Weighted Endpoint Estimates for Singular Integral Operators Associated with Zygmund Dilations

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Abstract. The main purpose of this paper is to study multi-parameter singular integral operators which commute with Zygmund dilations. We develop the theory of the weighted multi-parameter Hardy space $H^p_{\mathfrak{z},w}$ and prove the boundedness for these operators on $H^p_{\mathfrak{z},w}$ for certain $p \leq 1$, which provide endpoint estimates for those singular integral operators studied by Ricci-Stein [31] and Fefferman-Pipher [15]. We also establish the Calderón-Zygmund decomposition and interpolation theorem in this setting.

1. Introduction

Ricci-Stein [31] introduced multi-parameter singular integral operators and Fefferman-Pipher [15] considered specific singular integral operators associated with Zygmund dilations. The boundedness for these operators on L^p and weighted L^p_w, $1 < p < \infty$, was obtained by Ricci-Stein [31] and Fefferman-Pipher [15], respectively. In [21], the first four authors of this paper introduced a class of singular integral operators associated with Zygmund dilations and prove the boundedness for such a class on L^p and L^p_w, $1 < p < \infty$. The operators in this class and the boundedness results for these operators generalize some those studied by Ricci-Stein [31], Fefferman-Pipher [15] and Nagel-Wainger [29]. In the endpoint case, it is natural to expect that Hardy space bounds are available.

The main purpose of this paper is to prove the boundedness of this class of operators on the weighed Hardy spaces $H^p_{\mathfrak{z},w}$ associated with Zygmund dilations for certain $p \leq 1$, which, in particular, provides the endpoint estimates for operators studied by Ricci-Stein [31] and Fefferman-Pipher [15]. We also establish the Calderón-Zygmund decomposition and deduce interpolation theorems in this setting.

To achieve our results, we use a "standard strategy", but one that has to be adapted to our special situation, of the structure with Zygmund dilations. More specifically, we

have to construct a Calderón reproducing formula associated with Zygmund dilations, to discretize the operators according to it, to apply almost orthogonal estimates, and to implement various stopping time arguments.

We now set our work in context. In their well-known theory, Calderón and Zygmund [1] introduced certain convolution singular integral operators on $\mathbb{R}^n$ which generalize the Hilbert transform on $\mathbb{R}^1$. They proved that if $T(f) = \mathcal{K} * f$, where $\mathcal{K}$ is defined on $\mathbb{R}^n$ and satisfies the analogous estimates as $1/x$ does on $\mathbb{R}^1$, namely

$$|\mathcal{K}(x)| \leq \frac{C}{|x|^n}, \quad |\nabla \mathcal{K}(x)| \leq \frac{C}{|x|^{n+1}} \quad \text{and} \quad \int_{a<|x|<b} \mathcal{K}(x)\,dx = 0 \quad \text{for all } 0 < a < b,$$

then T is bounded on $L^p(\mathbb{R}^n)$ for $1 < p < \infty$. The core of this theory is that the regularity and cancelation conditions are invariant with respect to the one-parameter family of dilations on $\mathbb{R}^n$ defined by $\delta(x_1, x_2, \ldots, x_n) = (\delta x_1, \ldots, \delta x_n)$, $\delta > 0$, in the sense that the kernel $\delta^n \mathcal{K}(\delta x)$ satisfies the same conditions with the same bound as $\mathcal{K}(x)$. Indeed, the classical singular integrals, maximal functions and multipliers are invariant with respect to such one-parameter dilations. The one-parameter theory is currently well understood.

The multiparameter theory of $\mathbb{R}^n$ began with Zygmund's study of the strong maximal function, which is defined by

$$\mathcal{M}_n(f)(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f(y)|\,dy,$$

where R are the rectangles in $\mathbb{R}^n$ with sides parallel to the axes, and then continued with Marcinkiewicz's proof of his multiplier theorem. The strong maximal function and Marcinkiewicz's multiplier are invariant with respect to the product dilations defined by $\delta(x_1, x_2, \ldots, x_n) = (\delta_1 x_1, \ldots, \delta_n x_n)$, $\delta_i > 0$, $i = 1, \ldots, n$. The multiparameter dilations are also associated with problems in the theory of differentiation of integrals. Jensen-Marcinkiewicz-Zygmund [22] proved that the strong maximal function in $\mathbb{R}^n$ is bounded from the Orlicz space $L(1 + (\log^+ L)^{n-1})$ to weak L^1. Zygmund further conjectured that if the rectangles in $\mathbb{R}^n$ had n side lengths which involve only k independent variables, then the resulting maximal operator should behave like $\mathcal{M}_k$, the k-parameter strong maximal operator. More precisely, for $1 \leq k \leq n$, and for positive functions $\phi_1, \ldots, \phi_n$ as the side-lengths of the given collection of rectangles where the maximal function is defined, each one depending on parameters $t_1 > 0, t_2 > 0, \ldots, t_k > 0$, assuming arbitrarily small values and increasing in each variable separately, then the resulting maximal function would be bounded from $L(1 + (\log^+ L)^{k-1})$ to weak L^1 according to Zygmund's conjecture. A. Córdoba [7] showed that for the unit cube Q in $\mathbb{R}^3$,

$$|\{x \in Q : \mathcal{M}f(x) > \lambda\}| \leq \frac{C}{\lambda} \|f\|_{L \log L(Q)},$$

where $\mathcal{M}f$ denotes the maximal function on $\mathbb{R}^3$ defined by

$$\mathcal{M}f(x) = \sup_{R \ni x} \frac{1}{|R|} \int_R |f(u)| \, du.$$

The supremum above is taken over all rectangles with sides parallel to the axes and side lengths of the form s, t, and $\phi(s,t)$. Córdoba's result was generalized to the case of $\phi_1(s,t)$, $\phi_2(s,t)$, $\phi_3(s,t)$ by Soria [32] with some assumptions on ϕ_1, ϕ_2, ϕ_3. Moreover, Soria showed that Zygmund's conjecture is not true even when $\phi_1(s,t) = s$, $\phi_2(s,t) = s\phi(t)$, $\phi_3(s,t) = s\psi(t)$, with ϕ, ψ being positive and increasing functions.

In [16] R. Fefferman and Stein generalized the singular integral operator theory to the product space. They took the space $\mathbb{R}^n \times \mathbb{R}^m$ along with the two-parameter family of dilations $(x,y) \mapsto (\delta_1 x, \delta_2 y)$, $(x,y) \in \mathbb{R}^n \times \mathbb{R}^m$, $\delta_1, \delta_2 > 0$. Those operators considered in [16] generalize the double Hilbert transform on $\mathbb{R}^2$ given by $H(f) = f * \frac{1}{xy}$ and are of the form $T(f) = \mathcal{K} * f$, where the kernel $\mathcal{K}$ is characterized by the cancelation properties

(1.1) $$\int_{a<|x|<b} \mathcal{K}(x,y) \, dx = 0 \quad \text{for all } 0 < a < b \text{ and } y \in \mathbb{R}^m,$$

(1.2) $$\int_{a<|y|<b} \mathcal{K}(x,y) \, dy = 0 \quad \text{for all } 0 < a < b \text{ and } x \in \mathbb{R}^n,$$

and the regularity conditions

(1.3) $$|\partial_x^\alpha \partial_y^\beta \mathcal{K}(x,y)| \leq C_{\alpha,\beta} |x|^{-n-|\alpha|} |y|^{-m-|\beta|}.$$

Under the conditions (1.1)–(1.3), Fefferman and Stein proved the L^p, $1 < p < \infty$, boundedness of the product convolution operators $T(f) = \mathcal{K} * f$. See [16] for more details. Note that the kernel $\mathcal{K}$ satisfying the conditions (1.1)–(1.3) is invariant with respect to the product dilation in the sense that the kernel $\delta_1^n \delta_2^m \mathcal{K}(\delta_1 x, \delta_2 y)$ satisfies conditions (1.1)–(1.3) with the same bound. For more discussions about the multiparameter product theory, see [2–6, 8, 10–14, 20, 23–25, 30] among others and in particular the survey article of R. Fefferman [14] for development in this area. For the multiparameter flag theory, see [26–28] among others.

It has been widely considered that the next simplest multiparameter group of dilations after the product multiparameter dilations is the so-called Zygmund dilation defined on $\mathbb{R}^3$ by $\rho_{s,t}(x_1, x_2, x_3) = (sx_1, tx_2, stx_3)$ for $s, t > 0$. There are two operators associated with the Zgymund dilations. The first is the maximal operator $\mathcal{M}_{\mathfrak{z}}$ defined via

$$\mathcal{M}_{\mathfrak{z}}f(x) = \sup_{\substack{R \in \mathcal{R}_{\mathfrak{z}} \\ R \ni x}} \frac{1}{|R|} \int_R |f(y)| \, dy,$$

where $\mathcal{R}_{\mathfrak{z}}$ is the collection of all rectangles in $\mathbb{R}^3$ with side length t, s and ts, respectively, for $t, s > 0$. The other is the singular integral operator $f * \mathcal{K}$ introduced by Ricci and

Stein [31], where

$$\mathcal{K}(x_1, x_2, x_3) = \sum_{k,j \in \mathbb{Z}} 2^{-2(k+j)} \phi^{k,j} \left(\frac{x_1}{2^k}, \frac{x_2}{2^j}, \frac{x_3}{2^{k+j}} \right)$$

and the functions $\phi^{k,j}$ are supported in a unit cube in $\mathbb{R}^3$ satisfying a certain amount of uniform smoothness and cancelation conditions. It was shown in [31] that $T_{\mathfrak{z}}$ is bounded on $L^p(\mathbb{R}^3)$ for all $1 < p < \infty$. R. Fefferman and Pipher [15] further showed that this class of singular integral operators is bounded in L_w^p spaces for $1 < p < \infty$ when the weights w's satisfy an analogous condition of Muckenhoupt associated with Zygmund dilations. The link between the properties of maximal operators $\mathcal{M}_{\mathfrak{z}}$ and the boundary value problems for Poisson integrals on symmetric spaces, such as Siegel's upper half space, was discovered by E. Stein. See the survey article of R. Fefferman [12] for this link and for more research directions of multiparameter analysis on Zygmund dilations. Related to the theory of operators like $\mathcal{M}_{\mathfrak{z}}$ and $T_{\mathfrak{z}}$, several authors have considered singular integrals along surfaces. See, for example, Nagel-Wainger [29].

Recently, the first four authors [21] introduced a more general class of singular integral operators associated with Zygmund dilations, which we now recall. Suppose that $\mathcal{K}(x_1, x_2, x_3)$ is a function defined on $\mathbb{R}^3$ away from the union $\{0, x_2, x_3\} \cup \{x_1, 0, x_3\} \cup \{x_1, x_2, 0\}$ and all α, β and γ are integers taking only values 0 and 1. Define

$$\Delta_{x_1, h_1}^{\alpha} \mathcal{K}(x_1, x_2, x_3) = \alpha \mathcal{K}(x_1 + h_1, x_2, x_3) - \mathcal{K}(x_1, x_2, x_3), \quad \alpha = 0 \text{ or } 1;$$
$$\Delta_{x_2, h_2}^{\beta} \mathcal{K}(x_1, x_2, x_3) = \beta \mathcal{K}(x_1, x_2 + h_2, x_3) - \mathcal{K}(x_1, x_2, x_3), \quad \beta = 0 \text{ or } 1;$$

and

$$\Delta_{x_3, h_3}^{\gamma} \mathcal{K}(x_1, x_2, x_3) = \gamma \mathcal{K}(x_1, x_2, x_3 + h_3) - \mathcal{K}(x_1, x_2, x_3), \quad \gamma = 0 \text{ or } 1.$$

We denote $\Delta_{x_1, h_1} = \Delta_{x_1, h_1}^1$, $\Delta_{x_2, h_2} = \Delta_{x_2, h_2}^1$ and $\Delta_{x_3, h_3} = \Delta_{x_3, h_3}^1$ for simplicity. The "regularity" conditions are characterized by

(R) $$|\Delta_{x_1, h_1}^{\alpha} \Delta_{x_2, h_2}^{\beta} \Delta_{x_3, h_3}^{\gamma} \mathcal{K}(x_1, x_2, x_3)| \leq \frac{C|h_1|^{\alpha \theta_1} |h_2|^{\beta \theta_1} |h_3|^{\gamma \theta_1}}{|x_1|^{\alpha \theta_1 + 1} |x_2|^{\beta \theta_1 + 1} |x_3|^{\gamma \theta_1 + 1} \left(|\frac{x_1 x_2}{x_3}| + |\frac{x_3}{x_1 x_2}| \right)^{\theta_2}}$$

for all $0 \leq \alpha \leq 1$, $0 \leq \beta + \gamma \leq 1$ or $0 \leq \alpha + \gamma \leq 1$, $0 \leq \beta \leq 1$, and $|x_1| \geq 2|h_1| > 0$, $|x_2| \geq 2|h_2| > 0$, $|x_3| \geq 2|h_3| > 0$, $h_1, h_2, h_3 \in \mathbb{R}$ and some $0 < \theta_1 \leq 1$, $0 < \theta_2 < 1$. The cancelation conditions are given by

(C.a) $$\left| \int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_1 \leq |x_1| \leq r_1} \mathcal{K}(x_1, x_2, x_3) \, dx_1 dx_2 dx_3 \right| \leq C$$

uniformly for all $\delta_1, \delta_2, \delta_3, r_1, r_2, r_3 > 0$;

(C.b)
$$\left| \int_{\delta_1 \leq |x_1| \leq r_1} \Delta_{x_2,h_2}^{\beta} \Delta_{x_3,h_3}^{\gamma} \mathcal{K}(x_1, x_2, x_3) \, dx_1 \right|$$
$$\leq \frac{C|h_2|^{\beta\theta_1}|h_3|^{\gamma\theta_1}}{|x_2|^{\beta\theta_1+1}|x_3|^{\gamma\theta_1+1}} \left(\frac{1}{\left(|\frac{r_1 x_2}{x_3}| + |\frac{x_3}{r_1 x_2}|\right)^{\theta_2}} + \frac{1}{\left(|\frac{\delta_1 x_2}{x_3}| + |\frac{x_3}{\delta_1 x_2}|\right)^{\theta_2}} \right)$$

for all $\delta_1, r_1 > 0$, $0 \leq \beta + \gamma \leq 1$, $|x_2| \geq 2|h_2| > 0$, $|z| \geq 2|h_3| > 0$;

(C.c)
$$\left| \int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_2 \leq |x_2| \leq r_2} \Delta_{x_1,h_1}^{\alpha} \mathcal{K}(x_1, x_2, x_3) \, dx_2 dx_3 \right| \leq \frac{C|h_1|^{\alpha\theta_1}}{|x_1|^{\alpha\theta_1+1}}$$

uniformly for all $\delta_2, \delta_3, r_2, r_3 > 0$, $|x_1| \geq 2|h_1| > 0$ and $0 \leq \alpha \leq 1$. Or

(C.a′)
$$\left| \int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_2 \leq |x_2| \leq r_2} \int_{\delta_1 \leq |x_1| \leq r_1} \mathcal{K}(x_1, x_2, x_3) \, dx_1 dx_2 dx_3 \right| \leq C$$

uniformly for all $\delta_1, \delta_2, \delta_3, r_1, r_2, r_3 > 0$;

(C.b′)
$$\left| \int_{\delta_2 \leq |x_2| \leq r_2} \Delta_{x_1,h_1}^{\alpha} \Delta_{x_3,h_3}^{\gamma} \mathcal{K}(x_1, x_2, x_3) \, dx_2 \right|$$
$$\leq \frac{C|h_1|^{\alpha\theta_1}|h_3|^{\gamma\theta_1}}{|x_1|^{\alpha\theta_1+1}|x_3|^{\gamma\theta_1+1}} \left(\frac{1}{\left(|\frac{r_2 x_1}{x_3}| + |\frac{x_3}{r_2 x_1}|\right)^{\theta_2}} + \frac{1}{\left(|\frac{\delta_2 x_1}{x_3}| + |\frac{x_3}{\delta_2 x_1}|\right)^{\theta_2}} \right)$$

for all $\delta_2, r_2 > 0$, $0 \leq \alpha + \gamma \leq 1$, $|x_1| \geq 2|h_1| > 0$ and $|x_3| \geq 2|h_3| > 0$;

(C.c′)
$$\left| \int_{\delta_3 \leq |x_3| \leq r_3} \int_{\delta_1 \leq |x_1| \leq r_1} \Delta_{x_2,h_2}^{\beta} \mathcal{K}(x_1, x_2, x_3) \, dx_1 dx_3 \right| \leq \frac{C|h_2|^{\beta\theta_1}}{|x_2|^{\beta\theta_1+1}}$$

uniformly for all $\delta_1, \delta_3, r_1, r_3 > 0$, $|x_2| \geq 2|h_2| > 0$ and $0 \leq \beta \leq 1$.

