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Diophantine Approximation with Mixed Powers of Primes

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Abstract. Let k be an integer with $k \geq 3$. Let λ_1, λ_2, λ_3 be non-zero real numbers, not all negative. Assume that λ_1/λ_2 is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence, and $\delta > 0$. In this paper, we prove that, for any $\varepsilon > 0$, the number of $\upsilon \in \mathcal{V}$ with $\upsilon \leq X$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^k - \upsilon| < \upsilon^{-\delta}$$

has no solution in primes p_1, p_2, p_3 does not exceed $O(X^{1-2/(7m_2(k))+2\delta+\varepsilon})$, where $m_2(k)$ relies on k. This refines a recent result. Furthermore, we briefly describe how a similar method can refine a previous result on a Diophantine problem with two squares of primes, one cube of primes and one k-th power of primes.

1. Introduction

Diophantine approximation is an important topic in number theory. We first introduce the definition of a well-spaced sequence. We call an increasing sequence $\upsilon_1 < \upsilon_2 < \cdots$ of positive real numbers a well-spaced sequence if there exist positive constants $C > c > 0$ such that

$$0 < c < \upsilon_{i+1} - \upsilon_i < C, \quad i = 1, 2, \ldots.$$

Let k be an integer. In this paper, we first consider a Diophantine problem with two squares of primes and one k-th power of primes. Let

$$m_2(k) = \begin{cases} 4 & \text{if } k = 3, \\ 2^{k/2} & \text{if } k = 4, 6, 8, \\ \frac{1}{2}\left(2^{(k-1)/2} + 2^{(k+1)/2}\right) & \text{if } k = 5, 7, 9, \\ \min\left(2^{[(k+1)/2]}, [\frac{k+1}{2}]\left([\frac{k+1}{2}] + 1\right)\right) & \text{if } k \geq 10, \end{cases} \tag{1.1}$$

where $[x]$ denotes the greatest integer not exceeding x. Let λ_1, λ_2, λ_3 be non-zero real numbers, not all negative. Let $\mathcal{V}$ be a well-spaced sequence, and $\delta > 0$. Let $E_k(\mathcal{V}, X, \delta)$ denote the number of $\upsilon \in \mathcal{V}$ with $\upsilon \leq X$ such that the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^k - \upsilon| < \upsilon^{-\delta} \tag{1.2}$$

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has no solution in primes p_1, p_2, p_3. The inequality (1.2) with $k = 2$ is considered in Harman [6]. Throughout this paper, constants, both explicit and implicit, in Vinogradov symbols may depend on λ_1, λ_2, λ_3, λ_4. We study the inequality (1.2) with $k \geq 3$ and prove the following theorems.

Theorem 1.1. *Let k be an integer with $k \geq 3$. Let λ_1, λ_2, λ_3 be non-zero real numbers, not all negative. Assume that λ_1/λ_2 is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence, and $\delta > 0$. Then we have, for any $\varepsilon > 0$,*

$$E_k(\mathcal{V}, X, \delta) \ll X^{1-2/(7m_2(k))+2\delta+\varepsilon}, \tag{1.3}$$

where $m_2(k)$ is defined by (1.1).

Theorem 1.2. *Let k be an integer with $k \geq 3$. Let λ_1, λ_2, λ_3 be non-zero real numbers, not all negative. Assume that λ_1/λ_2 is irrational. Let $\mathcal{V}$ be a well-spaced sequence, and $\delta > 0$. Then there is a sequence $X_j \to \infty$ such that, for any $\varepsilon > 0$,*

$$E_k(\mathcal{V}, X_j, \delta) \ll X_j^{1-2/(7m_2(k))+2\delta+\varepsilon}, \tag{1.4}$$

where $m_2(k)$ is defined by (1.1). *Moreover, if the convergent denominators q_j for λ_1/λ_2 satisfy*

$$q_{j+1}^{1-w} \ll q_j \quad \text{for some } w \in [0, 1), \tag{1.5}$$

then we have, for all $X \geq 1$ and any $\varepsilon > 0$,

$$E_k(\mathcal{V}, X, \delta) \ll X^{1-(2-4\chi)/m_2(k)+2\delta+\varepsilon} \tag{1.6}$$

with

$$\chi = \max\left(\frac{5 - 3w + 2/m_2(k)}{12 - 8w + 4/m_2(k)}, \frac{3}{7}\right). \tag{1.7}$$

Results of this type were first obtained by Ge and Wang [5]. Let

$$\sigma(k) = \min\left(2^{[(k+1)/2]-1}, \frac{1}{2}\left[\frac{k+1}{2}\right]\left(\left[\frac{k+1}{2}\right] + 1\right)\right).$$

They proved that

$$\begin{aligned} E_k(\mathcal{V}, X, \delta) &\ll X^{1-1/(8\sigma(k))+2\delta+\varepsilon}, \\ E_k(\mathcal{V}, X_j, \delta) &\ll X_j^{1-1/(8\sigma(k))+2\delta+\varepsilon}, \\ E_k(\mathcal{V}, X, \delta) &\ll X^{1-(1-\chi)/\sigma(k)+2\delta+\varepsilon} \end{aligned}$$

with

$$\chi = \max\left(\frac{5+1/\sigma(k)-3w}{6+1/\sigma(k)-4w}, \frac{7}{8}\right),$$

in replace of (1.3), (1.4), (1.6) and (1.7), respectively. It is easy to verify that the results in Theorems 1.1 and 1.2 improve Ge and Wang's results.

Theorem 1.1 follows immediately from Theorem 1.2, since, in the case of λ_1/λ_2 algebraic, we can take $w = \varepsilon$.

To prove Theorem 1.2, we apply the Davenport-Heilbronn version of the Hardy-Littlewood method. On the one hand, the improvement not only derives from more carefully estimating the integral by using an optimal choice of different Hölder's inequalities from [5], but also the recent new breakthrough of Vinogradov's mean value theorem [1]. On the other hand, motivated by the work of Wang and Yao [13], we use the sieve functions $\rho^{\pm}(m)$ constructed in Harman [6], which also results in the improvement.

