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Finding Efficient Solutions for Multicriteria Optimization Problems with SOS-convex Polynomials

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Abstract. In this paper, we focus on the study of finding efficient solutions for a multicriteria optimization problem (MP), where both the objective and constraint functions are SOS-convex polynomials. By using the well-known ϵ-constraint method (a scalarization technique), we substitute the problem (MP) to a class of scalar ones. Then, a zero duality gap result for each scalar problem, its sum of squares polynomial relaxation dual problem, the semidefinite representation of this dual problem, and the dual problem of the semidefinite programming problem, is established, under a suitable regularity condition. Moreover, we prove that an optimal solution of each scalar problem can be found by solving its associated semidefinite programming problem. As a consequence, we show that finding efficient solutions for the problem (MP) is *tractable* by employing the ϵ-constraint method. A numerical example is also given to illustrate our results.

1. Introduction

Convex optimization has applications in a wide range of disciplines, such as estimation and signal processing, automatic control systems, finance, and statistics; see, for example, [5, 8] and the references therein. With recent developments and improvements in computing and optimization theory, some convex minimization problems, for instance, linear programming problems, second-order cone programming problems and semidefinite programming problems, are showed to be poly-time solvable by interior points methods. However, we mention here that a convex optimization problem is still NP-hard from the complexity point of view. Mathematically speaking, a convex optimization problem admits the following form:

$$\text{(CP)} \qquad \min\ f_0(x) \quad \text{subject to} \quad f_i(x) \leq 0, \quad i = 1, \ldots, m,$$

where f_i, $i = 0, 1, \ldots, m$ are convex functions. In particular, if f_i, $i = 0, 1, \ldots, m$ are *SOS-convex polynomials* (see Definition 2.1), then problem (CP) enjoys an exact SDP-relaxation in the light that the optimal values of problem (CP) and its relaxation dual

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problem are equal; furthermore, the relaxation dual problem attains its optimum under Slater's condition; see [24, 26]. The notion of SOS-convex polynomials attracts much attention; see, for example, [1,2,19,22–25] and the references therein, according to its many nice properties. For example, the class of SOS-convexity polynomials is a numerically tractable subclass of convex polynomials; moreover, the SOS-convexity of a polynomial can be tractably checked by solving a semidefinite programming problem; see [26–28]. Besides, the class of SOS-convex polynomials contains the classes of separable convex polynomials and convex quadratic functions; in addition, Helton and Nie [19] studied some other examples of SOS-convex polynomials.

On the other hand, a multicriteria optimization problem (for short, MOP) is a problem that involves more than one objective function to be optimized simultaneously. The MOPs have been applied in many fields of science, such as engineering, economics and logistics [9,10,12]. It is worth noting that for an MOP, usually no single solution exists that simultaneously optimizes every objective function. In that case, the objective functions are actually conflicted, and there exists a (possibly infinite) number of efficient solutions (see Definition 3.1). In particular, if the involving functions in MOPs are linear, then we say MOPs as linear multicriteria optimization problems, one can refer [4, 7, 11, 13, 14, 30] for a deep study; if the involving functions in MOPs are convex, then we say MOPs as multicriteria convex optimization problems, some related results are referred to the literatures [12, 15]. This paper aims to contribute a new result, i.e., the process of finding efficient solutions in multicriteria convex optimization problem with SOS-convex polynomials, notwithstanding the fact that the involving functions are limited.

More precisely, in this paper, we consider a multicriteria optimization problem with SOS-convex polynomials. By using the well-known ϵ-constraint method (a scalarization technique), we substitute the multicriteria optimization problem to a class of scalar objective problems. First, we give a zero duality gap result for each scalar problem, its sum of squares polynomial relaxation dual problem, the semidefinite representation of this dual problem, and the dual problem of the semidefinite programming problem under a suitable regularity condition. Then, we show that an optimal solution of each scalar problem can be found by solving its associated semidefinite programming problem. Finally, we observe that finding efficient solutions for the considered multicriteria optimization problem is *tractable* by employing the ϵ-constraint method.

The rest of the paper is organized as follows. Section 2 gives some basic notations and preliminaries that will be used in this paper. In Section 3, we give our main results, i.e., the method on how to find efficient solutions of the considered multicriteria optimization problem by the ϵ-constraint method. A numerical example is also given to illustrate our main results. Finally, we propose the conclusion in Section 4.

2. Preliminaries

We begin this section by fixing notation and preliminaries. We suppose $1 \le n \in \mathbb{N}$ ($\mathbb{N}$ is the set of nonnegative integers) and abbreviate $(x_1, x_2, \dots, x_n)$ by x. The Euclidean space $\mathbb{R}^n$ is equipped with the usual Euclidean norm $\|\cdot\|$. The nonnegative orthant of $\mathbb{R}^n$ is denoted by $\mathbb{R}^n_+$.

A function $f\colon \mathbb{R}^n \to \mathbb{R}$ is said to be convex if for all $\mu \in [0,1]$,

$$f((1-\mu)x + \mu y) \le (1-\mu)f(x) + \mu f(y)$$

for all $x, y \in \mathbb{R}^n$. We say that a real polynomial f is sum of squares if there exist real polynomials q_l, $l = 1, \dots, r$, such that $f = \sum_{l=1}^r q_l^2$. The set consisting of all sum of squares real polynomial is denoted by Σ^2. In addition, the set consisting of all sum of squares real polynomial with degree at most d is denoted by Σ^2_d. For a multi-index $\alpha \in \mathbb{N}^n$, let $|\alpha| := \sum_{i=1}^n \alpha_i$, and let $\mathbb{N}^n_d := \{\alpha \in \mathbb{N}^n : |\alpha| \le d\}$. x^α denotes the monomial $x_1^{\alpha_1} \cdots x_n^{\alpha_n}$. The canonical basis of $\mathbb{R}[x]_d$ is denoted by

$$v_d(x) = (x^\alpha)_{\alpha \in \mathbb{N}^n_d} = (1, x_1, \dots, x_n, x_1^2, x_1x_2, \dots, x_n^2, \dots, x_1^d, \dots, x_n^d)^T,$$

which has dimension $s(n,d) := \binom{n+d}{n}$. The space of all real polynomials on $\mathbb{R}^n$ is denoted by $\mathbb{R}[x]$. Moreover, the space of all real polynomials on $\mathbb{R}^n$ with degree at most d is denoted by $\mathbb{R}[x]_d$. The degree of a polynomial f is denoted by $\deg f$.

Let S^n be the set of $n \times n$ symmetric matrices. For $X \in S^n$, X is positive semidefinite denoted by $X \succeq 0$, if $z^T X z \ge 0$ for any $z \in \mathbb{R}^n$. Let S^n_+ be the set of $n \times n$ symmetric positive semidefinite matrices. For $M, N \in S^n$, $\langle M, N \rangle := \operatorname{tr}(MN)$, where "tr" denotes the trace (sum of diagonal elements) of a matrix.

We now recall the notion of SOS-convex polynomials.

Definition 2.1. [1,2,19] A real polynomial f on $\mathbb{R}^n$ is called *SOS-convex*, if there exists a matrix polynomial $F(x)$ such that $\nabla^2 f(x) = F(x)F(x)^T$, equivalently,

$$f(x) - f(y) - \nabla f(y)^T (x-y)$$

is a sum of squares polynomial in $\mathbb{R}[x; y]$ (with respect to variables x and y).

