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Existence and Multiplicity of Solutions for a Class of (p,q)-Laplacian Equations in $\mathbb{R}^N$ with Sign-changing Potential

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Abstract. In this paper, we use variational approaches to establish the existence of weak solutions for a class of (p,q)-Laplacian equations on $\mathbb{R}^N$, for $1 < q < p < q^* := Nq/(N-q)$, $p < N$, with a sign-changing potential function and a Carathéodory reaction term which do not satisfy the Ambrosetti-Rabinowitz type growth condition. By linking theorem with Cerami condition, the fountain theorem and dual fountain theorem with Cerami condition, we obtain some existence of weak solutions for the above equations under our considerations which are different from those used in related papers.

1. Introduction

The (p,q)-Laplacian equations appear in various branches of mathematical physics, for example, as the stationary version of a general reaction-diffusion equation:

$$u_t = \operatorname{div}[(|\nabla u|^{p-2} + |\nabla u|^{q-2})\nabla u] + b(x,u),$$

where u describes a concentration, $D(u) := (|\nabla u|^{p-2} + |\nabla u|^{q-2})$ is the diffusion coefficient and $b(x,u)$ is the reaction term connected with source and loss mechanisms. Typically, in chemical and biological applications, the reaction term $b(x,u)$ is a polynomial of u with variable coefficients (see [13, 15, 30]).

The purpose of this paper is to show the existence of weak solutions for the following nonlinear elliptic equation

$$-\Delta_p u - \Delta_q u + g(x)|u|^{p-2}u + h(x)|u|^{q-2}u = \lambda f(x,u), \quad x \in \mathbb{R}^N, \tag{1.1}$$

where $\lambda > 0$, $1 < q < p < q^* := Nq/(N-q)$, $p < N$, $\Delta_m u := \operatorname{div}(|\nabla u|^{m-2}\nabla u)$ for $1 < m < \infty$, $g(x)$ and $h(x)$ are potential functions on $\mathbb{R}^N$, and $f\colon \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function which don't satisfy the Ambrosetti-Rabinowitz condition.

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When $p = q = 2$, the equation (1.1) turns out to be a semilinear Schrödinger one of the form

$$-\Delta u + V(x)u = a(x,u), \quad u \in H^1(\mathbb{R}^N). \tag{1.2}$$

The differential operator $\Delta_p + \Delta_q$ is known as the (p,q)-Laplacian operator, if $p \neq q$. The single p-Laplacian operator has been studied for at least four decades (see [1, 7, 8, 10, 11, 19, 20]), whereas a deeper research involving the (p,q)-Laplacian operator has only arisen in the last decade (see [9, 14, 15, 21, 28, 30, 31]).

As known to all, the (p,q)-Laplacian operator is not homogeneous, some technical difficulties appear when using the common methods of the theory of elliptic equations. In the case of g and h constants, minimax type theorems are the main tools to prove existence of solutions (see [13, 30]). In such a variational approach it is usual to assume the Ambrosetti-Robinowitz (A-R) condition on the nonlinearity. In general, the (A-R) condition not only guarantees that the functional has a mountain pass geometry, but also ensures boundedness of Palais-Smale sequences associated with the functional. Although the (A-R) condition is very useful to obtain existence of weak solutions of elliptic equations via variational methods, it is not satisfied by some exceptional nonlinearities, such as

$$f_1(t) := |t|^{p-2}t\log(|t|+1)$$

and

$$f_2(t) := \begin{cases} |t|^{p-2}t - \left(\frac{p-1}{p}\right)|t|^{l-2} & \text{if } |t| \leq 1, \\ |t|^{p-2}t\left(\log|t| + \frac{1}{p}\right) & \text{if } |t| > 1, \end{cases} \quad p < l < p^* := \frac{Np}{N-p}$$

(see [19], [21], respectively).

We realize a few contributions concerning problem (1.1) with sign-changing potential functions and without assuming (A-R) condition. Superlinear (p,q)-Laplacian equations without the Ambrosetti-Robinowitz condition have been studied both in bounded domains (see [21]) and on $\mathbb{R}^N$ but when the potential functions $g(x)$ and $h(x)$ are continuous, coercive and positive (see [9]). In [28], the authors dealt with the case of $g \in L^1_{\text{loc}}(\mathbb{R})$ and $h \in L^1_{\text{loc}}(\mathbb{R})$. For the unbounded case we refer also to [3, 17], where the set of conditions on f includes (A-R) and Concentration-Compactness Principle is used (see [15, 18]). Finally. we refer to [2, 22] and references therein for the special case of $(p,2)$-Laplacian.

Even through problem (1.1) has a variational structure, the main difficulties in the application of classical variational arguments are due to the lack both of homogeneity of (p,q)-Laplacian operator and compactness of the Sobolev's embeddings on the whole space $\mathbb{R}^N$. Here, we overcome the first defect by using a sharp decomposition of the ambient space and solve the lack of compactness by adding extra some properties on the potential functions, particularly, one of which may change sign. Finally, we use the linking theorem with Cerami condition, as well as, the fountain and dual fountain theorem with Cerami condition to get the existence of weak solutions for problem (1.1) under our considerations, which motivated by R. Bartolo, E. Juárez Hurtado and L. Shao (see [3, 16, 26]).

We introduce the hypotheses on the function $f(x,t)$ and the potential functions $g(x)$ and $h(x)$:

(f$_1$) $f(x,t)$ is a Carathéodory function such that $f(x,0)=0$ and $F(x,t)>0$ for a.e. $x\in\mathbb{R}^N$, all $t\in\mathbb{R}\setminus\{0\}$, where

$$F(x,s):=\int_0^s f(x,t)\,dt,\quad (x,t)\in\mathbb{R}^N\times\mathbb{R};$$

(f$_2$) There exist $\tau\in(q,p^*)$, $K(x)\in L^\infty(\mathbb{R}^N)_+\cap L^{\tau/(\tau-1)}(\mathbb{R}^N)$ and $c>0$, such that

$$|f(x,t)|\le K(x)+c|t|^{l-1}$$

for a.e. $x\in\mathbb{R}^{\mathrm{N}}$, all $t\in\mathbb{R}$, where $l\in(p,p^*)$ and $p^*:=Np/(N-p)$;

(f$_3$) $\lim_{t\to+\infty}F(x,t)/|t|^p=+\infty$, uniformly in $x\in\mathbb{R}^N$;

(f$_4$) There exists $\zeta(x)\in L^1(\mathbb{R}^N)_+$ such that

$$\hbar(x,t)\le\hbar(x,s)+\zeta(x)$$

for a.e. $x\in\mathbb{R}^N$ and all $0\le t\le s$ or $s\le t\le 0$, where

$$\hbar(x,t):=f(x,t)t-pF(x,t);$$

(f$_5$) $f(x,-t)=-f(x,t)$ for a.e. $x\in\mathbb{R}^N$;

(gh) The potentials $g,h\colon\mathbb{R}^N\to\mathbb{R}$ are Lebesgue measurable functions such that

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^N}g(x)>0,\quad \operatorname*{ess\,inf}_{x\in\mathbb{R}^N}h(x)>-\infty$$

and

$$\lim_{|x|\to+\infty}\int_{B_1(x)}\frac{1}{g(y)}\,dy=0,\quad \lim_{|x|\to+\infty}\int_{B_1(x)}\frac{1}{h(y)}\,dy=0,$$

where $B_1(x)=\{y\in\mathbb{R}^N:|x-y|<1\}$.

