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An Analytic Version of Wiener-Itô Decomposition on Abstract Wiener Spaces

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In memory of Professor Hwai-Chiuan Wang

Abstract. In this paper, we first establish an analogue of Wiener-Itô theorem on finite-dimensional Gaussian spaces through the inverse S-transform, that is, the Gauss transform on Segal-Bargmann spaces. Based on this point of view, on infinite-dimensional abstract Wiener space (H, B), we apply the analyticity of the S-transform, which is an isometry from the L^2-space onto the Bargmann-Segal-Dwyer space, to study the regularity. Then, by defining the Gauss transform on Bargmann-Segal-Dwyer space and showing the relationship with the S-transform, an analytic version of Wiener-Itô decomposition will be obtained.

1. Introduction

The Wiener-Itô theorem is an important result concerning an orthogonal decomposition of the Hilbert space of square integrable functions on a Gaussian space. It was first proved in 1938 by N. Wiener [19] in terms of homogeneous chaos. It is a natural question to find an explicit description concerning homogeneous chaos. About this decomposition theorem, K. Itô [8] in 1951 provided a different proof from N. Wiener. In that paper, Itô defined multiple Wiener integrals, and shown that there is a one-to-one correspondence between the homogeneous chaoses and multiple Wiener integrals.

In [11], Lee studied the regularity of the heat semigroup generated from the abstract Wiener measure p_t on an abstract Wiener space (AWS, in short) (H, B). It was shown that the convolution $p_t * f$ is infinitely Fréchet differentiable for $f \in L^\alpha_c(B, p_t)$ with $\alpha > 1$. Further, Lee [12] applied the results concerning regularity of $p_t * f$ to show that there is one-to-one correspondence between the homogeneous chaoses of order n and the space of the Gauss transforms of n-linear continuous Hilbert-Schmidt operators on the complexification H_c of H. In that paper, Lee did not apply the analyticity of $p_t * f$ to study the regularity.

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In infinite-dimensional Gaussian analysis, the semigroup $p_t * f$ plays the role of the S-transform of f, which provides an isometric link between the L^2-space and the Bargmann-Segal-Dwyer spaces. If H is finite-dimensional, a well-known fact is that the Taylor expansion of an Bargmann analytic function is an orthogonal decomposition with respect to the Gauss measure on the complex Euclidean space. Then the Wiener-Itô decomposition is immediately achieved by taking the inverse S-transform, that is, the Gauss transform.

In this paper, we first study the relationship between the S-transform, the Gauss transform and Hermite polynomials in order to establish the analogue of Wiener-Itô theorem on finite-dimensional Gaussian spaces in Section 2. Our basic idea is to consider the S-transform of an L^2-function, and then expand it into a Taylor series; finally the analogue of Wiener-Itô decomposition is obtained by taking the Gauss transform. In Section 3, we review some basic properties of AWS. For the S-transform and Gauss transform, we will study in Sections 4 and 5. It is worth noting that every L^2-function on a general infinite-dimensional AWS can be regarded as an L^2-limit of a sequence of cylinder functions by taking the conditional expectation (see Lemma 3.4). Based on this point of view, we will apply the results obtained in Section 2 to cylinder functions, and then use Lemma 3.4 to arrive at the desired Wiener-Itô theorem on (H, B) in Section 5. And, by defining the Gauss transform on Bargmann-Segal-Dwyer space and showing the relationship with the S-transform, the Wiener-Itô decomposition will be expressed in terms of the Gauss transform in Theorem 5.3.

Notations. Throughout this paper, we always use boldface to represent multi-indices, and elements in multi-dimensional real or complex Euclidean spaces, or to emphasize elements in the complexification of a real locally convex space. In addition, we list some of shorthand notations that are often used in this paper.

(1) For a real locally convex space V, V_c denotes its complexification. If V is a real Hilbert space endowed with $\|\cdot\|_V$-norm induced by the inner product $\langle\langle \cdot, \cdot \rangle\rangle_V$, then V_c is a complex Hilbert space with the $\|\cdot\|_{V_c}$-norm given by $\|\phi_1 + \mathrm{i}\phi_2\|_{V_c}^2 = \|\phi_1\|_V^2 + \|\phi_2\|_V^2$ induced by the inner product defined by $\langle\langle \phi_1 + \mathrm{i}\phi_2, \psi_1 + \mathrm{i}\psi_2 \rangle\rangle_{V_c} = \langle\langle \phi_1, \psi_1 \rangle\rangle_V + \langle\langle \phi_2, \psi_2 \rangle\rangle_V + \mathrm{i}(\langle\langle \phi_2, \psi_1 \rangle\rangle_V - \langle\langle \phi_1, \psi_2 \rangle\rangle_V)$ for any ϕ_1, ϕ_2, ψ_1 and ψ_2 in V. In addition, we define $(\cdot, \cdot)_{V_c}$ on $V_c \times V_c$ as a bilinear form given by

$$(\phi_1 + \mathrm{i}\phi_2, \psi_1 + \mathrm{i}\psi_2)_{V_c} = \langle\langle \phi_1, \psi_1 \rangle\rangle_V - \langle\langle \phi_2, \psi_2 \rangle\rangle_V + \mathrm{i}(\langle\langle \phi_2, \psi_1 \rangle\rangle_V + \langle\langle \phi_1, \psi_2 \rangle\rangle_V).$$

If $V = \mathbb{R}^k$, $V_c = \mathbb{C}^k$ endowed with the dot product in this paper.

(2) If T is an n-linear operator on $X \times \cdots \times X$ (n-times) ($X = V$ or V_c), $Tx_1 \cdots x_n$ means $T(x_1, \ldots, x_n)$ for $x_1, \ldots, x_n \in X$, and if $x_1 = \cdots = x_n = x$, we write $Tx \cdots x$ as Tx^n.

Let $\mathbb{N}_0$ be the set of all nonnegative integers. For any multi-index $\boldsymbol{\alpha} = (n_1, \ldots, n_k) \in \mathbb{N}_0^k$ and $\mathbf{z} = (z_1, \ldots, z_k) \in \mathbb{C}^k$, define

(3) $|\boldsymbol{\alpha}| = n_1 + \cdots + n_k$, $\boldsymbol{\alpha}! = n_1! \cdots n_k!$,

(4) $\mathbf{z}^{\boldsymbol{\alpha}} = z_1^{n_1} \cdots z_k^{n_k}$.

Similarly, we define the differential operator $\big(\frac{\partial}{\partial \mathbf{x}}\big)^{\boldsymbol{\alpha}}$ by

(5) $\big(\frac{\partial}{\partial \mathbf{x}}\big)^{\boldsymbol{\alpha}} = \big(\frac{\partial}{\partial x_1}\big)^{n_1} \cdots \big(\frac{\partial}{\partial x_k}\big)^{n_k}$ for $\mathbf{x} = (x_1, \ldots, x_k) \in \mathbb{R}^k$ or $\mathbb{C}^k$.

2. Finite-dimensional Gaussian space

Let $\mu_1(du)$ be the standard Gaussian measure on $(\mathbb{R}, \mathscr{B}(\mathbb{R}))$ with mean 0 and variance 1. Apply the orthogonalization procedure to the monomials $\{1, u, u^2, \ldots, u^n, \ldots\}$ in $L_c^2(\mathbb{R}, \mu_1)$, in this order, to get Hermite polynomials $\{H_0(u), H_1(u), \ldots, H_n(u), \ldots\}$. Here H_n is a polynomial of degree n with leading coefficient 1, and the generating function of $\{H_n\}$ is given by

$$\sum_{n=0}^{\infty} \frac{H_n(u)}{n!} z^n = e^{uz - \frac{1}{2}z^2}, \quad \forall\, u \in \mathbb{R},\ z \in \mathbb{C}. \tag{2.1}$$

And, by substituting the formula $e^{-\frac{1}{2}z^2} = \int_{-\infty}^{\infty} e^{\mathrm{i}zv}\, \mu_1(dv)$, $\mathrm{i} = \sqrt{-1}$ and $z \in \mathbb{C}$, into (2.1), we have

$$\sum_{n=0}^{\infty} \frac{H_n(u)}{n!} z^n = \sum_{n=0}^{\infty} \frac{z^n}{n!} \int_{-\infty}^{\infty} (u + \mathrm{i}v)^n\, \mu_1(dv), \quad \forall\, u, z \in \mathbb{C}. \tag{2.2}$$

For other related properties, we refer the reader to [17].