It is clearly that an SOS-convex polynomial is convex; but the converse is not true, which means that there exists a convex polynomial which is not SOS-convex [1,2].

The following lemma, which plays a key role for our main result in the paper, shows an useful existence result of solutions of convex polynomial programs.

Lemma 2.2. [3] *Let $f, g_1, \dots, g_m$ be convex polynomials on $\mathbb{R}^n$. Let $K := \{x \in \mathbb{R}^n : g_i(x) \le 0, i = 1, \dots, m\}$. Suppose that* $\inf_{x \in K} f(x) > -\infty$. *Then,* $\operatorname{argmin}_{x \in K} f(x) \ne \emptyset$.

Proposition 2.3. [27] *A polynomial $f \in \mathbb{R}[x]_{2d}$ is a sum of squares if and only if there exists a matrix $Q \in S_+^{s(n,d)}$ such that $f(x) = \langle v_d(x)v_d(x)^T, Q\rangle$ for all $x \in \mathbb{R}^n$.*

Let $v_d(x)v_d(x)^T := \sum_{\alpha\in\mathbb{N}^n_{2d}} x^\alpha B_\alpha$, where B_α are $s(n,d) \times s(n,d)$ real symmetric matrices. Then, $f(x) := \sum_{\alpha\in\mathbb{N}^n_{2d}} f_\alpha x^\alpha$ is a sum of squares if and only if solving the following semidefinite feasibility problem [27]:

$$\text{Find } Q \in S_+^{s(n,d)} \text{ such that } \langle Q, B_\alpha\rangle = f_\alpha,\ \forall\, \alpha \in \mathbb{N}^n_{2d}.$$

3. Multicriteria SOS-convex polynomial optimization problems: Finding efficient solutions

Consider the following multicriteria optimization problem with SOS-convex polynomials:

$$\text{(MP)} \qquad \min\ (f_1(x), \ldots, f_p(x)) \quad \text{subject to} \quad g_i(x) \le 0, \quad i = 1, \ldots, m,$$

where f_j, $j = 1, \ldots, p$, and $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \ldots, m$, are SOS-convex polynomials. Let $2d := \max\{\deg f_1, \ldots, \deg f_p, \deg g_1, \ldots, \deg g_m\}$. Let $K := \{x \in \mathbb{R}^n : g_i(x) \le 0, i = 1, \ldots, m\}$ be the feasible set of (MP).

Below, we recall the concept of an efficient solution of (MP).

Definition 3.1. A point $\overline{x} \in K$ is said to be an *efficient solution* of (MP) if

$$f(x) - f(\overline{x}) \notin -\mathbb{R}^p_+ \setminus \{0\}, \quad \forall\, x \in K,$$

where $f(x) := (f_1(x), \ldots, f_p(x))$.

Actually, there are many methods studying the multicriteria optimization problem (the assumptions on the involving functions are not necessary to be SOS-convex polynomials); among them, the scalarization method (such as the weighted-sum method and the ϵ-constraint method) is showed to be an important one. The relevance of using scalarization methods to solve multicriteria optimization problems is that scalar problems can have more effective means of finding optimal solutions than multicriteria problems. For more details, the reader is referred to the books [10, 12, 21] and the papers [20, 29] and the references therein.

In our research, we are interested in the *ϵ-constraint method*, which was minutely introduced by Chankong and Haimes [10]. The ϵ-constraint method is based on a scalarization, where one of the objective functions is minimized while all the other objective functions are bounded from above by means of additional constraints:

$$(\mathrm{P}_j(\epsilon)) \qquad \min\ f_j(x) \quad \text{subject to} \quad f_k(x) \le \epsilon_k, \quad g_i(x) \le 0, \quad k \ne j,\ i = 1, \ldots, m,$$

where $\epsilon = (\epsilon_1, \ldots, \epsilon_{j-1}, \epsilon_{j+1}, \ldots, \epsilon_p) \in \mathbb{R}^{p-1}$ is given. For each $j = 1, \ldots, p$, let $K_j(\epsilon) := \{x \in K : f_k(x) \leq \epsilon_k, k \neq j\}$ be the feasible set of $(\mathrm{P}_j(\epsilon))$, which is assumed to be nonempty for the given ϵ. Besides, let $v(\cdot)$ be the optimal value of the problem $(\cdot)$, correspondingly. For example, $v(\mathrm{P}_j(\epsilon))$ stands for the optimal value of the problem $(\mathrm{P}_j(\epsilon))$.

It is worth noting that the ϵ-constraint method has several advantages over the weighted-sum method. For example, along with the ϵ-constraint method, we can control the number of the generated efficient solutions by properly adjusting the number of grid points in each one of the objective function ranges, however this may not so easy with the weighted-sum method, even if it may take an increased solution time to use the ϵ-constraint method rather than the weighted-sum method, for problems with several (more than two) objective functions; see [31, Section 2] for more details.

On the other hand, for each given $j \in \{1, \ldots, p\}$, the problem $(\mathrm{P}_j(\epsilon))$ can be solved (such as gradient methods, see [6]) by means of *approximating* its exact solutions. However, we aim to find efficient solutions of (MP) by solving $(\mathrm{P}_j(\epsilon))$, and the exact solutions of $(\mathrm{P}_j(\epsilon))$ are essential. To this end, motivated by [23–25], we consider the following dual relaxation problems of the problem $(\mathrm{P}_j(\epsilon))$, since solving its SDP relaxation problems can provide exact solutions of problem $(\mathrm{P}_j(\epsilon))$.

Let $j \in \{1, \ldots, p\}$ be any fixed. Then, the Lagrangian dual problem $(\mathrm{D}_j(\epsilon))$ for $(\mathrm{P}_j(\epsilon))$ is given by

$$(\mathrm{D}_j(\epsilon)) \qquad \sup_{\substack{\gamma_j \in \mathbb{R} \\ \mu_k \geq 0, \lambda_i \geq 0}} \left\{ \gamma_j : f_j(x) + \sum_{k \neq j} \mu_k (f_k(x) - \epsilon_k) + \sum_{i=1}^{m} \lambda_i g_i(x) - \gamma_j \geq 0, \forall\, x \in \mathbb{R}^n \right\}.$$

A sum of squares relaxation problem of $(\mathrm{D}_j(\epsilon))$ is stated as follows:

$$(\mathrm{D}_j(\epsilon)^{\mathrm{sos}}) \qquad \sup_{\substack{\gamma_j \in \mathbb{R} \\ \mu_k \geq 0, \lambda_i \geq 0}} \left\{ \gamma_j : f_j + \sum_{k \neq j} \mu_k (f_k - \epsilon_k) + \sum_{i=1}^{m} \lambda_i g_i - \gamma_j \in \Sigma^2_{2d} \right\}.$$

