $G_{ijkl} \leq C$.*

For details, we refer to (3.13) in [7].

3. Partial Birkhoff normal form

Due to Lemma 2.3, the Hamiltonian (2.1) turns into

$$H = \Lambda + G = \frac{1}{2}\sum_{j\geq 1}\lambda_j |q_j|^2 + \frac{1}{4}\sum_{i,j,k,l\geq 1} G_{ijkl} q_i q_j \overline{q}_k \overline{q}_l. \tag{3.1}$$

Since the quadratic part of the Hamiltonian does not provide any "twist" required by KAM theory, we shall use the normal form technique to get the "twisted" integrable terms from the fourth order terms. To get a two-dimensional KAM tori, for simplicity, we choose (q_1, q_2) as tangential variables. All the other variables are called normal ones. In this part, the fourth order terms with at most two normal variables will be cancelled, while the other fourth order terms are left since they have no effect on the tori. Then we define the index sets $\triangle_*$, $* = 0, 1, 2$ and $\triangle_3$ in the following way: $\triangle_*$ is the set of index (i,j,k,l) such that there exist right $*$ components not in $\{1,2\}$. $\triangle_3$ is the set of index (i,j,k,l) such that there exist at least three components not in $\{1,2\}$. Define the resonance sets $\mathcal{N} = \{(i,j,i,j)\}$. For our convenience, rewrite $G = \overline{G} + \widetilde{G} + \widehat{G}$, where

$$\overline{G} = \frac{1}{4}\sum_{(i,j,k,l)\in(\triangle_0\cup\triangle_1\cup\triangle_2)\cap\mathcal{N}} G_{ijkl} q_i q_j \overline{q}_k \overline{q}_l, \quad \widetilde{G} = \frac{1}{4}\sum_{(i,j,k,l)\in(\triangle_0\cup\triangle_1\cup\triangle_2)\setminus\mathcal{N}} G_{ijkl} q_i q_j \overline{q}_k \overline{q}_l$$

and

$$\widehat{G} = \frac{1}{4} \sum_{(i,j,k,l) \in \triangle_3} G_{ijkl} q_i q_j \overline{q}_k \overline{q}_l.$$

To remove the term $\widetilde{G}$, we need the following lemma.

Lemma 3.1. *If $(i,j,k,l) \in (\triangle_0 \cup \triangle_1 \cup \triangle_2) \setminus \mathcal{N}$, then*

$$\lambda_i + \lambda_j - \lambda_k - \lambda_l \neq 0.$$

Proof. Set $\delta(i,j,k,l) = \frac{1}{2}(\lambda_i + \lambda_j - \lambda_k - \lambda_l) = 2(i^2 + j^2 - k^2 - l^2) - (i + j - k - l)$, it is clear that

$$\delta(i,j,k,l) = \delta(j,i,k,l) = \delta(i,j,l,k) = -\delta(k,l,i,j).$$

Therefore, we only consider the case: $i \leq j$, $k \leq l$ and $i \leq k$. If $j < l$, it follows

$$\begin{aligned}\delta &= 2(i^2 - k^2) - (i - k) + 2(j^2 - l^2) - (j - l) \\ &= (i - k)(2i + 2k - 1) + (j - l)(2j + 2l - 1) < 0.\end{aligned}$$

Then we assume that $i \leq k \leq l \leq j$, and divide it into two cases to prove the lemma.

Case 1: $i = k = 1$ (or 2). Since (i,j,k,l) is not in $\mathcal{N}$, it is easy to see $\delta \neq 0$.

Case 2: $i = 1$, $k = 2$. $\delta = 2(1^2 + j^2 - l^2 - 2^2) - (1 + j - 2 - l) = (j - l)(2j + 2l - 1) - 5$. If $j = l$, $\delta = -5$, otherwise, $\delta \geq 2j + 2l - 6 \geq 2$, since $j \geq l \geq 2$. □

In view of the following lemma, we cannot remove some special terms whose indexes (i,j,k,l) lie in $\triangle_3$.