Many authors also considered the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^3 + \lambda_4 p_4^k - v| < v^{-\delta}. \tag{1.8}$$

Let λ_1, λ_2, λ_3, λ_4 be positive real numbers. Assume that λ_1/λ_2 is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence, and $\delta > 0$. Let $E_k^*(\mathcal{V}, X, \delta)$ denote the number of $v \in \mathcal{V}$ with $v \le X$ such that the inequality (1.8) has no solution in primes p_1, p_2, p_3, p_4. In 2010, Li and Wang [8] proved that, for any $\varepsilon > 0$,

$$E_3^*(\mathcal{V}, X, \delta) \ll X^{20/21+2\delta+\varepsilon}.$$

Subsequently, the exponent 20/21 was improved to 67/72 by Mu and Lü [9]. Recently Mu and Lü [10] refined the exponent to 29/33. In the same paper [10], Mu and Lü also considered the case $k \ge 4$ and proved that

$$E_k^*(\mathcal{V}, X, \delta) \ll \begin{cases} X^{1-1/11+2\delta+\varepsilon} & \text{if } k = 4, \\ X^{1-2/(11k)-16/(11k^2(k+1))+2\delta+\varepsilon} & \text{if } k \ge 5. \end{cases}$$

It is not hard to show that the argument proving Theorem 1.2 leads also to improvements on the results established by Mu and Lü [10]. Let

$$m_3(k) = \begin{cases} 2^{[(k-1)/6]+1} & \text{if } 5 \le k \le 48, \\ 2\left[\frac{k^2+k+6}{12}\right] & \text{if } k \ge 49. \end{cases} \tag{1.9}$$

We have the following theorem.

Theorem 1.3. *Let k be an integer with $k \ge 3$. Let λ_1, λ_2, λ_3, λ_4 be positive real numbers. Assume that λ_1/λ_2 is irrational and algebraic. Let $\mathcal{V}$ be a well-spaced sequence, and $\delta > 0$. Then we have, for any $\varepsilon > 0$,*

$$E_k^*(\mathcal{V}, X, \delta) \ll X^{1-\sigma^*(k)+2\delta+\varepsilon},$$

where

$$\sigma^*(k) = \begin{cases} 1/7 & \text{if } k = 3, 4, \\ 1/14 + 1/(7m_3(k)) & \text{if } 5 \le k \le 48, \\ 1/14 + 2/(7m_2(k)) & \text{if } k \ge 49, \end{cases} \tag{1.10}$$

and $m_2(k)$, $m_3(k)$ *are defined by* (1.1) *and* (1.9), *respectively.*

Since the proof of Theorem 1.3 is similar to that of Theorem 1.2, we prove Theorem 1.2 in the following Sections 2–6 and sketch the proof of Theorem 1.3 in Section 7.

Notation. Throughout this paper, the letter p, with or without a subscript, always denotes a prime. We use ε to denote a sufficiently small positive number, and the value of ε may change from statement to statement. We abbreviate $\log x$ to L.

2. Outline of the method

To use the Davenport-Heilbronn method, we first introduce some notations. Suppose that $k \ge 3$ is an integer and η is a fixed sufficiently small positive number. Let $0 < \tau < 1$, indeed we shall chose $\tau = X^{-\delta}$ in Section 6. Let

$$K_\tau(\alpha) = \left(\frac{\sin(\pi\tau\alpha)}{\pi\alpha} \right)^2$$

for $\tau > 0$ and $\alpha \ne 0$. By continuity, we define $K_\tau(0) = \tau^2$. Then we have

$$K_\tau(\alpha) \ll \min(\tau^2, |\alpha|^{-2}), \tag{2.1}$$

and

$$\widehat{K}_\tau(t) := \int_{\mathbb{R}} e(t\alpha) K_\tau(\alpha)\, d\alpha = \max(0, \tau - |t|). \tag{2.2}$$

Assume that a/q is a convergent to λ_1/λ_2, with the denominator q sufficiently large. Fix $X = q^{7/3}$. We write

$$I_1 = \left[\left(\frac{\eta X}{\lambda_1} \right)^{1/2}, \left(\frac{2\eta X}{\lambda_1} \right)^{1/2} \right],\ I_2 = \left[\left(\frac{\eta X}{\lambda_2} \right)^{1/2}, \left(\frac{2\eta X}{\lambda_2} \right)^{1/2} \right],\ I_3 = \left[\left(\frac{\eta X}{\lambda_3} \right)^{1/k}, \left(\frac{2\eta X}{\lambda_3} \right)^{1/k} \right].$$

Let $\rho_0(m)$ denote the characteristic function of the set of primes. To prove Theorem 1.2, we suppose that we have arithmetic functions $\rho^\pm(m)$ such that, for $m \in I_i$, $i = 1, 2$,

$$\rho^-(m) \le \rho_0(m) \le \rho^+(m).$$

Our choice of $\rho^\pm(m)$ is borrowed from Harman [6]. Namely, ρ^- and ρ^+ are the functions b_0 and b_1 constructed in Section 8 of [6], respectively. Here we just state some properties of the sieve functions $\rho^{\pm}(m)$, one can refer to [6] (see also [7]) for their construction in detail. In many ways, the functions $\rho^{\pm}(m)$ imitate the characteristic functions of primes. In particular, as in Section 8 of [6], for any subinterval $\mathcal{I} \in I_i$, $i = 1, 2$, one has

$$\sum_{m\in\mathcal{I}} \rho^{\pm}(m) = \kappa_i^{\pm}|\mathcal{I}|L^{-1} + O(X^{1/2}L^{-2}), \tag{2.3}$$

where $\kappa_i^{\pm} > 0$ are absolute constants satisfying

$$\kappa_i^{-} > 0.9, \quad \kappa_i^{+} < 1.7, \quad i = 1, 2. \tag{2.4}$$

Then the vector sieve of Brüdern and Fouvry [2] gives

$$\rho_0(m_1)\rho_0(m_2) \geq \rho^{-}(m_1)\rho^{+}(m_2) + \rho^{+}(m_1)\rho^{-}(m_2) - \rho^{+}(m_1)\rho^{+}(m_2). \tag{2.5}$$

We define

$$S_1(\alpha, \rho) = \sum_{m\in I_1} \rho(m)e(m^2\alpha), \quad S_2(\alpha, \rho) = \sum_{m\in I_2} \rho(m)e(m^2\alpha), \quad S_k(\alpha) = \sum_{p\in I_3} (\log p)e(p^k\alpha).$$

For any measurable subset $\mathfrak{X}$ of $\mathbb{R}$, write

$$I(\tau, \upsilon, \mathfrak{X}, \rho_1, \rho_2) = \int_{\mathfrak{X}} S_1(\lambda_1\alpha, \rho_1)S_2(\lambda_2\alpha, \rho_2)S_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\, d\alpha.$$

Then from (2.2), we have

$$\begin{aligned} I(\tau, \upsilon, \mathbb{R}, \rho_0, \rho_0) &= \sum_{\substack{m_i\in I_i, i=1,2\\ p_3\in I_3}} \rho_0(m_1)\rho_0(m_2)\log p_3 \\ &\qquad \times \int_{-\infty}^{\infty} e\big((\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 p_3^k - \upsilon)\alpha\big)K_\tau(\alpha)\, d\alpha \\ &= \sum_{\substack{m_i\in I_i, i=1,2\\ p_3\in I_3}} \rho_0(m_1)\rho_0(m_2)\log p_3 \\ &\qquad \times \max(0, \tau - |\lambda_1 m_1^2 + \lambda_2 m_2^2 + \lambda_3 p_3^k - \upsilon|). \end{aligned}$$