According to Proposition 2.3, it is clear that the constraints of problem $(\mathrm{D}_j(\epsilon)^{\mathrm{sos}})$, i.e.,

$$f_j + \sum_{k \neq j} \mu_k (f_k - \epsilon_k) + \sum_{i=1}^{m} \lambda_i g_i - \gamma_j,$$

which is a sum of squares polynomial, can be rewritten as solving the following semidefinite feasibility problem: Find $X \in S_+^{s(n,d)}$ such that

$$(f_j)_0 + \sum_{k \neq j} \mu_k ((f_k)_0 - \epsilon_k) + \sum_{i=1}^{m} \lambda_i (g_i)_0 - \gamma_j = \langle B_0, X \rangle,$$

$$(f_j)_\alpha + \sum_{k \neq j} \mu_k ((f_k)_\alpha - \epsilon_k) + \sum_{i=1}^{m} \lambda_i (g_i)_\alpha = \langle B_\alpha, X \rangle, \quad \alpha \neq 0.$$

In other words, problem $(\mathrm{D}_j(\epsilon)^{\mathrm{sos}})$ is equivalent to the following semidefinite programming problem:

$$(\mathrm{SDD}_j(\epsilon)) \qquad \begin{aligned} &\sup_{\gamma_j, X, \mu, \lambda} \gamma_j \quad \text{subject to} \\ &(f_j)_0 + \sum_{k\neq j} \mu_k((f_k)_0 - \epsilon_k) + \sum_{i=1}^m \lambda_i (g_i)_0 - \gamma_j = \langle B_0, X\rangle, \\ &(f_j)_\alpha + \sum_{k\neq j} \mu_k((f_k)_\alpha - \epsilon_k) + \sum_{i=1}^m \lambda_i (g_i)_\alpha = \langle B_\alpha, X\rangle, \quad \alpha \neq 0, \\ &\gamma_j \in \mathbb{R}, \quad X \in S_+^{s(n,d)}, \quad \mu_k \geq 0, \quad k \neq j, \quad \lambda_i \geq 0, \quad i = 1, \ldots, m. \end{aligned}$$

The dual problem of $(\mathrm{SDD}_j(\epsilon))$ is the following semidefinite programming problem:

$$(\mathrm{SDP}_j(\epsilon)) \qquad \begin{aligned} &\inf_{y \in \mathbb{R}^{s(n,2d)}} \sum_{\alpha \in \mathbb{N}^n_{2d}} (f_j)_\alpha y_\alpha \quad \text{subject to} \\ &\sum_{\alpha \in \mathbb{N}^n_{2d}} (f_k)_\alpha y_\alpha - \epsilon_k \leq 0, \quad k \neq j, \\ &\sum_{\alpha \in \mathbb{N}^n_{2d}} (g_i)_\alpha y_\alpha \leq 0, \quad i = 1, \ldots, m, \\ &\sum_{\alpha \in \mathbb{N}^n_{2d}} y_\alpha B_\alpha \succeq 0, \quad y_0 = 1. \end{aligned}$$

Definition 3.2. For each fixed $j = 1, \ldots, p$, we say that *Slater condition* holds for $(\mathrm{P}_j(\epsilon))$, if there exists $\widehat{x} \in \mathbb{R}^n$ such that $f_k(\widehat{x}) - \epsilon_k < 0$, $k \neq j$, and $g_i(\widehat{x}) < 0$, $i = 1, \ldots, m$.

The perturbation function $w_j(\cdot)$, which is associated with $(\mathrm{P}_j(\epsilon))$, is defined on $\mathbb{R}^{p-1} \times \mathbb{R}^m$ as

$$w_j(z, z') = \inf_{x \in \mathbb{R}^n} \{f_j(x) : f_k(x) - \epsilon_k \leq z_k, k \neq j, g_i(x) \leq z'_i, i = 1, \ldots, m\}.$$

Note that $w_j(0)$ is the optimal value of $(\mathrm{P}_j(\epsilon))$.

Definition 3.3. [16] For each fixed $j = 1, \ldots, p$, we say that $(\mathrm{P}_j(\epsilon))$ is *stable*, if $w_j(0)$ is finite and there exists $M > 0$ such that for all $(z, z') \neq 0$,

$$\frac{w_j(0) - w_j(z, z')}{||(z, z')||} \leq M.$$

Lemma 3.4. [16, Theorem 6] *For each fixed* $j = 1, \ldots, p$, *if* $v(\mathrm{P}_j(\epsilon))$ *is finite and* $(\mathrm{P}_j(\epsilon))$ *satisfies the Slater condition, then* $(\mathrm{P}_j(\epsilon))$ *is stable.*

We now give a zero duality gap result for $(\mathrm{P}_j(\epsilon))$, $(\mathrm{D}_j(\epsilon)^{\mathrm{sos}})$, $(\mathrm{SDD}_j(\epsilon))$ and $(\mathrm{SDP}_j(\epsilon))$.

Theorem 3.5. *For each fixed* $j = 1, \dots, p$, *if* $(\mathrm{P}_j(\epsilon))$ *is stable, then*

$$v(\mathrm{P}_j(\epsilon)) = v(\mathrm{D}_j(\epsilon)^{\mathrm{sos}}) = v(\mathrm{SDD}_j(\epsilon)) = v(\mathrm{SDP}_j(\epsilon)).$$

Proof. Let $j \in \{1, \dots, p\}$ be any fixed. Since $(\mathrm{P}_j(\epsilon))$ is stable, by [16, Theorem 3], $v(\mathrm{P}_j(\epsilon)) = v(\mathrm{D}_j(\epsilon))$, and $(\mathrm{D}_j(\epsilon))$ attaints its supremum. Let $(\gamma_j, \mu, \lambda) \in \mathbb{R} \times \mathbb{R}^{p-1}_+ \times \mathbb{R}^m_+$ be any feasible for $(\mathrm{D}_j(\epsilon))$. Then, $f_j(x) + \sum_{k\neq j} \mu_k(f_k(x) - \epsilon_k) + \sum_{i=1}^m \lambda_i g_i(x) - \gamma_j \geq 0$ for all $x \in \mathbb{R}^n$. As f_l, $l = 1, \dots, p$, and g_i, $i = 1, \dots, m$, are SOS-convex ploynomials, $f_j(x)+\sum_{k\neq j} \mu_k(f_k(x)-\epsilon_k)+\sum_{i=1}^m \lambda_i g_i(x)-\gamma_j$ is also SOS-convex, which takes nonnegative values. Hence, it follows from [24, Remark 2.3] that $f_j+\sum_{k\neq j} \mu_k(f_k-\epsilon_k)+\sum_{i=1}^m \lambda_i g_i-\gamma_j$ is a sum of squares in $\mathbb{R}[x]_{2d}$. Thus, we have

$$v(\mathrm{D}_j(\epsilon)) = v(\mathrm{D}_j(\epsilon)^{\mathrm{sos}}).$$

Moreover, $v(\mathrm{D}_j(\epsilon)^{\mathrm{sos}}) = v(\mathrm{SDD}_j(\epsilon))$ obviously holds by the construction of $(\mathrm{D}_j(\epsilon)^{\mathrm{sos}})$ and $(\mathrm{SDD}_j(\epsilon))$.