Lemma 3.2. *If $(1,j,k,l) \in \triangle_3$, and j, k, l take the following form*

$$l = a(4b + 1) + 1, \quad k = 8ab^2 + 4ab + 3b + 1, \quad j = k + a,$$

where $a, b \in \mathbb{N}^+$, then the corresponding $\delta(1,j,k,l) = 0$.

Proof. Without loss of generality, we assume that

$$j > k \geq l. \tag{3.2}$$

Let $a = j - k$, thus δ turns into

$$\delta = 2a^2 + (4k - 1)a - (2l^2 - l - 1).$$

If $\delta = 0$, it is easy to get that

$$a(2a + 4k - 1) = (l - 1)(2l + 1). \tag{3.3}$$

From (3.2) and (3.3), it is clear that $1 \leq a \leq l-1$. Now we only consider a special case, that is, $a \mid (l-1)$. So there exists a positive integer denoted by $\bar{b}$ such that

$$a\bar{b} = l - 1. \tag{3.4}$$

Substituting (3.4) into (3.3) to obtain that

$$4k = 2a\bar{b}^2 + 3\bar{b} - 2a + 1 \tag{3.5}$$

which implies that $\bar{b}$ is an odd number. Suppose that $\bar{b} = 2\widehat{b} + 1$, then by (3.5), we have

$$4k = 8a\widehat{b}^2 + 8a\widehat{b} + 6\widehat{b} + 4. \tag{3.6}$$

Due to (3.6), it deduces that $\widehat{b}$ is even. So replacing $\widehat{b}$ by $2b$ in (3.6), we obtain

$$k = 8ab^2 + 4ab + 3b + 1.$$

Finally, from (3.4), we get $l = a(4b+1) + 1$. □

Next we shall transform the Hamiltonian (3.1) into the partial Birkhoff form of order four so that the KAM Theorem can be applied.

Proposition 3.3. *For the Hamiltonian $H = \Lambda + G$ with nonlinearity (2.2), there exists a real analytic, symplectic change of coordinates Γ in some neighbourhood of the origin in ℓ_4^2 that for all real values of m takes it into*

$$H \circ \Gamma = \Lambda + \overline{G} + \widehat{G} + K,$$

where $X_{\overline{G}}$, $X_{\widehat{G}}$ and X_K are real analytic vector fields in a neighborhood of the origin in ℓ_4^2,

$$\overline{G} = \frac{1}{2} \sum_{\min(i,j) \leq 2} \overline{G}_{ij} |q_i|^2 |q_j|^2, \quad |K| = O(\|q\|_4^6)$$

with uniquely determined coefficients

$$\overline{G}_{1j} = \overline{G}_{j1} = \frac{3(2-\delta_{1j})(8j^2 - 4j - 1)}{4(4j-3)(4j+1)}, \tag{3.7}$$

$$\overline{G}_{2j} = \overline{G}_{j2} = \frac{7(2-\delta_{2j})(1088j^6 - 1632j^5 - 2440j^4 + 3120j^3 + 1406j^2 - 1110j - 225)}{4(4j-7)(4j-5)(4j-3)(4j+1)(4j+3)(4j+5)}. \tag{3.8}$$

Proof. Let Γ be the time-1-map of the flow of a Hamiltonian vector field X_F given by a Hamiltonian

$$F = \frac{1}{4} \sum_{i,j,k,l \geq 1} F_{ijkl} q_i q_j \bar{q}_k \bar{q}_l$$

with coefficients

$$\mathbf{i}F_{ijkl} = \begin{cases} \frac{G_{ijkl}}{\lambda_i+\lambda_j-\lambda_k-\lambda_l} & \text{for } (i,j,k,l) \in (\triangle_0 \cup \triangle_1 \cup \triangle_2) \setminus \mathcal{N}, \\ 0 & \text{otherwise.} \end{cases}$$

As we will show in a moment, X_F is a real analytic vector field on ℓ_4^2 of order three at the origin. Hence Γ is a real analytic, symplectic change of coordinates defined at least in a neighbourhood of the origin in ℓ_4^2.