The L^p, $1 < p < \infty$, boundedness of the singular integral operators was proved in [21]. The purpose of this paper is to establish the endpoint estimates of the singular integral operators $T_{\mathfrak{z}}$ on $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ when the kernel of $T_{\mathfrak{z}}$ satisfies the above cancelation conditions. These results in particular provide the endpoint estimates for operators studied by Ricci-Stein [31] and Fefferman-Pipher [15].

Let $\mathscr{S}(\mathbb{R}^3)$ denote the set of Schwartz functions in $\mathbb{R}^3$ and $x = (x_1, x_2, x_3) \in \mathbb{R}^3$. A Schwartz function f defined on $\mathbb{R}^3$ is said to be a test function in $\mathscr{S}_{\mathfrak{z}}(\mathbb{R}^3)$ if $f \in \mathscr{S}(\mathbb{R}^3)$ and
$$\int_{\mathbb{R}} f(x_1, x_2, x_3) x_1^{\alpha} \, dx_1 = \int_{\mathbb{R}^2} f(x_1, x_2, x_3) x_2^{\beta} x_3^{\gamma} \, dx_2 dx_3 = 0$$

for all indices α, β, γ of nonnegative integers. The seminorms of f in $\mathscr{S}_{\mathfrak{z}}(\mathbb{R}^3)$ are the Schwartz seminorms. We denote by $\mathscr{S}_{\mathfrak{z}}'(\mathbb{R}^3)$ the dual of $\mathscr{S}_{\mathfrak{z}}(\mathbb{R}^3)$.

Let $\psi^{(1)} \in \mathscr{S}(\mathbb{R})$ satisfy

$$(1.4) \qquad \operatorname{supp} \widehat{\psi^{(1)}}(\xi_1) \subset \{\xi_1 : 1/2 < |\xi_1| \leq 2\}$$

and

$$(1.5) \qquad \sum_{j \in \mathbb{Z}} \widehat{\psi^{(1)}}(2^j \xi_1)^2 = 1 \quad \text{for all } \xi_1 \in \mathbb{R} \setminus \{0\},$$

and let $\psi^{(2)} \in \mathscr{S}(\mathbb{R}^2)$ satisfy

$$(1.6) \qquad \operatorname{supp} \widehat{\psi^{(2)}}(\xi_2, \xi_3) \subset \{(\xi_2, \xi_3) : 1/2 < |(\xi_2, \xi_3)| \leq 2\}$$

and

$$(1.7) \qquad \sum_{k \in \mathbb{Z}} \widehat{\psi^{(2)}}(2^k \xi_2, 2^k \xi_3)^2 = 1 \quad \text{for all } (\xi_2, \xi_3) \in \mathbb{R}^2 \setminus \{0\}.$$

Set $\psi_{j,k}(x_1, x_2, x_3) = 2^{-2(j+k)} \psi^{(1)}(x_1/2^j) \psi^{(2)}(x_2/2^k, x_3/2^{j+k})$.

By taking the Fourier transform, it is easy to see the *continuous* Calderón's reproducing formula on $L^2(\mathbb{R}^3)$

$$(1.8) \qquad f(x) = \sum_{j,k \in \mathbb{Z}} \psi_{j,k} * \psi_{j,k} * f(x).$$

This continuous Calderón's reproducing formula was used to prove the $L^p(\mathbb{R}^3)$, $1 < p < \infty$, boundedness of singular integral operators. However, to develop the Hardy space, we feel that it is more convenient to use the following discrete Calderón's reproducing formula.

Theorem 1.1. *Suppose that $\psi_{j,k}$ are the same as in* (1.8). *Then*

$$(1.9) \qquad f(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\tilde{3}}^{j,k}} |R| \psi_{j,k}(x - x_R)(\psi_{j,k} * f)(x_R),$$

where $\mathcal{R}_{\tilde{3}}^{j,k}$ is the collection of rectangles in $\mathbb{R}^3$ and $R \in \mathcal{R}_{\tilde{3}}^{j,k}$ means that $R = I \times J \times S$ with the side length $|I| = 2^j$, $|J| = 2^k$ and $|S| = 2^{j+k}$, x_R denotes the "left lower corner" of R (i.e., the corner of R with the least value of each coordinate component), and the series converges in both $\mathscr{S}_{\tilde{3}}(\mathbb{R}^3)$ and $\mathscr{S}'_{\tilde{3}}(\mathbb{R}^3)$.

Remark 1.2. As mentioned in [21], we would like to point out that both cancelation conditions (C.a)–(C.c) and (C.a′)–(C.c′) and regularity condition (R) are invariant with respect to Zygmund dilations. Indeed (see [21, Theorem 4.1]), if $\mathcal{K}(x_1, x_2, x_3) = \sum_{j,k \in \mathbb{Z}} \psi_{j,k} * \psi_{j,k}(x_1, x_2, x_3)$, then $\mathcal{K}(x_1, x_2, x_3)$ satisfies all cancelation conditions (C.a)–(C.c) and (C.a′)–(C.c′) and the regularity condition (R). This means that the Calderón reproducing formula above, as a special Calderón-Zygmund operator, is invariant with respect to Zygmund dilations.

Based on the above discrete Calderón's reproducing formula, the *Littlewood-Paley square function* of $f \in \mathscr{S}'_{\mathfrak{z}}(\mathbb{R}^3)$ is defined by

$$g_{\mathfrak{z}}^{\psi}(f)(x) = \left(\sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j,k}} |(\psi_{j,k} * f)(x_R)|^2 \chi_R(x) \right)^{1/2},$$

where χ_R is the indicator function for the rectangle R.

Note that the functions $\psi_{j,k}$ constructed above belong to $\mathscr{S}_{\mathfrak{z}}(\mathbb{R}^3)$, so the Littlewood-Paley square function $g_{\mathfrak{z}}^{\psi}$ is well defined for all distributions in $\mathscr{S}'_{\mathfrak{z}}(\mathbb{R}^3)$.

The natural class of A_p weights with respect to Zygmund dilations was introduced by R. Fefferman and Pipher [15], which we now recall. For $1 < p < \infty$, a nonnegative measurable function w on $\mathbb{R}^3$ is called a Muckenhoupt weight in $A_p^{\mathfrak{z}}(\mathbb{R}^3)$ if

$$\sup_{R \in \mathcal{R}_{\mathfrak{z}}} \left(\frac{1}{|R|} \int_R w(x)\, dx \right) \left(\frac{1}{|R|} \int_R w(x)^{-1/(p-1)}\, dx \right)^{p-1} < \infty.$$

A nonnegative measurable function w on $\mathbb{R}^3$ is called a Muckenhoupt weight in $A_1^{\mathfrak{z}}(\mathbb{R}^3)$ if $\mathcal{M}_{\mathfrak{z}}(w)(x) \leq Cw(x)$ for almost every $x \in \mathbb{R}^3$. And $A_{\infty}^{\mathfrak{z}}(\mathbb{R}^3) = \bigcup_{p \geq 1} A_p^{\mathfrak{z}}(\mathbb{R}^3)$. If $w \in A_{\infty}^{\mathfrak{z}}(\mathbb{R}^3)$, the critical index q_w of w is defined by

$$q_w := \inf\{q : w \in A_q^{\mathfrak{z}}(\mathbb{R}^3)\}.$$

Now we formally define the weighted multi-parameter Hardy space associated with Zygmund dilations as follows.

Definition 1.3. Let $0 < p < \infty$ and $w \in A_{\infty}^{\mathfrak{z}}(\mathbb{R}^3)$. The *weighted Hardy space* $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ associated with Zygmund dilations is defined by

$$H_{\mathfrak{z},w}^p(\mathbb{R}^3) = \{f \in \mathscr{S}'_{\mathfrak{z}}(\mathbb{R}^3) : g_{\mathfrak{z}}^{\psi}(f) \in L_w^p(\mathbb{R}^3)\}$$

with quasi-norm $\|f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)} := \|g_{\mathfrak{z}}^{\psi}(f)\|_{L_w^p(\mathbb{R}^3)}$.

To see that the definition of $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ is independent of the choice of $\{\psi_{j,k}\}$, we will prove the following

Theorem 1.4. *Let* $0 < p < \infty$ *and* $w \in A_{\infty}^{\mathfrak{z}}(\mathbb{R}^3)$. *Suppose* $\{\psi_{j,k}\}$, $\{\varphi_{j,k}\}$ *satisfy conditions* (1.4)–(1.7). *Then*

$$\|g_{\mathfrak{z}}^{\psi}(f)\|_{L_w^p(\mathbb{R}^3)} \approx \|g_{\mathfrak{z}}^{\varphi}(f)\|_{L_w^p(\mathbb{R}^3)}.$$

Our first main result is the boundedness of singular integral operators associated with Zygmund dilations, which is given by the following.

Theorem 1.5. *Let $w \in A_\infty^{\mathfrak{z}}(\mathbb{R}^3)$ with critical index q_w satisfying $q_w < \frac{2 + \frac{1}{2}\min(\theta_1, \theta_2)}{2}$. Suppose that $\mathcal{K}$ is a function defined on $\mathbb{R}^3$ and satisfies conditions* (R) *and* (C.a)–(C.c) *(or* (R), (C.a′)–(C.c′)*). Then the operator $T_{\mathfrak{z}} f := \mathcal{K} * f$ defined initially on $L^2(\mathbb{R}^3) \cap H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ extends to a bounded operator on $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ for p satisfying $q_w \cdot \frac{2}{2+\frac{1}{2}\min(\theta_1,\theta_2)} < p \leq 1$ and, moreover,*

$$\|\mathcal{K} * f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)} \leq A\|f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)}$$

with the positive constant A being independent of f.

It is worthwhile to point out that the operators considered in this paper are invariant with respect to Zygmund dilations and cover the ones studied in [31] and [15] associated with Zygmund dilations. Thus, Theorem 1.5 (also Theorem 1.6 below) in particular provides the endpoint estimates for the operators studied in [31] and [15].

In many applications, singular integral operators are of the form $\mathcal{K} * f$ where $\mathcal{K}$ is a distribution that equals a function $\mathcal{K}$ on $\mathbb{R}^3$ away from the union $\{0, x_2, x_3\} \cup \{x_1, 0, x_3\} \cup \{x_1, x_2, 0\}$ and the cancelation conditions are described in terms of bump functions. Associated with Zygmund dilations, the cancelation conditions defined using bump functions were introduced in [21]. It was shown that such kind of cancelation conditions are weaker than the ones considered in Theorem 1.5, and hence the same boundedness result also holds for singular integrals with these cancelation conditions.

We point out that in the classical case, the main tool to get the $H^p - L^p$ (or $H_w^p - L_w^p$) boundedness of the classical singular integrals is the atomic decomposition. However, in the present situation one even does not know whether such an atomic decomposition exists. We will show the following more general result without using atomic decomposition and, as a consequence, obtain the $H_{\mathfrak{z},w}^p(\mathbb{R}^3) - L_w^p(\mathbb{R}^3)$ boundedness for the singular integrals given in Theorem 1.5.

Theorem 1.6. *Suppose $0 < p \leq 1$ and $w \in A_\infty^{\mathfrak{z}}(\mathbb{R}^3)$ with critical index q_w satisfying $q_w < \frac{2+\frac{1}{2}\min(\theta_1,\theta_2)}{2}$. Let T be a linear operator mapping $L^2(\mathbb{R}^3)$ into $L^2(\mathbb{R}^3)$. If T is bounded on $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$, then T can be extended to a bounded operator from $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ to $L_w^p(\mathbb{R}^3)$. As a consequence, the operator considered in Theorem 1.5 is bounded from $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ to $L_w^p(\mathbb{R}^3)$ for $q_w \cdot \frac{2}{2+\frac{1}{2}\min(\theta_1,\theta_2)} < p \leq 1$.*

We would like to remark that the range of p given in Theorems 1.5 and 1.6 can be extended to be $0 < p < \infty$ if more regularity and cancelation conditions are added to the kernel for operators considered in these theorems.

Our last main results are the Calderón-Zygmund decomposition and interpolation.