Let $\boldsymbol{\mu}_r(d\mathbf{v})$, $r > 0$, be the Gausssian measure on $(\mathbb{R}^k, \mathscr{B}(\mathbb{R}^k))$ given by

$$\boldsymbol{\mu}_r(d\mathbf{v}) = (\sqrt{2\pi r})^{-k} e^{-\frac{1}{2r}\mathbf{v}\cdot\mathbf{v}}\, d\mathbf{v},$$

where $d\mathbf{v}$ denotes k-dimensional Lebesgue measure on $\mathbb{R}^k$. Note that $(\mathbb{R}^k, \mathscr{B}(\mathbb{R}^k), \boldsymbol{\mu}_1)$ is the product measure space $(\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu_1) \times \cdots \times (\mathbb{R}, \mathscr{B}(\mathbb{R}), \mu_1)$. For any multi-index $\boldsymbol{\alpha} = (n_1, \ldots, n_k) \in \mathbb{N}_0^k$ and $\mathbf{u} = (u_1, \ldots, u_k) \in \mathbb{R}^k$, set the shorthand notation

$$H_{\boldsymbol{\alpha}}(\mathbf{u}) = H_{n_1}(u_1) \cdots H_{n_k}(u_k).$$

Comparing the coefficients on both sides of (2.2) yields that

$$H_{\boldsymbol{\alpha}}(\mathbf{u}) = \int_{\mathbb{R}^k} (\mathbf{u} + \mathrm{i}\mathbf{v})^{\boldsymbol{\alpha}}\, \boldsymbol{\mu}_1(d\mathbf{v}), \quad \forall\, \boldsymbol{\alpha} \in \mathbb{N}_0^k,\ \mathbf{u} \in \mathbb{R}^k, \tag{2.3}$$

and then, through direct calculation,

$$\int_{\mathbb{R}^k} H_{\boldsymbol{\alpha}}(\mathbf{v}) H_{\boldsymbol{\beta}}(\mathbf{v})\, \boldsymbol{\mu}_1(d\mathbf{v}) = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}} \cdot \boldsymbol{\alpha}!, \tag{2.4}$$

where $\delta_{\boldsymbol{\alpha},\boldsymbol{\beta}} = 1$ as $\boldsymbol{\alpha} = \boldsymbol{\beta}$; otherwise, 0.

2.1. The S-transform on $L_c^2(\mathbb{R}^k, \boldsymbol{\mu}_1)$

Let $\phi \in L_c^2(\mathbb{R}^k, \boldsymbol{\mu}_1)$ be arbitrarily given. The S-transform $S\phi$ of ϕ is a complex-valued function on $\mathbb{R}^k$ defined by $S\phi(\mathbf{u}) = \boldsymbol{\mu}_1 * \phi(\mathbf{u})$, $\mathbf{u} \in \mathbb{R}^k$, where $\boldsymbol{\mu}_1 * \phi$ is the convolution of ϕ with $\boldsymbol{\mu}_1$. It is easily seen by checking their characteristic functions that $\boldsymbol{\mu}_1(d\mathbf{v} - \mathbf{u}) = e^{\mathbf{v}\cdot\mathbf{u} - \frac{1}{2}|\mathbf{u}|^2}\, \boldsymbol{\mu}_1(d\mathbf{v})$ for any $\mathbf{u} \in \mathbb{R}^k$. Then we can immediately extend the definition of the S-transform of ϕ to $\mathbb{C}^k$, still denoted by $S\phi$, by

$$S\phi(\mathbf{z}) = \int_{\mathbb{R}^k} \phi(\mathbf{v}) e^{(\mathbf{v},\mathbf{z}) - \frac{1}{2}(\mathbf{z},\mathbf{z})}\, \boldsymbol{\mu}_1(d\mathbf{v}), \quad \forall\, \mathbf{z} \in \mathbb{C}^k,$$

where $(\,\cdot\,,\cdot\,)$ represents $(\,\cdot\,,\cdot\,)_{\mathbb{C}^k}$. By applying Hartogs' theorem (see [2]), $S\phi(\mathbf{z})$ is a holomorphic function of $\mathbf{z} \in \mathbb{C}^k$; therefore, it enjoys the Taylor series expansion

$$S\phi(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^k} \frac{\mathbf{z}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \left(\frac{\partial}{\partial \mathbf{z}}\right)^{\boldsymbol{\alpha}} S\phi(\mathbf{0}), \quad \forall\, \mathbf{z} \in \mathbb{C}^k. \tag{2.5}$$

For any $\mathbf{u} \in \mathbb{R}^k$ and $\mathbf{z} \in \mathbb{C}^k$, it follows from (2.1) that

$$e^{(\mathbf{u},\mathbf{z}) - \frac{1}{2}(\mathbf{z},\mathbf{z})} = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^k} \frac{H_{\boldsymbol{\alpha}}(\mathbf{u})}{\boldsymbol{\alpha}!} \mathbf{z}^{\boldsymbol{\alpha}}.$$

Let $\mathbf{z} \in \mathbb{C}^k$ be fixed and consider both sides of the above formula as a function of $\mathbf{u}$. Observe that $\sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^k} \frac{|\mathbf{z}^{\boldsymbol{\alpha}}|^2}{\boldsymbol{\alpha}!} = e^{|\mathbf{z}|^2}$ is finite. Then, by (2.4),

$$e^{(\,\cdot\,,\mathbf{z}) - \frac{1}{2}(\mathbf{z},\mathbf{z})} = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^k} \frac{\mathbf{z}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} H_{\boldsymbol{\alpha}} \quad \text{in } L_c^2(\mathbb{R}^k, \boldsymbol{\mu}_1),$$

and thus

$$S\phi(\mathbf{z}) = \sum_{\boldsymbol{\alpha} \in \mathbb{N}_0^k} \frac{\mathbf{z}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!} \int_{\mathbb{R}^k} \phi(\mathbf{u}) H_{\boldsymbol{\alpha}}(\mathbf{u})\, \boldsymbol{\mu}_1(d\mathbf{u}). \tag{2.6}$$

Compare (2.5) with (2.6) to get that

$$\int_{\mathbb{R}^k} \phi(\mathbf{u}) H_{\boldsymbol{\alpha}}(\mathbf{u})\, \boldsymbol{\mu}_1(d\mathbf{u}) = \left(\frac{\partial}{\partial \mathbf{z}}\right)^{\boldsymbol{\alpha}} S\phi(\mathbf{0}), \quad \forall\, \boldsymbol{\alpha} \in \mathbb{N}_0^k. \tag{2.7}$$

2.2. The Segal-Bargmann space of $\mathbb{C}^k$

A Bargmann analytic function defined on $\mathbb{C}^k$, introduced in [1], is a holomorphic function, which is square-integrable with respect to the Gausssian measure $\boldsymbol{\mu}_r$, $r > 0$, on $(\mathbb{C}^k, \mathscr{B}(\mathbb{C}^k))$, $\boldsymbol{\mu}_r(d\mathbf{z})$ being given by

$$\boldsymbol{\mu}_r(d\mathbf{z}) = (\sqrt{2\pi r})^{-k} e^{-\frac{1}{2r}\mathbf{z}\cdot\mathbf{z}}\, d\mathbf{z},$$

where $d\mathbf{z}$ denotes $2k$-dimensional Lebesgue measure on $\mathbb{C}^k$. Denote the class of Bargmann analytic functions defined above by $\mathcal{K}^r(\mathbb{C}^k)$, called the Segal-Bargmann space of $\mathbb{C}^k$ (see [6]). One notes that $\mathcal{K}^r(\mathbb{C}^k)$ is a complex separable Hilbert space with $\|\cdot\|_{\mathcal{K}^r(\mathbb{C}^k)}$-norm induced by the inner product $\langle\!\langle\,\cdot\,,\cdot\,\rangle\!\rangle_{\mathcal{K}^r(\mathbb{C}^k)}$ defined by

$$\langle\!\langle F, G\rangle\!\rangle_{\mathcal{K}^r(\mathbb{C}^k)} = \int_{\mathbb{C}^k} F(\mathbf{z})\overline{G(\mathbf{z})}\,\boldsymbol{\mu}_r(d\mathbf{z}).$$

For any $F \in \mathcal{K}^r(\mathbb{C}^k)$, it enjoys the Taylor expansion $\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k} \frac{\mathbf{z}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\left(\frac{\partial}{\partial\mathbf{z}}\right)^{\boldsymbol{\alpha}} F(\mathbf{0})$, and for $r = 1/2$, we have, through direct calculation, that

$$\int_{\mathbb{C}^k} \mathbf{z}^{\boldsymbol{\alpha}}\,\overline{\mathbf{z}}^{\boldsymbol{\beta}}\,\boldsymbol{\mu}_{1/2}(d\mathbf{z}) = \delta_{\boldsymbol{\alpha},\boldsymbol{\beta}}\cdot\boldsymbol{\alpha}!, \quad \forall\,\boldsymbol{\alpha},\boldsymbol{\beta}\in\mathbb{N}_0^k. \tag{2.8}$$

Then it immediately from (2.5), (2.7) and (2.8) that $S\phi \in \mathcal{K}^{1/2}(\mathbb{C}^k)$, where

$$\|S\phi\|^2_{\mathcal{K}^{1/2}(\mathbb{C}^k)} = \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k} \frac{1}{\boldsymbol{\alpha}!}\left|\int_{\mathbb{R}^k} \phi(\mathbf{u})H_{\boldsymbol{\alpha}}(\mathbf{u})\,\boldsymbol{\mu}_1(d\mathbf{u})\right|^2 < +\infty. \tag{2.9}$$

For any holomorphic function F on $\mathbb{C}^k$ and $n \in \mathbb{N}$, let $D^nF(\mathbf{0})$ be the function defined by

$$(\mathbf{z}_1,\ldots,\mathbf{z}_n) \in \mathbb{C}^k\times\cdots\times\mathbb{C}^k \mapsto \left.\frac{\partial^n}{\partial\xi_1\cdots\partial\xi_n}\right|_{\xi_1=\cdots=\xi_n=0} F(\xi_1\mathbf{z}_1+\cdots+\xi_n\mathbf{z}_n).$$