Expanding at $t = 0$ and using Taylor's formula we obtain

$$\begin{aligned} H \circ \Gamma &= H \circ X_F^t\big|_{t=1} \\ &= H + \{H, F\} + \int_0^1 (1-t)\big\{\{H,F\},F\big\} \circ X_F^t \,\mathrm{d}t \\ &= \Lambda + \overline{G} + \widetilde{G} + \{\Lambda, F\} + \widehat{G} + \{G, F\} + \int_0^1 (1-t)\big\{\{H,F\},F\big\} \circ X_F^t \,\mathrm{d}t, \end{aligned}$$

where $\{H, F\}$ denotes the Poisson bracket of H and F. The last line consists of terms of order six or more in q and constitutes the higher order term K. In the second to last line,

$$\widetilde{G} + \{\Lambda, F\} = \sum_{(i,j,k,l)\in(\triangle_0\cup\triangle_1\cup\triangle_2)\setminus\mathcal{N}} (G_{ijkl} - \mathbf{i}(\lambda_i + \lambda_j - \lambda_k - \lambda_l)F_{ijkl}) q_i q_j \overline{q}_k \overline{q}_l = 0.$$

Hence, we have $H \circ \Gamma = \Lambda + \overline{G} + \widehat{G} + K$ as claimed.

To prove the analyticity of map Γ, we introduce the "threshold function"

$$\widetilde{F} = \int_{-1}^1 v^4(x)\,\mathrm{d}x, \quad \text{where } v(x) = \sum_{j\ge1} |q_j|\phi_j(x).$$

Clearly,

$$\widetilde{F} = \sum_{i,j,k,l\ge1} G_{ijkl}|q_i||q_j||q_k||q_l|.$$

Then it follows that

$$\left|\frac{\partial\widetilde{F}}{\partial\overline{q}_l}\right| = \left|\frac{\partial\widetilde{F}}{\partial|\overline{q}_l|}\cdot\frac{\partial|\overline{q}_l|}{\partial\overline{q}_l}\right| = \left|\frac{\partial\widetilde{F}}{\partial|\overline{q}_l|}\right| = \left|\sum_{i,j,k} G_{ijkl}|q_i||q_j||q_k|\right|. \tag{3.9}$$

While,

$$\begin{aligned} \left|\frac{\partial F}{\partial\overline{q}_l}\right| &\le \sum_{(i,j,k,l)\in(\triangle_0\cup\triangle_1\cup\triangle_2)\setminus\mathcal{N}} |F_{ijkl}||q_i||q_j||q_k| \\ &\le \sum_{(i,j,k,l)\in(\triangle_0\cup\triangle_1\cup\triangle_2)\setminus\mathcal{N}} G_{ijkl}|q_i||q_j||q_k|, \end{aligned} \tag{3.10}$$

in which the second inequality has used Lemma 3.1 and the fact that $G_{ijkl} > 0$. Hence, by (3.9) and (3.10), we get that

$$\left|\frac{\partial F}{\partial \overline{q}_l}\right| \le \left|\frac{\partial \widetilde{F}}{\partial \overline{q}_l}\right|,$$

which means

$$\|F_{\overline{q}}\|_4 \le \|\widetilde{F}_{\overline{q}}\|_4.$$

On the other hand,

$$\left|\frac{\partial \widetilde{F}}{\partial \overline{q}_l}\right| = \left|\int_{-1}^{1} 4v^3 \frac{\partial v}{\partial |\overline{q}_l|} \cdot \frac{\partial |\overline{q}_l|}{\partial \overline{q}_l}\,\mathrm{d}x\right| = \left|\int_{-1}^{1} 4v^3 \phi_l \cdot \frac{\partial |\overline{q}_l|}{\partial \overline{q}_l}\,\mathrm{d}x\right| \lesssim |\langle v^3, \phi_l\rangle|,$$

hence,

$$\|\widetilde{F}_{\overline{q}}\|_4 \lesssim \|(\langle v^3, \phi_j\rangle)_{j\ge 1}\|_4 \sim \|A^2 v^3\|_{L^2[-1,1]} \lesssim \|A^2 v\|^3_{L^2[-1,1]} \sim \|q\|_4^3.$$