Theorem 1.7. *Let $w \in A_\infty^{\mathfrak{z}}(\mathbb{R}^3)$, $p_1 \in (0,1]$ and $p_1 < p < p_2 < \infty$. Given $f \in H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ and $\alpha > 0$, we have the decomposition $f = g + b$, where $g \in H_{\mathfrak{z},w}^{p_2}(\mathbb{R}^3)$ and $b \in H_{\mathfrak{z},w}^{p_1}(\mathbb{R}^3)$ with $\|g\|_{H_{\mathfrak{z},w}^{p_2}(\mathbb{R}^3)}^{p_2} \leq C\alpha^{p_2-p}\|f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)}^p$ and $\|b\|_{H_{\mathfrak{z},w}^{p_1}(\mathbb{R}^3)}^{p_1} \leq C\alpha^{p_1-p}\|f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)}^p$.*

We would like to point out that the above result was first proved by Chang and R. Fefferman [5] for the product Hardy space H^1. As an application of Theorem 1.7, we immediately have the following interpolation of operators.

Theorem 1.8. *Let $w \in A_{\infty}^{\mathfrak{z}}(\mathbb{R}^3)$ and $0 < p_1 < p_2 < \infty$. If T is a sublinear operator bounded from $H_{\mathfrak{z},w}^{p_1}(\mathbb{R}^3)$ to $L_w^{p_1}(\mathbb{R}^3)$ and bounded from $H_{\mathfrak{z},w}^{p_2}(\mathbb{R}^3)$ to $L_w^{p_2}(\mathbb{R}^3)$, then T is bounded from $H_{\mathfrak{z},w}^{p}(\mathbb{R}^3)$ to $L_w^{p}(\mathbb{R}^3)$ for all $p \in (p_1, p_2)$. Similarly, if T is bounded both on $H_{\mathfrak{z},w}^{p_1}(\mathbb{R}^3)$ and $H_{\mathfrak{z},w}^{p_2}(\mathbb{R}^3)$, then T is bounded on $H_{\mathfrak{z},w}^{p}(\mathbb{R}^3)$ for all $p \in (p_1, p_2)$.*

The organization of this paper is as follows. In Section 2, we prove the discrete Calderón reproducing formula associated with Zygmund dilations. Section 3 is devoted to proving that the weighted Hardy spaces are well defined. The boundedness of singular integral operators on the weighted Hardy spaces is proved in Section 4. We establish a new discrete Calderón-type identity and show the $H_{\mathfrak{z},w}^p - L_w^p$, $1 < p < \infty$, boundedness in Section 5. The last section contains the proofs of Theorems 1.7 and 1.8.

2. The Calderón reproducing formula

The following almost orthogonal estimates can be found in [21, Lemma 3.3].

Lemma 2.1. *Suppose that $\psi_{j,k}$ and $\phi_{j',k'}$ satisfy (1.4)–(1.7). Then*

$$|\psi_{j,k} * \phi_{j',k'}(x)| \leq C 2^{-|j-j'|L} 2^{-|k-k'|L} \frac{2^{M(j\vee j')}}{(2^{j\vee j'} + |x_1|)^{1+M}} \frac{2^{M(k\vee k')}}{2^{j^{\star}}(2^{k\vee k'} + |x_2| + 2^{-j^{\star}}|x_3|)^{2+M}}$$

for any fixed $L, M > 0$, where $x = (x_1, x_2, x_3) \in \mathbb{R}^3$, $j^{\star} = j$ if $k \geq k'$ and $j^{\star} = j'$ if $k < k'$.

We now use Lemma 2.1 to prove Theorem 1.1.

Proof of Theorem 1.1. As in the classical case, by the Fourier transform, we have the following continuous version of Calderón reproducing formula

$$f = \sum_{j,k \in \mathbb{Z}^2} \psi_{j,k} * \psi_{j,k} * f,$$

where the series converges in $L^2(\mathbb{R}^3)$ norm. To get a discrete version of Calderón reproducing formula, we need to decompose $\psi_{j,k} * \psi_{j,k} * f$. We use some ideas in [17, 18]. Set $g = \psi_{j,k} * f$ and $h = \psi_{j,k}$. For $\xi \in \mathbb{R}^3$, the Fourier transforms of g and h are respectively given by

$$\widehat{g}(\xi_1, \xi_2, \xi_3) = \widehat{\psi^{(1)}}(2^j \xi_1) \widehat{\psi^{(2)}}(2^k \xi_2, 2^{j+k} \xi_3) \widehat{f}(\xi_1, \xi_2, \xi_3),$$
$$\widehat{h}(\xi_1, \xi_2, \xi_3) = \widehat{\psi^{(1)}}(2^j \xi_1) \widehat{\psi^{(2)}}(2^k \xi_2, 2^{j+k} \xi_3).$$

Note that the Fourier transforms of g and h are both compactly supported. More precisely, both $\mathrm{supp}\,\widehat{g}$ and $\mathrm{supp}\,\widehat{h}$ are contained in

$$R_{j,k} := \left\{ \xi \in \mathbb{R}^3 : |\xi_1| \le 2^{-j}\pi, |\xi_2| \le 2^{-k}\pi, |\xi_3| \le 2^{-(j+k)}\pi \right\}.$$

Now we first expand $\widehat{g}$ in a Fourier series on the rectangle $R_{j,k}$

$$\widehat{g}(\xi) = \sum_{\ell \in \mathbb{Z}^3} 2^{2(j+k)}(2\pi)^{-3} \left(\int_{R_{j,k}} \widehat{g}(\xi')e^{i[(2^j\ell_1\xi_1' + 2^k\ell_2\xi_2' + 2^{j+k}\ell_3\xi_3')]}\, d\xi' \right)$$
$$\times\, e^{-i[(2^j\ell_1\xi_1' + 2^k\ell_2\xi_2' + 2^{j+k}\ell_3\xi_3')]}$$

and then replace $R_{j,k}$ by $\mathbb{R}^3$ since $\widehat{g}$ is supported in $R_{j,k}$. We have

$$\widehat{g}(\xi) = \sum_{\ell \in \mathbb{Z}^3} 2^{2(j+k)} g(2^j\ell_1, 2^k\ell_2, 2^{j+k}\ell_3)e^{-i[(2^j\ell_1\xi_1' + 2^k\ell_2\xi_2' + 2^{j+k}\ell_3\xi_3')]}.$$

Multiplying both sides by $\widehat{h}(\xi)$ and noting

$$\widehat{h}(\xi)e^{-i[(2^j\ell_1\xi_1' + 2^k\ell_2\xi_2' + 2^{j+k}\ell_3\xi_3')]} = [h(\,\cdot\, - 2^j\ell_1, \,\cdot\, - 2^k\ell_2, \,\cdot\, - 2^{j+k}\ell_3)]\widehat{}(\xi),$$

we obtain

$$(g * h)(x) = \sum_{\ell \in \mathbb{Z}^3} 2^{2(j+k)} g(2^j\ell_1, 2^k\ell_2, 2^{j+k}\ell_3)h(x_1 - 2^j\ell_1, x_2 - 2^k\ell_2, x_3 - 2^{j+k}\ell_3).$$

Substituting g by $\psi_{j,k} * f$ and h by $\psi_{j,k}$ into the above identity gives the discrete Calderón reproducing formula in Theorem 1.1 and the convergence in $L^2(\mathbb{R}^3)$.

To finish the proof of Theorem 1.1, it remains to show that the series in (1.9) converges in $\mathscr{S}_{\mathfrak{z}}(\mathbb{R}^3)$. Note that $\psi_{j,k} \in \mathscr{S}_{\mathfrak{z}}$, it suffices to prove

$$\sum_{|j_1|,|j_2|,|j_3|>L} \sum_{\ell \in \mathbb{Z}^3} 2^{2(j+k)}\psi_{j,k}(x_1 - 2^j\ell_1, x_2 - 2^k\ell_2, x_3 - 2^{j+k}\ell_3)(\psi_{j,k} * f)(2^j\ell_1, 2^k\ell_2, 2^{j+k}\ell_3)$$

tends to zero in $\mathscr{S}_{\mathfrak{z}}(\mathbb{R}^3)$ as $L \to \infty$. It is easy to see that this will follow from the following inequality

(2.1)
$$\left| \sum_{\ell \in \mathbb{Z}^3} 2^{2(j+k)}[\partial^\alpha\psi_{j,k}(x_1 - 2^j\ell_1, x_2 - 2^k\ell_2, x_3 - 2^{j+k}\ell_3)](\psi_{j,k} * f)(2^j\ell_1, 2^k\ell_2, 2^{j+k}\ell_3) \right|$$
$$\le C2^{-(|j_1|+|j_2|+|j_3|)}(1 + |x|)^{-M}$$

for all $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $M > 0$. Thus, it remains to verify (2.1).

We have

$$|(\partial^\alpha \psi_{j,k})(x)|$$

$$= 2^{-2(j+k)} \left| 2^{-j\alpha_1} \left(\frac{d^{\alpha_1}}{dx_1^{\alpha_1}} \psi^{(1)} \right) \left(\frac{x_1}{2^j} \right) 2^{-k\alpha_2} 2^{-(j+k)\alpha_3} \left(\partial_{x_2}^{\alpha_2} \partial_{x_3}^{\alpha_3} \psi^{(2)} \right) \left(\frac{x_2}{2^k}, \frac{x_3}{2^{j+k}} \right) \right|$$

$$(2.2) \qquad \leq 2^{-j(2+\alpha_1+\alpha_3)} 2^{-k(2+\alpha_2+\alpha_3)} \frac{1}{(1+\frac{|x_1|}{2^j})^M} \frac{1}{(1+\frac{|x_2|}{2^k})^M} \frac{1}{(1+\frac{|x_3|}{2^{j+k}})^M}$$

$$\leq 2^{(|j|+|k|)(2+|\alpha|)} \frac{2^{|j|M}}{(1+|x_1|)^M} \frac{2^{|k|M}}{(1+|x_2|)^M} \frac{2^{(|j|+|k|)M}}{(1+|x_3|)^M}$$

$$\leq 2^{(|j|+|k|)(2+|\alpha|+2M)} \frac{1}{(1+|x|)^M}.$$

Also, from Lemma 2.1,

$$(2.3) \qquad |(\psi_{j,k} * f)(x)| \leq C 2^{-|j|L} 2^{-|k|L} \frac{2^{M(j \vee 0)}}{(2^{j \vee 0} + |x_1|)^{1+M}} \frac{2^{M(k \vee 0)}}{2^{j^\star}(2^{k \vee 0} + |x_2| + 2^{-j^*}|x_3|)^{2+M}}$$

$$\leq 2^{-|j|(L-2M-1)} 2^{-|k|(L-M)} \frac{1}{(1+|x|)^M}.$$

Using (2.2) and (2.3), we have

$$\left| \sum_{\ell \in \mathbb{Z}^3} 2^{2(j+k)} [\partial^\alpha \psi_{j,k} (x_1 - 2^j \ell_1, x_2 - 2^k \ell_2, x_3 - 2^{j+k} \ell_3)] (\psi_{j,k} * f)(2^j \ell_1, 2^k \ell_2, 2^{j+k} \ell_3) \right|$$

$$\leq C 2^{-(|j|+|k|)L'} \sum_{\ell \in \mathbb{Z}^3} 2^{2(j+k)} \frac{1}{(1 + |2^j \ell_1| + |2^k \ell_2| + |2^{j+k} \ell_3|)^M}$$

$$\times \frac{1}{(1 + |x_1 - 2^j \ell_1| + |x_2 - 2^k \ell_2| + |x_3 - 2^{j+k} \ell_3|)^M}.$$

Let $R_{j,k,\ell}$ denote the Zygmund rectangle centered at $(2^j \ell_1, 2^k \ell_2, 2^{j+k} \ell_3)$ of sidelength 2^j, 2^k and 2^{j+k}. Observe that, for any $x' \in R_{j,k,\ell}$,

$$1 + |2^j \ell_1| + |2^k \ell_2| + |2^{j+k} \ell_3| \approx 1 + |x'|$$

$$1 + |x_1 - 2^j \ell_1| + |x_2 - 2^k \ell_2| + |x_3 - 2^{j+k} \ell_3| \approx 1 + |x - x'|.$$

We then have

$$\left| \sum_{\ell \in \mathbb{Z}^3} 2^{2(j+k)} [\partial^\alpha \psi_{j,k} (x_1 - 2^j \ell_1, x_2 - 2^k \ell_2, x_3 - 2^{j+k} \ell_3)] (\psi_{j,k} * f)(2^j \ell_1, 2^k \ell_2, 2^{j+k} \ell_3) \right|$$

$$\leq C 2^{-(|j|+|k|)L'} \sum_{j,k,\ell \in \mathbb{Z}^3} \int_{R_{j,k,\ell}} \frac{dx'}{(1+|x'|)^M (1+|x-x'|)^M}$$

$$= 2^{-(|j|+|k|)L'} \int_{\mathbb{R}^3} \frac{dx'}{(1+|x'|)^M (1+|x-x'|)^M}$$

$$\leq C 2^{-(|j|+|k|)L'} (1+|x|)^{-M}.$$

This proves (2.1) and hence Theorem 1.1 follows. □

3. The definition of weighted Hardy spaces

The following lemmas will be needed to prove Theorem 1.4.

Lemma 3.1 ([15]). *Suppose $1 < p < \infty$. Then $\mathcal{M}_{\mathfrak{z}}$ is bounded on $L^p_w(\mathbb{R}^3)$ if and only if $w \in A^{\mathfrak{z}}_p(\mathbb{R}^3)$.*

Using Lemma 3.1 and the idea of Rubio de Francia's extrapolation, Cruz-Uribe, Martell, and Pérez [9] obtained the following weighted version of vector-valued maximal inequality associated with Zygmund dilations.

Lemma 3.2. *Suppose $1 < p, q < \infty$. If $w \in A^{\mathfrak{z}}_p(\mathbb{R}^3)$, then*

$$\int_{\mathbb{R}^3} \left(\sum_{j \in \mathbb{Z}} |\mathcal{M}_{\mathfrak{z}}(f_j)(x)|^q \right)^{p/q} w(x)\, dx \leq C \int_{\mathbb{R}^3} \left(\sum_{j \in \mathbb{Z}} |f_j(x)|^q \right)^{p/q} w(x)\, dx.$$

The following lemma will be used in the proof of Theorems 1.4 and 5.1.