Then $D^nF(\mathbf{0})$ is a symmetric n-linear form (see [7]), and

$$\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k,|\boldsymbol{\alpha}|=n} \frac{\mathbf{z}^{\boldsymbol{\alpha}}}{\boldsymbol{\alpha}!}\left(\frac{\partial}{\partial\mathbf{z}}\right)^{\boldsymbol{\alpha}} F(\mathbf{0}) = \frac{1}{n!}D^nF(\mathbf{0})\mathbf{z}^n, \quad \forall\,\mathbf{z}\in\mathbb{C}^k, \tag{2.10}$$

where $\mathbf{z}^n$ means $(z_1, z_2, \ldots, z_n)$ with $z_1 = z_2 = \cdots = z_n$. By (2.8) and together with (2.10), we see that for any $F, G \in \mathcal{K}^{1/2}(\mathbb{C}^k)$,

$$\begin{aligned}\langle\!\langle F, G\rangle\!\rangle_{\mathcal{K}^{1/2}(\mathbb{C}^k)} &= \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k,|\boldsymbol{\alpha}|=n}\frac{n!}{\boldsymbol{\alpha}!}\left(\frac{\partial}{\partial\mathbf{z}}\right)^{\boldsymbol{\alpha}}F(\mathbf{0})\left(\frac{\partial}{\partial\mathbf{z}}\right)^{\boldsymbol{\alpha}}G(\mathbf{0})\\ &= \sum_{n=0}^{\infty}\frac{1}{n!}\langle\!\langle D^nF(\mathbf{0}), D^nG(\mathbf{0})\rangle\!\rangle_{\mathcal{HS}},\end{aligned} \tag{2.11}$$

where $\langle\!\langle\,\cdot\,,\cdot\,\rangle\!\rangle_{\mathcal{HS}}$ means the Hilbert-Schmidt inner product (see [16] for the definition). In particular, for any $\phi \in L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1)$,

$$\|S\phi\|^2_{\mathcal{K}^{1/2}(\mathbb{C}^k)} = \sum_{n=0}^{\infty}\frac{1}{n!}\|D^nS\phi(\mathbf{0})\|^2_{\mathcal{HS}}, \tag{2.12}$$

where $\|\cdot\|_{\mathcal{HS}}$ denotes the Hilbert-Schmidt operator norm induced by $\langle\!\langle\,\cdot\,,\cdot\,\rangle\!\rangle_{\mathcal{HS}}$.

2.3. The Gauss transform on $\mathcal{K}^{1/2}(\mathbb{C}^k)$

Let $F \in \mathcal{K}^{1/2}(\mathbb{C}^k)$ be given. The Gauss transform $\sigma(F)$ of F is a function on $\mathbb{R}^k$ defined by

$$\sigma(F)(\cdot) = \int_{\mathbb{R}^k} F(\,\cdot + \mathrm{i}\mathbf{v})\, \boldsymbol{\mu}_1(d\mathbf{v}) \quad \text{(if it exists).}$$

It follows from (2.3) that for any $\boldsymbol{\alpha} \in \mathbb{N}_0^k$ and $\mathbf{u} \in \mathbb{R}^k$,

$$\sigma(\mathbf{z}^{\boldsymbol{\alpha}})(\mathbf{u}) = \int_{\mathbb{R}^k} (\mathbf{u} + \mathrm{i}\mathbf{v})^{\boldsymbol{\alpha}}\, \boldsymbol{\mu}_1(d\mathbf{v}) = H_{\boldsymbol{\alpha}}(\mathbf{u}), \tag{2.13}$$

and therefore, by (2.6), $S\sigma(\mathbf{z}^{\boldsymbol{\alpha}}) = \mathbf{z}^{\boldsymbol{\alpha}}$. Now, for any $n \in \mathbb{N}$, set

$$\psi_n \equiv \sigma\left(\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k, |\boldsymbol{\alpha}|\le n} \frac{1}{\boldsymbol{\alpha}!}\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\boldsymbol{\alpha}} F(\mathbf{0})\mathbf{z}^{\boldsymbol{\alpha}}\right) \overset{\text{(by (2.13))}}{=} \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k, |\boldsymbol{\alpha}|\le n} \frac{1}{\boldsymbol{\alpha}!}\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\boldsymbol{\alpha}} F(\mathbf{0}) \cdot H_{\boldsymbol{\alpha}}.$$

As n tends to infinity, since, by (2.11),

$$\sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k} \frac{1}{\boldsymbol{\alpha}!}\left|\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\boldsymbol{\alpha}} F(\mathbf{0})\right|^2 = \|F\|^2_{\mathcal{K}^{1/2}(\mathbb{C}^k)} < +\infty,$$

it follows by (2.4) that $\psi_n \to \psi$ in $L_c^2(\mathbb{R}^k, \boldsymbol{\mu}_1)$, where

$$\psi = \sum_{n=0}^{\infty} \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k, |\boldsymbol{\alpha}|= n} \frac{1}{\boldsymbol{\alpha}!}\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\boldsymbol{\alpha}} F(\mathbf{0}) \cdot H_{\boldsymbol{\alpha}}. \tag{2.14}$$

Consequently, by (2.6), $S\psi = F$.

2.4. Wiener-Itô decomposition of $L_c^2(\mathbb{R}^k, \boldsymbol{\mu}_1)$

Let $\phi \in L_c^2(\mathbb{R}^k, \boldsymbol{\mu}_1)$ be given. Combine (2.5) with (2.14) to get that

$$S\left(\sum_{n=0}^{\infty} \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k, |\boldsymbol{\alpha}|= n} \frac{1}{\boldsymbol{\alpha}!}\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\boldsymbol{\alpha}} S\phi(\mathbf{0}) \cdot H_{\boldsymbol{\alpha}}\right) = S\phi,$$

which implies that

$$S\left(\phi - \sum_{n=0}^{\infty} \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k, |\boldsymbol{\alpha}|= n} \frac{1}{\boldsymbol{\alpha}!}\left(\frac{\partial}{\partial \mathbf{z}}\right)^{\boldsymbol{\alpha}} S\phi(\mathbf{0}) \cdot H_{\boldsymbol{\alpha}}\right)(\mathbf{z}) = 0, \quad \forall\, \mathbf{z} \in \mathbb{C}^k. \tag{2.15}$$

Then we need the following

Lemma 2.1. *The S-transform on $L_c^2(\mathbb{R}^k, \boldsymbol{\mu}_1)$ is injective.*

Proof. Let $\psi \in L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1)$. If $S\psi(\mathbf{z}) = 0$ for any $\mathbf{z} \in \mathbb{C}^k$, then $S\psi(\mathrm{i}\mathbf{u}) = 0$ for any $\mathbf{u} \in \mathbb{R}^k$. Therefore

$$\int_{\mathbb{R}^k} \psi(\mathbf{v}) e^{\mathrm{i}\mathbf{v}\cdot\mathbf{u}}\, \boldsymbol{\mu}_1(d\mathbf{v}) = 0, \quad \forall\, \mathbf{u} \in \mathbb{R}^k.$$

This implies that $\psi(\mathbf{v})e^{-\frac{1}{2}\mathbf{v}\cdot\mathbf{v}} = 0$ and hence $\psi(\mathbf{v}) = 0$ for almost every $\mathbf{v} \in \mathbb{R}^k$ with respect to Lebesgue measure $\mathbf{m}_k$ on $\mathbb{R}^k$. Since $\boldsymbol{\mu}_1$ is absolutely continuous with respect to $\mathbf{m}_k$, $\psi = 0$ almost everywhere on $\mathbb{R}^k$ with respect to $\boldsymbol{\mu}_1$. The proof is complete. □

Applying Lemma 2.1 together with (2.3), (2.10) and (2.15), we see that

$$\begin{aligned}
\phi &= \sum_{n=0}^{\infty} \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k, |\boldsymbol{\alpha}|=n} \frac{1}{\boldsymbol{\alpha}!} \left(\frac{\partial}{\partial \mathbf{z}}\right)^{\boldsymbol{\alpha}} S\phi(\mathbf{0}) \cdot H_{\boldsymbol{\alpha}} \\
&= \sum_{n=0}^{\infty} \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k, |\boldsymbol{\alpha}|=n} \frac{1}{\boldsymbol{\alpha}!} \left(\frac{\partial}{\partial \mathbf{z}}\right)^{\boldsymbol{\alpha}} S\phi(\mathbf{0}) \int_{\mathbb{R}^k} (\,\cdot + \mathrm{i}\mathbf{v})^{\boldsymbol{\alpha}}\, \boldsymbol{\mu}_1(d\mathbf{v}) \\
&= \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^k} D^n S\phi(\mathbf{0})(\,\cdot + \mathrm{i}\mathbf{v})^n\, \boldsymbol{\mu}_1(d\mathbf{v}) \quad \text{in } L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1).
\end{aligned} \tag{2.16}$$

As an immediate consequence of the first equality of (2.16), $\{(\boldsymbol{\alpha}!)^{-1/2} H_{\boldsymbol{\alpha}}; \boldsymbol{\alpha} \in \mathbb{N}_0^k\}$ forms a complete orthonormal basis (CONB, in short) of $L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1)$; therefore, the sum on the right side of (2.9) is exactly equal to $\|\phi\|^2_{L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1)}$.