Here, we provide some examples of the functions $f(x,t)$, $g(x)$ and $h(x)$. We set $f(x,t):=|t|^{p-2}t\log(|t|+1)$. It is easy to verify that $f(t)$ satisfies (f$_1$)–(f$_5$). But it does not satisfy

$$F(x,t)\ge c_1|t|^\mu-c_2,\quad x\in\mathbb{R}^N,\ t\in\mathbb{R},$$

where $c_1, c_2 > 0$ and $\mu > p$, which is consequence of (A-R) condition. (gh) is also satisfied when choose $g(x) = 1 + |x|^2$ and $h(x) = -1 + |x|^2$.

The assumptions in (gh) were provided in [6] in the study of linear Schrödinger equation and used in [4] for the single p-Laplacian. Notice that in [24] the authors show the existence of a nontrivial solution of (1.2) by the mountain pass theorem when $V(x) \in C^1(\mathbb{R}^N, \mathbb{R})$ is positive and coercive. In [5], by means of symmetric mountain pass theorem (see [1, Theorem 2.8]), Bartsch and Wang find infinitely many solutions if $f(x,t)$ is odd in t and $V(x)$ is a positive continuous function such that

$$\operatorname{meas}(\{x \in \mathbb{R}^N : V(x) \le M\}) < \infty \quad \text{for all } M > 0.$$

As shown in Proposition 3.1 of [25], the hypotheses on $V(x)$ both in [5] and in [24] imply that

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^N} V(x) > 0 \quad \text{and} \quad \lim_{|x|\to+\infty} \int_{B_1(x)} \frac{1}{V(y)}\,dy = 0.$$

Consequently, for problem (1.1) the assumptions on the potential functions $g(x)$ and $h(x)$ in (gh) are weaker than those ones in [9], where the potential functions $g(x)$ and $h(x)$ are continuous, coercive and positive.

Our main results are the following theorems:

Theorem 1.1. *Assume that* (f_1)–(f_4) *and* (gh) *hold. Then for each* $\lambda \in (0, 1/(p2^p))$, *the problem* (1.1) *has at least one nontrivial weak solution* u *in* W *(*W *is defined in Section* 2*).*

Theorem 1.2. *Assume that* (f_1)–(f_5) *and* (gh) *hold. Then for each* $\lambda \in (0, 1/(p2^p))$, *the problem* (1.1) *has a sequence of weak solutions* $\{u_n\}_{n\in\mathbb{N}} \subset W$ *such that* $I_\lambda(u_n) \to +\infty$ *as* $n \to +\infty$ *(*I_λ *is defined in Section* 3*).*

Theorem 1.3. *Assume that* (f_1)–(f_5) *and* (gh) *hold. Then for each* $\lambda \in (0, 1/(p2^p))$, *the problem* (1.1) *has a sequence of weak solutions* $\{v_n\}_{n\in\mathbb{N}} \subset W$ *such that* $I_\lambda(v_n) < 0$ *and* $I_\lambda(v_n) \to 0$ *as* $n \to \infty$.

This paper is organized as follows. In Section 2 we give some auxiliary lemmas and results used in our work. In Section 3 we will prove the main theorems of this paper.

2. Preliminaries

First, let $g\colon \mathbb{R}^N \to \mathbb{R}$ be a Lebesgue measurable function such that

$$\operatorname*{ess\,inf}_{x\in\mathbb{R}^N} g(x) > 0 \quad \text{and} \quad \lim_{|x|\to+\infty} \int_{B_1(x)} \frac{1}{g(y)}\,dy = 0.$$

For any $m > 1$ we consider the weighted Sobolev space

$$\mathbb{E}_{m,g} := W_g^{1,m}(\mathbb{R}^N) = \left\{ u \in W^{1.m}(\mathbb{R}^N) : \int_{\mathbb{R}^N} g(x)|u|^m\,dx < +\infty \right\}$$

endowed with the norm

$$\|u\|_{\mathbb{E}_{m,g}} := \left(\int_{\mathbb{R}^N} |\nabla u|^m + g(x)|u|^m \, dx \right)^{1/m}.$$

The space $(\mathbb{E}_{m,g}, \|u\|_{\mathbb{E}_{m,g}})$ is a separable and reflexive Banach space (see [4, Proposition 2.1]).

We recall the following compact embedding lemma (see [6, Theorem 3.1]).

Lemma 2.1. *If $\mathbb{E}_{m,g}$ is defined in the above, the embedding $\mathbb{E}_{m,g} \hookrightarrow L^s(\mathbb{R}^N)$ is continuous if $m \le s \le m^*$ and compact if $m \le s < m^*$, where $m^* = Nm/(N - m)$.*

Next, let us consider two potential functions $g(x)$ and $h(x)$ such that (gh) holds. Take $\alpha > 0$ such that

$$\operatorname*{ess\,inf}_{x \in \mathbb{R}^N} (h(x) + \alpha) > 0.$$

Therefore, we can define the spaces $(\mathbb{E}_{p,g}, \|u\|_{\mathbb{E}_{p,g}})$ and $(\mathbb{E}_{q,h+\alpha}, \|u\|_{\mathbb{E}_{q,h+\alpha}})$. From the above statements, we know the spaces $(\mathbb{E}_{p,g}, \|u\|_{\mathbb{E}_{p,g}})$ and $(\mathbb{E}_{q,h+\alpha}, \|u\|_{\mathbb{E}_{q,h+\alpha}})$ are both reflexive and separable Banach space. From now on, we take into account the Banach space

$$W := \mathbb{E}_{p,g} \cap \mathbb{E}_{q,h+\alpha}$$

as our working space, and endow the norm

$$\|u\|_W := \|u\|_{\mathbb{E}_{p,g}} + \|u\|_{\mathbb{E}_{q,h+\alpha}}.$$

The following corollary is an immediate consequence of both Lemma 2.1 and the definition of W.

Corollary 2.2. *W is a separable and reflexive Banach space and the embedding $W \hookrightarrow L^s(\mathbb{R}^{\mathbb{N}})$ is continuous if $q \le s \le p^*$ and compact if $q \le s < p^*$.*

Finally, since W is a real, reflexive and separable Banach space. It is well-known (see [12, Chapter 4] or [32, Section 17]) that there exist $\{e_j\}_{j \in \mathbb{N}} \subset W$ and $\{e_j^*\}_{j \in \mathbb{N}} \subset W^*$ (W^* is the dual space of W) such that

$$W = \overline{\operatorname{span}\{e_j : j = 1, 2, \ldots\}}, \quad W^* = \overline{\operatorname{span}\{e_j^* : j = 1, 2, \ldots\}}^{w^*}$$

and

$$\langle e_i^*, e_j \rangle = \begin{cases} 1 & \text{if } i = j, \\ 0 & \text{if } i \ne j. \end{cases}$$

We denote

$$X_j = \operatorname{span}\{e_j\}, \quad Y_k = \bigoplus_{j=1}^{k} X_j = \operatorname{span}\{e_1, \ldots, e_k\}, \quad Z_k = \overline{\bigoplus_{j=k}^{\infty} X_j} = \overline{\operatorname{span}\{e_k, e_{k+1}, \ldots\}}.$$

In the end of this section, we will prove two essential lemmas. In order to simplify the presentation we will denote the norm $\|\cdot\|$ and $\|\cdot\|_{L^p}$ instead of $\|\cdot\|_W$ and $\|\cdot\|_{L^p(\mathbb{R}^N)}$.