In the end, we obtain

$$\|F_{\overline{q}}\|_4 \lesssim \|q\|_4^3.$$

The analyticity of $F_{\overline{q}}$ follows from the analyticity of each component function and its local boundedness. The analogue claims for $X_{\overline{G}}$, $X_{\widehat{G}}$ and X_K are obvious.

Finally, we calculate the coefficients $\overline{G}_{1j}$ and $\overline{G}_{2j}$. For $i = 1, 2$, it is clear that

$$\overline{G}_{ij} = \frac{2-\delta_{ij}}{2} G_{ijij} = \frac{2-\delta_{ij}}{2} \int_{-1}^{1} \phi_i\phi_j\phi_i\phi_j\,\mathrm{d}x,$$

where δ_{ij} is the Dirac notation. Recall that $\phi_j = \sqrt{2j-1/2}P_{2j-1}$ in (1.4), we have

$$\overline{G}_{ij} = \frac{2-\delta_{ij}}{2}\left(2i-\frac{1}{2}\right)\left(2j-\frac{1}{2}\right)\int_{-1}^{1} P_{2i-1}P_{2j-1}P_{2i-1}P_{2j-1}\,\mathrm{d}x.$$

We still denote the integral $\int_{-1}^{1} P_{2i-1}P_{2j-1}P_{2i-1}P_{2j-1}\,\mathrm{d}x$ by $\mathrm{P}(2i-1, 2j-1)$. It follows that

$$\overline{G}_{1j} = \frac{3(2-\delta_{1j})}{4}\left(2j-\frac{1}{2}\right)\mathrm{P}(1, 2j-1) \quad \text{and} \quad \overline{G}_{2j} = \frac{7(2-\delta_{1j})}{4}\left(2j-\frac{1}{2}\right)\mathrm{P}(3, 2j-1).$$

With the help of the recursion formula of the Legendre sequences $P(m, n)$ in Theorem 3.2 of [7], one has

$$\begin{aligned} P(1,n) &= \frac{2(2n^2+2n-1)}{(2n-1)(2n+1)(2n+3)}, \\ P(3,n) &= \frac{34n^6+102n^5-305n^4-780n^3+703n^2+1110n-450}{(2n-5)(2n-3)(2n-1)(2n+1)(2n+3)(2n+5)(2n+7)}, \end{aligned}$$

based on which one obtains (3.7) and (3.8). $\square$

4. Proof of main theorem

In this section, with aid of the KAM Theorem for infinite-dimensional Hamiltonian systems [23], we shall establish the existence of quasi-periodic solutions for the equation (1.3).

Firstly, we introduce symplectic polar and real coordinate by setting

$$q_j = \begin{cases} \sqrt{2(\xi_j + y_j)}e^{-\mathbf{i}x_j} & \text{if } j = 1, 2, \\ u_j + \mathbf{i}v_j & \text{if } j \geq 3 \end{cases}$$

depending on parameters $\xi = (\xi_1, \xi_2) \in \Pi = [0, 1]^2$. The precise domains will be specified later when they become important. Then we have

$$\frac{\mathbf{i}}{2} \sum_{j\geq 1} dq_j \wedge d\overline{q}_j = \sum_{1\leq j\leq 2} dx_j \wedge dy_j + \sum_{j\geq 3} du_j \wedge dv_j.$$