We now claim that $v(\mathrm{SDD}_j(\epsilon)) \leq v(\mathrm{SDP}_j(\epsilon))$. Let $(\gamma, X, \mu, \lambda)$ and y be any feasible points of $(\mathrm{SDD}_j(\epsilon))$ and $(\mathrm{SDP}_j(\epsilon))$, respectively. Then, we have

$$\begin{aligned}
\gamma_j &= (f_j)_0 + \sum_{k\neq j} \mu_k((f_k)_0 - \epsilon_k) + \sum_{i=1}^m \lambda_i (g_i)_0 - \langle B_0, X\rangle \\
&\leq (f_j)_0 + \sum_{k\neq j} \mu_k((f_k)_0 - \epsilon_k) + \sum_{i=1}^m \lambda_i (g_i)_0 + \left\langle \sum_{\alpha\neq 0} y_\alpha B_\alpha, X \right\rangle \\
&= \sum_{\alpha\in\mathbb{N}^n_{2d}} (f_j)_\alpha y_\alpha + \sum_{k\neq j} \mu_k \left(\sum_{\alpha\in\mathbb{N}^n_{2d}} (f_k)_\alpha y_\alpha - \epsilon_k \right) + \sum_{i=1}^m \lambda_i \sum_{\alpha\in\mathbb{N}^n_{2d}} (g_i)_\alpha y_\alpha \\
&\leq \sum_{\alpha\in\mathbb{N}^n_{2d}} (f_j)_\alpha y_\alpha.
\end{aligned}$$

Therefore, $v(\mathrm{SDD}_j(\epsilon)) \leq v(\mathrm{SDP}_j(\epsilon))$.

To complete the proof of this theorem, it remains to show that $v(\mathrm{P}_j(\epsilon)) \geq v(\mathrm{SDP}_j(\epsilon))$. Let $\widetilde{x}$ be any feasible for $(\mathrm{P}_j(\epsilon))$. Then, $f_k(\widetilde{x}) \leq \epsilon_k$, $k \neq j$, and $g_i(\widetilde{x}) \leq 0$, $i = 1, \dots, m$. Let $\widetilde{y} := (1, \widetilde{x}_1, \dots, \widetilde{x}_n, \widetilde{x}_1^2, \widetilde{x}_1\widetilde{x}_2, \dots, \widetilde{x}_1^{2d}, \dots, \widetilde{x}_n^{2d})$. Then, we have

$$f_k(\widetilde{x}) = \sum_{\alpha\in\mathbb{N}^n_{2d}} (f_k)_\alpha \widetilde{x}^\alpha = \sum_{\alpha\in\mathbb{N}^n_{2d}} (f_k)_\alpha \widetilde{y}_\alpha \leq \epsilon_k, \quad k \neq j$$

and

$$g_i(\widetilde{x}) = \sum_{\alpha\in\mathbb{N}^n_{2d}} (g_i)_\alpha \widetilde{x}^\alpha = \sum_{\alpha\in\mathbb{N}^n_{2d}} (g_i)_\alpha \widetilde{y}_\alpha \leq 0, \quad i = 1, \dots, m.$$

Moreover, since $\widetilde{y}\widetilde{y}^T = \sum_{\alpha\in\mathbb{N}^n_{2d}} \widetilde{y}_\alpha B_\alpha \succeq 0$ with $y_0 = 1$, $\widetilde{y}$ is feasible for $(\mathrm{SDP}_j(\epsilon))$. Hence, it follows that

$$f_j(\widetilde{x}) = \sum_{\alpha} (f_j)_\alpha \widetilde{x}^\alpha = \sum_{\alpha\in\mathbb{N}^n_{2d}} (f_j)_\alpha \widetilde{y}_\alpha \geq v(\mathrm{SDP}_j(\epsilon)).$$

Since $\widetilde{x}$ is any feasible point of $(\mathrm{P}_j(\epsilon))$, we have $v(\mathrm{P}_j(\epsilon)) \geq v(\mathrm{SDP}_j(\epsilon))$. Thus, we obtain the desired result. □

In what follows, we give a relationship of the optimal solutions of $(\mathrm{P}_j(\epsilon))$ and $(\mathrm{SDP}_j(\epsilon))$.

Theorem 3.6. *For each fixed $j = 1, \ldots, p$, if $(\mathrm{P}_j(\epsilon))$ is stable, then the following statements are equivalent:*

(i) *$\overline{x}$ is an optimal solution of $(\mathrm{P}_j(\epsilon))$;*

(ii) *the vector*

$$\overline{y} := (\overline{x}_1, \ldots, \overline{x}_n, \overline{x}_1^2, \overline{x}_1\overline{x}_2, \ldots, \overline{x}_1^{2d}, \ldots, \overline{x}_n^{2d}) \tag{3.1}$$

is an optimal solution of $(\mathrm{SDP}_j(\epsilon))$.

Proof. (i) ⇒ (ii). Let $j \in \{1, \ldots, p\}$ be any fixed. Assume that $\overline{x}$ is an optimal solution of $(\mathrm{P}_j(\epsilon))$. Then $f_k(\overline{x}) \leq \epsilon_k$, $k \neq j$, and $g_i(\overline{x}) \leq 0$, $i = 1, \ldots, m$. Let $\overline{y} = (\overline{y}_\alpha)_{\alpha \in \mathbb{N}^n_{2d}} = (\overline{x}_1, \ldots, \overline{x}_n, \overline{x}_1^2, \overline{x}_1\overline{x}_2, \ldots, \overline{x}_1^{2d}, \ldots, \overline{x}_n^{2d})$. Then, $f_k(\overline{x}) = \sum_{\alpha \in \mathbb{N}^n_{2d}} (f_k)_\alpha \overline{x}^\alpha = \sum_{\alpha \in \mathbb{N}^n_{2d}} (f_k)_\alpha \overline{y}_\alpha \leq \epsilon_k$, $k \neq j$, and $g_i(\overline{x}) = \sum_{\alpha \in \mathbb{N}^n_{2d}} (g_i)_\alpha \overline{x}^\alpha = \sum_{\alpha \in \mathbb{N}^n_{2d}} (g_i)_\alpha \overline{y}_\alpha \leq 0$, $i = 1, \ldots, m$. Moreover, $\overline{y}\overline{y}^T = \sum_{\alpha \in \mathbb{N}^n_{2d}} \overline{y}_\alpha B_\alpha \succeq 0$ with $y_0 = 1$. So, $\overline{y}$ is feasible for $(\mathrm{SDP}_j(\epsilon))$. Since

$$v(\mathrm{P}_j(\epsilon)) = f_j(\overline{x}) = \sum_{\alpha \in \mathbb{N}^n_{2d}} (f_j)_\alpha \overline{y}_\alpha = v(\mathrm{SDP}_j(\epsilon))$$

by Theorem 3.5, $\overline{y}$ is an optimal solution of $(\mathrm{SDP}_j(\epsilon))$.