Lemma 2.3. *If $s \in [q, p^*)$, denote*

$$\beta_{k,s} := \sup\{\|u\|_{L^s} : \|u\| = 1, u \in Z_k\},$$

then $\lim_{k\to\infty} \beta_{k,s} = 0$.

Proof. Clearly, $0 \leq \beta_{k+1,s} \leq \beta_{k,s}$ and $\beta_{k,s} \to \beta_s \geq 0$, $k \to +\infty$. For every $k \geq 0$, there exists $u_k \in Z_k$ such that $\|u_k\| = 1$ and $\|u_k\|_{L^s} > \beta_{k,s}/2$. By definition of Z_k, $u_k \rightharpoonup 0$ in W. Corollary 2.2 implies that $u_k \to 0$ in $L^s(\mathbb{R}^N)$. Thus, we have proved that $\beta_s = 0$. □

Lemma 2.4. *Assume that $\Theta\colon W \to \mathbb{R}$ is weakly continuous and $\Theta(0) = 0$. Then for each $r > 0$ and $k \in \mathbb{N}$ there exists*

$$\theta_k := \sup\{|\Theta(u)| : \|u\| \leq r, u \in Z_k\} < +\infty.$$

Moreover, $\lim_{k\to\infty} \theta_k = 0$.

Proof. It is obvious that $0 \leq \theta_{k+1} \leq \theta_k$ and $\theta_k \to \theta \geq 0$, $k \to +\infty$. For each $k \geq 0$, we can take $u_k \in Z_k$, $\|u_k\| \leq r$ such that $0 \leq \theta_k - |\Theta(u_k)| < 1/k$. By definition of Z_k, we have $u_k \rightharpoonup 0$ in W. The weakly continuity of Θ guarantees $\Theta(u_k) \to \Theta(0) = 0$. This proves that $\theta = 0$. □

3. Proofs of theorems

The Euler-Lagrange functional associated with problem (1.1) is

$$\begin{aligned}
I_\lambda(u) :=& \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + g(x)|u|^p)\,dx + \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + h(x)|u|^q)\,dx - \lambda\int_{\mathbb{R}^N} F(x,u)\,dx \\
\equiv& \frac{1}{p}\int_{\mathbb{R}^N} (|\nabla u|^p + g(x)|u|^p)\,dx + \frac{1}{q}\int_{\mathbb{R}^N} (|\nabla u|^q + (h(x)+\alpha)|u|^q)\,dx \\
&- \frac{\alpha}{q}\int_{\mathbb{R}^N} |u|^q\,dx - \lambda\int_{\mathbb{R}^N} F(x,u)\,dx.
\end{aligned}$$

As consequence of the hypotheses (f_1)–(f_4) and (gh), it is clear that the functional I_λ is well-defined in W and of class C^1.

In order to prove our theorems we need some technical lemmas presented below.

Lemma 3.1. *Suppose that* (f_1)–(f_3) *and* (gh) *hold. Then, for any finite dimensional subspace $\widehat{W} \subset W$ and for $u \in \widehat{W}$*

$$I_\lambda(u) \to -\infty \quad \text{as } \|u\| \to +\infty. \tag{3.1}$$

Proof. Arguing by contradiction, we can assume that a finite dimensional subspace $\widehat{W} \subset W$ exists which does not satisfy (3.1). Hence, a sequence $\{u_n\}_{n\in\mathbb{N}} \subset \widehat{W}$ can be found such that

$$\|u_n\| \to +\infty \quad \text{as } n \to +\infty \tag{3.2}$$

and for some $M > 0$ it is $I_\lambda(u_n) \geq -M$ for all $n \in \mathbb{N}$. Setting $w_n = u_n/\|u_n\|$, it follows that $\|w_n\| = 1$ and $w_n \rightharpoonup w$ weakly in W, up to subsequences, or better, since $\dim \widehat{W} < +\infty$, $w_n \to w$ strongly in $\widehat{W}$ and almost everywhere in $\mathbb{R}^N$. Thus, $\|w\| = 1$ and $\operatorname{meas}\Omega^* > 0$ where $\Omega^* := \{x \in \mathbb{R}^N : w(x) \neq 0\}$. Hence, $\lim_{n\to+\infty} |w_n(x)| = |w(x)| > 0$ for a.e. $x \in \Omega^*$, so by (3.2) it follows $|u_n.(x)| \to +\infty$. (f_1) and (f_3) imply

$$\lim_{n\to+\infty} \frac{F(x,u_n)}{|u_n|^p}|w_n| = +\infty \quad \text{for a.e. } x \in \Omega^*.$$

We can deduce that

$$\int_{\mathbb{R}^N} \lim_{n\to+\infty} \frac{F(x,u_n)}{|u_n|^p}|w_n| = +\infty. \tag{3.3}$$

On the other hand, for n sufficiently large, by standard calculations we get

$$\frac{\|u_n\|^p_{\mathbb{E}_{p,g}}}{p\|u_n\|^p} + \frac{\|u_n\|^q_{\mathbb{E}_{q,h+\alpha}}}{q\|u_n\|^p} \leq \frac{\|u_n\|^p_{\mathbb{E}_{p,g}} + \|u_n\|^q_{\mathbb{E}_{q,h+\alpha}}}{q\|u_n\|^p} \leq \frac{2\|u_n\|^p}{q\|u_n\|^p} = \frac{2}{q}$$

(without loss of generality, by (3.2) we assume $\|u_n\| \geq 1$ for all $n \in \mathbb{N}$). Then, by Fatou's lemma and (f_1), we get

$$\begin{aligned}
0 &= \lim_{n\to+\infty} \frac{-M}{\|u_n\|^p} \leq \limsup_{n\to+\infty} \frac{I_\lambda(u_n)}{\|u_n\|^p} \\
&= \limsup_{n\to+\infty} \frac{1}{\|u_n\|^p}\left(\frac{1}{p}\|u\|^p_{\mathbb{E}_{p,g}} + \frac{1}{q}\|u\|^q_{\mathbb{E}_{q,h+\alpha}} - \frac{\alpha}{q}\|u\|^q_{L^q} - \lambda\int_{\mathbb{R}^N} F(x,u_n)\,dx\right) \\
&\leq \limsup_{n\to+\infty}\left(\frac{2}{q} - \lambda\int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^p}\,dx\right) \\
&\leq \frac{2}{q} - \lambda\int_{\mathbb{R}^N} \liminf_{n\to+\infty} \frac{F(x,u_n)}{|u_n|^p}|w_n|^p\,dx \\
&= \frac{2}{q} - \lambda\int_{\mathbb{R}^N} \lim_{n\to+\infty} \frac{F(x,u_n)}{|u_n|^p}|w_n|^p\,dx,
\end{aligned}$$

whence

$$\lambda\int_{\mathbb{R}^N} \lim_{n\to+\infty} \frac{F(x,u_n)}{|u_n|^p}|w_n|^p\,dx \leq \frac{2}{q}$$

in contradiction with (3.3) as $\operatorname{meas}\Omega^* > 0$. □

Lemma 3.2. *Suppose that* (f_1), (f_2) *and* (gh) *hold. Then, for any $c > 0$ there exist $\rho > 0$ and $k \in \mathbb{N}$ such that*

$$I_\lambda(u) \geq c \quad \text{for all } u \in \partial B_\rho \cap Z_k,$$

where $B_\rho := \{u \in W : \|u\| < \rho\}$.