Lemma 3.3. *Given any nonnegative integer N and integers j, k, j', k', let $R \in \mathcal{R}^{j-N,k-N}_{\mathfrak{z}}$ and $R' = I' \times J' \times K' \in \mathcal{R}^{j'-N,k'-N}_{\mathfrak{z}}$. Let $\{a_{R'}\}$ be any given sequence and let $x^*_{R'} = (x^*_{I'}, x^*_{J'}, x^*_{K'})$ be any point in R'. Then for any $u^* = (u^*_1, u^*_2, u^*_3), v^* = (v^*_1, v^*_2, v^*_3) \in \mathbb{R}^3$, we have*

$$\sum_{R' \in \mathcal{R}^{j'-N,k'-N}_{\mathfrak{z}}} \frac{2^{(j \vee j')M_1} 2^{(k \vee k')M_2} |R'|}{(2^{j \vee j'} + |u^*_1 - x^*_{I'}|)^{1+M_1} 2^{j^*} (2^{k \vee k'} + |u^*_2 - x^*_{J'}| + 2^{-j^*}|u^*_3 - x^*_{K'}|)^{2+M_2}} |a_{R'}|$$

$$\leq C 2^{[4N + 2(j-j')_+ + 2(k-k')_+](1/r - 1)} 2^{|j-j'|} \left\{ \mathcal{M}_{\mathfrak{z}} \left(\sum_{R' \in \mathcal{R}^{j'-N,k'-N}_{\mathfrak{z}}} |a_{R'}|^r \chi_{R'} \right) (v^*) \right\}^{1/r},$$

where $(j-j')_+ = \max(j-j', 0)$, $j^* = j$ if $k < k'$ and $j^* = j'$ if $k \geq k'$ and $\max\{2/M_1, 2/M_2\} < r \leq 1$.

Proof. For $R' = I' \times J' \times K' \in \mathcal{R}^{j'-N,k'-N}_{\mathfrak{z}}$, we set

$$A_0 = \{I' : 2^{j \vee j'} |u^*_1 - x^*_{I'}| \leq 1\},$$
$$B_0 = \{J' \times K' : 2^{k \vee k'} (|u^*_2 - x^*_{J'}| + 2^{-j^*}|u^*_3 - x^*_{K'}|) \leq 1\}$$

and for $\ell \geq 1$, $i \geq 1$,

$$A_\ell = \{I' : 2^{\ell-1} < 2^{j \vee j'} |u^*_1 - x^*_{I'}| \leq 2^\ell\},$$
$$B_i = \{J' \times K' : 2^{i-1} < 2^{k \vee k'} (|u^*_2 - x^*_{J'}| + 2^{-j^*}|u^*_3 - x^*_{K'}|) \leq 2^i\}.$$

For any fixed $\ell, i \geq 0$, denote

$$E_{\ell,i} = \big\{ (w_1, w_2, w_3) \in \mathbb{R}^3 : |w_1 - x^*_{I'}| \leq 2^{\ell+1} 2^{(j \vee j')}, |w_2 - x^*_{J'}| \leq 2^{i+1} 2^{(k \vee k')},$$
$$|w_3 - x^*_{K'}| \leq 2^{\ell+i+2} 2^{j \vee j'} 2^{k \vee k'} \big\}.$$

Note that every $E_{\ell,i}$ is a Zygmund rectangle. For $R' \in A_\ell \times B_i$, we have $R' \subset E_{\ell,i}$ and obviously

$$|E_{\ell,i}| = 2^{2(\ell+i+2)} 2^{2(j \vee j')} 2^{2(k \vee k')}.$$

Then, for any $v^* \in R$, $v^* \in E_{\ell,i}$,

$$\sum_{R \in \mathcal{R}_3^{j'-N, k'-N}} \frac{2^{(j \vee j')M_1 + (k \vee k')M_2} |R'| |a_{R'}|}{\left(2^{j \vee j'} + |u^*_1 - x^*_{I'}| \right)^{1+M_1} 2^{j^*} \left(2^{k \vee k'} + |u^*_2 - x^*_{J'}| + 2^{-j^*} |u^*_3 - x^*_{K'}| \right)^{2+M_2}}$$

$$\leq 2^{-j^* - j \vee j' - 2(k \vee k') + 2(j'+k') - 4N} \sum_{\ell, i \geq 0} 2^{-\ell(1+M_1)} 2^{-i(2+M_2)} \left(\sum_{I' \in A_\ell, J' \times K' \in B_i} |a_{R'}|^r \right)^{1/r}$$

$$= 2^{-j^* - j \vee j' - 2(k \vee k')} \sum_{\ell, i \geq 0} 2^{-\ell(1+M_1)} 2^{-i(2+M_2)}$$

$$\times |R'| |E_{\ell,i}|^{-1/r} \left(\frac{1}{|E_{\ell,i}|} \int_{E_{\ell,i}} |R'|^{-1} \sum_{I' \in A_\ell, J' \times K' \in B_i} |a_{R'}|^r \chi_{R'} \right)^{1/r}$$

$$\leq C 2^{4N(1/r-1)} 2^{(j \vee j') - j^*} 2^{2(1/r-1)[(j-j')_+ + (k-k')_+]}$$

$$\times \sum_{\ell, i \geq 0} 2^{-\ell(3+M_1-2/r)} 2^{-i(4+M_2-2/r)} \left(\mathcal{M}_3 \left(\sum_{I' \in A_\ell, J' \times K' \in B_i} |a_{R'}|^r \chi_{R'} \right) (v^*) \right)^{1/r}$$

$$\leq C 2^{[4N + 2(j-j')_+ + 2(k-k')_+](1/r-1)} 2^{|j-j'|} \left(\mathcal{M}_3 \left(\sum_{R' \in \mathcal{R}_3^{j'-N, k'-N}} |a_{R'}|^r \chi_{R'} \right) (v^*) \right)^{1/r}.$$

We note that the last inequality in the above string follows from $1 \geq r > \max\{2/M_1, 2/M_2\}$, which can be done by the assumption and choosing M_1 and M_2 large enough. Hence the proof of Lemma 3.3 is concluded. $\qquad \square$

Proof of Theorem 1.4. We begin with estimating $(\psi_{j',k'} * f)(x_{R'})$. To do this, applying Theorem 1.1 implies the following pointwise identity:

$$(\psi_{j',k'} * f)(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_3^{j,k}} |R| (\psi_{j',k'} * \varphi_{j,k})(x - x_R)(\varphi_{j,k} * f)(x_R).$$

For any sufficiently large constant M, applying Lemma 2.1 for $(\psi_{j',k'} * \varphi_{j,k})(x_{R'} - x_R)$

with $L > M$ and then Lemma 3.3 with $M_1 = M_2 = M$ and $N = 0$, we have

$$|(\psi_{j',k'} * f)(x_{R'})| \leq C \sum_{j,k \in \mathbb{Z}} 2^{-(|j-j'|+|k-k'|)L} \sum_{R \in \mathcal{R}_3^{j,k}} |R| \frac{2^{(j \vee j')M}}{(2^{(j \vee j')} + |u' - x_I|)^{1+M}}$$

$$\times \frac{2^{(k \vee k')M}}{2^{j^*}(2^{(k \vee k')} + |v' - x_J| + 2^{-j^*}|w' - x_K|)^{2+M}} |(\varphi_{j,k} * f)(x_R)|$$

$$\leq C \sum_{j,k \in \mathbb{Z}} 2^{-(|j-j'|+|k-k'|)L} 2^{[2(j-j')_+ + 2(k-k')_+](1/r-1)} 2^{|j-j'|}$$

$$\times \left\{ \mathcal{M}_3 \left(\sum_{R \in \mathcal{R}_3^{j,k}} |(\varphi_{j,k} * f)(x_R)|^2 \chi_R \right)^{r/2} (x) \right\}^{1/r}$$

for any $x \in R'$, where $2/(2 + M) < r < \min\{p, 2\}$, and the fact that $\left(\sum_{R \in \mathcal{R}_3^{j,k}} |(\varphi_{j,k} * f)(x_R)|^2 \chi_R \right)^{r/2} = \sum_{R \in \mathcal{R}_3^{j,k}} |(\varphi_{j,k} * f)(x_R)|^r \chi_R$ is used. Applying Cauchy-Schwarz's inequality and summing over j', k' and R' yield

$$\left\{ \sum_{j',k' \in \mathbb{Z}} \sum_{R' \in \mathcal{R}_3^{j',k'}} |(\psi_{j',k'} * f)(x_{R'})|^2 \chi_{R'}(x) \right\}^{1/2}$$

$$\leq C \left\{ \sum_{j,k \in \mathbb{Z}} \left\{ \mathcal{M}_3 \left(\sum_{R \in \mathcal{R}_3^{j,k}} |(\varphi_{j,k} * f)(x_R)|^2 \chi_R \right)^{r/2} (x) \right\}^{2/r} \right\}^{1/2}$$

since

$$\sum_{j \in \mathbb{Z}} 2^{-(|j-j'|+|k-k'|)L} 2^{[2(j-j')_+ + 2(k-k')_+](1/r-1)} 2^{|j-j'|} \leq C$$

and

$$\sum_{k \in \mathbb{Z}} 2^{-(|j-j'|+|k-k'|)L} 2^{[2(j-j')_+ + 2(k-k')_+](1/r-1)} 2^{|j-j'|} \leq C.$$

Hence, if $w \in A_\infty^3$, taking M big enough, we may choose r with $r < p$ so that $w \in A_{p/r}^3$. Applying the weighted vector-valued maximal inequality on $L_w^{p/r}(\ell^{2/r})$, we get

$$\left\| \left\{ \sum_{j',k' \in \mathbb{Z}} \sum_{R' \in \mathcal{R}_3^{j',k'}} |(\psi_{j',k'} * f)(x_{R'})|^2 \chi_{R'} \right\}^{1/2} \right\|_{L_w^p}$$

$$\leq C \left\| \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_3^{j,k}} |(\varphi_{j,k} * f)(x_R)|^2 \chi_R \right\}^{1/2} \right\|_{L_w^p},$$

and this completes the proof of Theorem 1.4. □

As an immediate consequence of Theorem 1.4, the definition of $H_{3,w}^p(\mathbb{R}^3)$ is independent of the choice of $\psi_{j,k}$. Henceforth, we usually write $g_3(f)$ and $\| \cdot \|_{H_{3,w}^p}$ to stand for $g_3^\psi(f)$ and $\| \cdot \|_{H_{3,w}^p}^\psi$ without specifying ψ.

The following density result of $H_{\mathfrak{z},w}^p$ is useful for the proof of the boundedness of operators on $H_{\mathfrak{z},w}^p$, which follows from the proof of Theorem 1.4.

Corollary 3.4. *Let* $w \in A_\infty^{\mathfrak{z}}(\mathbb{R}^3)$ *and* $0 < p < \infty$. *Then* $L^2 \cap H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ *is dense in* $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$.

Proof. Since $\mathscr{S}_{\mathfrak{z}}(\mathbb{R}^3) \subset L^2(\mathbb{R}^3)$, we only need to show that $\mathscr{S}_{\mathfrak{z}}(\mathbb{R}^3)$ is dense in $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ for $0 < p < \infty$. Let $f \in H_{\mathfrak{z},w}^p(\mathbb{R}^3)$. By the discrete Calderón reproducing formula in Theorem 1.1,

$$f(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j,k}} |R| \psi_{j,k}(x - x_R)(\psi_{j,k} * f)(x_R).$$

For any fixed $L > 0$, denote

$$E_L = \{(j,k,R) : |j|, |k| \leq L, R \subset B(0,L)\}$$

and

$$f_L(x) = \sum_{(j,k,R) \in E_L} |R| \psi_{j,k}(x - x_R)(\psi_{j,k} * f)(x_R).$$

Since $\psi_{j,k} \in \mathscr{S}_{\mathfrak{z}}(\mathbb{R}^3)$, it is obvious that $f_L \in \mathscr{S}_{\mathfrak{z}}(\mathbb{R}^3)$. Repeating the proof of Theorem 1.4, we conclude that

$$\|f_L\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)} \leq C\|f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)}.$$

To see that f_L tends to f in $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$, we again apply the discrete Calderón reproducing formula in Theorem 1.1 to write

$$(f - f_L)(x) = \sum_{(j,k,R) \in E_L^c} |R| \psi_{j,k}(x - x_R)(\psi_{j,k} * f)(x_R),$$

where the series converges in $\mathscr{S}_{\mathfrak{z}}'(\mathbb{R}^3)$. Thus, $[g_{\mathfrak{z}}(f - f_L)(x)]^2$ equals

$$\sum_{j',k' \in \mathbb{Z}} \sum_{R' \in \mathcal{R}_{\mathfrak{z}}^{j',k'}} \left| \sum_{(j,k,R) \in E_L^c} |R| (\psi_{j',k'} * \psi_{j,k})(x_{R'} - x_R)(\psi_{j,k} * f)(x_R) \right|^2 \chi_{R'}(x).$$

Now repeating the same argument as in the proof of Theorem 1.4 again, we get

$$\|g_{\mathfrak{z}}(f - f_L)\|_{L_w^p(\mathbb{R}^3)} \leq C \left\| \left\{ \sum_{(j,k,R) \in E_L^c} |(\psi_{j,k} * f)(x_R)|^2 \chi_R \right\}^{1/2} \right\|_{L_w^p(\mathbb{R}^3)},$$

where the last term tends to 0 as L goes to infinity. This implies that f_L tends to f in $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ norm and hence the proof of Corollary 3.4 is concluded. $\qquad\square$

4. The boundedness of singular integral operators on $H^p_{\mathfrak{z},w}(\mathbb{R}^3)$

The following lemma is a variant of Lemma 3.3, which will be needed in the proof of Theorem 1.5.

Lemma 4.1. *Given any integers j, k, j', k', let $R \in \mathcal{R}^{j,k}_{\mathfrak{z}}$ and $R' = I' \times J' \times K' \in \mathcal{R}^{j',k'}_{\mathfrak{z}}$. Let $\{a_{R'}\}$ be any given sequence and let $x^*_{R'} = (x^*_{I'}, x^*_{J'}, x^*_{K'})$ be any point in R'. Then for any $u^* = (u^*_1, u^*_2, u^*_3), v^* = (v^*_1, v^*_2, v^*_3) \in \mathbb{R}^3$, we have*

$$\sum_{R' \in \mathcal{R}^{j',k'}_{\mathfrak{z}}} \frac{2^{(j \vee j')\lambda}}{(2^{(j \vee j')} + |u^*_1 - x^*_{I'}|)^{1+\lambda}} \frac{2^{(k \vee k')\lambda/2}}{(2^{(k \vee k')} + |u^*_2 - x^*_{J'}|)^{1+\lambda/2}}$$

$$\times \frac{2^{[(j \vee j')+(k \vee k')]\lambda/2}}{(2^{(j \vee j')+(k \vee k')} + |u^*_3 - x^*_{K'}|)^{1+\lambda/2}} |R'||a_{R'}|$$

$$\leq C 2^{[2(j-j')_+ + 2(k-k')_+](1/r-1)} \left\{ \mathcal{M}_{\mathfrak{z}} \left(\sum_{R' \in \mathcal{R}^{j',k'}_{\mathfrak{z}}} |a_{R'}|^r \chi_{R'} \right)(v^*) \right\}^{1/r},$$

where $2/(2+\lambda) < r \leq 1$ and $\lambda = \frac{1}{2} \min(\theta_1, \theta_2)$.