Now, we sum up all arguments in this section toghther with (2.9), (2.12), (2.14) and (2.16) to obtain the following

Theorem 2.2. (i) *The S-transform S is an isometry from $L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1)$ onto $\mathcal{K}^{1/2}(\mathbb{C}^k)$.*

(ii) *(Wiener-Itô Theorem on $L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1)$) Let $\phi \in L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1)$ be given. Then*

$$\phi = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^k} D^n S\phi(\mathbf{0})(\,\cdot + \mathrm{i}\mathbf{v})^n\, \boldsymbol{\mu}_1(d\mathbf{v}) \quad \text{in } L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1).$$

Moreover,

$$\|\phi\|^2_{L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1)} = \sum_{n=0}^{\infty} \frac{1}{n!} \|D^n S\phi(\mathbf{0})\|^2_{\mathcal{HS}}.$$

Corollary 2.3. *Let $\lambda > 1$. For $\phi \in L^2_c(\mathbb{R}^k, \boldsymbol{\mu}_1)$, if*

$$A_\lambda(\phi) \equiv \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} \|D^n S\phi(\mathbf{0})\|^2_{\mathcal{HS}} < +\infty,$$

then the Wiener-Itô decomposition of ϕ, that is the right-hand series of Theorem 2.2(ii), is defined everywhere in $\mathbb{C}^k$ and converges absolutely and uniformly on bounded subsets of $\mathbb{C}^k$.

Proof. By using the Cauchy-Schwarz inequality,

$$\sum_{n=0}^{\infty} \frac{1}{n!} \int_{\mathbb{R}^k} |D^n S\phi(\mathbf{0})(\mathbf{z}+\mathrm{i}\mathbf{v})^n| \, \boldsymbol{\mu}_1(d\mathbf{v}) \le \sqrt{A_\lambda(\phi)} \int_{\mathbb{R}^k} e^{\frac{1}{2\lambda}(|\mathbf{z}|+|\mathbf{v}|)^2} \, \boldsymbol{\mu}_1(d\mathbf{v}),$$

from which the assertion follows. □

3. Abstract Wiener spaces

Let H be a real separable Hilbert space with $|\cdot|$-norm induced by the inner product $\langle \cdot, \cdot \rangle$, and $\|\cdot\|$ be an another norm defined on H which is weaker than the $|\cdot|$-norm and measurable on H. Then the triple (i, H, B) is called an abstract Wiener space (AWS, in short), where B is the completion of H with respect to $\|\cdot\|$-norm and i is the canonical embedding of H into B. The AWS was introduced by L. Gross in his celebrated paper [4]. Readers interested in details and other related results concerning AWS may consult [4,5,9].

As H is identified as a dense subspace of B, we also identify B^*, the dual of B, as a dense subspace of the dual H^* of H under the adjoint operator i^* of i by the following way: For any $x \in H$ and $\eta \in B^*$, $\langle x, i^*(\eta)\rangle = (i(x), \eta)$, where $(\cdot, \cdot)$ denotes the B-B^* dual pairing in the subsequent study. Applying the Riesz representation theorem to identify H^* with H, there are continuous inclusion maps $B^* \subset H \subset B$. Then L. Gross [4] proved that B carries a probability measure p_t, known as the abstract Wiener measure with variance parameter $t > 0$, which is characterized as the unique Borel measure on B such that for any $\eta \in B^*$,

$$\int_B e^{\mathrm{i}(x,\eta)} \, p_t(dx) = e^{-\frac{t}{2}|\eta|^2}. \tag{3.1}$$

Remark 3.1. If $H = \mathbb{R}^k$, then $B^* = H = B$. For any $\mathbf{u}, \mathbf{v} \in \mathbb{R}^k$, $(\mathbf{u}, \mathbf{v}) = \langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v}$, and p_t is exactly the Gaussian measure μ_t on $(\mathbb{R}^k, \mathscr{B}(\mathbb{R}^k))$ given by

$$p_t(d\mathbf{v}) = (\sqrt{2\pi t})^{-k} e^{-\frac{1}{2t}\mathbf{v}\cdot\mathbf{v}} \, d\mathbf{v},$$

where $d\mathbf{v}$ denotes k-dimensional Lebesgue measure on $\mathbb{R}^k$.

From (3.1), it follows that $(\cdot, \eta)$, $\eta \in B^*$, is a random variable on $(B, \mathscr{B}(B), p_1)$, distributed by the law of $N(0, |\eta|^2)$. For any $h \in H$, let $\{\eta_n\}$ be a sequence in B^* such that $|\eta_n - h| \to 0$ as $n \to \infty$. Then $\{(\cdot, \eta_n)\}$ forms a Cauchy sequence in $L^2_c(B, p_1)$, the $L^2(B, p_1)$-limit of which is denoted by $\mathbf{n}(h)$. As $h \in B^*$, $\mathbf{n}(h) = (\cdot, h)$, and for any $h \in H$, $\mathbf{n}(h)$ is independent of the choice of $\{\eta_n\}$ and distributed by the law of $N(0, |h|^2)$. Regard $\mathbf{n}$ as a function from H into $L^2(B, p_1)$. Then it can be further extended to H_c by defining, for any $h_1, h_2 \in H$, $\mathbf{n}(h_1 + \mathrm{i}h_2) = \mathbf{n}(h_1) + \mathrm{i}\mathbf{n}(h_2)$. It is easy to check that $\mathbf{n}$ is linear from H_c into $L^2_c(B, p_1)$, and for any $\mathbf{h}, \mathbf{k} \in H_c$,

$$(\mathbf{h}, \mathbf{k})_{H_c} = \int_B \mathbf{n}(\mathbf{h})(x)\mathbf{n}(\mathbf{k})(x) \, p_1(dx). \tag{3.2}$$

Example 3.2. Let $\mathcal{C}$ denote the Banach space, consisting of all real-valued continuous functions $x(t)$ in the unit interval $[0,1]$ with $x(0)=0$, endowed with the supremum norm $|\cdot|_\infty$. Let $\mathcal{C}'$ be the Cameron-Martin space which is a Hilbert space, consisting of all absolutely continuous functions $x\in\mathcal{C}$ with a square integrable derivative $\dot{x}$, with the norm $|\cdot|_0$ induced by the inner product $\langle\cdot,\cdot\rangle_0$ defined by $\langle x,y\rangle_0=\int_0^1\dot{x}(t)\dot{y}(t)\,dt$ for any $x,y\in\mathcal{C}'$. The space $\mathcal{C}'$ is usually called the Cameron-Martin space and it is well-known that $(i,\mathcal{C}',\mathcal{C})$ forms an AWS (see [4,9]). The triple $(i,\mathcal{C}',\mathcal{C})$ is known as the classical Wiener space. As $\mathcal{C}'$ is a dense subspace of $\mathcal{C}$, we identify the dual space $\mathcal{C}^*$ of $\mathcal{C}$ as a dense subspace of the dual space $(\mathcal{C}')^*$ of $\mathcal{C}'$ under the adjoint operator i^* of i. Then $\mathcal{C}^*$ is regarded as the set of all $y\in\mathcal{C}'$ satisfying the properties that $\dot{y}$ is right continuous and of bounded variation with $\dot{y}(1)=0$. Moreover, for any $x\in\mathcal{C}$ and $y\in\mathcal{C}^*$,

$$(x,y)=-\int_0^1 x(t)\,d\dot{y}(t),$$

where $(\cdot,\cdot)$ is the $\mathcal{C}$-$\mathcal{C}^*$ pairing. Let μ be the associated abstract Wiener measure. Then a standard Brownian motion $B(t)$ can be represented by $B(t;x)=x(t)=(x,\alpha_t)$ for any $x\in\mathcal{C}$ and $0\le t\le 1$, where $\alpha_t(s)=\min\{t,s\}$. Moreover, apply the integration by parts formula to see that for any $y\in\mathcal{C}^*$ and $x\in\mathcal{C}$, $(x,y)=\int_0^1\dot{y}(t)\,dB(t;x)$, where the right-hand integral is a Riemann-Stieltjes integral. From this equality it immediately follows that the random variable $\mathbf{n}(y)$ is the Wiener integral $\int_0^1\dot{y}(t)\,dB(t)$. For the further details, we refer the reader to [9,14].

