

General Decay Rates for a Laminated Beam with Memory

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Abstract. In previous work [23], Mustafa considered a viscoelastic laminated beam system with structural damping in the case of equal-speed wave propagations, and established explicit energy decay formula which gives the best decay rates. In this paper, we continue to consider the similar problems and establish the general decay result for the energy, to system with structural damping in the case of non-equal wave speeds and to system without structural damping in the case of equal wave speeds, respectively. For the first case, we use the second-order energy method to overcome the difficulty of estimating the non-equal speeds term. For the second case, we construct an appropriated perturbed functional to estimate $\|w_t\|_2^2$ so as to overcome the absence of structural damping.

1. Introduction

In previous work [23], Mustafa considered the following viscoelastic laminated beam system with structural damping

$$(1.1) \quad \begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x = 0 & (x, t) \in (0, 1) \times (0, \infty), \\ I_\rho(3w - \psi)_{tt} - G(\psi - \varphi_x) - D(3w - \psi)_{xx} \\ \quad + \int_0^t g(t-s)(3w - \psi)_{xx}(s) ds = 0 & (x, t) \in (0, 1) \times (0, \infty), \\ I_\rho w_{tt} - D w_{xx} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t = 0 & (x, t) \in (0, 1) \times (0, \infty) \end{cases}$$

under initial conditions

$$(1.2) \quad \begin{cases} \varphi(x, 0) = \varphi_0(x), & \psi(x, 0) = \psi_0(x), & w(x, 0) = w_0(x) & x \in (0, 1), \\ \varphi_t(x, 0) = \varphi_1(x), & \psi_t(x, 0) = \psi_1(x), & w_t(x, 0) = w_1(x) & x \in (0, 1) \end{cases}$$

and boundary conditions

$$(1.3) \quad \varphi_x(0, t) = \varphi_x(1, t) = \psi(0, t) = \psi_x(1, t) = w(0, t) = w_x(1, t) = 0, \quad t \in (0, \infty),$$

where φ denotes the transverse displacement of the beam which departs from its equilibrium position, ψ denotes the rotation angle, w is proportional to the amount of slip along

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the interface at time t and longitudinal spatial variable x , $3w - \psi$ represents the effective rotation angle. Moreover, the third equation of (1.1) describes the dynamics of the slip; ρ , G , I_ρ , D , γ , β are positive constant coefficients and denote the density of the beams, the shear stiffness, the mass moment of inertia, the flexural rigidity, the adhesive stiffness of the beams, and the adhesive damping parameter, respectively. If $\beta \neq 0$, the adhesion at the interface supplies a restoring force proportion to the interfacial slip. If $\beta = 0$, the third equation of (1.1) describes the coupled laminated beams without structural damping at the interface. In that paper, the author established the general decay result under the equal-speed wave propagation case: $G/\rho = D/I_\rho$. As for the previous results and developments of the viscoelastic laminated beam system, the authors have stated and summarized in great detail [23], thus we just omit it here. The readers, for a better understanding of present work, are recommended to see [1, 6, 10–13, 16, 20, 25–27, 29] and the references therein.

Apart from studying the laminated beam system itself, many people have been interested in the relationship between the laminated beam and the thermal conditions. For example, Liu and Zhao [18] considered the stabilization of a thermoelastic laminated beam with past history as

$$(1.4) \quad \begin{cases} \rho\varphi_{tt} + G(\psi - \varphi_x)_x + \theta_x = 0 & (x, t) \in (0, 1) \times (0, \infty), \\ I_\rho(3w - \psi)_{tt} - G(\psi - \varphi_x) - D(3w - \psi)_{xx} \\ \quad + \int_0^\infty g(s)(3w - \psi)_{xx}(x, t - s) ds - \theta = 0 & (x, t) \in (0, 1) \times (0, \infty), \\ I_\rho w_{tt} - Dw_{xx} + G(\psi - \varphi_x) + \frac{4}{3}\gamma w + \frac{4}{3}\beta w_t = 0 & (x, t) \in (0, 1) \times (0, \infty), \\ k\theta_t - \tau\theta_{xx} + \varphi_{xt} + (3w - \psi)_t = 0 & (x, t) \in (0, 1) \times (0, \infty), \end{cases}$$

in which, g is assumed to satisfy, for a positive constant ξ and $1 \leq p \leq 3/2$,

$$g'(t) \leq -\xi g^p(t), \quad \forall t \leq 0.$$

For system (1.4) with structural damping and without any restriction on the speeds of wave propagations, they obtained the exponential and polynomial stabilities. For system (1.4) without structural damping, they established the exponential and polynomial stabilities in case of equal speeds and lack of exponential stability in case of non-equal speeds. In the recent work [20], Apalara investigated a laminated beam with structural damping under Cattaneo's law of heat conduction, and proved the exponential and polynomial stability results depend on a stability number. However, the case of the absence of structural damping was left as an open problem. We refer the reader to [3–5, 7–9, 14, 15, 17, 19, 24, 28] for some other related results.