Proof. We provide a detailed proof for the reader's convenience. For $R' = I' \times J' \times K' \in \mathcal{R}^{j',k'}_{\mathfrak{z}}$, we set

$$A_0 := \{I' : |u^*_1 - x^*_{I'}| \leq 2^{j \vee j'}\}, \quad B_0 := \{J' : |u^*_2 - x^*_{J'}| \leq 2^{k \vee k'}\}$$

and

$$C_0 := \{J' : |u^*_3 - x^*_{K'}| \leq 2^{(j \vee j')+(k \vee k')}\}.$$

Moreover, and for integers $\ell_1, \ell_2, \ell_3 \geq 1$, we set

$$A_{\ell_1} := \{I' : 2^{\ell_1-1}2^{j \vee j'} < |u^*_1 - x^*_{I'}| \leq 2^{\ell_1}2^{j \vee j'}\},$$
$$B_{\ell_2} := \{I' : 2^{\ell_2-1}2^{j \vee j'} < |u^*_1 - x^*_{I'}| \leq 2^{\ell_2}2^{j \vee j'}\}$$

and

$$C_{\ell_3} := \{K' : 2^{\ell_3-1}2^{(j \vee j')+(k \vee k')} < |u^*_3 - x^*_{K'}| \leq 2^{\ell_3}2^{(j \vee j')+(k \vee k')}\}.$$

For each fixed $\ell_1, \ell_2, \ell_3 \geq 0$, denote

$$E_{\ell_1,\ell_2,\ell_3} := \big\{ (w_1, w_2, w_3) \in \mathbb{R}^3 : |w_1 - x^*_{I'}| \leq 2^{\ell_1+1}2^{(j \vee j')}, |w_2 - x^*_{J'}| \leq 2^{\ell_2+1}2^{(k \vee k')},$$
$$|w_3 - x^*_{K'}| \leq 2^{\ell_3+1}2^{(j \vee j')+(k \vee k')} \big\}.$$

Obviously, we have

$$|E_{\ell_1,\ell_2,\ell_3}| = 2^{\ell_1+1}2^{\ell_2+1}2^{\ell_3+1}2^{2(j \vee j')}2^{2(k \vee k')}.$$

Here we point out that E_{ℓ_1,ℓ_2,ℓ_3} may not be a Zygmund rectangle any more. Also note that for $R' \in A_{\ell_1} \times B_{\ell_2} \times C_{\ell_3}$, we have $R' \subset E_{\ell_1,\ell_2,\ell_3}$, which yields that for any $v^* \in R'$, we have $v^* \in E_{\ell_1,\ell_2,\ell_3}$.

Now we have

$$\sum_{R' \in \mathcal{R}_{\mathfrak{z}}^{j',k'}} \frac{2^{(j \vee j')\lambda}}{(2^{(j \vee j')} + |u_1^* - x_{I'}^*|)^{1+\lambda}} \frac{2^{(k \vee k')\lambda/2}}{(2^{(k \vee k')} + |u_2^* - x_{J'}^*|)^{1+\lambda/2}}$$

$$\times \frac{2^{[(j \vee j')+(k \vee k')]\lambda/2}}{(2^{(j \vee j')+(k \vee k')} + |u_3^* - x_{K'}^*|)^{1+\lambda/2}} |R'||a_{R'}|$$

$$\leq \sum_{\substack{\ell_1,\ell_2,\ell_3 \geq 0}} \sum_{\substack{R'=I' \times J' \times K' \in \mathcal{R}_{\mathfrak{z}}^{j',k'} \\ I' \in A_{\ell_1}, J' \in B_{\ell_1}, K' \in C_{\ell_3}}} \frac{2^{(j \vee j')\lambda}}{(2^{(j \vee j')} + |u_1^* - x_{I'}^*|)^{1+\lambda}} \frac{2^{(k \vee k')\lambda/2}}{(2^{(k \vee k')} + |u_2^* - x_{J'}^*|)^{1+\lambda/2}}$$

$$\times \frac{2^{[(j \vee j')+(k \vee k')]\lambda/2}}{(2^{(j \vee j')+(k \vee k')} + |u_3^* - x_{K'}^*|)^{1+\lambda/2}} |R'||a_{R'}|$$

$$\leq C \sum_{\substack{\ell_1,\ell_2,\ell_3 \geq 0}} \sum_{\substack{R'=I' \times J' \times K' \in \mathcal{R}_{\mathfrak{z}}^{j',k'} \\ I' \in A_{\ell_1}, J' \in B_{\ell_1}, K' \in C_{\ell_3}}} 2^{-(j \vee j')} 2^{-\ell_1(1+\lambda)} 2^{-(k \vee k')} 2^{-\ell_2(1+\lambda/2)}$$

$$\times 2^{-[(j \vee j')+(k \vee k')]} 2^{-\ell_3(1+\lambda/2)} 2^{2j'} 2^{2k'} |a_{R'}|$$

$$\leq C 2^{-2(j \vee j')} 2^{-2(k \vee k')} 2^{2j'} 2^{2k'} \sum_{\ell_1,\ell_2,\ell_3 \geq 0} 2^{-\ell_1(1+\lambda)} 2^{-\ell_2(1+\lambda/2)} 2^{-\ell_3(1+\lambda/2)}$$

$$\times \left(\sum_{\substack{R'=I' \times J' \times K' \in \mathcal{R}_{\mathfrak{z}}^{j',k'} \\ I' \in A_{\ell_1}, J' \in B_{\ell_1}, K' \in C_{\ell_3}}} |a_{R'}|^r \right)^{1/r}$$

$$= C 2^{-2(j \vee j')} 2^{-2(k \vee k')} 2^{2j'} 2^{2k'} \sum_{\ell_1,\ell_2,\ell_3 \geq 0} 2^{-\ell_1(1+\lambda)} 2^{-\ell_2(1+\lambda/2)} 2^{-\ell_3(1+\lambda/2)} (2^{2j'} 2^{2k'})^{-1/r} |E_{\ell_1,\ell_2,\ell_3}|^{1/r}$$

$$\times \left(\frac{1}{|E_{\ell_1,\ell_2,\ell_3}|} \int_{E_{\ell_1,\ell_2,\ell_3}} \sum_{\substack{R'=I' \times J' \times K' \in \mathcal{R}_{\mathfrak{z}}^{j',k'} \\ I' \in A_{\ell_1}, J' \in B_{\ell_1}, K' \in C_{\ell_3}}} |a_{R'}|^r \chi_{R'}(x) \, dx \right)^{1/r}$$

$$\leq C 2^{-2(j \vee j')(1-1/r)} 2^{-2(k \vee k')(1-1/r)} (2^{2j'} 2^{2k'})^{1-1/r}$$

$$\times \sum_{\ell_1,\ell_2,\ell_3 \geq 0} 2^{-\ell_1(1+\lambda-1/r)} 2^{-\ell_2(1+\lambda/2-1/r)} 2^{-\ell_3(1+\lambda/2-1/r)} \left(\mathcal{M}_{\mathfrak{z}} \left(\sum_{R' \in \mathcal{R}_{\mathfrak{z}}^{j',k'}} |a_{R'}|^r \chi_{R'} \right)(v^*) \right)^{1/r}$$

$$\leq 2^{2(1-1/r)[-(j \vee j')+j']} 2^{2(1-1/r)[-(k \vee k')+k']} \left(\mathcal{M}_{\mathfrak{z}} \left(\sum_{R' \in \mathcal{R}_{\mathfrak{z}}^{j',k'}} |a_{R'}|^r \chi_{R'} \right)(v^*) \right)^{1/r}$$

$$= 2^{[2(j-j')_+ + 2(k-k')_+](1/r-1)} \left(\mathcal{M}_{\mathfrak{z}} \left(\sum_{R' \in \mathcal{R}_{\mathfrak{z}}^{j',k'}} |a_{R'}|^r \chi_{R'} \right)(v^*) \right)^{1/r}.$$

Here v^* is any fixed point in R. We note that the last inequality above follows from the assumption that $1 \geq r > 2/(2+\lambda)$. The proof of Lemma 4.1 is concluded. $\qquad\square$

We now prove Theorem 1.5, one of the main results in this paper.

Proof of Theorem 1.5. By the discrete Calderón reproducing formula in Theorem 1.1,

$$\|T_{\mathfrak{z}}(f)\|_{H^p_{\mathfrak{z},w}(\mathbb{R}^3)} = \left\|\left\{\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}^{j,k}_{\mathfrak{z}}}|(\psi_{j,k}*\mathcal{K}*f)(x_R)|^2\chi_R\right\}^{1/2}\right\|_{L^p_w(\mathbb{R}^3)}$$

$$= \left\|\left\{\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}^{j,k}_{\mathfrak{z}}}\left|\psi_{j,k}*\mathcal{K}*\left(\sum_{j',k'\in\mathbb{Z}}\sum_{R'\in\mathcal{R}^{j,k}_{\mathfrak{z}}}|R'|\psi_{j',k'}*f(x_{R'})\right.\right.\right.\right.$$

$$\left.\left.\left.\left.\times\,\psi_{j',k'}(\,\cdot\,-x_{R'})\right)(x_R)\right|^2\chi_R\right\}^{1/2}\right\|_{L^p_w(\mathbb{R}^3)}$$

$$= \left\|\left\{\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}^{j,k}_{\mathfrak{z}}}\left|\sum_{j',k'\in\mathbb{Z}}\sum_{R'\in\mathcal{R}^{j,k}_{\mathfrak{z}}}|R'|(\psi_{j',k'}*f)(x_{R'})\right.\right.\right.$$

$$\left.\left.\left.\times\,(\psi_{j,k}*\mathcal{K}*\psi_{j',k'})(x_R-x_{R'})\right|^2\chi_R\right\}^{1/2}\right\|_{L^p_w(\mathbb{R}^3)},$$

where $\mathcal{R}^{j,k}_{\mathfrak{z}}$ is the collection of rectangles in $\mathbb{R}^3$ and $R\in\mathcal{R}^{j,k}_{\mathfrak{z}}$ means that $R=I\times J\times S$ with the side length $|I|=2^j$, $|J|=2^k$ and $|S|=2^{j+k}$, x_R is the left lower corner of R, similar for the set of rectangles R'.

[21, Proposition 3.1] showed that $\psi_{j,k}*\mathcal{K}*\psi_{j',k'}$ satisfies the following orthogonality estimate

$$|(\psi_{j,k}*\mathcal{K}*\psi_{j',k'})(x_1,x_2,x_3)|$$

$$\leq C2^{-|j-j'|}2^{-|k-k'|}\frac{2^{-(j\vee j')}}{(1+2^{-(j\vee j')}|x_1|)^{1+\lambda}}\frac{2^{-(k\vee k')}}{(1+2^{-(k\vee k')}|x_2|)^{1+\lambda/2}}$$

$$\times\frac{2^{-(j\vee j')-(k\vee k')}}{(1+2^{-(j\vee j')-(k\vee k')}|x_3|)^{1+\lambda/2}}$$

for $\lambda=\frac{1}{2}\min(\theta_1,\theta_2)$.

Thus, using Lemma 4.1 and repeating the same argument as the proof of Theorem 1.4, we obtain

$$\|Tf\|_{H^p_{\mathfrak{z},w}(\mathbb{R}^3)} \leq C\left\|\left\{\sum_{j',k'\in\mathbb{Z}}\left[\mathcal{M}_{\mathfrak{z}}\left(\sum_{R'\in\mathcal{R}^{j,k}_{\mathfrak{z}}}|(\psi_{j',k'}*f)(x_{R'})|^2\chi_{R'}\right)^{r/2}\right]^{2/r}\right\}^{1/2}\right\|_{L^p_w(\mathbb{R}^3)},$$

where $1\geq r>2/(2+\lambda)$.

Now since $q_w\cdot\frac{2}{2+\frac{1}{2}\min(\theta_1,\theta_2)}<p\leq 1$, we choose $r<p$ such that $p/r>q_w$. Applying the $L^{p/r}_w(\ell^{2/r})$ boundedness of $\mathcal{M}_{\mathfrak{z}}$ in Lemma 3.2, we obtain

$$\left\|\left\{\sum_{j',k'\in\mathbb{Z}}\left[\mathcal{M}_{\mathfrak{z}}\left(\sum_{R'\in\mathcal{R}^{j,k}_{\mathfrak{z}}}|(\psi_{j',k'}*f)(x_{R'})|^2\chi_{R'}\right)^{r/2}\right]^{2/r}\right\}^{1/2}\right\|_{L^p_w(\mathbb{R}^3)} \leq C\|f\|_{H^p_{\mathfrak{z},w}(\mathbb{R}^3)}.$$

This concludes the proof of Theorem 1.5. $\qquad\square$

5. A new discrete Calderón-type identity and $H_{\mathfrak{z},w}^p(\mathbb{R}^3) - L_w^p(\mathbb{R}^3)$ boundedness

To prove the boundedness for singular integral operator from the Hardy space $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ to $L_w^p(\mathbb{R}^3)$ for $0 < p \leq 1$, we need the following discrete Calderón-type identity in terms of bump functions.

Theorem 5.1. *Let $0 < p < \infty$ and $w \in A_\infty^{\mathfrak{z}}(\mathbb{R}^3)$. Suppose $M_0 \geq 10(N[q_w/p - 1] + 1)$ (here we use $[\cdot]$ to denote the greatest integer function). Given $f \in L^2 \cap H_{\mathfrak{z},w}^p(\mathbb{R}^3)$, there exists $h \in L^2 \cap H_{\mathfrak{z},w}^p(\mathbb{R}^3)$ such that, for a sufficiently large $N \in \mathbb{N}$,*

$$(5.1) \qquad f(x) \overset{L^2}{=} \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} |R| \phi_{j,k}(x - \vartheta_R)(\phi_{j,k} * h)(\vartheta_R),$$

where ϑ_R denotes any fixed point in R. Moreover,

$$(5.2) \qquad \|f\|_{H_{\mathfrak{z},w}^p} \approx \|h\|_{H_{\mathfrak{z},w}^p}, \quad \|f\|_2 \approx \|h\|_2.$$

If $1 < p < \infty$ and $w \in A_p^{\mathfrak{z}}$, then (5.1) holds for $f \in L^2 \cap L_w^p$ with

$$\|f\|_{L_w^p(\mathbb{R}^3)} \approx \|h\|_{L_w^p(\mathbb{R}^3)}.$$

Proof. For $f \in L^2(\mathbb{R}^3)$, the Fourier transform gives

$$f = \sum_{j,k \in \mathbb{Z}} \phi_{j,k} * \phi_{j,k} * f,$$

where the series converges in $L^2(\mathbb{R}^3)$ norm. Applying Coifman's idea of the decomposition for the identity operator, we have

$$f(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} |R|(\phi_{j,k} * f)(\vartheta_R) \phi_{j,k}(x - \vartheta_R)$$

$$+ \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} \int_R \phi_{j,k}(x - x')(\phi_{j,k} * f)(x') - \phi_{j,k}(x - \vartheta_R)(\phi_{j,k} * f)(\vartheta_R) \, dx'$$

$$:= T_N(f)(x) + R_N(f)(x),$$

where N is a fixed large integer to be determined later.