Example 3.3. Let $\mathbf{A}=-\left(\frac{d}{du}\right)^2+1+u^2$ be a densely defined self-adjoint operator on $L^2(\mathbb{R},du)$ with respect to the Lebesgue measure du, and $\{h_n;n\in\mathbb{N}_0\}$ be a CONB for $L^2(\mathbb{R},du)$, consisting of all Hermite functions on $\mathbb{R}$, formed by the eigenfunctions of $\mathbf{A}$ with corresponding eigenvalues $2n+2$, $n\in\mathbb{N}_0$, where $h_n(u)=(\sqrt{\sqrt{\pi}n!})^{-1}H_n(\sqrt{2}u)e^{-\frac{1}{2}u^2}$. Let $\mathcal{S}$ be the Schwartz space of real-valued, rapidly decreasing, and infinitely differentiable functions on $\mathbb{R}$ with its dual $\mathcal{S}'$, the spaces of tempered distributions. For each $p\in\mathbb{R}$, let $\mathcal{S}_p$ denote the space of all functions f in $\mathcal{S}'$ having $|f|_p^2\equiv\sum_{n=0}^\infty(2n+2)^{2p}|(f,h_n)|^2<+\infty$, where $(\cdot,\cdot)$ always denotes the $\mathcal{S}'$-$\mathcal{S}$ pairing. Then $\mathcal{S}_p$, $p\in\mathbb{R}$, forms a real Hilbert space with the inner product $\langle\cdot,\cdot\rangle_p$ induced by $|\cdot|_p$. The dual space $\mathcal{S}_p'$, $p\in\mathbb{R}$, is unitarily equivalent to $\mathcal{S}_{-p}$. Then we have the continuous inclusions:

$$\mathcal{S}\subset\mathcal{S}_q\subset\mathcal{S}_p\subset L^2(\mathbb{R},du)=\mathcal{S}_0\subset\mathcal{S}_{-p}\subset\mathcal{S}_{-q}\subset\mathcal{S}',$$

where $0<p<q<+\infty$. One notes that $\mathcal{S}$ is the projective limit of $\{\mathcal{S}_p;p>0\}$. In fact, $\mathcal{S}$ is a nuclear space, and thus $\mathcal{S}'$ is the inductive limit of $\{\mathcal{S}_{-p};p>0\}$. The well-known Minlos theorem (see [3]) guarantees the existence of the white noise measure μ on $(\mathcal{S}',\mathscr{B}(\mathcal{S}'))$, the characteristic functional of which is given by

$$\int_{\mathcal{S}'}e^{\mathrm{i}(x,\eta)}\,\mu(dx)=e^{-\frac{1}{2}|\eta|_0^2}\quad\text{for all }\eta\in\mathcal{S}.$$

For $p > 1/2$, it is easy to see that $(i, \mathcal{S}_0, \mathcal{S}_{-p})$ forms an AWS (see [10,15]), the measurable support of μ is contained in $\mathcal{S}_{-p}$ and μ coincides with the associated abstract Wiener measure. The probability space $(\mathcal{S}', \mathscr{B}(\mathcal{S}'), \mu)$ is known as the white noise space, and the Brownian motion $B = \{B(t); t \in \mathbb{R}\}$ on $(\mathcal{S}', \mathscr{B}(\mathcal{S}'), \mu)$ can be represented by

$$B(t; x) = \begin{cases} \mathbf{n}(\mathbf{1}_{[0,t]})(x) & \text{if } t \geq 0, \\ -\mathbf{n}(\mathbf{1}_{[t,0]})(x) & \text{if } t < 0,\ x \in \mathcal{S}', \end{cases}$$

where $\mathbf{1}_A$ means the indicator function of the set A. Moreover, for any $\eta \in \mathcal{S}_0$, $\mathbf{n}(h)$ is exactly the Wiener integral $\int_{-\infty}^{\infty} \eta(t)\, dB(t)$.

Next, assume that H is infinite-dimensional, and $\{e_j \in B^*; j \in \mathbb{N}\}$ is a CONB of H. Then, for any $\mathbf{h} \in H_c$, we have

$$\mathbf{n}(\mathbf{h}) = \sum_{j=1}^{\infty} \langle \mathbf{h}, e_j \rangle\, \mathbf{n}(e_j) \quad \text{in } L_c^2(B, p_1).$$

Let $\mathscr{B}_k$ be the σ-field generated by the random variables $(\,\cdot\,, e_j)$ for $j = 1, 2, \ldots, k$. Then $\{\mathscr{B}_k; k \in \mathbb{N}\}$ is a filtration. It is worth noting that the σ-field $\mathscr{B}_\infty \equiv \bigvee_{k=1}^{\infty} \mathscr{B}_k$ coincides with $\mathscr{B}(B)$ (see [9]). The following lemma plays an important role, which will enable us to extend the results of Section 2 from finite dimensions to infinite dimensions.

Lemma 3.4. *For any $f \in L_c^2(B, p_1)$, $\mathbb{E}[f|\mathscr{B}_k]$ converges to f in $L_c^2(B, p_1)$ as $k \to \infty$, where $\mathbb{E}[\varphi|\mathscr{B}_k]$ means the conditional expectation of φ relative to $\mathscr{B}_k$.*

Proof. Let $\varepsilon > 0$ be given. By applying the Carathéodory-Hahn extension theorem (see [18]), there exists a simple function $f_\varepsilon = \sum_{j=1}^{m} a_j \mathbf{1}_{E_j}$, a_j's $\in \mathbb{C}$, such that E_j's $\in \bigcup_{k=1}^{\infty} \mathscr{B}_k$ and $\|f - f_\varepsilon\|_{L_c^2(B,p_1)} < \varepsilon/2$. Let $N \in \mathbb{N}$ such that E_j's $\in \mathscr{B}_N$. Then, for any $k \geq N$, $f_\varepsilon \in \mathscr{B}_k$ and

$$\begin{aligned} \|f - \mathbb{E}[f|\mathscr{B}_k]\|_{L_c^2(B,p_1)}^2 &\leq 2\left(\|f - f_\varepsilon\|_{L_c^2(B,p_1)}^2 + \|f_\varepsilon - \mathbb{E}[f|\mathscr{B}_k]\|_{L_c^2(B,p_1)}^2\right) \\ &= 2\left(\|f - f_\varepsilon\|_{L_c^2(B,p_1)}^2 + \|\mathbb{E}[f_\varepsilon - f|\mathscr{B}_k]\|_{L_c^2(B,p_1)}^2\right) \\ &\leq 4\|f - f_\varepsilon\|_{L_c^2(B,p_1)}^2 < \varepsilon, \end{aligned}$$

where the last inequality is obtained by Jensen's inequality. The proof is complete. □

4. The S-transform on $L_c^2(B, p_1)$

In this and next section, (i, H, B) denotes a fixed but arbitrary AWS as given in Section 3, where H is always assumed to be infinitely dimensional. Roughly speaking, our basic idea of deducing the relevant formulas in an infinite-dimensional space is to consider the cylinder functions in the beginning; then apply the results obtained in Section 2 to these functions, and finally use Lemma 3.4 to arrive at the results by taking the limits. For example, the arguments in (4.2) and the procedure from (5.1) to (5.4) is a typical process of our basic idea.

Let $f \in L_c^2(B, p_1)$ be given. Define the S-transform $S_\infty f$ of f be a function by $S_\infty f(h) = p_1 * f(h)$, $h \in H$, where $p_1 * f$ is the convolution of f with p_1. For any $h \in H$, define $p_1(h, E) = p_1(E - h)$, $E \in \mathscr{B}(B)$. It is well-known (see [9]) that $p_1(dx) = p_1(-dx)$ and $p_1(h, \cdot)$ is absolutely continuous with respect to p_1, the associated Radon-Nikodym derivative being given by $e^{\mathbf{n}(h) - \frac{1}{2}|h|^2}$. Then, for any $h \in H$,

$$S_\infty f(h) = \int_B f(x+h)\, p_1(dx) = \int_B f(x) e^{\mathbf{n}(h)(x) - \frac{1}{2}|h|^2}\, p_1(dx).$$

And, there is a natural extension of $S_\infty f$ to H_c, still denoted by $S_\infty f$, given by

$$S_\infty f(\mathbf{h}) = \int_B f(x) e^{\mathbf{n}(\mathbf{h})(x) - \frac{1}{2}(\mathbf{h},\mathbf{h})_{H_c}}\, p_1(dx), \quad \mathbf{h} \in H_c.$$

In fact, if $\mathbf{h} \in H_c$, say $\mathbf{h} = h_1 + \mathrm{i}h_2 \in H_c$ with $h_1, h_2 \in H$, then

$$|S_\infty f(\mathbf{h})| \le \int_B |f(x)|\, e^{\mathbf{n}(h_1)(x) - \frac{1}{2}(|h_1|^2 - |h_2|^2)}\, p_1(dx) \le \|f\|_{L_c^2(B,p_1)} \cdot e^{\frac{1}{2}|\mathbf{h}|^2},$$

which implies that $S_\infty f$ is locally bounded in H_c. In addition, for any $\mathbf{h}, \mathbf{k} \in H_c$, it is easily seen that the function $z \in \mathbb{C} \mapsto S_\infty f(\mathbf{h} + z\mathbf{k})$ is holomorphic. Then $S_\infty f(\mathbf{h})$ is an analytic function of $\mathbf{h} \in H_c$ (see [7]); therefore, $S_\infty f$ is Fréchet differentiable in H_c, and $D^n S_\infty f(\mathbf{0})$, $n \in \mathbb{N}$, is a continuous symmetric n-linear form, where D is the Fréchet derivative of $S_\infty f$. Moreover, $S_\infty f$ enjoys the Taylor expansion:

$$S_\infty f(\mathbf{h}) = \sum_{n=0}^{\infty} \frac{1}{n!} D^n S_\infty f(\mathbf{0}) \mathbf{h}^n, \quad \forall\, \mathbf{h} \in H_c.$$