In this paper, we first investigate the general decay rate of the solutions for problem (1.1)–(1.3) with structural damping ($\beta \neq 0$) in the case of non-equal wave speeds:

$G/\rho \neq D/I_\rho$. For our this purpose, we use the second-order energy method to overcome the difficulty of estimating the non-equal speeds term. We then consider system (1.1)–(1.3) without structural damping ($\beta = 0$), and prove the general stability in the case of equal wave speeds $G/\rho = D/I_\rho$ by using the perturbed energy method. To overcome the absence of structural damping ($\beta = 0$), we shall construct an appropriated perturbed functional $J(t)$ (see Lemma 5.1 below) to estimate $\|w_t\|_2^2$.

This paper is organized as follows. In Section 2, we give some assumptions. In Section 3, we state and prove some technical lemmas that are useful in the entire work. In Section 4, we state and prove decay result for system (1.1)–(1.3) with structural damping in the case of non-equal wave speeds. Finally, Section 5 is devoted to proving the decay result for system (1.1)–(1.3) without structural damping in the case of equal wave speeds.

2. Preliminaries and main results

In this section, we give some assumptions and state some main theorems. We use $c > 0$ to denote a positive constant which does not depend on the initial data. First, we consider the following assumptions as in [23]:

(A1) $g: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a non-increasing differentiable function such that

$$g(0) > 0, \quad D - \int_0^{+\infty} g(s) \, ds = l > 0.$$

(A2) There exists an non-increasing differentiable function $\xi: \mathbb{R}^+ \rightarrow \mathbb{R}^+$ and a C^1 function $H: [0, \infty) \rightarrow [0, \infty)$ which is either linear or strictly increasing and strictly convex C^2 function on $(0, r]$, $r \leq g(0)$, with $H(0) = H'(0) = 0$, such that

$$g'(t) \leq -\xi(t)H(g(t)), \quad \forall t \geq 0.$$

Remark 2.1. [23, Remark 2.8] (1) From assumption (A1), we deduce that

$$g(t) \rightarrow 0 \text{ as } t \rightarrow +\infty \quad \text{and} \quad g(t) \leq \frac{D-l}{t}, \quad \forall t > 0.$$

Furthermore, from the assumption (A2), we obtain that there exists $t_0 > 0$ such that

$$g(t_0) = r \quad \text{and} \quad g(t) \leq r, \quad \forall t \geq t_0.$$

The non-increasing property of g gives

$$0 < g(t_0) \leq g(t) \leq g(0), \quad \forall t \in [0, t_0].$$

A combination of this with the continuity of H , for two constants $a, d > 0$, yields

$$a \leq H(g(t)) \leq d, \quad \forall t \in [0, t_0].$$

Consequently, for any $t \in [0, t_0]$, we get

$$g'(t) \leq -\xi(t)H(g(t)) \leq -a\xi(t) = -\frac{a}{g(0)}\xi(t)g(0) \leq -\frac{a}{g(0)}\xi(t)g(t)$$

and, hence,

$$(2.1) \quad \xi(t)g(t) \leq -\frac{g(0)}{a}g'(t), \quad \forall t \in [0, t_0].$$

(2) If H is a strictly increasing and strictly convex C^2 function on $(0, r]$, with $H(0) = H'(0) = 0$, then it has an extension \bar{H} , which is a strictly increasing and strictly convex C^2 function on $(0, \infty)$. For example, if $H(r) = A$, $H'(r) = B$, $H''(r) = C$, we can define \bar{H} , for any $t > r$, by

$$\bar{H}(t) = \frac{C}{2}t^2 + (B - Cr)t + \left(A + \frac{C}{2}r^2 - Br \right).$$

Next, we introduce the vector function

$$U = (\varphi, 3w - \psi, w, \varphi_t, 3w_t - \psi_t, w_t)^T$$

and

$$U_0 = (\varphi_0, 3w_0 - \psi_0, w_0, \varphi_1, 3w_1 - \psi_1, w_1)^T.$$

Then, we consider the following Hilbert spaces

$$\mathcal{H}_0 = \tilde{H}_*^1(0, 1) \times (H_*^1(0, 1))^2 \times (L^2(0, 1))^3$$

and

$$\begin{aligned} \mathcal{H}_1 = \{U \in \mathcal{H}_0 \mid \varphi \in \tilde{H}_*^2(0, 1), 3w - \psi, w \in H_*^2(0, 1), \varphi_t \in \tilde{H}_*^1(0, 1), \\ 3w_t - \psi_t, w_t \in H_*^1(0, 1), \varphi_x(0, t) = 0, \psi_x(1, t) = w_x(1, t) = 0\}, \end{aligned}$$

where

$$\begin{aligned} H_*^1(0, 1) &= \{\eta \mid \eta \in H^1(0, 1), \eta(0) = 0\}, & \tilde{H}_*^1(0, 1) &= \{\eta \mid \eta \in H^1(0, 1), \eta(1) = 0\}, \\ H_*^2(0, 1) &= H^2(0, 1) \cap H_*^1(0, 1), & \tilde{H}_*^2(0, 1) &= H^2(0, 1) \cap \tilde{H}_*^1(0, 1). \end{aligned}$$

For completeness, we state, without proof, the following global existence and regularity result which can be easily proved by the standard Galerkin method.