We can decompose $R_N(f)$ further as

$$R_N(f)(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} \int_R [\phi_{j,k}(x - x') - \phi_{j,k}(x - \vartheta_R)](\phi_{j,k} * f)(x') \, dx'$$

$$+ \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} \int_R \phi_{j,k}(x - x')[(\phi_{j,k} * f)(x') - (\phi_{j,k} * f)(\vartheta_R)] \, dx'$$

$$:= R_N^1(f)(x) + R_N^2(f)(x).$$

We claim that for $i = 1, 2$,

$$\|R_N^i(f)\|_{H_{\mathfrak{z},w}^p} \le C 2^{-N} \|f\|_{H_{\mathfrak{z},w}^p},$$

where C is a constant independent of f and N.

Assume the claim for the moment. Then by choosing sufficiently large N, both T_N and $T_N^{-1} = \sum_{n=0}^{\infty}(R_N)^n$ are bounded on $L^2(\mathbb{R}^3)$ and on $H_{\mathfrak{z},w}^p(\mathbb{R}^3)$. Set $h = T_N^{-1}(f)$. The estimate (5.5) implies (5.2). Moreover,

$$f = T_N(T_N^{-1}(f)) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} |R|\phi_{j,k}(\cdot - \vartheta_R)(\phi_{j,k} * h)(\vartheta_R),$$

where the series converges in $L^2(\mathbb{R}^3)$. Thus, it suffices to verify the claim to finish the proof of Theorem 5.1.

Since the proofs for R_N^1 and R_N^2 are similar, we give the proof for R_N^1 only. Let $f \in L^2(\mathbb{R}^3) \cap H_{\mathfrak{z},w}^p(\mathbb{R}^3)$. By Theorem 1.1, we write

$$[\psi_{j',k'} * R_N^1(f)](x)$$

$$= \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} \int_R (\psi_{j',k'} * [\phi_{j,k}(\cdot - x') - \phi_{j,k}(\cdot - \vartheta_R)])(x)(\phi_{j,k} * f)(x')\, dx'$$

(5.3)
$$= \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} \int_R (\psi_{j',k'} * [\phi_{j,k}(\cdot - x') - \phi_{j,k}(\cdot - \vartheta_R)])(x)$$

$$\times \left(\sum_{j'',k'' \in \mathbb{Z}} \sum_{R'' \in \mathcal{R}_{\mathfrak{z}}^{j'',k''}} |R''| \cdot (\psi_{j'',k''} * f)(x_{R''})(\phi_{j,k} * \psi_{j'',k''})(x' - x_{R''}) \right) dx',$$

where $x_{R''} = (x_{I''}, x_{J''}, x_{K''})$ is the left lower corner of R''. Set $\widetilde{\phi}_{j,k}(u) = \phi_{j,k}(u - x') - \phi_{j,k}(u - \vartheta_R)$. Applying Lemma 2.1 with M sufficiently large (which will be determined later) and $L = 10M$, we obtain that for some constant C (depending only on M, ψ and ϕ, but independent of N),

$$|(\psi_{j',k'} * \widetilde{\phi}_{j,k})(x)| \le C 2^{-N} 2^{-100M(|j-j'|+|k-k'|)}$$

$$\times \frac{2^{M(j \vee j')}}{(2^{j \vee j'} + |x_1 - \vartheta_I|)^{1+M}} \frac{2^{M(k \vee k')}}{2^{j^\star}(2^{k \vee k'} + |x_2 - \vartheta_{l,m}| + 2^{-j^\star}|x_3 - \vartheta_K|)^{2+M}}$$

$$\le C 2^{-N} 2^{-10M(|j-j'|+|k-k'|)}$$

$$\times \frac{2^{Mj'}}{(2^{j'} + |x_1 - x_1'|)^{1+M}} \frac{2^{Mk'}}{2^{j'}(2^{k'} + |x_2 - x_2'| + 2^{-j'}|x_3 - x_3'|)^{2+M}}$$

for any $x' = (x_1', x_2', x_3') \in \mathbb{R}^3$. Similarly,

$$|(\phi_{j,k} * \psi_{j'',k''})(x' - x_{R''})| \le C 2^{-10M(|j-j''|+|k-k''|)} \frac{2^{Mj''}}{(2^{j''} + |x_1' - x_{I''}|)^{1+M}}$$

$$\times \frac{2^{Mk''}}{2^{j''}(2^{k''} + |x_2' - x_{J''}| + 2^{-j''}|x_3' - x_{K''}|)^{2+M}}.$$

Substituting both estimates into the last term of (5.3) yields

(5.4)

$$|(\psi_{j',k'} * R_N^1 f)(x)|$$

$$\leq C \sum_{j'',k''\in\mathbb{Z}} \sum_{R''\in\mathcal{R}_{\mathfrak{z}}^{j'',k''}} |R''||(\psi_{j'',k''} * f)(x_{R''})| \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} \int_R 2^{-N} 2^{-|j-j'|3M} 2^{-|k-k'|3M}$$

$$\times \frac{2^{Mj'}}{(2^{j'}+|x_1-x_1'|)^{1+M}} \frac{2^{Mk'}}{2^{j'}(2^{k'}+|x_2-x_2'|+2^{-j'}|x_3-x_3'|)^{2+M}} 2^{-|j-j''|3M} 2^{-|k-k''|3M}$$

$$\times \frac{2^{Mj''}}{(2^{j''}+|x_1'-x_{I''}|)^{1+M}} \frac{2^{Mk''}}{2^{j''}(2^{k''}+|x_2'-x_{J''}|+2^{-j''}|x_3'-x_{K''}|)^{2+M}} \, dx'$$

$$\leq C2^{-N} \sum_{j'',k''\in\mathbb{Z}} \sum_{R''\in\mathcal{R}_{\mathfrak{z}}^{j'',k''}} 2^{-|j'-j''|3M} 2^{-|k'-k''|3M} |R''| \frac{2^{M(j'\vee j'')}}{(2^{j'\vee j''}+|x_1-x_{I''}|)^{1+M}}$$

$$\times \frac{2^{M(k'\vee k'')}}{2^{\overline{j^*}}(2^{k'\vee k''}+|x_2-x_{J''}|+2^{-\overline{j^*}}|x_3-x_{K''}|)^{2+M}} |(\psi_{j'',k''} * f)(x_{R''})|,$$

where $\overline{j^*} = j'$ if $k' \geq k''$ and $\overline{j^*} = j''$ if $k' < k''$. Now we choose $M = N[q_w/p - 1] + 1$, $L = 10M$ and $3/(N+M) < \delta < 1$. Then $p/\delta > q_w$ so that $w \in A_{p/\delta}^{\mathfrak{z}}(\mathbb{R}^3)$. We apply R'' to a similar estimate to Lemma 3.3 and get

$$\|R_N^1(f)\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)}$$

$$\leq C\|g_{\mathfrak{z}}[R_N^1(f)]\|_{L_w^p(\mathbb{R}^3)}$$

$$\leq C2^{-N}\left\|\left\{\sum_{j'',k''\in\mathbb{Z}}\left\{\mathcal{M}_{\mathfrak{z}}\left(\sum_{R''\in\mathcal{R}_{\mathfrak{z}}^{j'',k''}}|(\psi_{j'',k''}*f)(x_{R''})|\chi_{R''}\right)^\delta\right\}^{2/\delta}\right\}^{1/2}\right\|_{L_w^p(\mathbb{R}^3)}.$$

Using Lemma 3.2, we obtain the $L_w^{p/\delta}(\ell^{2/\delta})$ boundedness of $\mathcal{M}_{\mathfrak{z}}$ and hence

$$\|R_N^1(f)\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)} \leq C2^{-N}\left\|\left\{\sum_{j'',k''\in\mathbb{Z}}\sum_{R''\in\mathcal{R}_{\mathfrak{z}}^{j'',k''}}|(\psi_{j'',k''}*f)(x_{R''})|^2\chi_{R''}\right\}^{1/2}\right\|_{L_w^p(\mathbb{R}^3)}$$

$$\approx 2^{-N}\|f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)}.$$

Therefore, the claim (5.5) is concluded.

For $1 < p < \infty$ and $w \in A_p^{\mathfrak{z}}$, we assume $f \in L^2 \cap L_w^p$. Arguing as above, we see that the desired conclusions follows if we can show

(5.5) $$\|R_N^i(f)\|_{L_w^p} \leq C2^{-N}\|f\|_{L_w^p},$$

where C is a constant independent of f and N. We only give the proof for R_N^1 as R_N^2 can

be handled in the same manner. By continuous Calderón's reproducing formula,

$$
\begin{aligned}
&|[\psi_{j',k'} * R_N^1(f)](x)| \\
&= \left| \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_\mathfrak{z}^{j-N,k-N}} \int_R \psi_{j',k'} * [\phi_{j,k}(\cdot-x') - \phi_{j,k}(\cdot-\vartheta_R)](x)(\phi_{j,k}*f)(x')\,dx' \right| \\
&= \left| \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_\mathfrak{z}^{j-N,k-N}} \sum_{j'',k''\in\mathbb{Z}} \int_R (\psi_{j',k'}*\widetilde{\phi}_{j,k})(x)(\phi_{j,k}*\psi_{j'',k''}*\psi_{j'',k''}*f)(x')\,dx' \right|.
\end{aligned}
$$

Arguing as in (5.4), we obtain

$$
\begin{aligned}
|[\psi_{j',k'} * R_N^1(f)](x)| &\le C2^{-N} \sum_{j'',k''\in\mathbb{Z}} 2^{-(|j'-j''|+|k'-k''|)M} \int_{\mathbb{R}^3} \frac{2^{M(j'\vee j'')}}{(2^{j'\vee j''}+|x_1-x_1''|)^{1+M}} \\
&\quad \times \frac{2^{M(k'\vee k'')}}{2^{\overline{j^*}}(2^{k'\vee k''}+|x_2-x_2''|+2^{-\overline{j^*}}|x_3-x_3''|)^{2+M}}|(\psi_{j'',k''}*f)(x'')|\,dx'' \\
&\le C2^{-N} \sum_{j'',k''\in\mathbb{Z}} 2^{-(|j'-j''|+|k'-k''|)M} \mathcal{M}_\mathfrak{z}(\psi_{j'',k''}*f)(x),
\end{aligned}
$$

where in the last inequality we have used Lemma 3.3. Squaring both sides and applying Cauchy-Schwarz's inequality, we obtain

$$
|[\psi_{j',k'} * R_N^1(f)](x)|^2 \le C2^{-N} \sum_{j'',k''\in\mathbb{Z}} 2^{-(|j'-j''|+|k'-k''|)M}[\mathcal{M}_\mathfrak{z}(\psi_{j'',k''}*f)(x)]^2.
$$

It follows from [14, Theorem 2.9] with $w \in A_p^\mathfrak{z}(\mathbb{R}^3)$, Rubio de Francia's extrapolation theorem, and weighted Fefferman-Stein's vector-valued inequality that

$$
\begin{aligned}
\|R_N^1(f)\|_{L_w^p} \approx \|\overline{g}_\mathfrak{z}(R_N^1(f))\|_{L_w^p} &= \left\| \left(\sum_{j',k'\in\mathbb{Z}} |\psi_{j',k'}*R_N^1(f)|^2 \right)^{1/2} \right\|_{L_w^p} \\
&\le C2^{-N} \left\| \left(\sum_{j'',k''\in\mathbb{Z}} [\mathcal{M}_\mathfrak{z}(\psi_{j'',k''}*f)]^2 \right)^{1/2} \right\|_{L_w^p} \\
&\le C2^{-N} \left\| \left(\sum_{j'',k''\in\mathbb{Z}} |\psi_{j'',k''}*f|^2 \right)^{1/2} \right\|_{L_w^p} \\
&= 2^{-N} \|\overline{g}_\mathfrak{z}(f)\|_{L_w^p} \approx 2^{-N}\|f\|_{L_w^p},
\end{aligned}
$$

where $\overline{g}_\mathfrak{z}(f)(x) := \left(\sum_{j,k\in\mathbb{Z}} |(\psi_{j,k}*f)(x)|^2 \right)^{1/2}$. This verifies (5.5) and hence Theorem 5.1 follows. $\qquad\square$

We would like to point out that the difference between the discrete Calderón reproducing formula in Theorem 1.1 and the discrete Calderón-type identity in Theorem 5.1 is

that the functions $\psi_{j,k}$ in Theorem 1.1 have non-compact supports and all moments vanish, while the functions $\phi_{j,k}$ in Theorem 5.1 have compact supports but only have finite number of vanishing moments. The crucial fact is that the points x_R in Theorem 1.1 are fixed, namely the left lower corner points, and, however, the points ϑ_R in Theorem 5.1 are any fixed points. Note that in Theorem 1.1, the sum runs over all Zygmund rectangles $\mathcal{R}_{\mathfrak{z}}^{j,k}$, while in Theorem 5.1 the sum runs over Zygmund rectangles at the smaller scale $\mathcal{R}_{\mathfrak{z}}^{j-N,k-N}$. These two facts, compact supports for functions $\phi_{j,k}$ and the arbitrary fixed points ϑ_R, are very important to prove Proposition 5.3 below.

Lemma 5.2. (i) *If $1 < p < \infty$, then $w \in A_p^{\mathfrak{z}}$ if and only if $w^{-1/(p-1)} \in A_{p'}^{\mathfrak{z}}$.*

(ii) *If $1 \leq p < q < \infty$, then $A_p^{\mathfrak{z}} \subset A_q^{\mathfrak{z}}$.*

(iii) *If $1 < p < \infty$ and $w \in A_p^{\mathfrak{z}}$, then $w \in A_{p-\epsilon}^{\mathfrak{z}}$ for some $\epsilon > 0$.*

The conclusions (i) and (ii) in Lemma 5.2 are similar to the classical case (see [19]), while (iii) was proved in [15, page 347].

We are now ready to prove the following result, which, combined with Theorem 1.4, will imply Theorem 5.1.