Proof. Above all, by (f_2), there exists $C > 0$ such that

$$F(x,t) \le K(x)|t| + C|t|^l \quad \text{for a.e. } x \in \Omega, \text{ all } t \in \mathbb{R}.$$

Hence, for any $u \in Z_k$ with $\|u\| > 1$, we have

$$\begin{aligned} I_\lambda(u) &\ge \frac{1}{p}\|u\|^p_{\mathbb{E}_{p,g}} + \frac{1}{q}\|u\|^q_{\mathbb{E}_{q,h+\alpha}} - \frac{\alpha}{q}\|u\|^q_{L^q} - \lambda\left(\int_{\mathbb{R}^N} K(x)|u|\,dx + C\int_{\mathbb{R}^N}|u|^l\,dx\right) \\ &\ge \frac{1}{p}\|u\|^p_{\mathbb{E}_{p,g}} + \frac{1}{q}\|u\|^q_{\mathbb{E}_{q,h+\alpha}} - \frac{\alpha}{q}\|u\|^q_{L^q} - \lambda\big(\|K\|_{L^{\tau^*}}\|u\|_{L^\tau} + C\|u\|^l_{L^l}\big) \\ &\ge \frac{1}{p}\|u\|^p_{\mathbb{E}_{p,g}} + \frac{1}{q}\|u\|^q_{\mathbb{E}_{q,h+\alpha}} - \frac{\alpha}{q}\beta^q_{k,q}\|u\|^q - \lambda\beta_{k,\tau}\|K\|_{L^{\tau^*}}\|u\| - \lambda C\beta^l_{k,l}\|u\|^l, \end{aligned}$$

where $\beta_{k,\theta} := \sup\{\|w\|_{L^\theta} : \|w\| = 1, w \in Z_k\}$, for $\theta \in [q, p^*)$.

We can assume, without loss of generality, that $\|u\|_{\mathbb{E}_{p,g}} \ge \|u\|/2 \ge \|u\|_{\mathbb{E}_{q,h+\alpha}}$. So, we can get

$$I_\lambda(u) \ge I_{\lambda,p}(u) := \frac{1}{p2^p}\|u\|^p - \frac{\alpha}{q}\beta^q_{k,q}\|u\|^q - \lambda\beta_{k,\tau}\|K\|_{L^{\tau^*}}\|u\| - \lambda C\beta^l_{k,l}\|u\|^l. \tag{3.4}$$

Since $1 < q < p < l$, by using Lemma 2.3, it is easy to see that $r_{p,k} := (C\beta^l_{k,l})^{1/(p-l)} \to +\infty$ as $k \to \infty$. Then, for k large enough, $u \in Z_k$ with $\|u\| = r_{p,k} > 1$, and by (3.4), we have

$$I_{\lambda,p}(u) = \left(\frac{1}{p2^p} - \lambda\right) r^p_{p,k} - \frac{\alpha}{q}\beta^q_{k,q} r^q_{p,k} - \lambda\|K\|_{L^{\tau^*}}\beta_{k,\tau} r_{p,k}.$$

Thus,

$$I_{\lambda,p}(u) \to +\infty \quad \text{as } k \to +\infty.$$

Analogously, considering $\|u\|_{\mathbb{E}_{p,g}} \le \|u\|/2 \le \|u\|_{\mathbb{E}_{q,h+\alpha}}$, we get

$$I_\lambda(u) \ge I_{\lambda,q}(u) := \left(\frac{1}{q2^q} - \frac{\alpha}{q}\beta^q_{k,q}\right)\|u\|^q - \lambda\beta_{k,\tau}\|K\|_{L^{\tau^*}}\|u\| - \lambda C\beta^l_{k,l}\|u\|^l$$

and also have $r_{q,k} := (C\beta^l_{k,l})^{1/(q-l)} \to +\infty$. For k sufficiently large, $u \in Z_k$ with $\|u\| = r_{q,k} > 1$, we can also obtain

$$I_{\lambda,q}(u) = \left(\frac{1}{q2^q} - \lambda - \frac{\alpha}{q}\beta^q_{k,q}\right) r^q_{q,k} - \lambda\|K\|_{L^{\tau^*}}\beta_{k,\tau} r_{q,k} \to +\infty \quad \text{as } k \to +\infty.$$

Since $1 < q < p$ and $\lambda < 1/(p2^p)$, setting $r_k = \min\{r_{p,k}, r_{q,k}\}$, for $u \in Z_k$ and $\|u\| = r_k$, we have

$$I_\lambda(u) \to +\infty \quad \text{as } k \to +\infty.$$

The proof is completed once we fix $\|u\| = r_k = \rho$ large enough. □

Here, we recall the well-known Cerami's variant of the Palais-Smale condition (see [23, 27]).

Definition 3.3. Let X be a real Banach space and $I \in C^1(X, \mathbb{R})$. The functional I satisfies the Cerami's variant of the Palais-Smale condition, briefly (C), if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that

$$\{I(u_n)\}_{n\in\mathbb{N}} \text{ is bounded and } \lim_{n\to+\infty} \|I'(u_n)\|_{X^*}(1 + \|u_n\|_X) = 0 \tag{3.5}$$

converges in X, up to a subsequence. We say that $\{u_n\}_{n\in\mathbb{N}}$ is a (C) sequence if it verifies (3.5).

Lemma 3.4. *Suppose that* (f_1)–(f_4) *and* (gh) *hold. Then the functional* I_λ *satisfies* (C) *condition.*

Proof. Let $\{u_n\} \subset W$ be a Cerami sequence associated with I_λ. Then, there exists some constant $C > 0$, which does not depend on n, such that

$$|I_\lambda(u_n)| \leq C \tag{3.6}$$

and

$$(1 + \|u_n\|)I'_\lambda(u_n) \to 0 \quad \text{as } n \to +\infty. \tag{3.7}$$

As a consequence of (3.7) there exists $\epsilon_n \to 0$ such that

$$|\langle I'_\lambda(u_n), v\rangle| \leq \frac{\epsilon_n\|v\|}{1 + \|u_n\|} \tag{3.8}$$

for all v in W and all $n \in \mathbb{N}$. From (3.6) and (3.8), we obtain

$$\begin{aligned} &\|u_n\|^p_{\mathbb{E}_{p,g}} + \|u_n\|^q_{\mathbb{E}_{q,h+\alpha}} - \alpha\|u\|^q_{L^q} - \lambda \int_{\mathbb{R}^N} f(x, u_n)u_n\,dx \\ &= \langle I'_\lambda(u_n), u_n\rangle \leq \frac{\epsilon_n\|u_n\|}{1 + \|u_n\|} \leq \epsilon_n \end{aligned} \tag{3.9}$$

and

$$-pC \leq \|u_n\|^p_{\mathbb{E}_{p,g}} + \frac{p}{q}\|u_n\|^q_{\mathbb{E}_{q,h+\alpha}} - \frac{\alpha p}{q}\|u\|^q_{L^q} - p\lambda \int_{\mathbb{R}^N} F(x, u_n)\,dx \leq pC. \tag{3.10}$$

Now we are going to prove, by contradiction, that $\{u_n\}$ is bounded in W. Let us assume that $\|u_n\| \to \infty$. Let $w_n := u_n/\|u_n\|$. By Corollary 2.2 we can also assume that, up to a subsequence, $w_n \to w$ in $L^s(\mathbb{R}^N)$, with $s \in [q, p^*)$.