Thus the new Hamiltonian, still denoted by H, up to a constant depending only on ξ, is given by

$$H = \Lambda + \overline{G} + \widehat{G} + K = \langle \omega(\xi), y\rangle + \frac{1}{2}\langle \Omega(\xi), u^2 + v^2\rangle + \check{G} + \widehat{G} + K \tag{4.1}$$

with frequencies

$$\omega(\xi) = \alpha + A\xi, \quad \Omega(\xi) = \beta + B\xi, \tag{4.2}$$

where

$$\alpha = (\lambda_1, \lambda_2), \quad \beta = (\lambda_j)_{j\geq 3}, \quad A = \begin{pmatrix} \overline{G}_{11} & \overline{G}_{12} \\ \overline{G}_{21} & \overline{G}_{22} \end{pmatrix},$$

$$B_{jk} = \overline{G}_{jk}, \quad j \geq 3,\ k = 1, 2, \tag{4.3}$$

and the remainders

$$\check{G} = O(y^2 + |y|(u^2 + v^2)), \quad \widehat{G} = O(|\xi|^{1/2}(u^2 + v^2)^{3/2}), \quad K = O(|\xi|^3). \tag{4.4}$$

From (4.2), we know that the frequencies are affine functions of the parameters ξ. To prove our main theorem, by Theorem D in [23], we only check that the new Hamiltonian (4.1) satisfies the assumptions of Theorem A in [23]. For the readers' convenience, we shall list these assumptions in the following and then check them one by one.

Assumption A: Nondegeneracy. The map $\xi \mapsto \omega(\xi)$ is lipeomorphism between Π and its image, that is, a homeomorphism which is Lipschitz continuous in both directions. Moreover, for all integer vectors $(k, l) \in \mathbb{Z}^n \times \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$,

$$\big|\{\xi : \langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle = 0\}\big| = 0 \quad \text{and} \quad \langle l, \Omega(\xi)\rangle \neq 0 \quad \text{on } \Pi,$$

where $|\cdot|$ denotes Lebesgue measure for sets, $|l| = \sum_j |l_j|$ for integer vectors, and $\langle\,\cdot\,,\cdot\,\rangle$ is the usual scalar product.

Assumption B: Spectral Asymptotics. There exist $d \geq 1$ and $\delta < d-1$ such that

$$\Omega(\xi) = j^d + \cdots + O(j^\delta),$$

where the dots stands for the fixed lower order terms in j, allowing also negative exponents. More precisely, there exists a fixed, parameter-independent sequence $\overline{\Omega}$ with $\overline{\Omega}_j = j^d + \cdots$ such that the tails $\widetilde{\Omega}_j = \Omega_j - \overline{\Omega}_j$ give rise to a Lipschitz map

$$\widetilde{\Omega}\colon \Pi \to \ell_\infty^{-\delta},$$

where ℓ_∞^p is the space of all the real sequences with finite norm $|w|_p = \sup_j |w_j| j^p$. Note that the coefficient of j^d can always be normalized to one by rescaling the time.

Assumption C: Regularity. The perturbation P is real analytic in the space coordinates and Lipschitz in the parameters, and for each $\xi \in \Pi$ its Hamiltonian vector field $X_P = (P_y, -P_x, P_v, -P_u)$ defines a real analytic map

$$X_P\colon \mathcal{P}^{a,p} \to \mathcal{P}^{a,\overline{p}}, \qquad \begin{cases} \overline{p} \geq p & \text{for } d > 1, \\ \overline{p} > p & \text{for } d = 1 \end{cases}$$

with $\mathcal{P}^{a,\overline{p}} := \mathbb{T}^n \times \mathbb{R}^n \times \ell^{a,p} \times \ell^{a,p}$ where $\mathbb{T}^n$ is the usual n-torus and $\ell^{a,p}$ is the real Hilbert space of all real sequences $w = (w_1, w_2, \ldots)$ with $\sum_{j\geq 1} |w_j|^2 j^{2p} e^{2aj} < +\infty$, where $a \geq 0$ and $p > 0$.