(ii) ⇒ (i). Let $j \in \{1, \ldots, p\}$ be any fixed. Assume that $\overline{y}$ in (3.1) is an optimal solution of $(\mathrm{SDP}_j(\epsilon))$. Then $\sum_{\alpha \in \mathbb{N}^n_{2d}} (f_k)_\alpha \overline{y}_\alpha \leq \epsilon_k$, $\sum_{\alpha \in \mathbb{N}^n_{2d}} (g_i)_\alpha \overline{y}_\alpha \leq 0$ and $\sum_{\alpha \in \mathbb{N}^n_{2d}} \overline{y}_\alpha B_\alpha \succeq 0$. It means that

$$0 \geq \sum_{\alpha \in \mathbb{N}^n_{2d}} (f_k)_\alpha \overline{y}_\alpha - \epsilon_k = \sum_{\alpha \in \mathbb{N}^n_{2d}} (f_k)_\alpha \overline{x}^\alpha - \epsilon_k = f_k(\overline{x}) - \epsilon_k, \quad k \neq j,$$
$$0 \geq \sum_{\alpha \in \mathbb{N}^n_{2d}} (g_i)_\alpha \overline{y}_\alpha = \sum_{\alpha \in \mathbb{N}^n_{2d}} (g_i)_\alpha \overline{x}^\alpha = g_i(\overline{x}), \quad i = 1, \ldots, m.$$

So, $\overline{x}$ is feasible for $(\mathrm{P}_j(\epsilon))$. It follows from Theorem 3.5 that

$$v(\mathrm{SDP}_j(\epsilon)) = \sum_{\alpha \in \mathbb{N}^n_{2d}} (f_j)_\alpha \overline{y}_\alpha = \sum_{\alpha \in \mathbb{N}^n_{2d}} (f_j)_\alpha \overline{x}^\alpha = f_j(\overline{x}) = v(\mathrm{P}_j(\epsilon)).$$

Thus, $\overline{x}$ is an optimal solution of $(\mathrm{P}_j(\epsilon))$. □

Proposition 3.7. [12] *A feasible solution $\overline{x} \in K$ is an efficient solution of (MP) if and only if there exists $\overline{\epsilon} \in \mathbb{R}^p$ such that $\overline{x}$ is an optimal solution of $(\mathrm{P}_j(\epsilon))$ for all $j = 1, \ldots, p$, where $\overline{\epsilon} = (\overline{\epsilon}_1, \ldots, \overline{\epsilon}_{j-1}, \overline{\epsilon}_{j+1}, \ldots, \overline{\epsilon}_p) \in \mathbb{R}^{p-1}$.*

The following theorem shows that how to find efficient solutions of (MP) by the ϵ-constraint method.

Theorem 3.8. *Let* $f_j\colon \mathbb{R}^n \to \mathbb{R}$, $j = 1, \dots, p$, *and* $g_i\colon \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$, *be convex polynomials. Let* $\overline{x}_{(0)} \in K$ *be any given, and for* $j = 1, \dots, p$, *let*

$$\overline{x}_{(j)} \in \operatorname*{argmin}_{x \in K_j(\overline{\epsilon}_{(j)})} f_j(x),$$

where $(\overline{\epsilon}_{(j)})_k = f_k(\overline{x}_{(j-1)})$, $k \neq j$. *Assume that for each* $j = 1, \dots, p$, $(\mathrm{P}_j(\overline{\epsilon}_{(j)}))$ *has a finite optimal value. Then, the following statements are equivalent:*

(i) $\overline{x} \in K$ *is an efficient solution of* (MP)*;*

(ii) $\overline{x}$ *is an optimal solution of* $(\mathrm{P}_j(\overline{\epsilon}_{(p)}))$, $j = 1, \dots, p$.

Proof. Let $\overline{x}_{(0)} \in K$ be any given. We first claim that for each $j = 1, \dots, p$,

$$\operatorname*{argmin}_{x \in K_j(\overline{\epsilon}_{(j)})} f_j(x) \neq \emptyset.$$

Note that f_l, $l = 1, \dots, p$, and g_i, $i = 1, \dots, m$, are convex polynomials, and for each $j = 1, \dots, p$, $\inf_{x \in K_j(\overline{\epsilon}_{(j)})} f_j(x) > -\infty$. It follows from Lemma 2.2 that $\operatorname{argmin}_{x \in K_j(\overline{\epsilon}_{(j)})} f_j(x) \neq \emptyset$, $j = 1, \dots, p$.

For $j = 1, \dots, p$, let $\overline{x}_{(j)} \in \operatorname{argmin}_{x \in K_j(\overline{\epsilon}_{(j)})} f_j(x)$. Note that for each $j = 1, \dots, p$, the feasible set of $(\mathrm{P}_j(\overline{\epsilon}_{(j)}))$ is as follows:

$$K_j(\overline{\epsilon}_{(j)}) := \{x \in K : f_k(x) \leq f_k(\overline{x}_{(j-1)}), k \neq j\}.$$

Since for each $j = 1, \dots, p$, $\overline{x}_{(j)} \in K_j(\overline{\epsilon}_{(j)})$,

$$f_k(\overline{x}_{(j)}) \leq f_k(\overline{x}_{(j-1)}), \quad k \neq j. \tag{3.2}$$

Moreover, since for each $j = 1, \dots, p$, $\overline{x}_{(j)} \in \operatorname{argmin}_{x \in K_j(\overline{\epsilon}_{(j)})} f_j(x)$

$$f_j(\overline{x}_{(j)}) \leq f_j(x) \tag{3.3}$$

for any $x \in K_j(\overline{\epsilon}_{(j)})$. Since for each $j = 1, \dots, p$, $\overline{x}_{(j-1)} \in K_j(\overline{\epsilon}_{(j)})$, from (3.3), we see that

$$f_j(\overline{x}_{(j)}) \leq f_j(\overline{x}_{(j-1)}), \quad j = 1, \dots, p. \tag{3.4}$$

So, by (3.2) and (3.4), we obtain

$$f_j(\overline{x}_{(p)}) \leq f_j(\overline{x}_{(p-1)}) \leq \cdots \leq f_j(\overline{x}_{(1)}) \leq f_j(\overline{x}_{(0)}), \quad j = 1, \dots, p. \tag{3.5}$$

Hence, we see that for each $j = 1, \ldots, p$, $\overline{x}_{(p)} \in K_j(\overline{\epsilon}_{(p)}) \subseteq K_j(\overline{\epsilon}_{(j)})$. So, we have for each $j = 1, \ldots, p$,

$$f_j(\overline{x}_{(p)}) \geq \min_{x \in K_j(\overline{\epsilon}_{(p)})} f_j(x) \geq \min_{x \in K_j(\overline{\epsilon}_{(j)})} f_j(x) = f_j(\overline{x}_{(j)}).$$

Since, by (3.5), for each $j = 1, \ldots, p$, $f_j(\overline{x}_{(p)}) \leq f_j(\overline{x}_{(j)})$, we have for each $j = 1, \ldots, p$,

$$f_j(\overline{x}_{(p)}) = f_j(\overline{x}_{(j)}) \quad \text{and} \quad \overline{x}_{(p)} \in \operatorname*{argmin}_{x \in K_j(\overline{\epsilon}_{(p)})} f_j(x).$$

So, $\overline{x} := \overline{x}_{(p)}$ is an optimal solution of $(\mathrm{P}_j(\overline{\epsilon}_{(p)}))$ for all $j = 1, \ldots, p$. Thus, by Proposition 3.7, we obtain the desired result. □

Remark 3.9. Note that in the proof of Theorem 3.8, the assumption of convexity (SOS-convexity is not necessary) ensures the existence of solutions of $(\mathrm{P}_j(\overline{\epsilon}_{(j)}))$, $j = 1, \ldots, p$, based on Lemma 2.2. Moreover, it is obvious that Theorem 3.8 still holds whenever the involving functions are SOS-convex polynomials.

By Theorems 3.6 and 3.8, we obtain the following theorem.