Theorem 2.2. [13] *Let $U_0 \in \mathcal{H}_0$ be given. Assume that g satisfies hypothesis (A1). Then problem (1.1)–(1.3) has a unique weak solution*

$$\begin{aligned} \varphi &\in C(\mathbb{R}^+; \tilde{H}_*^1(0, 1)) \cap C^1(\mathbb{R}^+; L^2(0, 1)), \\ 3w - \psi, w &\in C(\mathbb{R}^+; H_*^1(0, 1)) \cap C^1(\mathbb{R}^+; L^2(0, 1)). \end{aligned}$$

Moreover, if $U_0 \in \mathcal{H}_1$, then the solution satisfies

$$\begin{aligned} \varphi &\in C(\mathbb{R}^+; \tilde{H}_*^2(0, 1)) \cap C^1(\mathbb{R}^+; \tilde{H}_*^1(0, 1)) \cap C^2(\mathbb{R}^+; L^2(0, 1)), \\ 3w - \psi, w &\in C(\mathbb{R}^+; H_*^2(0, 1)) \cap C^1(\mathbb{R}^+; H_*^1(0, 1)) \cap C^2(\mathbb{R}^+; L^2(0, 1)). \end{aligned}$$

Now, we introduce the following energy functional:

$$\begin{aligned} E(t) &= \frac{1}{2} \left(\rho \|\varphi_t\|_2^2 + I_\rho \|3w_t - \psi_t\|_2^2 + 3I_\rho \|w_t\|_2^2 + G \|\psi - \varphi_x\|_2^2 \right. \\ &\quad \left. + \left(D - \int_0^t g(s) \, ds \right) \|3w_x - \psi_x\|_2^2 + 3D \|w_x\|_2^2 + 4\gamma \|w\|_2^2 \right) \\ &\quad + \frac{1}{2} (g \circ (3w_x - \psi_x))(t), \end{aligned}$$

where, for any $v \in L^2(0, 1)$,

$$(g \circ v)(t) = \int_0^1 \int_0^t g(t-s)(v(t) - v(s))^2 \, ds \, dx.$$

The following lemmas play an important role in the proof of our main results.

Lemma 2.3. [13] *The following inequalities hold:*

$$\begin{aligned} &\left(\int_0^t g(t-s)((3w_x - \psi_x)(t) - (3w_x - \psi_x)(s)) \, ds \right)^2 \\ &\leq g_0(t) \int_0^t g(t-s)((3w_x - \psi_x)(t) - (3w_x - \psi_x)(s))^2 \, ds, \\ &\left(\int_0^t g'(t-s)((3w_x - \psi_x)(t) - (3w_x - \psi_x)(s)) \, ds \right)^2 \\ &\leq -g(0) \int_0^t g'(t-s)((3w_x - \psi_x)(t) - (3w_x - \psi_x)(s))^2 \, ds, \end{aligned}$$

where $g_0(t) = \int_0^t g(s) \, ds$.

Lemma 2.4 (Jensen’s inequality). *Let $P: [c, d] \rightarrow \mathbb{R}$ be a convex function. Assume that the functions $f: \Omega \rightarrow [c, d]$ and $h: \Omega \rightarrow \mathbb{R}$ are integrable such that $h(x) \geq 0$ for any $x \in \Omega$ and $\int_\Omega h(x) \, dx = k > 0$. Then,*

$$P \left(\frac{1}{k} \int_\Omega f(x)h(x) \, dx \right) \leq \frac{1}{k} \int_\Omega P(f(x))h(x) \, dx.$$

Lemma 2.5. [13] *Let $(\varphi, 3w - \psi, w)$ be the solution of (1.1)–(1.3). Then*

$$\frac{d}{dt} E(t) = -4\beta \|w_t\|_2^2 - \frac{g(t)}{2} \|3w_x - \psi_x\|_2^2 + \frac{1}{2} (g' \circ (3w_x - \psi_x))(t) \leq 0.$$

In [23], Mustafa presented the following general decay result for problem (1.1)–(1.3) with structural damping ($\beta \neq 0$) in the case of equal wave speeds:

Theorem 2.6. [23] *Let $U_0 \in \mathcal{H}_0$. Assume that (A1) and (A2) hold and $G/\rho = D/I_\rho$. Then, there exist positive constants k_1 and k_2 such that the energy functional associated to problem (1.1)–(1.3) satisfies*

$$E(t) \leq k_2 H_1^{-1} \left(k_1 \int_{t_0}^t \xi(s) \, ds \right), \quad \forall t > t_0,$$

where H_1 is given by $H_1(t) = \int_t^T \frac{1}{sH'(s)} \, ds$ and $t_0 = g^{-1}(r)$.