Now, firstly, we start to verify the nondegeneracy assumptions. Since the frequencies are affine functions of the parameters ξ, it is equivalent to proving that the following three conditions hold true.

(A$_1$) $\det A \neq 0$,

(A$_2$) $\langle l, \beta\rangle \neq 0$,

(A$_3$) $\langle k, \omega(\xi)\rangle + \langle l, \Omega(\xi)\rangle \not\equiv 0$

for all $(k,l) \in \mathbb{Z}^2 \times \mathbb{Z}^\infty$ with $1 \leq |l| \leq 2$.

By (3.7) and (3.8), we obtain

$$\overline{G}_{11} = \frac{9}{20}, \quad \overline{G}_{12} = \overline{G}_{21} = \frac{23}{30}, \quad \overline{G}_{22} = \frac{7 \times 241}{20 \times 11 \times 13}.$$

So

$$\det A = \overline{G}_{11}\overline{G}_{22} - \overline{G}_{12}\overline{G}_{21} = -\frac{165941}{514800}.$$

The condition (A_2) is easy to check since λ_j or $\lambda_i \pm \lambda_j$ $(i \neq j)$ are not equal to zero.

Next, we shall check the non-degeneracy condition (A_3). It is equivalent to showing that either $\langle \alpha, k\rangle + \langle \beta, l\rangle \neq 0$ or $Ak + B^T l \neq 0$ for all (k, l) with $1 \le |l| \le 2$. Suppose that $Ak + B^T l = 0$. It is easy to obtain that

$$k = -A^{-1}B^T l = -\frac{1}{\det A}\begin{pmatrix} \overline{G}_{22} & -\overline{G}_{12} \\ -\overline{G}_{21} & \overline{G}_{11} \end{pmatrix} B^T l. \tag{4.5}$$

If $|l| = 1$, without loss of generality, assume that $l = (0, \ldots, \underset{(j-2)\text{-th}}{\underset{\uparrow}{1}}, 0, \ldots)$ with $j \ge 3$, then (4.5) turns into

$$k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} = -\frac{1}{\det A}\begin{pmatrix} \overline{G}_{22} & -\overline{G}_{12} \\ -\overline{G}_{21} & \overline{G}_{11} \end{pmatrix}\begin{pmatrix} \overline{G}_{1j} \\ \overline{G}_{2j} \end{pmatrix}.$$

Clearly, we have

$$k_2 = -\frac{1}{\det A}(\overline{G}_{11}\overline{G}_{2j} - \overline{G}_{21}\overline{G}_{1j}) = \frac{514800}{165941}\left(\frac{9}{20}\overline{G}_{2j} - \frac{23}{30}\overline{G}_{1j}\right).$$

Due to the monotonicity of $\overline{G}_{1j}$, $\overline{G}_{2j}$ with respect to j (≥ 3), this implies that

$$\overline{G}_{1j} \in (0.75, 0.7575), \quad \overline{G}_{2j} \in (0.927, 0.98), \quad \forall\, j \ge 3. \tag{4.6}$$

Therefore, $k_2 \in (-0.506, -0.416)$. This is impossible because k_2 is an integer. For the case $|l| = 2$, it is enough to consider the following two subcases.

Subcase 1. Assume that $l = (0, \ldots, \underset{(i-2)\text{-th}}{\underset{\uparrow}{1}}, 0, \ldots, 0, \underset{(j-2)\text{-th}}{\underset{\uparrow}{1}}, 0, \ldots)$ with $j, i \ge 3$. If $i = j$, $l = (0, \ldots, 0, 2, 0, \ldots)$. By (4.5), one finds that

$$\begin{aligned} k_1 &= -\frac{1}{\det A}\big(\overline{G}_{22}(\overline{G}_{1i} + \overline{G}_{1j}) - \overline{G}_{12}(\overline{G}_{2i} + \overline{G}_{2j})\big) \\ &= \frac{514800}{165941}\left(\frac{1687}{2860}(\overline{G}_{1i} + \overline{G}_{1j}) - \frac{23}{30}(\overline{G}_{2i} + \overline{G}_{2j})\right). \end{aligned}$$

From (4.6), one concludes that $k_1 \in (-1.92, -1.61)$, which cannot happen.