Thus we have

$$I(\tau, \upsilon, \mathbb{R}, \rho_0, \rho_0) \ll \tau \log X\mathfrak{N}(X),$$

where $\mathfrak{N}$(X) denotes the number of solutions of the inequality

$$|\lambda_1 p_1^2 + \lambda_2 p_2^2 + \lambda_3 p_3^k - \upsilon| < \tau$$

with $p_j \in I_j$ for $1 \leq j \leq 3$. Recalling (2.5), we have

$$I(\tau, \upsilon, \mathbb{R}, \rho_0, \rho_0) \geq I(\tau, \upsilon, \mathbb{R}, \rho^{-}, \rho^{+}) + I(\tau, \upsilon, \mathbb{R}, \rho^{+}, \rho^{-}) - I(\tau, \upsilon, \mathbb{R}, \rho^{+}, \rho^{+}). \tag{2.6}$$

Then it is sufficient to give a positive lower bound for the right side of (2.6). Due to a dyadic dissection argument, we focus on those υ such that $X/2 \le \upsilon \le X$. To do this, we divide the real line into the major arc $\mathfrak{M}$, the minor arc $\mathfrak{m}$ and the trivial arc $\mathfrak{t}$. We define

$$\mathfrak{M} = \{\alpha : |\alpha| \le P/X\}, \quad \mathfrak{m} = \{\alpha : P/X < |\alpha| \le R\}, \quad \mathfrak{t} = \{\alpha : |\alpha| > R\},$$

where $P = X^{1/4}$ and $R = \tau^{-2}X^{1/2+2\varepsilon}$. Thus we can write

$$I(\tau,\upsilon,\mathbb{R},\rho_1,\rho_2) = I(\tau,\upsilon,\mathfrak{M},\rho_1,\rho_2) + I(\tau,\upsilon,\mathfrak{m},\rho_1,\rho_2) + I(\tau,\upsilon,\mathfrak{t},\rho_1,\rho_2).$$

We write

$$H(\alpha) = S_1(\lambda_1\alpha,\rho^-)S_2(\lambda_2\alpha,\rho^+) + S_1(\lambda_1\alpha,\rho^+)S_2(\lambda_2\alpha,\rho^-) - S_1(\lambda_1\alpha,\rho^+)S_2(\lambda_2\alpha,\rho^+).$$

We estimate the integrals on the major arc, the minor arc and the trivial arc in the following Sections 3, 4 and 5, respectively. In Section 6, we complete the proof of Theorem 1.2.

3. The major arc

In this section, we estimate the contribution from the right side of (2.6) on the major arc. Since the methods of estimating $I(\tau,\upsilon,\mathfrak{M},\rho^-,\rho^+)$, $I(\tau,\upsilon,\mathfrak{M},\rho^+,\rho^-)$ and $I(\tau,\upsilon,\mathfrak{M},\rho^+,\rho^+)$ are similar, we can focus on $I(\tau,\upsilon,\mathfrak{M},\rho^-,\rho^+)$ in the following. As in [5], we first consider the standard major arc $\widetilde{\mathfrak{M}} = \{\alpha : |\alpha| \le \phi = X^{-1+5/(6k)-\varepsilon}\}$. Let

$$T_1(\alpha) = \int_{I_1} e(t^2\alpha)\,dt, \quad T_2(\alpha) = \int_{I_2} e(t^2\alpha)\,dt, \quad T_k(\alpha) = \int_{I_3} e(t^k\alpha)\,dt. \tag{3.1}$$

Then, using a trivial bound for S_j, and the first derivative estimate for trigonometric integrals (see Titchmarsh [11, Lemma 4.2]), one has

$$\begin{aligned} S_1(\alpha,\rho^-) &\ll X^{1/2}, & T_1(\alpha) &\ll X^{1/2-1}\min(X,|\alpha|^{-1}),\\ S_2(\alpha,\rho^+) &\ll X^{1/2}, & T_2(\alpha) &\ll X^{1/2-1}\min(X,|\alpha|^{-1}),\\ S_k(\alpha) &\ll X^{1/k}, & T_k(\alpha) &\ll X^{1/k-1}\min(X,|\alpha|^{-1}). \end{aligned} \tag{3.2}$$

Then we can rewrite $I(\tau,\upsilon,\widetilde{\mathfrak{M}},\rho^-,\rho^+)$ as follows:

$$\begin{aligned} &I(\tau,\upsilon,\widetilde{\mathfrak{M}},\rho^-,\rho^+)\\ &= \kappa_1^-\kappa_2^+L^{-2}\int_{\widetilde{\mathfrak{M}}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha\\ &\quad + \kappa_2^+L^{-1}\int_{\widetilde{\mathfrak{M}}} \big(S_1(\lambda_1\alpha,\rho^-) - \kappa_1^-L^{-1}T_1(\lambda_1\alpha)\big)T_2(\lambda_2\alpha)T_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha\\ &\quad + \int_{\widetilde{\mathfrak{M}}} S_1(\lambda_1\alpha,\rho^-)\big(S_2(\lambda_2\alpha,\rho^+) - \kappa_2^+L^{-1}T_2(\lambda_2\alpha)\big)T_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha\\ &\quad + \int_{\widetilde{\mathfrak{M}}} S_1(\lambda_1\alpha,\rho^-)S_2(\lambda_2\alpha,\rho^+)\big(S_k(\lambda_3\alpha) - T_k(\lambda_3\alpha)\big)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha\\ &:= J + J_1 + J_2 + J_3. \end{aligned} \tag{3.3}$$

In the following, we restrict our attention to estimate J, J_1 and J_2, since the computation for J_3 is similar to the corresponding one in Ge and Wang [5]. We first establish the lower bound for J. Note that

$$\begin{aligned} J &= \kappa_1^- \kappa_2^+ L^{-2} \int_{\mathbb{R}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha \\ &\quad + O\Big(L^{-1} \int_{|\alpha|>\phi} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha\Big). \end{aligned} \tag{3.4}$$

Putting this together with (2.1) and (3.2), we obtain that the error term in (3.4) is

$$\ll \tau^2 X^{1/k-2} \int_{|\alpha|>\phi} \frac{d\alpha}{|\alpha|^3} \ll \tau^2 X^{1/k-5/(3k)+2\varepsilon} = o(\tau^2 X^{1/k} L^{-2}). \tag{3.5}$$