Let $\{e_j \in B^*; j = 1, 2, \ldots\}$ be a CONB of H. For any $k \in \mathbb{N}$, let S_k be the S-transform on $L_c^2(\mathbb{R}^k, \boldsymbol{\mu}_1)$. Denote $\mathbb{E}[f|\mathscr{B}_k]$ by f_k, where $\mathscr{B}_k$ is a Borel σ-subfield of $\mathscr{B}(B)$ defined as in the previous section. A well-known fact is that f_k can be represented as $\phi_k(\mathbf{n}(e_1), \ldots, \mathbf{n}(e_k))$, where $\phi_k \in L_c^2(\mathbb{R}^k, \boldsymbol{\mu}_1)$. Here one notes that

$$p_1(\{(\mathbf{n}(e_1), \ldots, \mathbf{n}(e_k)) \in E\}) = \boldsymbol{\mu}_1(E), \quad \forall\, E \in \mathscr{B}(\mathbb{R}^k).$$

For any $k \in \mathbb{N}$ and $x, y \in B$, define $P_k(x + \mathrm{i}y) = \sum_{j=1}^k ((x, e_j) + \mathrm{i}(y, e_j)) e_j \in B^*$, and

$$\mathbf{z}_{x+\mathrm{i}y,k} = (\mathbf{n}(e_1)(x + \mathrm{i}y), \ldots, \mathbf{n}(e_k)(x + \mathrm{i}y)) \in \mathbb{C}^k.$$

Lemma 4.1. *Let f and ϕ_k be given as above. Then, for any $\mathbf{h} \in H_c$, we have*

$$D^n S_k \phi_k(\mathbf{0}) \mathbf{z}_{\mathbf{h},k}^n = D^n S_\infty f(0)(P_k(\mathbf{h}))^n.$$

Proof. Observe that

$$
\begin{aligned}
D^n S_k\phi_k(\mathbf{0})\mathbf{z}^n_{\mathbf{h},k} &= \frac{d^n}{d\xi^n}\bigg|_{\xi=0} S_k\phi_k(\xi\mathbf{z}_{\mathbf{h},k}) \\
&= \frac{d^n}{d\xi^n}\bigg|_{\xi=0} \int_{\mathbb{R}^k} \phi_k(\mathbf{v}) e^{\xi(\mathbf{v},\mathbf{z}_{\mathbf{h},k})_{\mathbb{C}^k} - \frac{1}{2}\xi^2(\mathbf{z}_{\mathbf{h},k},\mathbf{z}_{\mathbf{h},k})_{\mathbb{C}^k}}\, \boldsymbol{\mu}_1(d\mathbf{v}) \qquad (4.1)\\
&= \frac{d^n}{d\xi^n}\bigg|_{\xi=0} \int_B f_k(x) e^{\mathbf{n}(\xi P_k(\mathbf{h}))(x) - \frac{1}{2}(\xi P_k(\mathbf{h}),\xi P_k(\mathbf{h}))_{H_c}}\, p_1(dx).
\end{aligned}
$$

Since $e^{\mathbf{n}(\xi P_k(\mathbf{h}))(x) - \frac{1}{2}(\xi P_k(\mathbf{h}),\xi P_k(\mathbf{h}))_{H_c}}$ is $\mathscr{B}_k$-measurable, (4.1) can be transformed into

$$
\begin{aligned}
&\frac{d^n}{d\xi^n}\bigg|_{\xi=0} \int_B \mathbb{E}[f\cdot e^{\mathbf{n}(\xi P_k(\mathbf{h})) - \frac{1}{2}(\xi P_k(\mathbf{h}),\xi P_k(\mathbf{h}))_{H_c}}|\mathscr{B}_k](x)\, p_1(dx) \\
&= \frac{d^n}{d\xi^n}\bigg|_{\xi=0} \int_B f(x) e^{\mathbf{n}(\xi P_k(\mathbf{h}))(x) - \frac{1}{2}(\xi P_k(\mathbf{h}),\xi P_k(\mathbf{h}))_{H_c}}\, p_1(dx) = D^n S_\infty f(0)(P_k(\mathbf{h}))^n.
\end{aligned}
$$

The proof is complete. □

Applying Lemma 4.1, the explicit formula of $D^n S_\infty f(0)$ can be obtained as follows.

Proposition 4.2. *For any $f\in L^2_c(B,p_1)$ and $\mathbf{h}_1,\ldots,\mathbf{h}_n\in H_c$,*

$$
D^n S_\infty f(0)\mathbf{h}_1,\cdots\mathbf{h}_n = \int_B f(x)\bigg\{\int_B \prod_{j=1}^n (\mathbf{n}(\mathbf{h}_j)(x) + \mathrm{i}\mathbf{n}(\mathbf{h}_j)(y))\, p_1(dy)\bigg\}\, p_1(dx).
$$

Proof. First, we consider the case that $\mathbf{h}_1 = \cdots = \mathbf{h}_n = h \in H\setminus\{0\}$. By Lemma 4.1 and applying (2.3), (2.7) and (2.10), one can see that

$$
\begin{aligned}
D^n S_\infty f(0) h^n &= \lim_{k\to\infty} D^n S_\infty f(0)(P_k(h))^n \\
&= \lim_{k\to\infty} D^n S_k\phi_k(\mathbf{0})\mathbf{z}^n_{h,k} \\
&= \lim_{k\to\infty} n! \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k, |\boldsymbol{\alpha}|=n} \frac{\mathbf{z}^{\boldsymbol{\alpha}}_{h,k}}{\boldsymbol{\alpha}!}\left(\frac{\partial}{\partial\mathbf{z}}\right)^{\boldsymbol{\alpha}} S_k\phi_k(\mathbf{0}) \\
&= \lim_{k\to\infty} n! \sum_{\boldsymbol{\alpha}\in\mathbb{N}_0^k, |\boldsymbol{\alpha}|=n} \frac{\mathbf{z}^{\boldsymbol{\alpha}}_{h,k}}{\boldsymbol{\alpha}!} \int_{\mathbb{R}^k} \phi_k(\mathbf{u}) H_{\boldsymbol{\alpha}}(\mathbf{u})\, \boldsymbol{\mu}_1(d\mathbf{u}) \qquad (4.2)\\
&= \lim_{k\to\infty} \int_{\mathbb{R}^k} \phi_k(\mathbf{u}) \left\{\int_{\mathbb{R}^k} (\mathbf{z}_{h,k}, \mathbf{u} + \mathrm{i}\mathbf{v})^n_{\mathbb{C}^k}\, \boldsymbol{\mu}_1(d\mathbf{v})\right\} \boldsymbol{\mu}_1(d\mathbf{u}) \\
&= \lim_{k\to\infty} \int_B f_k(x) \left\{\int_B (x+\mathrm{i}y, P_k(h))^n\, p_1(dy)\right\} p_1(dx) \\
&= \lim_{k\to\infty} \int_B f(x) \left\{\int_B (x+\mathrm{i}y, P_k(h))^n\, p_1(dy)\right\} p_1(dx),
\end{aligned}
$$

where $T_{h,k} \equiv \int_B(\,\cdot + \mathrm{i}y, P_k(h))^n\, p_1(dy)$ is obviously $\mathscr{B}_k$-measurable. Note that

$$
T_{h,k} = |P_k(h)|^n H_n((\,\cdot\,, P_k(h)/|P_k(h)|)) \sim N(0, n!|P_k(h)|^{2n}).
$$

Let $T_h(x) = \int_B (\mathbf{n}(h)(x) + \mathrm{i}\mathbf{n}(h)(y))^n \, p_1(dy)$, $x \in B$. Then

$$T_h = |h|^n H_n(\mathbf{n}(h/|h|)) \sim N(0, n!|h|^{2n}).$$

As $k \to \infty$, since $|P_k(h)| \to |h|$ and $(\,\cdot\,, P_k(h)/|P_k(h)|) \to \mathbf{n}(h/|h|)$ in $L^2(B, p_1)$, $T_{h,k}$ converges in probability to T_h, and moreover $\|T_{h,k}\|^2_{L^2(B,p_1)}$ converges to $\|T_h\|^2_{L^2(B,p_1)}$. By the dominated convergence theorem, $T_{h,k} \to T_h$ in $L^2(B, p_1)$. Then it follows from (4.2) that

$$D^n S_\infty f(0) h^n = \int_B f(x) \left\{ \int_B (\mathbf{n}(h)(x) + \mathrm{i}\mathbf{n}(h)(y))^n \, p_1(dy) \right\} p_1(dx). \tag{4.3}$$

By the uniqueness of analytic continuation, the formula (4.3) still holds by replacing h by $h_1 + z h_2$, $h_1, h_2 \in H$ and $z \in \mathbb{C}$, especially by $\mathbf{h} \in H_c$. Finally, by applying the polarization formula (see Appendices of [15]), we complete the proof. $\square$

Remark 4.3. In order to study the regularity of heat semigroup generated from p_t, Lee [11] has already achieved the same formula as in Proposition 4.2 even for $f \in L^\alpha_c(B, p_t)$ with $\alpha > 1$. In the course of derivation, Lee proved it by using induction and more sophisticated calculation than the above proof.