We are now in a position to state our the first general decay result for problem (1.1)–(1.3) with structural damping ($\beta \neq 0$) in the case of non-equal wave speeds.

Theorem 2.7. *Let $U_0 \in \mathcal{H}_0$. Assume that (A1) and (A2) hold and $G/\rho \neq D/I_\rho$. Then, there exist positive constants k_1, k_2 and $t_1 > t_0 = g^{-1}(r)$ such that the energy functional associated to problem (1.1)–(1.3) satisfies the estimate*

$$E(t) \leq k_2(t - t_0)H_2^{-1} \left(\frac{k_1}{(t - t_0) \int_{t_1}^t \xi(s) \, ds} \right), \quad \forall t > t_1,$$

where H_2 is given by $H_2(t) = tH'(t)$.

Remark 2.8. Assume $H(s) = s^p, 1 \leq p < 2$ in (A2), then by simple calculations, we see that the decay rates of $E(t)$ is given by, for constants C and \bar{k} ,

$$(2.2) \quad E(t) \leq \begin{cases} C \left(\frac{1 + \int_0^t g(s) \, ds}{\int_0^t \xi(s) \, ds} \right) & \text{if } p = 1, \\ \bar{k}(t - t_0)^{1-1/p} \left(\frac{1}{\int_0^t \xi(s) \, ds} \right)^{1/p} & \text{if } 1 < p < 2. \end{cases}$$

Remark 2.9. Note that the estimate (2.2) was obtained by Li et al. in [13].

The second general decay result for problem (1.1)–(1.3) without structural damping ($\beta = 0$) in the case of equal wave speeds reads as follows.

Theorem 2.10. *Let $U_0 \in \mathcal{H}_0$. Assume that (A1) and (A2) hold, $\beta = 0$ and $G/\rho = D/I_\rho$. Then, there exist positive constants k_1 and k_2 such that the energy functional associated to problem (1.1)–(1.3) satisfies*

$$E(t) \leq k_2 H_1^{-1} \left(k_1 \int_{t_0}^t \xi(s) \, ds \right), \quad \forall t > t_0,$$

where H_1 is given by $H_1(t) = \int_t^T \frac{1}{sH'(s)} \, ds$ and $t_0 = g^{-1}(r)$.

Remark 2.11. Assume that $H(s) = s^p$, $1 \leq p < 2$ in (A2), then by simple calculations, we see that the decay rates of $E(t)$ is given by, for constants k_1, k_2 and k_3 ,

$$E(t) \leq \begin{cases} k_1 \exp\left(-k_2 \int_0^t \xi(s) \, ds\right) & \text{if } p = 1, \\ k_3 \left(1 + \int_0^t \xi(s) \, ds\right)^{-1/(p-1)} & \text{if } 1 < p < 2. \end{cases}$$

3. Technical lemmas

In this section, we establish several lemmas needed to prove our main results. For our purpose, we will adopt the functionals introduced in [23] with some modifications.

Lemma 3.1. *Assume that (A1) and (A2) hold. Then, the functional $F_1(t)$ defined as*

$$F_1(t) = -\rho \int_0^1 \varphi \varphi_t \, dx$$

satisfies, for any $\varepsilon_1 > 0$,

$$(3.1) \quad F_1'(t) \leq -\rho \|\varphi_t\|_2^2 + G\varepsilon_1 \|\psi - \varphi_x\|_2^2 + \frac{G}{2\varepsilon_1} \|3w_x - \psi_x\|_2^2 + \frac{9G}{2\varepsilon_1} \|w_x\|_2^2.$$

Proof. By differentiating $F_1(t)$ with respect to t , using (1.1) and integrating by parts, we obtain

$$F_1'(t) = -\rho \|\varphi_t\|_2^2 - G \int_0^1 \varphi_x (\psi - \varphi_x) \, dx.$$

Note that

$$-G \int_0^1 \varphi_x (\psi - \varphi_x) \, dx \leq G\varepsilon_1 \|\psi - \varphi_x\|_2^2 + \frac{G}{2\varepsilon_1} \|3w_x - \psi_x\|_2^2 + \frac{9G}{2\varepsilon_1} \|w_x\|_2^2.$$

This completes the proof. □

Lemma 3.2. *Under the conditions (A1) and (A2), the functional $F_2(t)$ defined by*

$$F_2(t) = I_\rho \int_0^1 (3w - \psi)(3w_t - \psi_t) \, dx$$

satisfies, for any $\varepsilon_2 > 0$,

$$(3.2) \quad \begin{aligned} F_2'(t) &\leq -(l - \varepsilon_2(G + 1)) \|3w_x - \psi_x\|_2^2 + I_\rho \|3w_t - \psi_t\|_2^2 \\ &\quad + \frac{G}{4\varepsilon_2} \|\psi - \varphi_x\|_2^2 + \frac{D - l}{4\varepsilon_2} (g \circ (3w_x - \psi_x))(t). \end{aligned}$$