We write

$$\begin{aligned} f(\upsilon) &= \int_{\mathbb{R}} T_1(\lambda_1\alpha)T_2(\lambda_2\alpha)T_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha \\ &= \int_{I_1\times I_2\times I_3} \max(0, \tau - |\lambda_1 t_1^2 + \lambda_2 t_2^2 + \lambda_3 t_3^k - \upsilon|)\,dt_1dt_2dt_3. \end{aligned} \tag{3.6}$$

Thus from (3.4), (3.5) and (3.6), we get

$$J = f(\upsilon)\kappa_1^-\kappa_2^+(1+o(1))L^{-2}. \tag{3.7}$$

Now we turn to establish the upper bound for the integral J_1. Using (2.1), we have

$$J_1 \ll \tau^2 L^{-1} \int_{\widetilde{\mathfrak{M}}} |S_1(\lambda_1\alpha, \rho^-) - \kappa_1^- L^{-1} T_1(\lambda_1\alpha)||T_2(\lambda_2\alpha)||T_k(\lambda_3\alpha)|\,d\alpha. \tag{3.8}$$

By partial summation, we obtain

$$S_1(\lambda_1\alpha, \rho^-) = \int_{I_1} e(\lambda_1 t^2\alpha)\,d\Big(\sum_{m\le t, m\in I_1} \rho^-(m)\Big).$$

From (2.3), we have

$$|S_1(\lambda_1\alpha, \rho^-) - \kappa_1^- L^{-1} T_1(\lambda_1\alpha)| \ll X^{1/2} L^{-2}(1+|\alpha|X). \tag{3.9}$$

Inserting (3.9) into (3.8), we obtain

$$\begin{aligned} J_1 &\ll \tau^2 X^{1/2} L^{-3} \int_0^{1/X} |T_2(\lambda_2\alpha)||T_k(\lambda_3\alpha)|\,d\alpha \\ &\quad + \tau^2 X^{3/2} L^{-3} \int_{1/X}^{\phi} |\alpha||T_2(\lambda_2\alpha)||T_k(\lambda_3\alpha)|\,d\alpha. \end{aligned}$$

Then from (3.2), we obtain

$$J_1 = o(\tau^2 X^{1/k} L^{-2}). \tag{3.10}$$

Arguing similarly, in spite of different sieve functions, we can also get

$$J_2 = o(\tau^2 X^{1/k} L^{-2}). \tag{3.11}$$

Following the argument in [5, Section 3.3], we can obtain

$$J_3 = o(\tau^2 X^{1/k} L^{-2}). \tag{3.12}$$

Therefore combining (3.3), (3.7), (3.10), (3.11) and (3.12), we have

$$I(\tau, \upsilon, \widetilde{\mathfrak{M}}, \rho^-, \rho^+) = f(\upsilon)\kappa_1^-\kappa_2^+(1+o(1))L^{-2}. \tag{3.13}$$

Arguing similarly, we can get

$$I(\tau, \upsilon, \widetilde{\mathfrak{M}}, \rho^+, \rho^-) = f(\upsilon)\kappa_1^+\kappa_2^-(1+o(1))L^{-2} \tag{3.14}$$

and

$$I(\tau, \upsilon, \widetilde{\mathfrak{M}}, \rho^+, \rho^+) = f(\upsilon)\kappa_1^+\kappa_2^+(1+o(1))L^{-2}. \tag{3.15}$$

From (3.5) in [5] or by a standard argument (see [3, Lemma 51]), we have

$$f(\upsilon) \gg \tau^2 X^{1/k}.$$

Thus from (3.13), (3.14) and (3.15), we have

$$\int_{\widetilde{\mathfrak{M}}} H(\alpha)S_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha \gg \big(\kappa_1^-\kappa_2^+ + \kappa_1^+\kappa_2^- - \kappa_1^+\kappa_2^+ + o(1)\big)\tau^2 X^{1/k} L^{-2}.$$

Recalling (2.4), since $1.7 \times 0.9 + 0.9 \times 1.7 - 1.7 \times 1.7 > 0$, we know that the coefficient in the right side of (3.1) is positive.

Note that $\widetilde{\mathfrak{M}} \supseteq \mathfrak{M}$ when $k = 3$. The remaining region needed to be handled is $\mathfrak{M} \setminus \widetilde{\mathfrak{M}}$ for $k \geq 4$. Although there are sieve functions here, following a similar argument to [5, Lemma 3.3], we can get, for $k \geq 4$,

$$\int_{\mathfrak{M}\setminus\widetilde{\mathfrak{M}}} |H(\alpha)S_k(\lambda_3\alpha)|K_\tau(\alpha)\,d\alpha \ll \tau^2 X^{1/k - 4^{1-k}/(2k)+\varepsilon} = o(\tau^2 X^{1/k} L^{-2}).$$

Therefore from the above analysis, we can conclude the following lemma.

Lemma 3.1. *We have*

$$\int_{\mathfrak{M}} H(\alpha)S_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha \gg \tau^2 X^{1/k} L^{-2}.$$

4. The minor arc

In this section, we give the estimate of the integral on the minor arc. Note that

$$\begin{aligned}
&\int_{\mathfrak{m}} |H(\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\,d\alpha \\
&\ll \int_{\mathfrak{m}} |S_1(\lambda_1\alpha,\rho^-)S_2(\lambda_2\alpha,\rho^+)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\,d\alpha \\
&\quad + \int_{\mathfrak{m}} |S_1(\lambda_1\alpha,\rho^+)S_2(\lambda_2\alpha,\rho^-)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\,d\alpha \\
&\quad + \int_{\mathfrak{m}} |S_1(\lambda_1\alpha,\rho^+)S_2(\lambda_2\alpha,\rho^+)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\,d\alpha.
\end{aligned} \tag{4.1}$$

From [6, Section 8], we know that the sieve functions can be expressed in terms of finitely many sums of the form

$$\sum_{m=rs} f_r g_s, \tag{4.2}$$

where either $X^{1/7} \le r \le X^{3/14}$ or $f_r \equiv 1$ and $r \ge X^{1/7}$. In either case, f_r, g_s are bounded by divisor functions at worst.

We write

$$S_i(\alpha) = \sum_{n\in I_i} a_n e(n^2\alpha), \quad i = 1, 2, \tag{4.3}$$

where a_n takes the form (4.2). Without loss of generality we need only to consider the integral

$$\int_{\mathfrak{m}} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\,d\alpha.$$

First we need the following lemmas.