5. Wiener-Itô theorem on abstract Wiener spaces

In this section, we would like to establish an analytic version of Wiener-Itô orthogonal decomposition of $L^2_c(B, p_1)$. Let $\{e_j \in B^*; j = 1, 2, \ldots\}$ be a CONB of H. For any $f \in L^2_c(B, p_1)$, f_k, ϕ_k are defined as in Section 4. First, we observe by Theorem 2.2(ii) and Lemma 4.1 that for any $k \in \mathbb{N}$,

$$\begin{aligned} f_k(x) &= \phi_k(\mathbf{u}_{x,k}) \\ &= \sum_{n=0}^\infty \frac{1}{n!} \int_{\mathbb{R}^k} D^n S_k \phi_k(\mathbf{0}) (\mathbf{u}_{x,k} + \mathrm{i}\mathbf{v})^n \, \boldsymbol{\mu}_1(d\mathbf{v}) \\ &= \sum_{n=0}^\infty \frac{1}{n!} \int_B D^n S_k \phi_k(\mathbf{0}) (\mathbf{u}_{x,k} + \mathrm{i}\mathbf{u}_{y,k})^n \, p_1(dy) \\ &= \sum_{n=0}^\infty \frac{1}{n!} \int_B D^n S_k \phi_k(\mathbf{0}) \mathbf{z}^n_{P_k(x+\mathrm{i}y),k} \, p_1(dy) \\ &= \sum_{n=0}^\infty \frac{1}{n!} \int_B D^n S_\infty f(0) (P_k(x + \mathrm{i}y))^n \, p_1(dy) \quad \text{in } L^2_c(B, p_1). \end{aligned} \tag{5.1}$$

And, for any $x \in B$, it follows by (2.3) and Lemma 4.1 that

$$\begin{aligned}
&\int_B D^n S_\infty f(0)(P_k(x+\mathrm{i}y))^n \, p_1(dy) \\
&= \sum_{1\le i_1,\dots,i_n\le k} D^n S_\infty f(0) e_{i_1}\cdots e_{i_n} \int_B \prod_{j=1}^n (x+\mathrm{i}y, e_{i_j})\, p_1(dy) \qquad (5.2)\\
&= n! \sum_\ell \sum_{\substack{1\le j_1<\cdots<j_\ell\le k\\ n_1+\cdots+n_\ell=n}} \frac{D^n S_\infty f(0) e_{j_1}^{n_1}\cdots e_{j_\ell}^{n_\ell}}{n_1!\cdots n_\ell!} H_{(n_1,\dots,n_\ell)}(Q_{j_1,\dots,j_\ell}(x)),
\end{aligned}$$

where

$$Q_{j_1,\dots,j_\ell}(x) = ((x,e_{j_1}),\dots,(x,e_{j_\ell})), \quad x\in B.$$

Since, by (2.4) and (3.2),

$$\left\{(\boldsymbol{\alpha}!)^{-1/2} H_{\boldsymbol{\alpha}}(Q_{j_1,\dots,j_\ell}(\cdot)); 1\le j_1<\cdots<j_\ell, \boldsymbol{\alpha}\in\mathbb{N}^\ell, \forall\,\ell\in\mathbb{N}\right\} \tag{5.3}$$

is an orthonormal set in $L_c^2(B,p_1)$, we combine (5.1) and (5.2) to obtain that

$$\begin{aligned}
\|f_k\|^2_{L_c^2(B,p_1)} &= \sum_{n=0}^\infty \sum_\ell \sum_{\substack{1\le j_1<\cdots<j_\ell\le k\\ n_1+\cdots+n_\ell=n}} \frac{|D^n S_\infty f(0) e_{j_1}^{n_1}\cdots e_{j_\ell}^{n_\ell}|^2}{n_1!\cdots n_\ell!} \\
&= \sum_{n=0}^\infty \frac{1}{n!} \sum_{1\le i_1,\dots,i_n\le k} |D^n S_\infty f(0) e_{i_1}\cdots e_{i_n}|^2.
\end{aligned}$$

Letting $k\to\infty$, it follows by Lemma 3.4 that

$$\begin{aligned}
\|f\|^2_{L_c^2(B,p_1)} &= \sum_{n=0}^\infty \frac{1}{n!} \sum_{1\le i_1,\dots,i_n\in\mathbb{N}} |D^n S_\infty f(0) e_{i_1}\cdots e_{i_n}|^2 \qquad (5.4)\\
&= \sum_{n=0}^\infty \frac{1}{n!} \|D^n S_\infty f(0)\|^2_{\mathcal{HS}} < +\infty.
\end{aligned}$$

Then the following are immediate consequences by Proposition 4.2 and (5.1)–(5.4).

Theorem 5.1. (i) $\left\{(\boldsymbol{\alpha}!)^{-1/2} H_{\boldsymbol{\alpha}}(Q_{j_1,\dots,j_\ell}(\cdot)); 1\le j_1<\cdots<j_\ell, \boldsymbol{\alpha}\in\mathbb{N}^\ell, \forall\,\ell\in\mathbb{N}\right\}$ *is a CONB of* $L_c^2(B,p_1)$.

(ii) *Let* $f\in L_c^2(B,p_1)$ *be given. Then*

$$f = \sum_{n=0}^\infty \sum_\ell \sum_{\substack{1\le j_1<\cdots<j_\ell\le k\\ n_1+\cdots+n_\ell=n}} \frac{D^n S_\infty f(0) e_{j_1}^{n_1}\cdots e_{j_\ell}^{n_\ell}}{n_1!\cdots n_\ell!} H_{(n_1,\dots,n_\ell)}(Q_{j_1,\dots,j_\ell}(\cdot)) \quad \text{in } L_c^2(B,p_1).$$

Moreover,

(iii) $\|f\|^2_{L^2_c(B,p_1)} = \sum_{n=0}^{\infty} \frac{1}{n!}\|D^n S_\infty f(0)\|^2_{\mathcal{HS}}$.

Recall that, for $r > 0$, denote by $\mathcal{F}^r(H_c)$, the class of analytic functionals on H_c with $\|\cdot\|_{\mathcal{F}^r(H_c)}$-norm satisfying

$$\|F\|^2_{\mathcal{F}^r(H_c)} = \sum_{n=0}^{\infty} \frac{r^n}{n!}\|D^n F(0)\|^2_{\mathcal{HS}^n(H_c)} < +\infty,$$

called the Bargmann-Segal-Dwyer space. Members of $\mathcal{F}^r(H_c)$ are called Bargmann-Segal analytic functionals. We refer the interested reader to [13] and the references cited therein. If $H_c = \mathbb{C}^k$, we remark that $\mathcal{F}^r(\mathbb{C}^k) = \mathcal{K}^{r/2}(\mathbb{C}^k)$ given as in Subsection 2.2. Consequently, it follows by Theorem 5.1(iii) that $S_\infty f \in \mathcal{F}^1(H_c)$ with $\|S_\infty f\|_{\mathcal{F}^1(H_c)} = \|f\|_{L^2_c(B,p_1)}$.

In order to express the right-hand side of Theorem 5.1(ii) as an integral form shown in (2.16), we need to introduce the Gauss transform Λ on $\mathcal{F}^1(H_c)$ as follows. For any $F \in \mathcal{F}^1(H_c)$, if F has an extension to B_c, the Gauss transform $\Lambda(F)$ of F is defined as a function on B such that for any $x \in B$,

$$\Lambda(F)(x) = \int_B F(x + \mathrm{i}y)\, p_1(dy) \quad \text{(if it exists)}.$$

Let $\mathcal{L}^n(B_c)$ be the space of n-linear continuous operators from $B_c \times \cdots \times B_c$ (n-times) into $\mathbb{C}$, and $\mathcal{L}^n_{(2)}(H_c)$ the space of n-linear Hilbert-Schmidt operators from $B_c \times \cdots \times B_c$ (n-times) into $\mathbb{C}$. Then, by applying Kuo's theorem (see Corollary 4.4, p. 85, in [9]), the restriction of $T \in \mathcal{L}^n(B_c)$ to $H_c \times \cdots \times H_c$ is a member of $\mathcal{L}^n_{(2)}(H_c)$. For $T \in \mathcal{L}^n(B_c)$, let f_T be a function on H_c given by $f_T(\mathbf{x}) = T\mathbf{x}^n$, $\mathbf{x} \in B_c$. Then f_T is obviously an analytic function on B_c; moreover, $D^n f_T(0) = n! \cdot \mathrm{sym}(T)$ and $D^m f_T(0) = 0$, if $m \neq n$. Hence $f_T \in \mathcal{F}^1(H_c)$ and

$$\Lambda(f_T)(x) = \int_B \mathrm{sym}(T)(x + \mathrm{i}y)^n\, p_1(dy), \quad x \in B,$$

where $\mathrm{sym}(T)$ denotes the symmetrization of T, and the existence of the right-hand integral is guaranteed by the Fernique theorem (see [9]). It is easy to check that $\Lambda(f_T) \in L^2_c(B, p_1)$. Moreover, we can apply Proposition 4.2 to see that for any $\mathbf{h}_1, \ldots, \mathbf{h}_m \in H_c$,

$$D^m S_\infty \Lambda(f_T)(0)\mathbf{h}_1, \cdots \mathbf{h}_m = \int_B \Lambda(f_T)(x)\Big\{\int_B \prod_{j=1}^{m}(\mathbf{n}(\mathbf{h}_j)(x) + \mathrm{i}\mathbf{n}(\mathbf{h}_j)(y))\, p_1(dy)\Big\}\, p_1(dx).$$