Theorem 3.10. *Let $\overline{x}_{(0)} \in K$ be any given, and for $j = 1, \ldots, p$, let*

$$\overline{x}_{(j)} \in \operatorname*{argmin}_{x \in K_j(\overline{\epsilon}_{(j)})} f_j(x),$$

where $(\overline{\epsilon}_{(j)})_k = f_k(\overline{x}_{(j-1)})$, $k \neq j$. For each fixed $j = 1, \ldots, p$, if $(\mathrm{P}_j(\overline{\epsilon}_{(j)}))$ is stable, then the following statements are equivalent:

(i) *$\overline{x} \in K$ is an efficient solution of* (MP)*;*

(ii) *the vector*

$$\overline{y} := (\overline{x}_1, \ldots, \overline{x}_n, \overline{x}_1^2, \overline{x}_1\overline{x}_2, \ldots, \overline{x}_1^{2d}, \ldots, \overline{x}_n^{2d})$$

is an optimal solution of $(\mathrm{SDP}_j(\overline{\epsilon}_{(p)}))$ *for all $j = 1, \ldots, p$.*

We finish the section by giving an example, which aims to illustrate Theorem 3.10.

Example 3.11. Consider the following multicriteria problem:

$$(\mathrm{MP}_1) \qquad \min\ (f_1(x_1, x_2), f_2(x_1, x_2)) \quad \text{subject to} \quad g_1(x_1, x_2) \leq 0,$$

where $f_1(x_1, x_2) = x_1^8 + x_1^2 + x_1x_2 + x_2^2$, $f_2(x_1, x_2) = x_1^4 - x_2$ and $g_1(x_1, x_2) = x_1^2 + x_2^2 - 1$. Clearly, $f_1(x_1, x_2)$ is an SOS-convex polynomial (since x_1^8 is SOS-convex and the Hessian of $x_1^2 + x_1x_2 + x_2^2$ is SOS, and hence sum of x_1^8 and $x_1^2 + x_1x_2 + x_2^2$ is SOS-convex polynomial); see also [18, 23, 25]. Observe that $K = \{(x_1, x_2) \in \mathbb{R}^2 : g_1(x_1, x_2) \leq 0\}$ is the feasible set of (MP_1).

We substitute the above problem (MP_1) by the ϵ-constraint problems as follows:

$$(\mathrm{P}_1(\overline{\epsilon}_{(1)})) \qquad \min\ f_1(x_1,x_2) \quad \text{subject to} \quad f_2(x_1,x_2) \le \overline{\epsilon}_{(1)}, \quad (x_1,x_2)\in K,$$

$$(\mathrm{P}_2(\overline{\epsilon}_{(2)})) \qquad \min\ f_2(x_1,x_2) \quad \text{subject to} \quad f_1(x_1,x_2) \le \overline{\epsilon}_{(2)}, \quad (x_1,x_2)\in K,$$

where for each $j = 1, 2$, $\overline{\epsilon}_{(j)}$ is given by $\overline{\epsilon}_{(1)} = f_2(\overline{x}_{(0)})$ and $\overline{\epsilon}_{(2)} = f_1(\overline{x}_{(1)})$. Here, $\overline{x}_{(0)} \in K$ is any given, and $\overline{x}_{(1)} \in \mathrm{argmin}_{x\in K_1(\overline{\epsilon}_{(1)})} f_1(x)$.

Now, we formulate the sum of squares relaxation dual problem for $(\mathrm{P}_j(\overline{\epsilon}_{(j)}))$ as follows:

$$(\mathrm{D}_j(\overline{\epsilon}_{(j)})^{\mathrm{sos}}) \qquad \sup_{\substack{\gamma_j\in\mathbb{R}\\ \mu_k\ge 0,\lambda_1\ge 0}} \{\gamma_j : f_j + \mu_k(f_k - \epsilon_k) + \lambda_1 g_1 - \gamma_j \in \Sigma^2_8\}.$$

Invoking Proposition 2.3, there exists $X \in S_+^{s(2,4)}$ $(= S_+^{15})$ such that

$$f_j(x) + \mu_k(f_k(x) - \overline{\epsilon}_{(j)}) + \lambda_1 h_1(x) - \gamma_j = \langle v_4(x)v_4(x)^T, X\rangle \tag{3.6}$$

for all $x \in \mathbb{R}^2$. Then, from [32, Theorem 1], we can reduce the dimension of $v_4(x)$ to 6, and so $X \in S^6_+$. In more detail, $v_4(x) = (1, x_1, x_2, x_1^2, x_1^3, x_1^4)^T$ in (3.6). With this fact, for each $j = 1, 2$, $(\mathrm{D}_j(\overline{\epsilon}_{(j)})^{\mathrm{sos}})$ can be rewritten as the following semidefinite programming problems:

$$(\mathrm{SDD}_1(\overline{\epsilon}_{(1)})) \qquad \begin{aligned} &\sup_{\gamma_1,X,\mu_2,\lambda_1} \gamma_1 \quad \text{subject to}\\ &-\mu_2\overline{\epsilon}_{(1)} - \lambda_1 - \gamma_1 = X_{11}, \quad -\mu_2 = 2X_{13}, \quad 1 = 2X_{23},\\ &1+\lambda_1 = 2X_{14} + X_{22} = X_{33}, \quad \mu_2 = 2X_{16} + 2X_{25} + X_{44},\\ &1 = X_{66}, \quad 0 = X_{12} = X_{34} = X_{35} = X_{36},\\ &0 = X_{15} + X_{24} = X_{26} + X_{45} = 2X_{46} + X_{55} = X_{56},\\ &\gamma_1\in\mathbb{R}, \quad X\in S^6_+, \quad \mu_2\ge 0, \quad \lambda_1 \ge 0, \end{aligned}$$

$$(\mathrm{SDD}_2(\overline{\epsilon}_{(2)})) \qquad \begin{aligned} &\sup_{\gamma_2,X,\mu_1,\lambda_1} \gamma_2 \quad \text{subject to}\\ &-\mu_1\overline{\epsilon}_{(2)} - \lambda_1 = X_{11}, \quad -1 = 2X_{13}, \quad \mu_1 = 2X_{23},\\ &\mu_1+\lambda_1 = 2X_{14} + X_{22} = X_{33}, \quad 1 = 2X_{16} + 2X_{25} + X_{44},\\ &\mu_1 = X_{66}, \quad 0 = X_{12} = X_{34} = X_{35} = X_{36},\\ &0 = X_{26} + X_{45} = X_{15} + X_{24} = 2X_{46} + X_{55} = X_{56},\\ &\gamma_2\in\mathbb{R}, \quad X\in S^6_+, \quad \mu_1\ge 0, \quad \lambda_1 \ge 0. \end{aligned}$$

Solving the above semidefinite programming problems using the MATLAB optimization package CVX [17] together with the SDP-solver SDPT3 [33], we can find the optimal solutions of $(\mathrm{SDP}_j(\epsilon))$, $j = 1, 2$. For example, let $x^*_{(0)} = (1, 0)$ and $\overline{\epsilon}_{(1)} = f_2(\overline{x}_{(0)}) = 1$.