Lemma 4.1. *Suppose that $X^{1/2} \ge Z \ge X^{3/7+\varepsilon/2}$ and $|S_j(\lambda_j\alpha)| > Z$ for $j = 1, 2$. Then there are coprime integers a_j, q_j satisfying*

$$1 \le q_j \ll \left(\frac{X^{1/2+\varepsilon/2}}{Z}\right)^4, \quad |q_j\lambda_j\alpha - a_j| \ll X^{-1}\left(\frac{X^{1/2+\varepsilon/2}}{Z}\right)^4.$$

Proof. This lemma is Lemma 1 in [13]. □

Lemma 4.2. *Let $m_2(k)$ and $m_3(k)$ be defined by* (1.1) *and* (1.9)*, respectively. Suppose that $F \in \{S_1^4, S_2^4, S_3^8, S_2^2 S_k^{m_2(k)}, S_2^2 S_3^2 S_k^{m_3(k)}\}$. Then we have*

$$\int_{-1}^{1} |F(\alpha)|\,d\alpha \ll X^{-1}(F(0))^{1+\varepsilon}, \quad \int_{-\infty}^{\infty} |F(\alpha)|K_\tau(\alpha)\,d\alpha \ll \tau X^{-1}(F(0))^{1+\varepsilon}.$$

Proof. All of these results follow from [12] by using Hua's lemma and the recent breakthrough of Bourgain-Demeter-Guth [1] on Vinogradov's mean value theorem. We can find this result in [4, Lemma 3] for $3 \le k \le 9$ and in [5, Lemma 5.1] for $k \ge 10$, respectively. $\square$

Let $\widetilde{\mathfrak{m}} = \mathfrak{m}_1 \cup \mathfrak{m}_2$, where

$$\mathfrak{m}_j = \{\alpha \in \mathfrak{m} : |S_j(\lambda_j\alpha)| \le X^{3/7+\varepsilon}\} \quad \text{for } j = 1, 2.$$

Lemma 4.3. *For $k \ge 3$, we have*

$$\int_{\widetilde{\mathfrak{m}}} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{1+2/k-2/(7m_2(k))+\varepsilon},$$

where $m_2(k)$ is defined by (1.1).

Proof. Using Hölder's inequality and Lemma 4.2, we can obtain

$$\begin{aligned}
&\int_{\mathfrak{m}_1} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\, d\alpha \\
&\ll \left(\sup_{\alpha\in\mathfrak{m}_1} |S_1(\lambda_1\alpha)|\right)^{4/m_2(k)} \left(\int_{\mathbb{R}} |S_1(\lambda_1\alpha)|^4 K_\tau(\alpha)\, d\alpha\right)^{1/2-1/m_2(k)} \\
&\quad \times \left(\int_{\mathbb{R}} |S_2(\lambda_2\alpha)|^4 K_\tau(\alpha)\, d\alpha\right)^{1/2-1/m_2(k)} \left(\int_{\mathbb{R}} |S_2(\lambda_2\alpha)|^2 |S_k(\lambda_3\alpha)|^{m_2(k)} K_\tau(\alpha)\, d\alpha\right)^{2/m_2(k)} \\
&\ll \tau X^{1+2/k-2/(7m_2(k))+\varepsilon}.
\end{aligned}$$

By symmetry we can get the same bound for the integral on $\mathfrak{m}_2$. Thus we have

$$\int_{\widetilde{\mathfrak{m}}} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{1+2/k-2/(7m_2(k))+\varepsilon}. \qquad \square$$

Now the remaining work is to handle the range $\mathfrak{m}^* = \mathfrak{m} \setminus \widetilde{\mathfrak{m}}$. We have the following lemma.

Lemma 4.4. *We have*

$$\int_{\mathfrak{m}^*} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{1+2/k-1/7+\varepsilon}.$$

Proof. Note that for any $\alpha \in \mathfrak{m}^*$, we have

$$|S_1(\lambda_1\alpha)| > X^{3/7+\varepsilon} \quad \text{and} \quad |S_2(\lambda_2\alpha)| > X^{3/7+\varepsilon}.$$

We divide $\mathfrak{m}^*$ into disjoint sets $S(Z_1, Z_2, y)$, such that for $\alpha \in S(Z_1, Z_2, y)$, we have

$$Z_1 < |S_1(\lambda_1\alpha)| \le 2Z_1, \quad Z_2 < |S_2(\lambda_2\alpha)| \le 2Z_2, \quad y < |\alpha| \le 2y,$$

where $Z_1 = 2^{k_1} X^{3/7+\varepsilon}$, $Z_2 = 2^{k_2} X^{3/7+\varepsilon}$ and $y = 2^{k_3} X^{-3/4}$ for some non-negative integers k_1, k_2, k_3. Then from Lemma 4.1, there exist two pairs of coprime integers (a_1, q_1) and (a_2, q_2) satisfying

$$1 \le q_i \ll \left(\frac{X^{1/2+\varepsilon/2}}{Z_i}\right)^4, \quad |q_i\lambda_i\alpha - a_i| \ll X^{-1}\left(\frac{X^{1/2+\varepsilon/2}}{Z_i}\right)^4, \quad i = 1, 2.$$

We remark that $a_1 a_2 \neq 0$, since otherwise we have $\alpha \in \mathfrak{M}$. Furthermore, we subdivide $S(Z_1, Z_2, y)$ into sets $S(Z_1, Z_2, y, Q_1, Q_2)$, where $Q_j < q_j \le 2Q_j$ on each set. Then we have

$$\begin{aligned}\left|a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2\right| &= \left|(q_1\lambda_1\alpha - a_1)\frac{a_2}{\lambda_2\alpha} - (q_2\lambda_2\alpha - a_2)\frac{a_1}{\lambda_2\alpha}\right| \\ &\ll Q_2 X^{-1}\left(\frac{X^{1/2+\varepsilon/2}}{Z_1}\right)^4 + Q_1 X^{-1}\left(\frac{X^{1/2+\varepsilon/2}}{Z_2}\right)^4 \\ &\ll X^{3+4\varepsilon} Z_1^{-4} Z_2^{-4} \\ &\ll X^{-3/7-4\varepsilon}.\end{aligned}$$

Recall that $q = X^{3/7}$. Thus

$$\left|a_2 q_1 \frac{\lambda_1}{\lambda_2} - a_1 q_2\right| = o(q^{-1}).$$

We also have $|a_2 q_1| \ll y Q_1 Q_2$. Thus, if $|a_2 q_1|$ took W distinct values, we could deduce the existence of n satisfying

$$\left\|n\frac{\lambda_1}{\lambda_2}\right\| \ll X^{-3/7-4\varepsilon}, \quad n \ll \frac{yQ_1Q_2}{W}.$$