Let $\lambda > 0$ be fixed. If w is not the null function, the set $\Omega^* := \{x \in \mathbb{R}^N : w(x) \neq 0\}$ has positive Lebesgue measure and, of course, $|u_n(x)| \to \infty$ for all $x \in \Omega^*$ (recall that we are assuming that $|u_n(x)| = |w_n(x)|\|u_n\|$, $\|u_n\| \to \infty$). By (f_2), we can get

$$\limsup_{n\to\infty} \frac{F(x, u_n)}{\|u_n\|^p} = \limsup_{n\to\infty} \frac{F(x, u_n)|w_n|^p}{|u_n|^p} = +\infty$$

for any $x \in \Omega^*$. We can deduce that

$$\limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^p}\,dx = \limsup_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n)|w_n|^p}{|u_n|^p}\,dx = +\infty. \tag{3.11}$$

It follows from (3.10) that

$$\begin{aligned}\lambda \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^p}\,dx &\le \frac{\|u_n\|^p_{\mathbb{E}_{p,g}}}{p\|u_n\|^p} + \frac{\|u_n\|^q_{\mathbb{E}_{q,h+\alpha}}}{q\|u_n\|^p} - \frac{\alpha\|u_n\|^q_{L^q}}{q\|u_n\|^p} + \frac{C}{\|u_n\|^p} \\ &\le \frac{1}{p} + \frac{1}{q\|u_n\|^{p-q}} + \frac{C}{\|u_n\|^p}.\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \frac{F(x,u_n)}{\|u_n\|^p}\,dx \le \frac{1}{\lambda p} < +\infty$$

and this contradicts (3.11).

We then are able to assume that $w = 0$ and again arrive at a contradiction. The continuity of the function $t \in [0,1] \mapsto I_\lambda(tu_n)$ for each $n \ge 1$, allows us to define the sequence $\{t_n\} \subset [0,1]$ by

$$I_\lambda(t_n u_n) = \max_{0\le t\le 1} I_\lambda(tu_n).$$

Let $v_n := (2\sigma)^{1/q} w_n = \frac{(2\sigma)^{1/q}}{\|u_n\|} u_n \in W$, where $\sigma > \frac{1}{2}\left(\frac{p}{q}\right)^{q/(p-q)}$. Then, $v_n \to 0$ a.e. in $\mathbb{R}^N$, and $v_n \to 0$ in $L^s(\mathbb{R}^N)$, for all $s \in [q, p^*)$. By Lebesgue theorem

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} F(x,v_n)\,dx = 0 \quad \text{and} \quad \lim_{n\to\infty} \|v_n\|^q_{L^q} = 0. \tag{3.12}$$

Since $\|u_n\| \to \infty$, there exists n_0 large enough such that $\frac{(2\sigma)^{1/q}}{\|u_n\|} \in (0,1)$ for all $n \ge n_0$. This implies that

$$\begin{aligned} I_\lambda(t_n u_n) &\ge I_\lambda(v_n) \\ &= \frac{(2\sigma)^{p/q}}{p}\|w_n\|^p_{\mathbb{E}_{p,g}} + \frac{2\sigma}{q}\|w_n\|^q_{\mathbb{E}_{q,h+\alpha}} - \frac{\alpha}{q}\|v_n\|^q_{L^q} - \lambda \int_{\mathbb{R}^N} F(x,v_n)\,dx \\ &\ge \frac{2\sigma}{q}\big(\|w_n\|^p_{\mathbb{E}_{p,g}} + \|w_n\|^q_{\mathbb{E}_{q,h+\alpha}}\big) - \frac{\alpha}{q}\|v_n\|^q_{L^q} - \lambda \int_{\mathbb{R}^N} F(x,v_n)\,dx \\ &\ge \frac{2\sigma}{q}\big(\|w_n\|^p_{\mathbb{E}_{p,g}} + \|w_n\|^p_{\mathbb{E}_{q,h+\alpha}}\big) - \frac{\alpha}{q}\|v_n\|^q_{L^q} - \lambda \int_{\mathbb{R}^N} F(x,v_n)\,dx \\ &\ge \frac{2\sigma}{q2^{p-1}}\big(\|w_n\|_{\mathbb{E}_{p,g}} + \|w_n\|_{\mathbb{E}_{q,h+\alpha}}\big)^p - \frac{\alpha}{q}\|v_n\|^q_{L^q} - \lambda \int_{\mathbb{R}^N} F(x,v_n)\,dx \\ &= \frac{2\sigma}{q2^{p-1}} - \frac{\alpha}{q}\|v_n\|^q_{L^q} - \lambda \int_{\mathbb{R}^N} F(x,v_n)\,dx, \end{aligned}$$

where we have used that $q < p$, $\|w_n\|_{\mathbb{E}_{q,h+\alpha}} \le \|w_n\| = 1$. By (3.12), we can choose $n_1 \ge n_0$ such that

$$\frac{\alpha}{q}\|v_n\|^q_{L^q} + \lambda \int_{\mathbb{R}^{\mathbb{N}}} F(x,v_n)\,dx < \frac{\sigma}{q2^{p-1}} \quad \text{for all } n \ge n_1.$$

It follows that

$$I_\lambda(t_n u_n) > \frac{2\sigma}{q2^{p-1}} - \frac{\sigma}{q2^{p-1}} = \frac{\sigma}{q2^{p-1}} \quad \text{for all } n \geq n_1$$

and since $\sigma > \frac{1}{2}\left(\frac{p}{q}\right)^{q/(p-q)}$ is arbitrary, we get

$$\lim_{n\to\infty} I_\lambda(t_n u_n) = +\infty. \tag{3.13}$$

Since $0 \leq t_n|u_n| \leq |u_n|$, (f_4) yields

$$\int_{\mathbb{R}^N} \hbar(x, t_n u_n)\,dx \leq \int_{\mathbb{R}^N} \hbar(x, u_n)\,dx + \int_{\mathbb{R}^N} \zeta(x)\,dx = \int_{\mathbb{R}^N} \hbar(x, u_n)\,dx + \|\zeta\|_{L^1} \tag{3.14}$$

for all $n \geq n_1$.

By taking new subsequence, if necessary, we can assume that $0 < t_n < 1$ for all $n \geq n_2 \geq n_1$. Indeed, (3.13) combined with (3.6) implies that $t_n \neq 1$, and the fact that $I_\lambda(0) = 0$ implies $t_n \neq 0$ for $n \geq n_2$. Thus, by the definition of t_n, we can obtain that

$$\begin{aligned} 0 &= t_n \frac{d}{dt} I_\lambda(t u_n)\Big|_{t=t_n} = \langle I'_\lambda(t_n u_n), t_n u_n\rangle \\ &= \|t_n u_n\|^p_{\mathbb{E}_{p,g}} + \|t_n u_n\|^q_{\mathbb{E}_{q,h+\alpha}} - \alpha\|t_n u_n\|^q_{L^q} - \lambda \int_{\mathbb{R}^N} f(x, t_n u_n) t_n u_n\,dx \end{aligned} \tag{3.15}$$

for all $n \geq n_2$.