Proof. It follows from (1.1) and integration by parts that

$$\begin{aligned} F_2'(t) &= I_\rho \|3w_t - \psi_t\|_2^2 + G \int_0^1 (3w - \psi)(\psi - \varphi_x) \, dx - D \|3w_x - \psi_x\|_2^2 \\ &\quad - \int_0^1 (3w - \psi) \int_0^t g(t - s)(3w - \psi)_{xx}(s) \, ds \, dx. \end{aligned}$$

Using Young's inequality and Lemma 2.3 with $\varepsilon_2 > 0$, we obtain (3.2). □

Lemma 3.3. *Assume that (A1) and (A2) hold. Then, the functional $F_3(t)$ defined by*

$$F_3(t) = I_\rho \int_0^1 ww_t \, dx$$

satisfies the estimate

$$(3.3) \quad \begin{aligned} F'_3(t) \leq & - \left(\frac{4}{3}\gamma - \varepsilon_3 \left(G + \frac{4\beta}{3} \right) \right) \|w\|_2^2 - D\|w_x\|_2^2 \\ & + \left(I_\rho + \frac{\beta}{3\varepsilon_3} \right) \|w_t\|_2^2 + \frac{G}{4\varepsilon_3} \|\psi - \varphi_x\|_2^2, \end{aligned}$$

where $\varepsilon_3 > 0$.

Proof. Differentiating $F_3(t)$ with respect to t , using the third equation of (1.1) and integrating by parts, we obtain

$$F'_3(t) = I_\rho \|w_t\|_2^2 - G \int_0^1 w(\psi - \varphi_x) \, dx - \frac{4\gamma}{3} \|w\|_2^2 - \frac{4\beta}{3} \int_0^1 ww_t \, dx - D\|w_x\|_2^2.$$

Next, it follows from Young's inequality that

$$F'_3(t) \leq - \left(\frac{4\gamma}{3} - \varepsilon_3 \left(G + \frac{4\beta}{3} \right) \right) \|w\|_2^2 - D\|w_x\|_2^2 + \left(I_\rho + \frac{\beta}{3\varepsilon_3} \right) \|w_t\|_2^2 + \frac{G}{4\varepsilon_3} \|\psi - \varphi_x\|_2^2,$$

which completes the proof of Lemma 3.3. □

Lemma 3.4. *Assume that (A1) and (A2) hold, after fixing $\varepsilon_4 > 0$, the functional $F_4(t)$ defined by*

$$F_4(t) = -I_\rho \int_0^1 (3w_t - \psi_t) \int_0^t g(t-s)[(3w - \psi)(t) - (3w - \psi)(s)] \, ds \, dx$$

satisfies the estimate

$$(3.4) \quad \begin{aligned} F'_4(t) \leq & - \frac{I_\rho g_0(t)}{2} \|3w_t - \psi_t\|_2^2 + \frac{G\varepsilon_4}{2} \|\psi - \varphi_x\|_2^2 + \frac{D\varepsilon_4}{2} \|3w_x - \psi_x\|_2^2 \\ & - \frac{I_\rho g(0)\lambda_0}{2g_0(t)} (g' \circ (3w_x - \psi_x))(t) \\ & + (D - l) \left(\frac{G\lambda_0}{2\varepsilon_4} + \frac{D}{2\varepsilon_4} \right) (g \circ (3w_x - \psi_x))(t), \end{aligned}$$

where $g_0(t) = \int_0^t g(s) \, ds$.

Proof. Taking the derivative of $F_4(t)$ with respect to t and using the second equation of (1.1), we get

$$F'_4(t) = -G \int_0^1 (\psi - \varphi_x) \int_0^t g(t-s)[(3w - \psi)(t) - (3w - \psi)(s)] \, ds \, dx$$

$$\begin{aligned}
 &+ \left(D - \int_0^t g(s) \, ds \right) \int_0^1 (3w - \psi)_x \int_0^t g(t-s)[(3w - \psi)_x(t) - (3w - \psi)_x(s)] \, ds dx \\
 &+ \left\| \int_0^t g(t-s)[(3w - \psi)_x(t) - (3w - \psi)_x(s)] \, ds \right\|_2^2 - I_\rho \left(\int_0^t g(s) \, ds \right) \|3w_t - \psi_t\|_2^2 \\
 &- I_\rho \int_0^1 (3w_t - \psi_t) \int_0^t g'(t-s)[(3w - \psi)(t) - (3w - \psi)(s)] \, ds dx.
 \end{aligned}$$

Now, we estimate the terms on the right-hand side of the above equation.