Proposition 5.3. *Suppose $0 < p \leq 1$ and $w \in A_{\infty}^{\mathfrak{z}}(\mathbb{R}^3)$. If $f \in H_{\mathfrak{z},w}^p(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, then $f \in L_w^p(\mathbb{R}^3)$ and there is a constant $C_p > 0$ independent of the $L^2(\mathbb{R}^3)$ norm of f such that*

$$\|f\|_{L_w^p(\mathbb{R}^3)} \leq C_p \|f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)}.$$

Proof. By Lemma 5.2(ii), we may assume $w \in A_q^{\mathfrak{z}}(\mathbb{R}^3)$ for some $q \in [2,\infty)$. Let

$$\widetilde{g}_{\mathfrak{z}}(f)(x) := \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} |(\phi_{j,k} * h)(x_R)|^2 \chi_R(x) \right\}^{1/2},$$

where $\{\phi_{j,k}\}$, h and N are the same as in Theorem 5.1 and x_R denotes the left lower corner of R. Applying Theorem 5.1 and repeating the same argument as in the proof of Theorem 1.4, we get

$$\|\widetilde{g}_{\mathfrak{z}}(f)\|_{L_w^p(\mathbb{R}^3)} \approx \|h\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)} \approx \|f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)}$$

for $f \in L^2(\mathbb{R}^3) \cap H_{\mathfrak{z},w}^p(\mathbb{R}^3)$.

Given $f \in L^2(\mathbb{R}^3) \cap H_{\mathfrak{z},w}^p(\mathbb{R}^3)$, applying the discrete Calderón-type identity in Theorem 5.1 with $\vartheta_R = x_R$, we can write

$$f = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} |R| \phi_{j,k}(\cdot - x_R)(\phi_{j,k} * h)(x_R) \quad \text{for } f \in L^2(\mathbb{R}^3) \cap H_{\mathfrak{z},w}^p(\mathbb{R}^3).$$

We now apply the method for atomic decomposition. For this purpose, set

$$\Omega_i = \{x \in \mathbb{R}^3 : \widetilde{g}_{\mathfrak{z}}(f)(x) > 2^i\}$$

and

$$B_i = \left\{(j, k, R) : j, k \in \mathbb{Z}, R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}, |R \cap \Omega_i| > \frac{1}{2}|R|, |R \cap \Omega_{i+1}| \le \frac{1}{2}|R|\right\}.$$

We rewrite

$$f(x) = \sum_{i \in \mathbb{Z}} \sum_{(j,k,R) \in B_i} |R|\phi_{j,k}(x - x_R)(\phi_{j,k} * h)(x_R),$$

where the series converges for almost every $x \in \mathbb{R}^3$. We claim that

$$(5.6) \qquad \left\| \sum_{(j,k,R) \in B_i} |R|\phi_{j,k}(\,\cdot - x_R)(\phi_{j,k} * h)(x_R) \right\|_{L_w^p(\mathbb{R}^3)}^p \le C 2^{pi} w(\Omega_i).$$

Since $0 < p \le 1$, the above claim and the definition of Ω_i yield

$$\|f\|_{L_w^p(\mathbb{R}^3)}^p \le \sum_{i \in \mathbb{Z}} \left\| \sum_{(j,k,R) \in B_i} |R|\widetilde{\phi}_{j,k}(\,\cdot - x_R)(\phi_{j,k} * h)(x_R) \right\|_{L_w^p(\mathbb{R}^3)}^p$$

$$\le C \sum_{i \in \mathbb{Z}} 2^{pi} w(\Omega_i) \le C \|\widetilde{g}_{\mathfrak{z}}(f)\|_{L_w^p(\mathbb{R}^3)}^p \approx \|f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)}^p$$

and Proposition 5.3 follows.

To show claim (5.6), we note that if $(j, k, R) \in B_i$, then $R \subset \widetilde{\Omega}_i := \{x : \mathcal{M}_{\mathfrak{z}}(\chi_{\Omega_i})(x) > 1/2\}$ and hence, the function $\phi_{j,k}(\,\cdot - x_R)$ is supported on $\widetilde{\Omega}_i := \{x : \mathcal{M}_{\mathfrak{z}}(\chi_{\Omega_i})(x) > 1/100\}$. By Hölder's inequality, we obtain that

$$(5.7)$$

$$\left\| \sum_{(j,k,R) \in B_i} |R|\phi_{j,k}(\,\cdot - x_R)(\phi_{j,k} * h)(x_R) \right\|_{L_w^p(\mathbb{R}^3)}^p$$

$$\le C w(\widetilde{\Omega}_i)^{1-(p/q)} \left\| \sum_{(j,k,R) \in B_i} |R|\phi_{j,k}(\,\cdot - x_R)(\phi_{j,k} * h)(x_R) \right\|_{L_w^q(\mathbb{R}^3)}^p.$$

We now estimate the last L_w^q-norm by the duality argument. For $\zeta \in L_{w^{1-q'}}^{q'}(\mathbb{R}^3)$ with $\|\zeta\|_{L_{w^{1-q'}}^{q'}(\mathbb{R}^3)} \le 1$,

$$\left| \left\langle \sum_{(j,k,R) \in B_i} |R|\phi_{j,k}(\,\cdot - x_R)(\phi_{j,k} * h)(x_R), \zeta \right\rangle \right|$$

$$= \left| \sum_{(j,k,R) \in B_i} \int (\overline{\phi}_{j,k} * \zeta)(x_R)(\phi_{j,k} * h)(x_R) \chi_R(x)\, dx \right|$$

$$\le \left\| \left\{ \sum_{(j,k,R) \in B_i} |(\phi_{j,k} * h)(x_R)|^2 \chi_R \right\}^{1/2} \right\|_{L_w^q(\mathbb{R}^3)} \left\| \left\{ \sum_{(j,k,R) \in B_i} |(\overline{\phi}_{j,k} * \zeta)(x_R)|^2 \chi_R \right\}^{1/2} \right\|_{L_{w^{1-q'}}^{q'}(\mathbb{R}^3)}$$

$$:= I_1 \times I_2,$$

where $\overline{\phi}_{j,k}(x) = \phi_{j,k}(-x)$.

We first estimate I_2. Applying [15, Theorem 2.9] with $w \in A_p^{\mathfrak{z}}(\mathbb{R}^3)$ and Rubio de Francia's extrapolation theorem, we get that if $1 < p < \infty$ and $w \in A_p^{\mathfrak{z}}(\mathbb{R}^3)$, then there exist constant $c, C > 0$ (depending only on $\|w\|_{A_p^{\mathfrak{z}}}$) so that

$$c\|f\|_{L_w^p(\mathbb{R}^3)} \leq \|g_{\mathfrak{z}}^c(f)\|_{L_w^p(\mathbb{R}^3)} \leq C\|f\|_{L_w^p(\mathbb{R}^3)}$$

for all $f \in L_w^p(\mathbb{R}^3)$.

Since $w \in A_q^{\mathfrak{z}}(\mathbb{R}^3)$ implies $w^{1-q'} \in A_{q'}^{\mathfrak{z}}(\mathbb{R}^3)$, applying the above estimates yields

$$(5.8) \quad I_2 \leq C\left\|\left\{\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_{\mathfrak{z}}^{j-N,k-N}}|(\overline{\phi}_{j,k}*\zeta)(x_R)|^2\chi_R\right\}^{1/2}\right\|_{L_{w^{1-q'}}^{q'}(\mathbb{R}^3)} \approx \|\zeta\|_{L_{w^{1-q'}}^{q'}(\mathbb{R}^3)} \leq 1.$$

As for I_1, note that $\Omega_i \subset \widetilde{\Omega}_i$ and $w(\widetilde{\Omega}_i) \leq Cw(\Omega_i)$ due to the $L_w^q(\mathbb{R}^3)$ boundedness of $\mathcal{M}_{\mathfrak{z}}$. For any $(j,k,R) \in B_i$ and $x \in R$, then $\mathcal{M}_{\mathfrak{z}}(\chi_{R\cap\widetilde{\Omega}_i\setminus\Omega_{i+1}})(x) > 1/2$. Applying Lemma 3.2 again, we have

$$(5.9) \quad \begin{aligned}
I_1^q &= \left\|\left\{\sum_{(j,k,R)\in B_i}|(\phi_{j,k}*h)(x_R)|^2\chi_R\right\}^{1/2}\right\|_{L_w^q(\mathbb{R}^3)}^q \\
&= \int_{\mathbb{R}^3}\left\{\sum_{(j,k,R)\in B_i}|(\phi_{j,k}*h)(x_R)|^2\chi_R(x)\right\}^{q/2}w(x)\,dx \\
&\leq C\int_{\mathbb{R}^3}\left\{\sum_{(j,k,R)\in B_i}|(\phi_{j,k}*h)(x_R)\mathcal{M}_{\mathfrak{z}}(\chi_{R\cap\widetilde{\Omega}_i\setminus\Omega_{i+1}})(x)|^2\right\}^{q/2}w(x)\,dx \\
&\leq C\int_{\mathbb{R}^3}\left\{\sum_{(j,k,R)\in B_i}|(\phi_{j,k}*h)(x_R)|^2\chi_{R\cap\widetilde{\Omega}_i\setminus\Omega_{i+1}}(x)\right\}^{q/2}w(x)\,dx \\
&\leq C\int_{\widetilde{\Omega}_i\setminus\Omega_{i+1}}\left\{\sum_{(j,k,R)\in B_i}|(\phi_{j,k}*h)(x_R)|^2\chi_R(x)\right\}^{q/2}w(x)\,dx \\
&\leq C2^{iq}w(\widetilde{\Omega}_i) \leq C2^{iq}w(\Omega_i).
\end{aligned}$$

Combining both estimates (5.8) and (5.9), we obtain

$$\left\|\sum_{(j,k,R)\in B_i}|R|\widetilde{\phi}_{j,k}(\cdot - x_R)(\phi_{j,k}*h)(x_R)\right\|_{L_w^q(\mathbb{R}^3)} \leq C2^{iq}w(\Omega_i).$$

Plugging this estimate into (5.7) yields claim (5.6), and hence Proposition 5.3 is concluded.

$\square$

We now show Theorem 1.6.

Proof of Theorem 1.6. If $f \in L^2(\mathbb{R}^3) \cap H^p_{\mathfrak{z},w}(\mathbb{R}^3)$, then $Tf \in L^2(\mathbb{R}^3) \cap H^p_{\mathfrak{z},w}(\mathbb{R}^3)$. Thus, by Proposition 5.3 and Theorem 1.5, we have

$$\|Tf\|_{L^p_w} \leq C\|Tf\|_{H^p_{\mathfrak{z},w}} \leq C\|f\|_{H^p_{\mathfrak{z},w}}$$

for $q_w \cdot \frac{2}{2 + \frac{1}{2}\min(\theta_1, \theta_2)} < p \leq 1$.

Since $L^2(\mathbb{R}^3) \cap H^p_{\mathfrak{z},w}(\mathbb{R}^3)$ is dense in $H^p_{\mathfrak{z},w}(\mathbb{R}^3)$, T extends to a bounded operator from $H^p_{\mathfrak{z},w}(\mathbb{R}^3)$ to $L^p_{\mathfrak{z},w}(\mathbb{R}^3)$ and hence the proof of Theorem 1.6 is complete. $\qquad\square$

As a consequence of Theorems 1.5 and 1.6, we obtain the following endpoint estimates for the operators studied by Ricci-Stein [31] and R. Fefferman-Pipher [15].

Corollary 5.4. *Suppose that the function ϕ is supported in an unit cube in $\mathbb{R}^3$ and satisfies a certain amount of uniform smoothness with the minimum cancelation conditions*

$$\int_{\mathbb{R}^2} \phi(x_1, x_2, x_3)\, dx = \int_{\mathbb{R}^2} \phi(x_1, x_2, x_3)\, dydz = 0.$$

Then the singular integral operator $T_{\mathfrak{z}}f = f * \mathcal{K}$ is bounded on $H^p_{\mathfrak{z},w}(\mathbb{R}^3)$ and bounded from $H^p_{\mathfrak{z},w}(\mathbb{R}^3)$ to L^p_w for $q_w \cdot \frac{2}{2 + \frac{1}{2}\min(\theta_1, \theta_2)} < p \leq 1$, where

$$\mathcal{K}(x_1, x_2, x_3) = \sum_{j,k \in \mathbb{Z}} 2^{-2(j+k)} \phi(2^{-j}x, 2^{-k}y, 2^{-(j+k)}z).$$

6. Calderón-Zygmund decompositions and interpolations

We first provide the

Proof of Theorem 1.7. According to Corollary 3.4, $L^2(\mathbb{R}^3) \cap H^p_{\mathfrak{z},w}(\mathbb{R}^3)$ is dense in $H^p_{\mathfrak{z},w}(\mathbb{R}^3)$. Thus it suffices to prove Theorem 1.7 for $f \in L^2(\mathbb{R}^3) \cap H^p_{\mathfrak{z},w}(\mathbb{R}^3)$. Given any fixed $\alpha > 0$, let

$$\Omega_l = \{x \in \mathbb{R}^3 : \widetilde{g}_{\mathfrak{z}}(f)(x) > \alpha 2^l\}, \quad l \in \mathbb{Z},$$

where

$$\widetilde{g}_{\mathfrak{z}}(f)(x) := \left\{ \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}^{j-N,k-N}_{\mathfrak{z}}} |(\phi_{j,k} * h)(x_R)|^2 \chi_R(x) \right\}^{1/2}.$$

For $j, k \in \mathbb{Z}$, let

$$\mathcal{R}^{j-N,k-N}_0 = \left\{ R \in \mathcal{R}^{j-N,k-N}_{\mathfrak{z}} : |R \cap \Omega_0| < \frac{1}{2}|R| \right\}$$

and

$$\mathcal{R}^{j-N,k-N}_l = \left\{ R \in \mathcal{R}^{j-N,k-N}_{\mathfrak{z}} : |R \cap \Omega_{l-1}| \geq \frac{1}{2}|R|, |R \cap \Omega_l| < \frac{1}{2}|R| \right\} \quad \text{for } l \geq 1,$$

where N is given in Theorem 5.1. It also follows from Theorem 5.1 (with $\vartheta_R = x_R$) that there exists $h \in L^2(\mathbb{R}^3) \cap H^p_{\mathfrak{z},w}(\mathbb{R}^3)$ such that

$$f(x) = \sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_0^{j-N,k-N}} |R|(\phi_{j,k} * h)(x_R)\phi_{j,k}(x - x_R)$$

$$+ \sum_{j,k \in \mathbb{Z}} \sum_{l \geq 1} \sum_{R \in \mathcal{R}_l^{j-N,k-N}} |R|(\phi_{j,k} * h)(x_R)\phi_{j,k}(x - x_R)$$

$$:= g(x) + b(x).$$

We first estimate $\|g\|_{H^{p_2}_{\mathfrak{z},w}}$ for $p_2 \in (0,\infty)$. Repeating the same argument as in the proof of Theorem 1.4, we deduce that for $3/(3+M) < \delta < (\{p_2/q_w\} \wedge 1)$,

$$\sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j,k}} |(\psi_{j,k} * g)(x_R)|^2 \chi_R(x)$$

$$\leq C \sum_{j',k' \in \mathbb{Z}} \left\{ \mathcal{M}_{\mathfrak{z}}\left[\left(\sum_{R' \in \mathcal{R}_0^{j'-N,k'-N}} |(\phi_{j',k'} * g)(x_{R'})|^2 \chi_{R'}\right)^{\delta/2}\right](x)\right\}^{2/\delta}.$$