By (f_4), (3.14) and (3.15), we obtain

$$\begin{aligned} &\|t_n u_n\|^p_{\mathbb{E}_{p,g}} + \|t_n u_n\|^q_{\mathbb{E}_{q,h+\alpha}} - \alpha\|t_n u_n\|^q_{L^q} \\ &\leq \lambda \left(\int_{\mathbb{R}^N} pF(x, t_n u_n)\,dx + \int_{\mathbb{R}^N} \hbar(x, t_n u_n)\,dx + \|\zeta\|_{L^1} \right) \end{aligned}$$

for all $n \geq n_2$. Therefore,

$$\begin{aligned} pI_\lambda(t_n u_n) &= \|t_n u_n\|^p_{\mathbb{E}_{p,g}} + \frac{p}{q}\|t_n u_n\|^q_{\mathbb{E}_{q,h+\alpha}} - \frac{p\alpha}{q}\|u_n\|^q_{L^q} - \lambda \int_{\mathbb{R}^N} pF(x, t_n u_n)\,dx \\ &\quad + \|t_n u_n\|^q_{\mathbb{E}_{q,h+\alpha}} - \|t_n u_n\|^q_{\mathbb{E}_{q,h+\alpha}} + \alpha\|t_n u_n\|^q_{L^q} - \alpha\|t_n u_n\|^q_{L^q} \\ &\leq \left(\frac{p}{q} - 1\right) \|u_n\|^q_{\mathbb{E}_{q,h+\alpha}} - \alpha\left(\frac{p}{q} - 1\right)\|u_n\|^q_{L^q} + \lambda \int_{\mathbb{R}^N} \hbar(x, u_n)\,dx + \lambda\|\zeta\|_{L^1}. \end{aligned}$$

Using (3.13) we obtain

$$\left(\frac{p}{q} - 1\right) \|u_n\|^q_{\mathbb{E}_{q,h+\alpha}} - \alpha\left(\frac{p}{q} - 1\right)\|u_n\|^q_{L^q} + \lambda \int_{\mathbb{R}^N} \hbar(x, u_n)\,dx \to \infty \tag{3.16}$$

as $n \to +\infty$. On the other hand, combining (3.9), (3.10) and (f_4) we get

$$\left(\frac{p}{q} - 1\right) \|u_n\|^q_{\mathbb{E}_{q,h+\alpha}} - \alpha\left(\frac{p}{q} - 1\right)\|u_n\|^q_{L^q} + \lambda \int_{\mathbb{R}^N} \hbar(x, u_n)\,dx \leq C,$$

which contradicts (3.16).

To sum up the above argument, we conclude that $\{u_n\}$ is bounded in W. Therefore, we can assume that $u_n \rightharpoonup u$ in W and $u_n \to u$ in $L^\theta(\mathbb{R}^N)$ for any $\theta \in [q, p^*)$. Furthermore, by hypothesis (f$_2$), Hölder inequality and Corollary 2.2, we can obtain that

$$\left| \int_{\mathbb{R}^N} |u_n|^{q-2} u_n (u_n - u)\, dx \right| \le \|u_n\|_{L^q}^{q-1} \|u_n - u\|_{L^q}$$

and

$$\begin{aligned} \left| \int_{\mathbb{R}^N} f(x, u_n)(u_n - u)\, dx \right| &\le \int_{\mathbb{R}^N} K(x)|u_n - u|\, dx + c \int_{\mathbb{R}^N} |u_n|^{l-1} |u_n - u|\, dx \\ &\le \|K\|_{L^{\tau^*}} \|u_n - u\|_{L^\tau} + c\|u_n\|_{L^l}^{l-1} \|u_n - u\|_{L^l}, \end{aligned}$$

where $\tau \in (q, p^*)$, $l \in (p, p^*)$ and $\tau^* = \tau/(\tau - 1)$. These imply that

$$\int_{\mathbb{R}^N} |u_n|^{q-2} u_n (u_n - u)\, dx \to 0 \quad \text{and} \quad \int_{\mathbb{R}^N} f(x, u_n)(u_n - u)\, dx \to 0$$

as $n \to +\infty$.

Hence, by taking into account that $(1 + \|u_n\|) I'_\lambda(u_n)(u_n - u) \to 0$ one has

$$\limsup_{n\to\infty} \langle -\Delta_p u_n - \Delta_q u_n + g\Phi_p(u_n) + (h + \alpha)\Phi_q(u_n), u_n - u\rangle = 0, \tag{3.17}$$

where $\Phi_m(t) = |t|^{m-2} t$.

Since the linear functional $W \ni \psi \mapsto \langle -\Delta_q u + g(x)\Phi_p(u) + (h(x) + \alpha)\Phi_q(u), \psi\rangle$ is bounded in W (it is straightforward to verify by Hölder inequality), it follows that

$$\lim_{n\to\infty} \langle -\Delta_q u + g\Phi_p(u) + (h + \alpha)\Phi_q(u), u_n - u\rangle = 0. \tag{3.18}$$

Using the monotonicity of the operator $-\Delta_q + g(x)\Phi_p + (h(x) + \alpha)\Phi_q$, we have

$$\begin{aligned} \langle -\Delta_p u_n, u_n - u\rangle &= \langle -\Delta_p u_n - \Delta_q u_n + g\Phi_p(u_n) + (h + \alpha)\Phi_q(u_n), u_n - u\rangle \\ &\quad - \langle -\Delta_q u_n + g\Phi_p(u_n) + (h + \alpha)\Phi_q(u_n), u_n - u\rangle \\ &\le \langle -\Delta_p u_n - \Delta_q u_n + g\Phi_p(u_n) + (h + \alpha)\Phi_q(u_n), u_n - u\rangle \\ &\quad - \langle -\Delta_q u + g\Phi_p(u) + (h + \alpha)\Phi_q(u), u_n - u\rangle. \end{aligned}$$

By combining this inequality with (3.17) and (3.18), we have

$$\limsup_{n\to\infty} \langle -\Delta_p u_n, u_n - u\rangle \le 0. \tag{3.19}$$

Arguing in the same way, we can also deduce that

$$\begin{aligned} &\limsup_{n\to\infty} \langle -\Delta_q u_n, u_n - u\rangle \le 0, \\ &\limsup_{n\to\infty} \langle g\Phi_p(u_n), u_n - u\rangle \le 0, \\ &\limsup_{n\to\infty} \langle (h + \alpha)\Phi_q(u_n), u_n - u\rangle \le 0. \end{aligned} \tag{3.20}$$

It is straightforward to verify that (3.19) and (3.20) yield

$$\limsup_{n\to\infty} \|u_n\| \le \|u\|,$$

which implies the strong convergence $u_n \to u$ in W, since this Banach space is uniformly convex. $\square$

Remark 3.5. $\Psi'(u) = -\Delta_p u - \Delta_q u + g\Phi_p(u) + (h+\alpha)\Phi_q(u)$ is of type (S_+).

Proof. It is an obvious consequence of the proof of Lemma 3.4. $\square$

3.1. Proof of Theorem 1.1

In this subsection, we give a proof of Theorem 1.1, which mainly relies on the following linking theorem.

Lemma 3.6. *Consider $a, b \in \mathbb{R}$ such that $a < b$. Assume that X is a real Banach space, let $I\colon X \to \mathbb{R}$ be a functional of class $C^1(X, \mathbb{R})$ that satisfies the* (C) *condition, and the following conditions hold:*

(i) *there exists a closed $S \subseteq X$ and $Q \subseteq Y$, Y a subspace of X, with boundary ∂Q in Y;*

(ii) *$I(u) \le a$ for all $u \in \partial Q$ and $I(u) \ge b$ for all $u \in S$;*

(iii) *S and ∂Q link, i.e. $S \cap \partial Q = \emptyset$ and $\phi(Q) \cap S \ne \emptyset$, for any $\phi \in C(X, X)$ such that $\phi|_{\partial Q} = \mathrm{id}$;*

(iv) $\sup_{u\in Q} I(u) < +\infty$.