Using Young’s inequality, Poincaré’s inequality $\|u\|_2^2 \leq \lambda_0 \|u_x\|_2^2$ with $\lambda_0 > 0$ and Lemma 2.3, we obtain, for $0 < \varepsilon_4 < 1$,

$$\begin{aligned}
 &- G \int_0^1 (\psi - \varphi_x) \int_0^t g(t-s)[(3w - \psi)(t) - (3w - \psi)(s)] \, ds dx \\
 &\leq \frac{G\varepsilon_4}{2} \|\psi - \varphi_x\|_2^2 + \frac{G}{2\varepsilon_4} \int_0^1 \left(\int_0^t \sqrt{g(t-s)} \sqrt{g(t-s)} [(3w - \psi)(t) - (3w - \psi)(s)] \, ds \right)^2 dx \\
 &\leq \frac{G\varepsilon_4}{2} \|\psi - \varphi_x\|_2^2 + \frac{Gg_0(t)}{2\varepsilon_4} (g \circ (3w - \psi))(t) \\
 &\leq \frac{G\varepsilon_4}{2} \|\psi - \varphi_x\|_2^2 + \frac{G(D-l)\lambda_0}{2\varepsilon_4} (g \circ (3w_x - \psi_x))(t).
 \end{aligned}$$

Also, we have, for $0 < \varepsilon_4 < 1$,

$$\begin{aligned}
 &\left(D - \int_0^t g(s) \, ds \right) \int_0^1 (3w - \psi)_x \int_0^t g(t-s)[(3w - \psi)_x(t) - (3w - \psi)_x(s)] \, ds dx \\
 &\leq \frac{(D - g_0(t))\varepsilon_4}{2} \|3w_x - \psi_x\|_2^2 + \frac{(D - g_0(t))(D-l)}{2\varepsilon_4} (g \circ (3w_x - \psi_x))(t) \\
 &\leq \frac{D\varepsilon_4}{2} \|3w_x - \psi_x\|_2^2 + \frac{D(D-l)}{2\varepsilon_4} (g \circ (3w_x - \psi_x))(t)
 \end{aligned}$$

and

$$\left\| \int_0^t g(t-s)[(3w - \psi)_x(t) - (3w - \psi)_x(s)] \, ds \right\|_2^2 \leq (D-l)(g \circ (3w_x - \psi_x))(t).$$

Exploiting Young’s inequality, Poincaré’s inequality and Lemma 2.3, we obtain, for any $0 < \varepsilon_4 < 1$,

$$\begin{aligned}
 &- I_\rho \int_0^1 (3w_t - \psi_t) \int_0^t g'(t-s)[(3w - \psi)(t) - (3w - \psi)(s)] \, ds dx \\
 &\leq \frac{I_\rho g_0(t)}{2} \|3w_t - \psi_t\|_2^2 + \frac{I_\rho}{2g_0(t)} \left\| \int_0^t g'(t-s)[(3w - \psi)(t) - (3w - \psi)(s)] \, ds \right\|_2^2 \\
 &\leq \frac{I_\rho g_0(t)}{2} \|3w_t - \psi_t\|_2^2 - \frac{I_\rho g(0)}{2g_0(t)} (g' \circ (3w - \psi))(t) \\
 &\leq \frac{I_\rho g_0(t)}{2} \|3w_t - \psi_t\|_2^2 - \frac{I_\rho g(0)\lambda_0}{2g_0(t)} (g' \circ (3w_x - \psi_x))(t).
 \end{aligned}$$

A combination of all the above estimates gives the desired result. □

Lemma 3.5. *Assume that (A1) and (A2) hold. Then for any $\varepsilon_5 > 0$, the functional $F_5(t)$ defined by*

$$F_5(t) = \frac{D\rho}{G} \int_0^1 \varphi_t(3w_x - \psi_x) \, dx - I_\rho \int_0^1 (3w_t - \psi_t)(\psi - \varphi_x) \, dx - \frac{\rho}{G} \int_0^1 \varphi_t \int_0^t g(t-s)(3w_x - \psi_x)(s) \, ds \, dx$$

satisfies the estimate

$$(3.5) \quad \begin{aligned} F_5'(t) &\leq -G\|\psi - \varphi_x\|_2^2 + \left(\frac{D\rho}{G} - I_\rho\right) \int_0^1 \varphi_t(3w_{xt} - \psi_{xt}) \, dx \\ &\quad + 18\varepsilon_5 I_\rho \|w_t\|_2^2 + I_\rho \left(2\varepsilon_5 + \frac{1}{4\varepsilon_5}\right) \|3w_t - \psi_t\|_2^2 \\ &\quad + \frac{\varepsilon_5 \rho}{G} (1 + g(0)) \|\varphi_t\|_2^2 + \frac{\rho g(t)}{4\varepsilon_5 G} \|3w_x - \psi_x\|_2^2 - \frac{\rho g(0)}{4\varepsilon_5 G} (g' \circ (3w_x - \psi_x))(t). \end{aligned}$$