Observe that the Slater condition for $(\mathrm{P}_1(\overline{\epsilon}_{(1)}))$ holds, and so, $(\mathrm{P}_1(\overline{\epsilon}_{(1)}))$ is stable. Solving $(\mathrm{SDD}_1(\overline{\epsilon}_{(1)}))$ with CVX [17], we find the dual variable of $(\mathrm{SDD}_1(\overline{\epsilon}_{(1)}))$, which is an optimal solution of $(\mathrm{SDP}_1(\overline{\epsilon}_{(1)}))$, as follows:

$$\overline{y} = (1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).$$

No.	$\overline{x}_{(0)}$	$\overline{\epsilon}_{(1)}$	$\overline{x}_{(1)}$	$\overline{\epsilon}_{(2)}$	$\overline{x}$: efficient sol.
1	$(1.0000, 0.0000)$	1.0000	$(0.0000, 0.0000)$	0.0000	$(0.0000, 0.0000)$
2	$(0.9877, 0.1564)$	0.7952	$(0.0000, 0.0000)$	0.0000	$(0.0000, 0.0000)$
3	$(0.9511, 0.3090)$	0.5091	$(0.0000, 0.0000)$	0.0000	$(0.0000, 0.0000)$
4	$(0.8910, 0.4540)$	0.1763	$(0.0000, 0.0000)$	0.0000	$(0.0000, 0.0000)$
5	$(0.8090, 0.5878)$	-0.1594	$(-0.0795, 0.1594)$	0.0254	$(-0.0916, 0.1840)$
6	$(0.7071, 0.7071)$	-0.4571	$(-0.2155, 0.4593)$	0.2048	$(-0.2389, 0.5220)$
7	$(0.5878, 0.8090)$	-0.6897	$(-0.2926, 0.6970)$	0.4531	$(-0.3109, 0.7723)$
8	$(0.4540, 0.8910)$	-0.8485	$(-0.3292, 0.8603)$	0.6737	$(-0.3428, 0.9365)$
9	$(0.3090, 0.9511)$	-0.9419	$(-0.3090, 0.9511)$	0.8017	$(-0.2025, 0.9793)$
10	$(0.1564, 0.9877)$	-0.9871	$(-0.1564, 0.9877)$	0.8700	$(-0.1312, 0.9914)$
11	$(0.0000, 1.0000)$	-1.0000	$(-0.0003, 1.0000)$	0.9997	$(-0.0003, 1.0000)$
12	$(-0.1564, 0.9877)$	-0.9871	$(-0.1564, 0.9877)$	0.8700	$(-0.1312, 0.9914)$
13	$(-0.3090, 0.9511)$	-0.9419	$(-0.3090, 0.9511)$	0.8017	$(-0.2025, 0.9793)$
14	$(-0.4540, 0.8910)$	-0.8485	$(-0.3292, 0.8603)$	0.6737	$(-0.3428, 0.9365)$
15	$(-0.5878, 0.8090)$	-0.6897	$(-0.2926, 0.6970)$	0.4531	$(-0.3109, 0.7723)$
16	$(-0.7071, 0.7071)$	-0.4571	$(-0.2155, 0.4593)$	0.2048	$(-0.2389, 0.5220)$
17	$(-0.8090, 0.5878)$	-0.1594	$(-0.0795, 0.1594)$	0.0254	$(-0.0916, 0.1840)$
18	$(-0.8910, 0.4540)$	0.1763	$(0.0000, 0.0000)$	0.0000	$(0.0000, 0.0000)$
19	$(-0.9511, 0.3090)$	0.5091	$(0.0000, 0.0000)$	0.0000	$(0.0000, 0.0000)$
20	$(-0.9877, 0.1564)$	0.7952	$(0.0000, 0.0000)$	0.0000	$(0.0000, 0.0000)$
21	$(-1.0000, 0.0000)$	1.0000	$(0.0000, 0.0000)$	0.0000	$(0.0000, 0.0000)$

Table 3.1: We give 21 points $\overline{x}_{(0)} = (\cos\theta, \sin\theta) \in K$, $0 \leq \theta \leq \pi$. If $\overline{x}_{(0)} = (\cos\theta, \sin\theta) \in K$, where $\pi \leq \theta \leq 2\pi$, then all of efficient solutions of (MP_1) are $(0, 0)$.

It means that $(\overline{x}_1, \overline{x}_2) = (0, 0)$ is an optimal solution of $(\mathrm{P}_1(\overline{\epsilon}_{(1)}))$. Now, let $\overline{\epsilon}_{(2)} = f_1(\overline{x}_{(1)}) = 0$. Then, we can see that $(\mathrm{P}_2(\overline{\epsilon}_{(2)}))$ is not stable. On the other hand, the feasible set $K_2(\overline{\epsilon}_{(2)})$ of $(\mathrm{P}_2(\overline{\epsilon}_{(2)}))$ is $\{(0,0)\}$, which is a singleton, so $(0,0)$ is an optimal solution of $(\mathrm{P}_2(\overline{\epsilon}_{(2)}))$, notwithstanding the fact that the stability of $(\mathrm{P}_2(\overline{\epsilon}_{(2)}))$ fails. It follows from Theorem 3.8 that $(\overline{x}_1, \overline{x}_2) = (0,0)$ is an efficient solution of (MP_1).

In order to find more efficient solutions of (MP_1), we give 21 points $\overline{x}_{(0)} = (\cos\theta, \sin\theta) \in K$, $0 \leq \theta \leq \pi$, and then we get the efficient solutions of (MP_1) in Table 3.1. An illustration of the found efficient solutions of (MP_1) is given in Figure 3.1. Moreover, we give 1000 points $\overline{x}_{(0)}$ (not only the boundary points but also the interior points) in K. The efficient solutions of (MP_1) for above points $\overline{x}_{(0)}$ described in Figure 3.2.

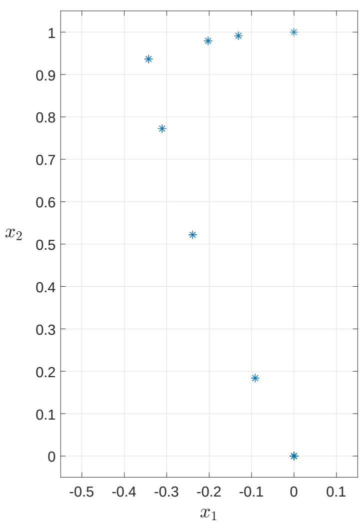

Figure 3.1: Efficient solutions of (MP_1) at Table 3.1.

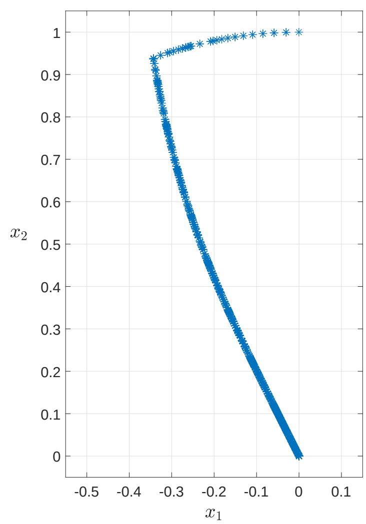

Figure 3.2: Efficient solutions of (MP_1) for given 1000 points.