Then, there exists a critical level c of I given by

$$c = \inf_{\phi\in\Gamma} \sup_{u\in Q} I(\phi(u)) \quad \textit{with } b \le c \le \sup_{u\in Q} I(u),$$

where $\Gamma = \{\phi \in C(X, X) : \phi|_{\partial Q} = \mathrm{id}\}$.

Proof of Theorem 1.1. By Lemmas 3.1 and 3.2, it follows that there exist some constants $\rho, c, R_1, R_2 > 0$ for k sufficiently large such that

$$R_2 > \rho, \quad \inf_{u\in S} I_\lambda(u) \ge c > 0 \quad \text{and} \quad \sup_{u\in\partial Q} I_\lambda(u) \le 0,$$

where $S = \partial B_\rho \cap Z_{k+1}$ and

$$Q = \{u + te \in W : u \in Y_k, e \in Z_{k+1}, \|u\| \le R_1, t \in [0, R_2]\}.$$

As a consequence, S and ∂Q link. By Lemma 3.4, the functional I_λ satisfies the (C) condition. Finally, Lemma 3.6 implies that problem (1.1) has a nontrivial weak solution. $\square$

3.2. Proof of Theorem 1.2

In order to prove Theorem 1.2, we will use the following fountain theorem.

Lemma 3.7. (Fountain theorem, [19, Theorem 2.9]) *Assume X is a Banach space, $I \in C^1(X,\mathbb{R})$ is an even functional. If for every $k \in \mathbb{N}$ there exist $\rho_k > r_k > 0$ such that*

(i) $b_k := \inf\{I(u) : u \in Z_k, \|u\| = r_k\} \to \infty$ *as* $k \to \infty$*;*

(ii) $a_k := \max\{I(u) : u \in Y_k, \|u\| = \rho_k\} \le 0$*;*

(iii) *I satisfies the* (C) *condition for every $c > 0$.*

Then I has an unbounded sequence of critical points such that $I(u_n) \to \infty$.

Proof of Theorem 1.2. By Lemmas 3.1 and 3.2, it was proven that if k is large enough, there exist $\rho_k > r_k > 0$ such that (i) and (ii) of Lemma 3.7 hold. This way, we have satisfied all the conditions of the fountain theorem. Hence, we obtain a sequence of critical points $\{u_n\}_{n\in\mathbb{N}} \subset W$ such that $I_\lambda(u_n) \to +\infty$ as $n \to \infty$. □

3.3. Proof of Theorem 1.3

Before proving Theorem 1.3, we will recall the $(\mathrm{C})_c^*$ condition and dual fountain theorem.

Definition 3.8. Let X be a separable and reflexive Banach space, $I \in C^1(X,\mathbb{R})$, $c \in \mathbb{R}$. The functional I satisfies the $(\mathrm{C})_c^*$ condition (with respect to Y_n which is defined in Section 2) if any sequence $\{u_n\}_{n\in\mathbb{N}} \subset X$ such that $u_n \in Y_n$, $I(u_n) \to c$ and $\|(I|_{Y_n})'(u_n)\|_{X^*}(1+\|u_n\|_X) \to 0$ as $n \to \infty$ contains a subsequence converging to a critical point of I.

Lemma 3.9. (Dual fountain theorem [29]) *Assume X is a Banach space and $I \in C^1(X,\mathbb{R})$ is an even functional. If there exists $k_0 \ge 1$, such that for each $k \ge k_0$, there exist ρ_k and r_k with $\rho_k > r_k > 0$ satisfying the following properties:*

(i) $a_k = \max\{I(u) : u \in Y_k, \|u\| = r_k\} < 0$*;*

(ii) $b_k = \inf\{I(u) : u \in Z_k, \|u\| = \rho_k\} \ge 0$*;*

(iii) $d_k = \inf\{I(u) : u \in Z_k, \|u\| \le \rho_k\} \to 0$ *as* $k \to +\infty$*;*

(iv) *I satisfies the* $(\mathrm{C})_c^*$ *condition for every* $c \in [d_{k_0}, 0)$.

Then I has a sequence of negative critical values converging to 0.

Lemma 3.10. *Suppose that* (f_1)–(f_4) *and* (gh) *hold, then I_λ satisfies the* $(\mathrm{C})_c^*$ *condition.*

Proof. Let $c \in \mathbb{R}$ and the sequence $\{u_n\}_{n\in\mathbb{N}} \subset W$ be such that $u_n \in Y_n$, for all $n \in \mathbb{N}$, $I_\lambda(u_n) \to c$ and $\|(I_\lambda|_{Y_n})'(u_n)\|_{W^*}(1 + \|u_n\|_W) \to 0$ as $n \to +\infty$. Therefore, we have

$$c = I_\lambda(u_n) + o_n(1) \quad \text{and} \quad \langle I'_\lambda(u_n), u_n\rangle = o_n(1),$$

where $o_n(1) \to 0$ as $n \to +\infty$.

Analogous to the proof of Lemma 3.4, we can prove that $\{u_n\}_{n\in\mathbb{N}}$ is bounded in W. Since W is a reflexive space, by the Eberlein-Šmulian theorem, we can choose a subsequence of $\{u_n\}_{n\in\mathbb{N}}$, denoted for $\{u_{n_k}\}_{k\in\mathbb{N}}$, such that $u_{n_k} \rightharpoonup u$ weakly in W.

On the other hand, as $W = \overline{\bigcup_n Y_n} = \overline{\text{span}\{e_n : n \geq 1\}}$, we can choose $v_{n_k} \in Y_{n_k}$ such that $v_{n_k} \to u$ strongly in W. Hence, by $(I_\lambda|_{Y_{n_k}})'(u_{n_k}) \to 0$ and $u_{n_k} - v_{n_k} \rightharpoonup 0$ in Y_{n_k} as $k \to +\infty$ (see [8, Proposition 3.5]), we get

$$\lim_{k\to+\infty} \langle I'_\lambda(u_{n_k}), u_{n_k} - u\rangle = \lim_{k\to+\infty} \big(\langle I'_\lambda(u_{n_k}), u_{n_k} - v_{n_k}\rangle + \langle I'_\lambda(u_{n_k}), v_{n_k} - u\rangle\big) = 0.$$

We notice that

$$\begin{aligned}\langle \Psi'(u_{n_k}), u_{n_k} - u\rangle = \alpha \int_{\mathbb{R}^N} |u_{n_k}|^{q-2} u_{n_k}(u_{n_k} - u)\, dx + \lambda \int_{\mathbb{R}^N} f(x, u_{n_k})(u_{n_k} - u)\, dx \\ + \langle I'_\lambda(u_{n_k}), u_{n_k} - u\rangle.\end{aligned}$$

Therefore, we have $\langle \Psi'(u_{n_k}), u_{n_k} - u\rangle \to 0$ as $k \to +\infty$, where $\Psi'(u) = -\Delta_p u - \Delta_q u + g\Phi_p(u) + (h + \alpha)\Phi_q(u)$. Since Ψ' is of type (S_+) (see Remark 3.5), it follows that $u_{n_k} \to u$ strongly in W. Then, we conclude that I_λ satisfies the $(\mathrm{C})^*_c$ condition. Furthermore, we have $I'_\lambda(u_{n_k}) \to I'_\lambda(u)$ as $k \to +\infty$.