Take the square root on both sides and apply Lemma 3.2 on $L_w^{p_2/\delta}(\ell^{2/\delta})$ (note that $w \in A_{p_2/\delta}^{\mathfrak{z}}$) to derive

$$\|g\|_{H^{p_2}_{\mathfrak{z},w}(\mathbb{R}^3)} = \left\|\left\{\sum_{j,k \in \mathbb{Z}} \sum_{R \in \mathcal{R}_{\mathfrak{z}}^{j,k}} |(\psi_{j,k} * g)(x_R)|^2 \chi_R\right\}^{1/2}\right\|_{L_w^{p_2}(\mathbb{R}^3)}$$

$$\leq C\left\|\left\{\sum_{j',k' \in \mathbb{Z}} \sum_{R' \in \mathcal{R}_0^{j'-N,k'-N}} |(\phi_{j',k'} * h)(x_{R'})|^2 \chi_{R'}\right\}^{1/2}\right\|_{L_w^{p_2}(\mathbb{R}^3)}.$$

We claim

(6.1)
$$\int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x) \leq \alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_2} w(x)\,dx$$

$$\gtrsim \left\|\left\{\sum_{j',k' \in \mathbb{Z}} \sum_{R \in \mathcal{R}_0^{j'-N,k'-N}} |(\phi_{j',k'} * h)(x_{R'})|^2 \chi_{R'}\right\}^{1/2}\right\|_{L_w^{p_2}(\mathbb{R}^3)}^{p_2},$$

which implies

$$\|g\|_{H^{p_2}_{\mathfrak{z},w}(\mathbb{R}^3)}^{p_2} \leq C \int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x) \leq \alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_2} w(x)\,dx$$

$$\leq \alpha^{p_2-p} \int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x) \leq \alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^p w(x)\,dx$$

$$\leq C\alpha^{p_2-p}\|f\|_{H^p_{\mathfrak{z},w}(\mathbb{R}^3)}^p.$$

To verify (6.1), we choose $\delta < \min\{p_2/q_w, 1\}$ and get

$$\int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x)\leq\alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_2} w(x)\, dx$$

$$= \int_{\complement\Omega_0} \left\{ \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} |(\phi_{j,k}*h)(x_R)|^2 \chi_R(x) \right\}^{p_2/2} w(x)\, dx$$

$$\gtrsim \int_{\mathbb{R}^3} \left\{ \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} |(\phi_{j,k}*h)(x_R)|^2 \chi_{R\cap\complement\Omega_0}(x) \right\}^{p_2/2} w(x)\, dx$$

$$\gtrsim \int_{\mathbb{R}^3} \left\{ \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} (|(\phi_{j,k}*h)(x_R)|^\delta \mathcal{M}_{\mathfrak{z}}(\chi_{R\cap\complement\Omega_0})(x))^{2/\delta} \right\}^{p_2/2} w(x)\, dx$$

$$\gtrsim \left\| \left\{ \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_{\mathfrak{z}}^{j-N,k-N}} |(\phi_{j,k}*h)(x_R)|^2 \chi_R \right\}^{1/2} \right\|_{L_w^{p_2}(\mathbb{R}^3)}^{p_2},$$

where in the last inequality we have used the estimate

$$\chi_R(x) \leq 2^{1/\delta} \mathcal{M}_{\mathfrak{z}}(\chi_{R\cap\complement\Omega_0})^{1/\delta}(x)$$

for $R \in \mathcal{R}_{\mathfrak{z}}^{j-N,k-N}$, and the second inequality follows from Lemma 3.2 with $q = 2/\delta$ and $p = p_2/\delta$.

Now, we turn to the estimate for $H_{\mathfrak{z},w}^{p_1}(\mathbb{R}^3)$ norm of b. Set

$$\widetilde{\Omega}_l = \{x \in \mathbb{R}^3 : \mathcal{M}_{\mathfrak{z}}(\chi_{\Omega_l}) > 1/2\}, \quad l \in \mathbb{Z}.$$

Then the desired estimate follows from

$$(6.2) \quad \left\| \sum_{j,k\in\mathbb{Z}} \sum_{R\in\mathcal{R}_l^{j-N,k-N}} |R|(\phi_{j,k}*h)(x_R)\phi_{j,k}(\cdot - x_R) \right\|_{H_{\mathfrak{z},w}^{p_1}(\mathbb{R}^3)}^{p_1} \leq C(2^l\alpha)^{p_1} w(\widetilde{\Omega}_{l-1})$$

for any $0 < p_1 \leq 1$ and $l \geq 1$. Indeed, by the $L_w^q(\mathbb{R}^3)$, $q > q_w$, boundedness of $\mathcal{M}_{\mathfrak{z}}$,

$$w(\widetilde{\Omega}_{l-1}) \leq C \int_{\mathbb{R}^3} [\mathcal{M}_{\mathfrak{z}}(\chi_{\Omega_{l-1}})(x)]^q w(x)\, dx \leq C w(\Omega_{l-1}).$$

This fact together with (6.2) yields

$$\|b\|_{H_{\mathfrak{z},w}^{p_1}(\mathbb{R}^3)}^{p_1} \leq C \sum_{l\geq 1} (2^l\alpha)^{p_1} w(\widetilde{\Omega}_{l-1}) \leq C \sum_{l\geq 1} (2^l\alpha)^{p_1} w(\Omega_{l-1})$$

$$\leq C \int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x)>\alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_1} w(x)\, dx$$

$$\leq C\alpha^{p_1-p} \int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x)>\alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^p w(x)\, dx$$

$$\leq C\alpha^{p_1-p} \|f\|_{H_{\mathfrak{z},w}^p(\mathbb{R}^3)}^p.$$

Thus to finish the proof, it remains to establish (6.2). Following the same argument as in the estimation of

$$\left\|\left\{\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_{\mathfrak{z}}^{j-N,k-N}}|(\psi_{j,k}*g)(x_R)|^2\chi_R\right\}^{1/2}\right\|_{L_w^{p_1}(\mathbb{R}^3)},$$

we get

$$\left\|\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_l^{j-N,k-N}}|R|(\phi_{j,k}*h)(x_R)\phi_{j,k}(\,\cdot-x_R)\right\|_{H_{\mathfrak{z},w}^{p_1}(\mathbb{R}^3)}$$

$$(6.3)\qquad \approx\left\|\left\{\sum_{j',k'\in\mathbb{Z}}\sum_{R'\in\mathcal{R}_{\mathfrak{z}}^{j'-N,k'-N}}\left|\sum_{j,k\in\mathbb{Z}}\right.\right.\right.$$

$$\left.\left.\left.\times\sum_{R\in\mathcal{R}_l^{j-N,k-N}}|R|(\phi_{j,k}*h)(x_R)(\psi_{j',k'}*\phi_{j,k})(x_{R'}-x_R)\right|^2\chi_{R'}\right\}^{1/2}\right\|_{L_w^{p_1}(\mathbb{R}^3)}$$

$$\leq C\left\|\left\{\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_l^{j-N,k-N}}|(\phi_{j,k}*h)(x_R)|^2\chi_R\right\}^{1/2}\right\|_{L_w^{p_1}(\mathbb{R}^3)}.$$

Note that $R\subset\widetilde{\Omega}_{l-1}$ for $R\in\mathcal{R}_l^{j-N,k-N}$. Thus, $|R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)|>\frac{1}{2}|R|$ which implies

$$\chi_R(x)\leq 2^{1/\delta}\mathcal{M}_{\mathfrak{z}}(\chi_{R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)})^{1/\delta}(x).$$

As in the proof of claim (6.1), choosing $0<\delta<\min\{2,p_1/q_w\}$ and applying Lemma 3.2, we have

$$(2^l\alpha)^{p_1}w(\widetilde{\Omega}_{l-1})\geq\int_{\widetilde{\Omega}_{l-1}\setminus\Omega_l}[\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_1}w(x)\,dx$$

$$=\int_{\mathbb{R}^3}\left\{\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_{\mathfrak{z}}^{j-N,k-N}}|(\phi_{j,k}*h)(x_R)|^2\chi_{R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)}(x)\right\}^{p_1/2}w(x)\,dx$$

$$\gtrsim\int_{\mathbb{R}^3}\left\{\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_l^{j-N,k-N}}|(\phi_{j,k}*h)(x_R)|^2\mathcal{M}_{\mathfrak{z}}(\chi_{R\cap(\widetilde{\Omega}_{l-1}\setminus\Omega_l)})^{2/\delta}(x)\right\}^{p_1/2}w(x)\,dx$$

$$\gtrsim\left\|\left\{\sum_{j,k\in\mathbb{Z}}\sum_{R\in\mathcal{R}_l^{j-N,k-N}}|(\phi_{j,k}*h)(x_R)|^2\chi_R\right\}^{1/2}\right\|_{L_w^{p_1}(\mathbb{R}^3)}^{p_1}.$$

Combining this with (6.3) yields (6.2), and hence Theorem 1.7 follows. $\qquad\square$

Finally, let us give the

Proof of Theorem 1.8. Let $w\in A_\infty^{\mathfrak{z}}(\mathbb{R}^3)$. In view of Corollary 3.4, we may assume $f\in L^2(\mathbb{R}^3)\cap H_{\mathfrak{z},w}^p(\mathbb{R}^3)$. Suppose that T is bounded from $H_{\mathfrak{z},w}^{p_1}(\mathbb{R}^3)$ to $L_w^{p_1}(\mathbb{R}^3)$ and bounded from $H_{\mathfrak{z},w}^{p_2}(\mathbb{R}^3)$ to $L_w^{p_2}(\mathbb{R}^3)$. The Calderón-Zygmund decomposition shows that $f=g+b$

with $\|g\|_{H^{p_2}_{\mathfrak{z},w}(\mathbb{R}^3)}^{p_2} \leq C\alpha^{p_2-p}\|f\|_{H^p_{\mathfrak{z},w}(\mathbb{R}^3)}^p$ and $\|b\|_{H^{p_1}_{\mathfrak{z},w}(\mathbb{R}^3)}^{p_1} \leq C\alpha^{p_1-p}\|f\|_{H^p_{\mathfrak{z},w}(\mathbb{R}^3)}^p$. Moreover, in the proof of Theorem 1.7, we have obtained

$$\|g\|_{H^{p_2}_{\mathfrak{z},w}(\mathbb{R}^3)}^{p_2} \leq C \int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x)\leq\alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_2} w(x)\,dx$$

and

$$\|b\|_{H^{p_1}_{\mathfrak{z},w}(\mathbb{R}^3)}^{p_1} \leq C \int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x)>\alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_1} w(x)\,dx.$$

Therefore,

$$\begin{aligned}
\|Tf\|_{L^p_w(\mathbb{R}^3)}^p &\leq p \int_0^\infty \alpha^{p-1} w(\{x : |T(g)(x)| > \alpha/2\})\,d\alpha \\
&\quad + p \int_0^\infty \alpha^{p-1} w(\{x : |T(b)(x)| > \alpha/2\})\,d\alpha \\
&\leq C \int_0^\infty \alpha^{p-1} \left(\frac{\|T(g)\|_{L^{p_2}_w}}{\alpha}\right)^{p_2} d\alpha + \int_0^\infty \alpha^{p-1} \left(\frac{\|T(b)\|_{L^{p_1}_w}}{\alpha}\right)^{p_1} d\alpha \\
&\leq C \int_0^\infty \alpha^{p-1} \int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x)\leq\alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_1} w(x)\,dx d\alpha \\
&\quad + \int_0^\infty \alpha^{p-1} \int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x)>\alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_2} w(x)\,dx d\alpha \\
&\leq C\|f\|_{H^p_{\mathfrak{z},w}(\mathbb{R}^3)}^p.
\end{aligned}$$

Thus, $\|Tf\|_{L^p_w(\mathbb{R}^3)} \leq C\|f\|_{H^p_{\mathfrak{z},w}(\mathbb{R}^3)}$ for any $p \in (p_2, p_1)$. Hence T is bounded from $H^p_{\mathfrak{z},w}(\mathbb{R}^3)$ to $L^p_w(\mathbb{R}^3)$.

To prove the second assertion that T is bounded on $H^p_{\mathfrak{z},w}(\mathbb{R}^3)$ for $p \in (p_2, p_1)$, for any given $\alpha > 0$ and $f \in H^p_{\mathfrak{z},w}(\mathbb{R}^3)$, we apply the Calderón-Zygmund decomposition again to obtain

$$\begin{aligned}
&w(\{x : |\widetilde{g}_{\mathfrak{z}}(Tf)(x)| > \alpha\}) \\
&\leq w(\{x : |\widetilde{g}_{\mathfrak{z}}(Tg)(x)| > \alpha/2\}) + w(\{x : |\widetilde{g}_{\mathfrak{z}}(Tb)(x)| > \alpha/2\}) \\
&\leq C\alpha^{-p_2}\|T(g)\|_{H^{p_2}_{\mathfrak{z},w}(\mathbb{R}^3)}^{p_2} + \alpha^{-p_1}\|T(b)\|_{H^{p_1}_{\mathfrak{z},w}(\mathbb{R}^3)}^{p_1} \\
&\leq C\alpha^{-p_2}\|g\|_{H^{p_2}_{\mathfrak{z},w}(\mathbb{R}^3)}^{p_2} + \alpha^{-p_1}\|b\|_{H^{p_1}_{\mathfrak{z},w}(\mathbb{R}^3)}^{p_1} \\
&\leq \alpha^{-p_2} \int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x)\leq\alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_2} w(x)\,dx + \alpha^{-p_1} \int_{\{\widetilde{g}_{\mathfrak{z}}(f)(x)>\alpha\}} [\widetilde{g}_{\mathfrak{z}}(f)(x)]^{p_1} w(x)\,dx,
\end{aligned}$$

which, as above, shows that

$$\|Tf\|_{H^p_{\mathfrak{z},w}(\mathbb{R}^3)} \leq C\|\widetilde{g}_{\mathfrak{z}}(Tf)\|_{L^p_w(\mathbb{R}^3)} \leq C\|f\|_{H^p_{\mathfrak{z},w}(\mathbb{R}^3)}$$

for any $p \in (p_2, p_1)$. The proof of Theorem 1.8 is complete. $\qquad\square$

Acknowledgments

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