This would contradict a/q being a convergent to λ_1/λ_2 if q is sufficiently large, unless

$$W \ll \frac{yQ_1Q_2}{q}.$$

From the upper bound for the divisor function, each value of $|a_2 q_1|$ corresponds to $O(X^{\varepsilon})$ values of a_2, q_1. Then we obtain that each set of $S(Z_1, Z_2, y, Q_1, Q_2)$ is made up of $O(WX^{\varepsilon})$ intervals of length

$$\begin{aligned}&\ll \min\left(Q_1^{-1} X^{-1}\left(\frac{X^{1/2+\varepsilon/2}}{Z_1}\right)^4, Q_2^{-1} X^{-1}\left(\frac{X^{1/2+\varepsilon/2}}{Z_2}\right)^4\right) \\ &\ll \frac{X^{1+2\varepsilon}}{Z_1^2 Z_2^2 Q_1^{1/2} Q_2^{1/2}}.\end{aligned}$$

Let $\mathcal{A}$ denote such a set $S(Z_1, Z_2, y, Q_1, Q_2)$. Then integrating over $\mathcal{A}$ gives

$$\begin{aligned}
&\int_{\mathcal{A}} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\,d\alpha \\
&\ll \min(\tau^2, y^{-2}) Z_1^2 Z_2^2 X^{2/k} \frac{X^{1+2\varepsilon}}{Z_1^2 Z_2^2 Q_1^{1/2} Q_2^{1/2}} \frac{X^\varepsilon y Q_1 Q_2}{q} \\
&\ll \tau X^{1+2/k-1/7+\varepsilon}.
\end{aligned} \tag{4.4}$$

Then summing over all possible values of Z_1, Z_2, y, Q_1, Q_2, we conclude that

$$\int_{\mathfrak{m}^*} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\,d\alpha \ll \tau X^{1+2/k-1/7+\varepsilon}. \qquad \square$$

Recalling (4.1), from Lemmas 4.3 and 4.4 we can get the following lemma immediately.

Lemma 4.5. *For $k \geq 3$, we have*

$$\int_{\mathfrak{m}} |H(\alpha)S_3(\lambda_3\alpha)S_k(\lambda_4\alpha)|^2 K_\tau(\alpha)\,d\alpha \ll \tau X^{1+2/k-2/(7m_2(k))+\varepsilon},$$

where $m_2(k)$ is defined by (1.1).

5. The trivial arc

In this section, we estimate the contribution of the right side of (2.6) from the trivial arc. Due to the similar reason as in Section 4, we also consider $S_i(\alpha)$ instead of $S_i(\alpha, \rho^\pm)$, $i = 1, 2$ in the trivial arc. Applying the trivial bounds for $S_k(\lambda_3\alpha)$ and Cauchy's inequality, recalling $R = \tau^{-2}X^{1/2+2\varepsilon}$ we have

$$\begin{aligned}
&\int_{\mathfrak{t}} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_3(\lambda_3\alpha)S_k(\lambda_4\alpha)| K_\tau(\alpha)\,d\alpha \\
&\ll X^{1/k} \left(\int_R^\infty |S_1(\lambda_1\alpha)|^2 K_\tau(\alpha)\,d\alpha\right)^{1/2} \left(\int_R^\infty |S_2(\lambda_2\alpha)|^2 K_\tau(\alpha)\,d\alpha\right)^{1/2} \\
&\ll X^{1/k} \left(\sum_{n=[R]}^\infty \int_n^{n+1} |S_1(\lambda_1\alpha)|^2 \frac{1}{\alpha^2}\,d\alpha\right)^{1/2} \left(\sum_{n=[R]}^\infty \int_n^{n+1} |S_2(\lambda_2\alpha)|^2 \frac{1}{\alpha^2}\,d\alpha\right)^{1/2} \\
&\ll X^{1/k} \left(\sum_{n=[R]}^\infty \frac{1}{n^2}\right) \left(\int_0^1 |S_1(\lambda_1\alpha)|^2\,d\alpha\right)^{1/2} \left(\int_0^1 |S_2(\lambda_2\alpha)|^2\,d\alpha\right)^{1/2} \\
&\ll R^{-1} X^{1/k+1/2+\varepsilon} \\
&\ll \tau^2 X^{1/k-\varepsilon}.
\end{aligned}$$

Therefore we have

$$\int_{\mathfrak{t}} |H(\alpha)S_k(\lambda_3\alpha)| K_\tau(\alpha)\,d\alpha \ll \tau^2 X^{1/k-\varepsilon}. \tag{5.1}$$

6. Proof of Theorem 1.2

In this section, we complete the proof of Theorem 1.2. Let $m_2(k)$ be defined as in (1.1). We take $\tau = X^{-\delta}$. Let $\mathcal{E}_k = \mathcal{E}_k(\mathcal{V}, X, \delta)$ denote the set of $\upsilon \in \left[\frac{1}{2}X, X\right] \cap \mathcal{V}$ such that the inequality (1.2) has no solution in primes p_1, p_2, p_3, and $E_k = E_k(\mathcal{V}, X, \delta) = |\mathcal{E}_k(\mathcal{V}, X, \delta)|$. Then from Lemma 3.1 and (5.1), we have

$$\left|\sum_{\upsilon\in\mathcal{E}_k}\int_{\mathfrak{m}} H(\alpha)S_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha\right| \gg \tau^2 X^{1/k}L^{-2}E_k. \tag{6.1}$$

Applying Cauchy's inequality, we have

$$\begin{aligned}
&\left|\sum_{\upsilon\in\mathcal{E}_k}\int_{\mathfrak{m}} H(\alpha)S_k(\lambda_3\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha\right| \\
&\ll \left(\int_{-\infty}^{+\infty}\left|\sum_{\upsilon\in\mathcal{E}_k} e(-\alpha\upsilon)\right|^2 K_\tau(\alpha)\,d\alpha\right)^{1/2}\left(\int_{\mathfrak{m}} |H(\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\,d\alpha\right)^{1/2} \\
&\ll \left(\tau X^{1+2/k-2/(7m_2(k))+\varepsilon}\right)^{1/2}\left(\sum_{\upsilon_1,\upsilon_2\in\mathcal{E}_k}\max(0,\tau-|\upsilon_1-\upsilon_2|)\right)^{1/2} \\
&\ll \tau E_k^{1/2}\left(X^{1+2/k-2/(7m_2(k))+\varepsilon}\right)^{1/2}.
\end{aligned} \tag{6.2}$$

Then combining (6.1) and (6.2), we have

$$E_k(\mathcal{V}, X_j, \delta) \ll X_j^{1-2/(7m_2(k))+2\delta+\varepsilon}. \tag{6.3}$$

Since λ_1/λ_2 is irrational, there are infinitely many q we could have taken and this gives the sequence $X_j \to \infty$.