Subcase 2. Assume that $l = (0, \ldots, \underset{(i-2)\text{-th}}{\underset{\uparrow}{1}}, 0, \ldots, 0, \underset{(j-2)\text{-th}}{\underset{\uparrow}{-1}}, 0, \ldots)$ with $j > i \ge 3$. Again, using (4.5), (4.6), we obtain

$$|k_1| = \frac{514800}{165941}\left|\frac{1687}{2860}(\overline{G}_{1i} - \overline{G}_{1j}) - \frac{23}{30}(\overline{G}_{2i} - \overline{G}_{2j})\right| < 0.5.$$

Since k_1 is an integer, this implies that $k_1 = 0$. Similarly, $k_2 = 0$. Then we have

$$\langle \alpha, k\rangle + \langle \beta, l\rangle = \beta_i - \beta_j \neq 0.$$

Thus we complete the proof of checking Assumption A.

Secondly, recalling the expression of $\Omega(\xi)$ in (4.2), (4.3) and Lemma 2.3, Assumption B is satisfied with $d = 2$ and $\delta = 0$. In the Hamiltonian (4.1), we consider $\check{G}$, $\widehat{G}$ and K as the perturbations, namely $P = \check{G} + \widehat{G} + K$. In view of Proposition 3.3, Assumption C holds true with $a = 0$ and $p = \overline{p} = 4$.

Finally, it remains to check the small perturbation assumption. To make this more precise we introduce complex neighbourhoods

$$D(s,r): \quad |\operatorname{Im} x| < s, \quad |y| < r^2, \quad \|u\|_4 + \|v\|_4 < r$$

of $\mathbb{T}^2 \times \{y = 0\} \times \{u = 0\} \times \{v = 0\}$ and weighted norms

$$|(x,y,u,v)|_r = |x| + \frac{|y|}{r^2} + \frac{\|u\|_4}{r} + \frac{\|v\|_4}{r},$$

where $|\cdot|$ is the max-norm for complex vectors. Then we assume that the Hamiltonian vector field X_G is real analytic on $D(s,r)$ for some positive s, r uniformly in ξ with finite norm $|X_G|_{r,D(s,r)} = \sup_{D(s,r)} |X_G|_r$, and that the same holds for its Lipschitz semi-norm

$$|X_G|_r^{\mathcal{L}} = \sup_{\xi \neq \zeta} \frac{|\Delta_{\xi\zeta} X_G|_{r,D(s,r)}}{|\xi - \zeta|},$$

where $\Delta_{\xi\zeta} X_G = X_G(\,\cdot\,,\xi) - X_G(\,\cdot\,,\zeta)$ and the sup is taken over Π.

Set $\Pi = \{\xi \in [0,1]^2 : 0 < |\xi| \leq r^{4/3}\}$. From the perturbation term (4.4), we easily get that

$$\begin{aligned} |X_{(\check{G}+\widehat{G}+K)}|_{r,D(s,r)} &\leq |X_{\check{G}}|_{r,D(s,r)} + |X_{\widehat{G}}|_{r,D(s,r)} + |X_K|_{r,D(s,r)} \\ &= O(r^2) + O(r^{5/3}) + O(r^2) = O(r^{5/3}). \end{aligned}$$

Since $X_{(\check{G}+\widehat{G}+K)}$ is analytic in ξ, one has

$$|X_{(\check{G}+\widehat{G}+K)}|_r^{\mathcal{L}} = O(r^{5/3} r^{-4/3}) = O(r^{1/3}).$$

If r is small enough, the small perturbation assumption for KAM is satisfied. Up to now, the proof of our main theorem is complete.

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