Proof. Taking the derivative of $F_5(t)$ with respect to t and using (1.1), we have

$$\begin{aligned} F_5'(t) &= -D \int_0^1 (\psi - \varphi_x)_x(3w_x - \psi_x) \, dx + \frac{D\rho}{G} \int_0^1 \varphi_t(3w_{xt} - \psi_{xt}) \, dx \\ &\quad - G \int_0^1 (\psi - \varphi_x)^2 \, dx - D \int_0^1 (3w_{xx} - \psi_{xx})(\psi - \varphi_x) \, dx \\ &\quad + \int_0^1 (\psi - \varphi_x) \int_0^t g(t-s)(3w_{xx} - \psi_{xx})(s) \, ds \, dx - I_\rho \int_0^1 (3w_t - \psi_t)(\psi - \varphi_x)_t \, dx \\ &\quad + \int_0^1 (\psi - \varphi_x)_x \int_0^t g(t-s)(3w_x - \psi_x)(s) \, ds \, dx \\ &\quad - \frac{\rho}{G} \int_0^1 \varphi_t \int_0^t g'(t-s)(3w_x - \psi_x)(s) \, ds \, dx - \frac{\rho g(0)}{G} \int_0^1 \varphi_t(3w_x - \psi_x)(t) \, dx. \end{aligned}$$

Next, making use of Young’s inequality, Lemma 2.1 and integrating by parts, we deduce that (3.5) (see [13] for details). This completes the proof. \square

As in [22], we use the functional

$$K(t) = \int_0^1 \int_0^t f(t-s)|(3w_x - \psi_x)(s)|^2 \, ds \, dx,$$

where $f(t) = \int_t^\infty g(s) \, ds$.

Lemma 3.6. [22, Lemma 3.6] *Under conditions (A1) and (A2), the functional $K(t)$ satisfies the estimate*

$$K'(t) \leq -\frac{1}{2}(g \circ (3w_x - \psi_x))(t) + 3(D - l)\|3w_x - \psi_x\|_2^2.$$

Lemma 3.7. *The functional \mathcal{L} defined by*

$$\mathcal{L}(t) := NE(t) + F_1(t) + N_2F_2(t) + N_3F_3(t) + N_4F_4(t) + N_5F_5(t)$$

satisfies, for a suitable choice of $N, N_2, N_3, N_4, N_5 \geq 0$,

$$\mathcal{L}(t) \sim E(t)$$

and the estimate

$$(3.6) \quad \begin{aligned} \mathcal{L}'(t) \leq & -c_1 [\|\varphi_t\|_2^2 + \|3w_t - \psi_t\|_2^2 + \|w_t\|_2^2 + \|w_x\|_2^2 + \|w\|_2^2 + \|\psi - \varphi_x\|_2^2 + \|3w_x - \psi_x\|_2^2] \\ & + c_2(g \circ (3w_x - \psi_x))(t) + N_5 \left(\frac{D\rho}{G} - I_\rho \right) \int_0^1 \varphi_t(3w_{xt} - \psi_{xt}) \, dx, \quad \forall t \geq t_0, \end{aligned}$$

where $c_1 > 0$, $c_2 > 0$ and t_0 has been introduced in Remark 2.1.

Proof. Let $g_0(t) = \int_0^t g(s) \, ds \geq \int_0^{t_0} g(s) \, ds = g_1 \geq 0$ for any $t \geq t_0$. According to (3.1)–(3.5), we have

$$(3.7) \quad \begin{aligned} \mathcal{L}'(t) \leq & -I_\rho \left(\frac{N_4g_1}{2} - N_2 - N_5 \left(2\varepsilon_5 + \frac{1}{4\varepsilon_5} \right) \right) \|3w_t - \psi_t\|_2^2 - \left(\rho - \frac{N_5\varepsilon_5\rho}{G}(1 + g(0)) \right) \|\varphi_t\|_2^2 \\ & - \left(4N\beta - N_3 \left(I_\rho + \frac{\beta}{3\varepsilon_3} \right) - 18N_5\varepsilon_5I_\rho \right) \|w_t\|_2^2 - \left(N_3 \left(\frac{4\gamma}{3} - G\varepsilon_3 \right) - N_3 \frac{4\beta\varepsilon_3}{3} \right) \|w\|_2^2 \\ & - \left(N_5G - G\varepsilon_1 - \frac{GN_2}{4\varepsilon_2} - \frac{N_4G\varepsilon_4}{2} - \frac{N_3G}{4\varepsilon_3} \right) \|\psi - \varphi_x\|_2^2 - \left(N_3D - \frac{9G}{2\varepsilon_1} \right) \|w_x\|_2^2 \\ & - \left(g(t) \left(\frac{N}{2} - \frac{N_5\rho}{4\varepsilon_5G} \right) + N_2(l - \varepsilon_2(G + 1)) - \frac{G}{2\varepsilon_1} - \frac{N_4D\varepsilon_4}{2} \right) \|3w_x - \psi_x\|_2^2 \\ & + (D - l) \left(\frac{N_2}{4\varepsilon_2} + N_4 \left(\frac{G\lambda_0}{2\varepsilon_4} + \frac{D}{2\varepsilon_4} \right) \right) (g \circ (3w_x - \psi_x))(t) \\ & + \left(\frac{N}{2} - \frac{N_4I_\rho g(0)\lambda_0}{2g_0(t)} - \frac{N_5\rho g(0)}{4\varepsilon_5G} \right) (g' \circ (3w_x - \psi_x))(t) \\ & + N_5 \left(\frac{D\rho}{G} - I_\rho \right) \int_0^1 \varphi_t(3w_{xt} - \psi_{xt}) \, dx. \end{aligned}$$