Now, if the convergent denominators for λ_1/λ_2 satisfy (1.5), then we can modify our works in Lemmas 4.3 and 4.4. We now assume that

$$\min(Z_1, Z_2) > X^{\chi+\varepsilon}$$

with χ given by (1.7). We then obtain

$$\left|a_2q_1\frac{\lambda_1}{\lambda_2} - a_1q_2\right| \ll X^{3-8\chi-4\varepsilon}.$$

However, we know from (1.5) that there is a convergent a/q to λ_1/λ_2 with

$$X^{(1-w)(8\chi-3)} \ll q \ll X^{8\chi-3}.$$

The expression corresponding to (4.4) is now

$$\begin{aligned}
&\int_{\mathcal{A}} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\,d\alpha \\
&\ll \tau X^{3+2/k-4\chi+\varepsilon}q^{-1} \ll \tau X^{3+2/k-4\chi-(1-w)(8\chi-3)+\varepsilon} \\
&\ll \tau X^{1+2/k-(2-4\chi)/m_2(k)+\varepsilon}
\end{aligned}$$

by our choice of χ. Thus

$$\int_{\mathfrak{m}} |S_1(\lambda_1\alpha)S_2(\lambda_2\alpha)S_k(\lambda_3\alpha)|^2 K_\tau(\alpha)\,d\alpha \ll \tau X^{1+2/k-(2-4\chi)/m_2(k)+\varepsilon}.$$

Working as (6.1)–(6.3), we can complete the proof of Theorem 1.2 easily.

7. Proof of Theorem 1.3

In this section, we sketch the proof of Theorem 1.3. We write

$$I_1' = \left[\left(\frac{\eta X}{\lambda_1}\right)^{1/2}, \left(\frac{2\eta X}{\lambda_1}\right)^{1/2}\right], \qquad I_2' = \left[\left(\frac{\eta X}{\lambda_2}\right)^{1/2}, \left(\frac{2\eta X}{\lambda_2}\right)^{1/2}\right],$$
$$I_3' = \left[\left(\frac{\eta X}{\lambda_3}\right)^{1/3}, \left(\frac{2\eta X}{\lambda_3}\right)^{1/3}\right], \qquad I_4' = \left[\left(\frac{\eta X}{\lambda_4}\right)^{1/k}, \left(\frac{2\eta X}{\lambda_4}\right)^{1/k}\right],$$

and define

$$S_1'(\alpha,\rho) = \sum_{m\in I_1} \rho(m)e(m^2\alpha), \qquad S_2'(\alpha,\rho) = \sum_{m\in I_2} \rho(m)e(m^2\alpha),$$
$$S_3'(\alpha) = \sum_{p\in I_3} (\log p)e(p^3\alpha), \qquad S_k'(\alpha) = \sum_{p\in I_4} (\log p)e(p^k\alpha).$$

For any measurable subset $\mathfrak{X}$ of $\mathbb{R}$, write

$$I'(\tau,\upsilon,\mathfrak{X},\rho_1,\rho_2) = \int_{\mathfrak{X}} S_1'(\lambda_1\alpha,\rho_1)S_2'(\lambda_2\alpha,\rho_2)S_3'(\lambda_3\alpha)S_k'(\lambda_4\alpha)e(-\alpha\upsilon)K_\tau(\alpha)\,d\alpha.$$

Then from (2.2), we have

$$\begin{aligned}
I'(\tau,\upsilon,\mathbb{R},\rho_0,\rho_0) &= \sum_{\substack{m_i\in I_i, i=1,2\\ p_j\in I_j, j=3,4}} \rho_0(m_1)\rho_0(m_2)(\log p_3)(\log p_4)\\
&\qquad \times \int_{-\infty}^{\infty} e\big((\lambda_1 m_1^2+\lambda_2 m_2^2+\lambda_3 p_3^3+\lambda_4 p_4^k-\upsilon)\alpha\big)K_\tau(\alpha)\,d\alpha\\
&= \sum_{\substack{m_i\in I_i, i=1,2\\ p_j\in I_j, j=3,4}} \rho_0(m_1)\rho_0(m_2)\log p_3\log p_4\\
&\qquad \times \max\big(0, \tau - |\lambda_1 m_1^2+\lambda_2 m_2^2+\lambda_3 p_3^3+\lambda_4 p_4^k-\upsilon|\big).
\end{aligned}$$

Thus we have

$$I'(\tau,\upsilon,\mathbb{R},\rho_0,\rho_0) \ll \tau(\log X)^2\mathfrak{N}'(X),$$

where $\mathfrak{N}'(X)$ denotes the number of solutions of the inequality

$$|\lambda_1 p_1^2+\lambda_2 p_2^2+\lambda_3 p_3^3+\lambda_4 p_4^k-\upsilon| < \tau$$

with $p_j \in I_j$ for $1 \le j \le 4$. Recalling that (2.5), we have

$$I'(\tau, \upsilon, \mathbb{R}, \rho_0, \rho_0) \ge I'(\tau, \upsilon, \mathbb{R}, \rho^-, \rho^+) + I'(\tau, \upsilon, \mathbb{R}, \rho^+, \rho^-) - I'(\tau, \upsilon, \mathbb{R}, \rho^+, \rho^+). \tag{7.1}$$

Then it is sufficient to give a positive lower bound for the right side of (7.1). To do this, we divide the real line into the major arc $\mathfrak{M}'$, the minor arc $\mathfrak{m}'$ and the trivial arc $\mathfrak{t}'$. We define

$$\mathfrak{M}' = \{\alpha : |\alpha| \le P/X\}, \quad \mathfrak{m}' = \{\alpha : P/X < |\alpha| \le R\}, \quad \mathfrak{t}' = \{\alpha : |\alpha| > R\},$$

where $P = X^{8/(11k)-2\varepsilon}$ and $R = \tau^{-2} X^{1/2+2\varepsilon}$.

We write

$$H'(\alpha) = S_1'(\lambda_1\alpha, \rho^-)S_2'(\lambda_2\alpha, \rho^+) + S_1'(\lambda_1\alpha, \rho^+)S_2'(\lambda_2\alpha, \rho^-) - S_1'(\lambda_1\alpha, \rho^+)S_2'(\lambda_2\alpha, \rho^+).$$

As in Sections 3 and 5, we can estimate the integral on $\mathfrak{M}'$ and $\mathfrak{t}'$. Here we just state the following lemmas without proof.

Lemma 7.1. *For $k \ge 3$, we have*

$$\int_{\mathfrak{M}'} H'(\alpha) S_3'(\lambda_3\alpha) S_k'(\lambda_4\alpha) e(-\alpha\upsilon) K_\tau(\alpha)\, d\alpha \gg c'\tau^2 X^{1/3+1/k} L^{-2},$$

where $c' > 0$ is an absolute constant.