4. Conclusions

The main techniques adopted in this paper are actually two, one is the discovery that the optimal value and optimal solutions of an SOS-convex optimization problem can be found by solving a single semidefinite programming problem; another is a powerful scalarization method, i.e., the ϵ-constraint method. As a consequence, we obtained our main results (Theorems 3.8 and 3.10) on finding efficient solutions for a multicriteria optimization problem with SOS-convex polynomials by using the ϵ-constraint method. The observation in this paper seems new in the literature [4, 7, 11, 13–15].

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References

[1] A. A. Ahmadi and P. A. Parrilo, *A convex polynomial that is not sos-convex*, Math. Program. **135** (2012), no. 1-2, 275–292. https://doi.org/10.1007/s10107-011-0457-z

[2] ______, *A complete characterization of the gap between convexity and sos-convexity*, SIAM J. Optim. **23** (2013), no. 2, 811–833. https://doi.org/10.1137/110856010

[3] E. G. Belousov and D. Klatte, *A Frank–Wolfe type theorem for convex polynomial programs*, Comput. Optim. Appl. **22** (2002), no. 1, 37–48. https://doi.org/10.1023/A:1014813701864

[4] H. P. Benson, *An outer approximation algorithm for generating all efficient extreme points in the outcome set of a multiple objective linear programming problem*, J. Glob. Optim. **13** (1998), no. 1, 1–24. https://doi.org/10.1023/A:1008215702611

[5] A. Ben-Tal and A. Nemirovskii, *Lectures on Modern Convex Optimization: Analysis, Algorithms, and Engineering Applications*, SIAM, Philadelphia, PA, 2001.

[6] D. P. Bertsekas, *Nonlinear Programming*: Third edition, Athena Scientific Optimization and Computation Series, Athena Scientific, Belmont, MA, 2016.

[7] V. Blanco, J. Puerto and S. El Haj Ben Ali, *A semidefinite programming approach for solving multiobjective linear programming*, J. Glob. Optim. **58** (2014), no. 3, 465–480. https://doi.org/10.1007/s10898-013-0056-z

[8] S. Boyd and L. Vandenberghe, *Convex Optimization*, Cambridge University Press, Cambridge, 2004.

[9] M. Caramia and P. Dell'Olmo, *Multi-objective Management in Freight Logistics: Increasing Capacity, Servie Level and Safety with Optimization Algorithms*, Springer-Verlag, London, 2008. https://doi.org/10.1007/978-1-84800-382-8

[10] V. Chankong and Y. Y. Haimes, *Multiobjective Decision Making*, North-Holland Series in System Science and Engineering **8**, North-Holland, Amsterdam, 1983.

[11] J. G. Ecker and I. A. Kouada, *Finding efficient points for linear multiple objective programs*, Math. Programming **8** (1975), 375–377. https://doi.org/10.1007/BF01580453

[12] M. Ehrgott, *Multicriteria Optimization*, Second edition, Springer, Berlin, 2005.

[13] M. Ehrgott, A. Löhne and L. Shao, *A dual variant of Benson's "outer approximation algorithm" for multiple objective linear programming*, J. Global Optim. **52** (2012), no. 4, 757–778. https://doi.org/10.1007/s10898-011-9709-y

[14] M. Ehrgott, J. Puerto and A. M. Rodriguez-Chia, *Primal-dual simplex method for multiobjective linear programming*, J. Optim. Theory Appl. **134** (2007), no. 3, 483–497. https://doi.org/10.1007/s10957-007-9232-y

[15] J. Fliege, *An efficient interior-point method for convex multicriteria optimization problems*, Math. Oper. Res. **31** (2006), no. 4, 825–845. https://doi.org/10.1287/moor.1060.0221

[16] A. M. Geoffrion, *Duality in nonlinear programming: A simplified applications-oriented development*, SIAM Rev. **13** (1971), no. 1, 1–37. https://doi.org/10.1137/1013001

[17] M. C. Grant and S. P. Boyd, *The CVX user's guide, release 2.0.*, User manual (2013). http://cvxr.com/cvx/doc/CVX

[18] J. W. Helton and J. Nie, *Structured semidefinite representation of some convex sets*, Proceedings of the 47th IEEE Conference on Decision and Control, Cancun, Mexico, Dec. 9-11, 2008, 4797–4800. https://doi.org/10.1109/CDC.2008.4738593

[19] ______, *Semidefinite representation of convex sets*, Math. Program. **122** (2010), no. 1, Ser. A, 21–64. https://doi.org/10.1007/s10107-008-0240-y

[20] J. Jahn, *Scalarization in vector optimization*, Math. Programming **29** (1984), no. 2, 203–218. https://doi.org/10.1007/BF02592221

[21] ______, *Vector Optimization: Theory, Applications, and Extensions*, Second edition, Springer-Verlag, Berlin, 2011.

[22] V. Jeyakumar, G. M. Lee and J. H. Lee, *Sums of squares characterizations of containment of convex semialgebraic sets*, Pac. J. Optim. **12** (2016), no. 1, 29–42.

[23] V. Jeyakumar and G. Li, *Exact SDP relaxations for classes of nonlinear semidefinite programming problems*, Oper. Res. Lett. **40** (2012), no. 6, 529–536. https://doi.org/10.1016/j.orl.2012.09.006

[24] ______, *A new class of alternative theorems for SOS-convex inequalities and robust optimization*, Appl. Anal. **94** (2015), no. 1, 56–74. https://doi.org/10.1080/00036811.2013.859251

[25] V. Jeyakumar, G. Li and J. Vicente-Pérez, *Robust SOS-convex polynomial optimization problems: exact SDP relaxations*, Optim. Lett. **9** (2015), no. 1, 1–18. https://doi.org/10.1007/s11590-014-0732-z

[26] J. B. Lasserre, *Convexity in semialgebraic geometry and polynomial optimization*, SIAM J. Optim. **19** (2009), no. 4, 1995–2014. https://doi.org/10.1137/080728214

[27] ______, *Moments, Positive Polynomials and Their Applications*, Imperial College Press Optimization Series **1**, Imperial College Press, London, 2010.

[28] M. Laurent, *Sums of squares, moment matrices and optimization over polynomials*, in: *Emerging Applications of Algebraic Geometry*, 157–270, IMA Vol. Math. Appl. **149**, Springer, Berlin, 2009. https://doi.org/10.1007/978-0-387-09686-5_7

[29] D. T. Luc, *Scalarization of vector optimization problems*, J. Optim. Theory Appl. **55** (1987), no. 1, 85–102. https://doi.org/10.1007/BF00939046

[30] ______, *Multiobjective Linear Programming: An Introduction*, Springer, Switzerland, 2016.

[31] G. Mavrotas, *Effective implementation of the ϵ-constraint method in multi-objective mathematical programming problems*, Appl. Math. Comput. **213** (2009), no. 2, 455–465. https://doi.org/10.1016/j.amc.2009.03.037

[32] B. Reznick, *Extremal PSD forms with few terms*, Duke Math. J. **45** (1978), no. 2, 363–374. https://projecteuclid.org/euclid.dmj/1077312822

[33] K. C. Toh, M. J. Todd and R. H. Tütüncü, *SDPT3—a MATLAB software package for semidefinite programming, Version 1.3. Interior point methods*, Optim. Methods Softw. **11** (1999), no. 1-4, 545–581. https://doi.org/10.1080/10556789908805762

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