Now, we claim that $I'_\lambda(u) = 0$. Indeed, taking $\psi_j \in Y_j$, if $n_k \geq j$, we have

$$\begin{aligned}\langle I'_\lambda(u), \psi_j\rangle &= \langle I'_\lambda(u) - I'_\lambda(u_{n_k}), \psi_j\rangle + \langle I'_\lambda(u_{n_k}), \psi_j\rangle \\ &= \langle I'_\lambda(u) - I'_\lambda(u_{n_k}), \psi_j\rangle + \langle (I_\lambda|_{Y_{n_k}})'(u_{n_k}), \psi_j\rangle.\end{aligned}$$

Hence, passing the limit on the right side of the above equation as $k \to +\infty$, we obtain $\langle I'_\lambda(u), \psi_j\rangle = 0$ for all $\psi_j \in Y_k$. Thus $I'_\lambda(u) = 0$ in W^*, and this shows that I_λ satisfies the $(\mathrm{C})^*_c$ condition for every $c \in \mathbb{R}$. □

Next, we will prove that I_λ satisfies the conditions (i)–(iv) of the dual fountain theorem under the hypotheses in Theorem 1.3 hold.

Claim 1: For each $k \in \mathbb{N}$ there exists r_k such that

$$\max\{I_\lambda(u) : u \in Y_k, \|u\| = r_k\} < 0.$$

Indeed, since Y_k is finite dimensional subspace, all norms are equivalent. For $\|u_0\| = 1$, by Lemma 3.1, we deduce

$$\lim_{t\to+\infty} I_\lambda(tu_0) = -\infty.$$

Therefore, there exists $t^* \in (1,+\infty)$ such that

$$I_\lambda(tu_0) < 0 \quad \text{for all } t \in [t^*,+\infty).$$

Hence, $I_\lambda(u) < 0$ for all $u \in Y_k$ with $\|u\| = t^*$. Taking $r_k = t^*$ for all $k \in \mathbb{N}$, we obtain

$$a_k = \max\{I_\lambda(u) : u \in Y_k, \|u\| = r_k\} < 0.$$

Claim 2: There exists $k_0 \in \mathbb{N}$ such that for each $k \geq k_0$, there exists $\rho_k > 0$ with $\rho_k > r_k > 0$ for which

$$b_k = \inf\{I_\lambda(u) : u \in Z_k, \|u\| = \rho_k\} \geq 0.$$

As in the proof of Lemma 3.2, since $0 < \lambda < 1/(p2^p)$ and $\beta_{k,q}, \beta_{k,l}, \beta_{k,\tau} \to 0$ as $k \to +\infty$ (see Lemma 2.3), we have

$$\lim_{k\to+\infty} \left(\left(\frac{1}{\eta 2^\eta} - \lambda \right) (C\beta_{k,l}^l)^{\eta/(\eta-l)} - \frac{\alpha}{q} \beta_{k,q}^q (C\beta_{k,l}^l)^{q/(\eta-l)} - \lambda \|K\|_{L^{\tau^*}} \beta_{k,\tau} (C\beta_{k,l}^l)^{1/(\eta-l)} \right)$$
$$= +\infty$$

for $\eta = p$ or q. Thus, there exists $k_0 \in \mathbb{N}$ such that

$$\left(\frac{1}{\eta 2^\eta} - \lambda \right) (C\beta_{k,l}^l)^{\eta/(\eta-l)} - \frac{\alpha}{q} \beta_{k,q}^q (C\beta_{k,l}^l)^{q/(\eta-l)} - \lambda \|K\|_{L^{\tau^*}} \beta_{k,\tau} (C\beta_{k,l}^l)^{1/(\eta-l)} \geq 0$$

for all $k \geq k_0$. So, for all $k \geq k_0$, $u \in Z_k$ with $\|u\| = \rho_k = (C\beta_{k,l}^l)^{1/(q-l)}$, we get $I_\lambda(u) \geq 0$. This implies

$$b_k = \inf\{I_\lambda(u) : u \in Z_k, \|u\| = \rho_k\} \geq 0.$$

Claim 3: $d_k = \inf\{I_\lambda(u) : u \in Z_k, \|u\| \leq \rho_k\} \to 0$ as $k \to +\infty$. Indeed, noticing that $Y_k \cap Z_k \neq \emptyset$ and $\rho_k > r_k > 0$, we have

$$d_k = \inf\{I_\lambda(u) : u \in Z_k, \|u\| \leq \rho_k\} \leq a_k = \max\{I_\lambda(u) : u \in Y_k, \|u\| = r_k\} < 0.$$

By (f_1), we can consider $\Upsilon_i \colon W \to \mathbb{R}$ ($i = 1,2,3$) defined by

$$\Upsilon_1(u) = \alpha \int_{\mathbb{R}^N} |u|^q \, dx, \quad \Upsilon_2(u) = \lambda C \int_{\mathbb{R}^N} |u|^l \, dx \quad \text{and} \quad \Upsilon_3(u) = \lambda \int_{\mathbb{R}^N} K(x)|u| \, dx.$$

It is easy to see that $\Upsilon_i(0) = 0$ for $i = 1,2,3$, and they are weakly continuous. Let us denote

$$\theta_{k,i} = \sup\{|\Upsilon_i(u)| : u \in Z_k, \|u\| = 1\}.$$

By the compact embedding $W \hookrightarrow\hookrightarrow L^s(\mathbb{R}^N)$ for $s \in [q, p^*)$ and Lemma 2.4, we get $\lim_{k\to+\infty} \theta_{k,i} = 0$. Furthermore, taking $u \in Z_k$ with $\|u\| = 1$ and $0 < t < \rho_k$, we have

$$\begin{aligned} I_\lambda(tu) &\geq -\alpha \int_{\mathbb{R}^N} |tu|^q \, dx - \lambda \int_{\mathbb{R}^N} F(x, tu) \, dx \\ &\geq \Upsilon_1(tu) - \Upsilon_2(tu) - \Upsilon_3(tu) \\ &= -t^q \Upsilon_1(u) - t^l \Upsilon_2(u) - t\Upsilon_3(u). \end{aligned}$$

So, for all $t \in (0, \rho_k)$ and $u \in Z_k$ with $\|u\| = 1$, we deduce

$$I_\lambda(tu) \geq -\rho_k^q \Upsilon_1(u) - \rho_k^l \Upsilon_2(u) - \rho_k \Upsilon_3(u) \geq -\rho_k^q \theta_{k,1} - \rho_k^l \theta_{k,2} - \rho_k \theta_{k,3}.$$

This implies,

$$d_k \geq -\rho_k^q \theta_{k,1} - \rho_k^l \theta_{k,2} - \rho_k \theta_{k,3},$$

and as $d_k < 0$ for all $k \geq k_0$, we have $\lim_{k\to+\infty} d_k = 0$.

Claim 4: I_λ satisfies the $(\mathrm{C})_c^*$ condition for $c \in [d_{k_0}, 0)$. The proof of Lemma 3.10 has shown that I_λ satisfies the $(\mathrm{C})_c^*$ condition for $c \in \mathbb{R}$.

Proof of Theorem 1.3. It is clear that I_λ is even and satisfies the $(\mathrm{C})_c^*$ condition by Lemma 3.10. Furthermore, we have proved that I_λ satisfies the all conditions of dual fountain theorem. By Lemma 3.9, there exists a sequence of negative critical values converging to 0, which concludes the proof of Theorem 1.3. □

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