Lemma 7.2. *For $k \ge 3$, we have*

$$\int_{\mathfrak{t}'} |H'(\alpha) S_3'(\lambda_3\alpha) S_k'(\lambda_4\alpha)| K_\tau(\alpha)\, d\alpha \ll \tau^2 X^{1/3+1/k-\varepsilon}.$$

For the minor arc, we follow the argument in Section 4. Also without loss of generality we only to consider the integral

$$\int_{\mathfrak{m}} |S_1'(\lambda_1\alpha) S_2'(\lambda_2\alpha) S_3'(\lambda_3\alpha) S_k'(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha,$$

where $S_i'(\lambda_i\alpha)$, $i = 1, 2$ takes the form (4.3). Let $\widetilde{\mathfrak{m}}' = \mathfrak{m}_1' \cup \mathfrak{m}_2'$, $\mathfrak{m}'^* = \mathfrak{m}' \setminus \widetilde{\mathfrak{m}}'$, where

$$\mathfrak{m}_j' = \{\alpha \in \mathfrak{m}' : |S_j(\lambda_j\alpha)| \le X^{3/7+\varepsilon}\} \quad \text{for } j = 1, 2.$$

Lemma 7.3. *For $k \ge 3$, we have*

$$\int_{\widetilde{\mathfrak{m}}'} |S_1'(\lambda_1\alpha) S_2'(\lambda_2\alpha) S_3'(\lambda_3\alpha) S_k'(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{1+2/3+2/k-\sigma^*(k)+\varepsilon},$$

where $\sigma^*(k)$ is defined by (1.10).

Proof. We first consider the cases $k = 3, 4$. Using Hölder's inequality and Lemma 4.2, we can obtain

$$\begin{aligned}
&\int_{\mathfrak{m}_1'} |S_1'(\lambda_1\alpha)S_2'(\lambda_2\alpha)S_3'(\lambda_3\alpha)S_k'(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \\
&\ll \left(\sup_{\alpha\in\mathfrak{m}_1'} |S_1'(\lambda_1\alpha)|\right)^2 \left(\int_{\mathbb{R}} |S_2'(\lambda_2\alpha)|^4 K_\tau(\alpha)\, d\alpha\right)^{1/4} \left(\int_{\mathbb{R}} |S_3'(\lambda_3\alpha)|^8 K_\tau(\alpha)\, d\alpha\right)^{1/4} \\
&\quad \times \left(\int_{\mathbb{R}} |S_2'(\lambda_2\alpha|^2 |S_k'(\lambda_4\alpha)|^4 K_\tau(\alpha)\, d\alpha\right)^{1/2} \\
&\ll \tau X^{1+2/3+2/k-1/7+\varepsilon}.
\end{aligned}$$

Now we turn to handle the case $5 \le k \le 48$. Using Hölder's inequality and Lemma 4.2 again, we have

$$\begin{aligned}
&\int_{\mathfrak{m}_1'} |S_1'(\lambda_1\alpha)S_2'(\lambda_2\alpha)S_3'(\lambda_3\alpha)S_k'(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \\
&\ll \left(\sup_{\alpha\in\mathfrak{m}_1{}'} |S_1'(\lambda_1\alpha)|\right)^{1+2/m_3(k)} \left(\int_{\mathbb{R}} |S_1'(\lambda_1\alpha)|^4 K_\tau(\alpha)\, d\alpha\right)^{1/4-1/(2m_3(k))} \\
&\quad \times \left(\int_{\mathbb{R}} |S_2'(\lambda_2\alpha)|^4 K_\tau(\alpha)\, d\alpha\right)^{1/2-1/m_3(k)} \left(\int_{\mathbb{R}} |S_3'(\lambda_3\alpha)|^8 K_\tau(\alpha)\, d\alpha\right)^{1/4-1/(2m_3(k))} \\
&\quad \times \left(\int_{\mathbb{R}} |S_2'(\lambda_2\alpha)|^2 |S_3'(\lambda_3\alpha)|^2 |S_k'(\lambda_4\alpha)|^{m_3(k)} K_\tau(\alpha)\, d\alpha\right)^{2/m_3(k)} \\
&\ll \tau X^{1+2/3+2/k-1/14-1/(7m_3(k))+\varepsilon}.
\end{aligned}$$

For $k \ge 49$, similarly we have

$$\begin{aligned}
&\int_{\mathfrak{m}_1'} |S_1'(\lambda_1\alpha)S_2'(\lambda_2\alpha)S_3'(\lambda_3\alpha)S_k'(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \\
&\ll \left(\sup_{\alpha\in\mathfrak{m}_1{}'} |S_1'(\lambda_1\alpha)|\right)^{1+4/m_2(k)} \left(\int_{\mathbb{R}} |S_1'(\lambda_1\alpha)|^4 K_\tau(\alpha)\, d\alpha\right)^{1/4-1/m_2(k)} \\
&\quad \times \left(\int_{\mathbb{R}} |S_2'(\lambda_2\alpha)|^4 K_\tau(\alpha)\, d\alpha\right)^{1/2-1/m_2(k)} \left(\int_{\mathbb{R}} |S_3'(\lambda_3\alpha)|^8 K_\tau(\alpha)\, d\alpha\right)^{1/4} \\
&\quad \times \left(\int_{\mathbb{R}} |S_2'(\lambda_2\alpha)|^2 S_k'(\lambda_4\alpha)|^{m_2(k)} K_\tau(\alpha)\, d\alpha\right)^{2/m_2(k)} \\
&\ll \tau X^{1+2/3+2/k-1/14-2/(7m_2(k))+\varepsilon}.
\end{aligned}$$

By symmetry we can get the same bound for the integral on $\mathfrak{m}_2'$. Thus we have

$$\int_{\widetilde{\mathfrak{m}}'} |S_1'(\lambda_1\alpha)S_2'(\lambda_2\alpha)S_3'(\lambda_3\alpha)S_k'(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{1+2/3+2/k-\sigma*(k)+\varepsilon}. \qquad \square$$

We handle $\mathfrak{m}'^*$ similarly to Lemma 4.4 and can get the following lemma.

Lemma 7.4. *We have*

$$\int_{\mathfrak{m}'^*} |S_1'(\lambda_1\alpha)S_2'(\lambda_2\alpha)S_3'(\lambda_3\alpha)S_k'(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{1+2/3+2/k-1/7+\varepsilon}.$$

Due to a similar reason to (4.1), from Lemmas 7.3 and 7.4 we can deduce that

Lemma 7.5. *For $k \geq 3$, we have*

$$\int_{\mathfrak{m}'} |H'(\alpha)S_3'(\lambda_3\alpha)S_k'(\lambda_4\alpha)|^2 K_\tau(\alpha)\, d\alpha \ll \tau X^{1+2/3+2/k-\sigma*(k)+\varepsilon},$$

where $\sigma^*(k)$ is defined by (1.10)*.*

Then from Lemmas 7.1, 7.2 and 7.5, arguing similarly to (6.1)–(6.3), we can complete the proof of Theorem 1.3 easily.

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