In fact, we have the following

Proposition 5.2. *Let $T \in \mathcal{L}^n(B_c)$ be arbitrarily given. Then, for any $m \in \mathbb{N}$, $D^m f_T(0) = m! \cdot \mathrm{sym}(T)$, if $m = n$; 0, if $m \neq n$, and for any $\mathbf{h}_1, \ldots, \mathbf{h}_m \in H_c$,*

$$D^m f_T(0)\mathbf{h}_1 \cdots \mathbf{h}_m = \int_B \Lambda(f_T)(x)\Big\{\int_B \prod_{j=1}^{m}(\mathbf{n}(\mathbf{h}_j)(x) + \mathrm{i}\mathbf{n}(\mathbf{h}_j)(y))\, p_1(dy)\Big\}\, p_1(dx).$$

Proof. It suffices to show that $S_\infty \Lambda(f_T)(\mathbf{h}) = \operatorname{sym}(T)\mathbf{h}^n$ for any $\mathbf{h} \in H_c$. Observe that

$$S_\infty \Lambda(f_T)(\mathbf{h}) = \int_B \int_B \operatorname{sym}(T)(\mathbf{h} + x + \mathrm{i}y)^n \, p_1(dy) p_1(dx).$$

Define a function g from $\mathbb{C}^2$ into $\mathbb{C}$ by

$$\begin{aligned} g(\alpha, \beta) &= \int_B \int_B \operatorname{sym}(T)(\mathbf{h} + \alpha x + \beta y)^n \, p_1(dy) p_1(dx) \\ &= n! \sum_{n_1+n_2+n_3=n} \frac{\alpha^{n_2} \beta^{n_3}}{n_1! n_2! n_3!} \int_B \int_B \operatorname{sym}(T)\mathbf{h}^{n_1} x^{n_2} y^{n_3} \, p_1(dy) p_1(dx). \end{aligned}$$

Then g is holomorphic on $\mathbb{C}^2$. For any $\alpha, \beta \in \mathbb{R}$,

$$\begin{aligned} g(\alpha, \beta) &= \int_B \int_B \operatorname{sym}(T)(\mathbf{h} + x + y)^n \, p_{\alpha^2}(dy) p_{\beta^2}(dx) \\ &= \int_B \operatorname{sym}(T)(\mathbf{h} + x)^n \, p_{\alpha^2+\beta^2}(dx) \\ &= \int_B \operatorname{sym}(T)(\mathbf{h} + \sqrt{\alpha^2+\beta^2}\, x)^n \, p_1(dx) \\ &= n! \sum_{n_1+n_2=n} \frac{(\sqrt{\alpha^2+\beta^2})^{n_2}}{n_1! n_2!} \int_B \operatorname{sym}(T)\mathbf{h}^{n_1} x^{n_2} \, p_1(dx) \\ &= n! \sum_{n_1+(2n_2')=n} \frac{(\alpha^2+\beta^2)^{n_2'}}{n_1!(2n_2')!} \int_B \operatorname{sym}(T)\mathbf{h}^{n_1} x^{2n_2'} \, p_1(dx), \end{aligned} \tag{5.5}$$

where $\int_B \operatorname{sym}(T)\mathbf{h}^{n_1} x^{n_2} \, p_1(dx) = 0$, if n_2 is an odd positive integer, since $p_1(dx) = p_1(-dx)$. The sum in the last equality of (5.5) is also holomorphic as a function of $(\alpha, \beta) \in \mathbb{C}^2$ and coincides with $g(\alpha, \beta)$ for any $\alpha, \beta \in \mathbb{R}$. By the uniqueness of analytic continuation, these two holomorphic functions are equal on $\mathbb{C}^2$. Substituting $\alpha = 1$ and $\beta = \mathrm{i}$ into (5.5), the proof is complete. □

By Theorem 5.1(i) and Proposition 5.2, we obtain that for any $T \in \mathcal{L}^n(B_c)$,

$$\Lambda(f_T) = \sum_{i_1,\ldots,i_n=0}^{\infty} T e_{i_1} \cdots e_{i_n} \int_B \prod_{j=1}^{n} (\,\cdot + \mathrm{i}y, e_{i_j}) \, p_1(dy), \tag{5.6}$$

where the sum is convergent in $L_c^2(B, p_1)$, and thus

$$\|\Lambda(f_T)\|_{L_c^2(B,p_1)} = \sqrt{n!} \|\operatorname{sym}(T)\|_{\mathcal{HS}}. \tag{5.7}$$

For $T \in \mathcal{L}_{(2)}^n(H_c)$, one can take a sequence $\{T_k\} \subset \mathcal{L}^n(B_c)$ such that $T_k \to T$ in $\mathcal{L}_{(2)}^n(H_c)$. Then it follows from (5.6) that

$$\lim_{k\to\infty} \Lambda(f_{T_k}) = \sum_{i_1,\ldots,i_n=0}^{\infty} T(e_{i_1}, \ldots, e_{i_n}) \int_B \prod_{j=1}^{n} (\,\cdot + \mathrm{i}y, e_{i_j}) \, p_1(dy), \tag{5.8}$$

where the limit and sum are taken in $L_c^2(B, p_1)$, and independent of the choice of $\{T_k\}$ and $\{e_j\}$. Accordingly, we can extend the definition of the Gauss transform to $\mathcal{L}_{(2)}^n(H_c)$ by defining $\Lambda(f_T)$ as the right-hand sum of (5.6).

By employing the Gauss transform, we now sum up all arguments concerning Theorem 5.1, Proposition 5.2 and (5.6)–(5.8) to give the following

Theorem 5.3. (i) *Let $f \in L_c^2(B, p_1)$. Then*

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \int_B D^n S_\infty f(0)(\,\cdot + \mathrm{i}y)^n \, p_1(dy) \quad \text{in } L_c^2(B, p_1),$$

where the integral means $\Lambda(f_{D^n S_\infty f(0)})$.

(ii) *For any $T_1 \in \mathcal{L}_{(2)}^n(H_c)$ and $T_2 \in \mathcal{L}_{(2)}^m(H_c)$,*

$$\int_B \Lambda(f_{T_1})(x)\, \overline{\Lambda(f_{T_2})(x)}\, p_1(dx) = \delta_{n,m} \cdot n!\, \langle\!\langle \mathrm{sym}(T_1), \mathrm{sym}(T_2) \rangle\!\rangle_{\mathcal{HS}},$$

where $\delta_{n,m} = 1$, if $n = m$; 0, if $n \neq m$.

(iii) *For any $T \in \mathcal{L}_{(2)}^n(H_c)$, $S_\infty \Lambda(f_T) = \mathrm{sym}(T)$ on H_c.*

(iv) *Let $\mathcal{H}_0 = \mathbb{C}$, and for any $n \in \mathbb{N}$, let $\mathcal{H}_n = \{\Lambda(f_T); T \in \mathcal{L}_{(2)}^n(H_c)\}$. Then $L_c^2(B, p_1)$ is the orthogonal direct sum $\sum_{n=0}^{\infty} \oplus \mathcal{H}_n$.*

The formula in Theorem 5.3(i) is a reformulation of Theorem 5.1(ii), called an analytic version of Wiener-Itô decomposition. The orthogonal direct sum in Theorem 5.3(iv) is called the Wiener-Itô theorem on abstract Wiener spaces. From (5.8), the homogeneous chaos $\mathcal{H}_n$ of order n is the completion in $L_c^2(B, p_1)$ of the span of Hermite polynomials of $\{(\,\cdot\,, e_i); i = 1, 2, \ldots\}$.

Corollary 5.4. *The S-transform S_∞ is an isometry from $L_c^2(B, p_1)$ onto $\mathcal{F}^1(H_c)$.*

Proof. By Theorem 5.1(iii), we have seen that S_∞ is an isometry from $L_c^2(B, p_1)$ into $\mathcal{F}^1(H_c)$. To show S_∞ is surjective, for any $F \in \mathcal{F}^1(H_c)$, let

$$f = \sum_{n=0}^{\infty} \frac{1}{n!} \Lambda(f_{D^n F(0)}) \quad \text{in } L_c^2(B, p_1).$$

Then it immediately follows from Theorem 5.3(iii) that $S_\infty f = F$. The proof is complete. □

Remark 5.5. Let (H, B) be the white noise space $(\mathcal{S}_0, \mathcal{S}_{-p})$ for $p > 1/2$ (see Example 3.3). For $f \in L_c^2(B, p_1)$, the integral in Theorem 5.3(i), that is,

$$\frac{1}{n!} \int_B D^n S_\infty f(0)(\,\cdot + \mathrm{i}y)^n \, p_1(dy),$$

is exactly the multiple Wiener integral $I_n(f_n)$ of order n with respect to the Brownian motion $B(t)$ given in Example 3.3, where $f_n \in \widehat{L_c^2}(\mathbb{R}^n)$, the space of symmetric complex-valued L^2-functions on $\mathbb{R}^n$, can be represented as

$$f_n(t_1, \ldots, t_n) = \frac{1}{n!} D^n S_\infty f(0)\delta_{t_1} \cdots \delta_{t_n},$$

where δ_t is the Dirac measure concentrated on the point t.

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