At this point, we need to choose our constants very carefully. First, we choose

$$\varepsilon_1 = G, \quad \varepsilon_2 = \frac{1}{N_2}, \quad \varepsilon_3 = \frac{1}{N_3}, \quad \varepsilon_4 = \frac{1}{N_4}, \quad \varepsilon_5 = \frac{G}{2N_5(1 + g(0))},$$

and (3.7) becomes

$$\begin{aligned} \mathcal{L}'(t) \leq & -\frac{\rho}{2}\|\varphi_t\|_2^2 - I_\rho \left(\frac{N_4g_1}{2} - N_2 - \frac{N_5^2(1 + g(0))}{2G} - \frac{G}{1 + g(0)} \right) \|3w_t - \psi_t\|_2^2 \\ & - \left(4N\beta - N_3 \left(I_\rho + \frac{N_3\beta}{3} \right) - \frac{9I_\rho G}{1 + g(0)} \right) \|w_t\|_2^2 - \left(\frac{4N_3\gamma}{3} - \left(G + \frac{4\beta}{3} \right) \right) \|w\|_2^2 \\ & - \left(N_5G - \frac{GN_2^2}{4} - \frac{GN_3^2}{4} - \frac{G}{2} - G^2 \right) \|\psi - \varphi_x\|_2^2 - \left(N_3D - \frac{9}{2} \right) \|w_x\|_2^2 \end{aligned}$$

$$\begin{aligned}
 & - \left(g(t) \left(\frac{N}{2} - \frac{N_5^2 \rho(1+g(0))}{2G^2} \right) + N_2 l - \left(\frac{3}{2} + G + \frac{D}{2} \right) \right) \|3w_x - \psi_x\|_2^2 \\
 & + (D - l) \left(\frac{N_2^2}{4} + N_4^2 \left(\frac{G\lambda_0}{2} + \frac{D}{2} \right) \right) (g \circ (3w_x - \psi_x))(t) \\
 & + \left(\frac{N}{2} - N_4 \frac{I_\rho g(0)\lambda_0}{2g_0(t)} - \frac{N_5^2 \rho g(0)(1+g(0))}{2G^2} \right) (g' \circ (3w_x - \psi_x))(t) \\
 & + N_5 \left(\frac{D\rho}{G} - I_\rho \right) \int_0^1 \varphi_t(3w_{xt} - \psi_{xt}) \, dx.
 \end{aligned}$$

Then, we select N_2 large enough so that

$$N_2 l - \left(\frac{3}{2} + G + \frac{D}{2} \right) > 0.$$

Next, we choose N_3 large enough so that

$$N_3 D - \frac{9}{2} > 0 \quad \text{and} \quad \frac{4N_3 \gamma}{3} - \left(G + \frac{4\beta}{3} \right) > 0.$$

Furthermore, we select N_5 large enough so that

$$N_5 G - G \frac{N_2^2}{4} - G \frac{N_3^2}{4} - \frac{G}{2} - G^2 > 0.$$

After that, we choose N_4 large enough so that

$$\frac{N_4 g_1}{2} - N_2 - \frac{N_5^2(1+g(0))}{2G} - \frac{G}{1+g(0)} > 0.$$

Finally, let us choose N large enough so that

$$4N\beta - N_3 \left(I_\rho + \frac{N_3 \beta}{3} \right) - \frac{9I_\rho G}{1+g(0)} > 0, \quad \frac{N}{2} - \frac{N_5^2 \rho(1+g(0))}{2G^2} > 0,$$

and

$$\frac{N}{2} - \frac{N_4 I_\rho g(0)\lambda_0}{2g_0(t)} - \frac{N_5^2 \rho g(0)(1+g(0))}{2G^2} > 0.$$

So we arrive at, for positive constants c_1 and c_2 ,

$$\begin{aligned}
 \mathcal{L}'(t) & \leq -c_1 \left[\|\varphi_t\|_2^2 + \|3w_t - \psi_t\|_2^2 + \|w_t\|_2^2 + \|w_x\|_2^2 + \|w\|_2^2 + \|\psi - \varphi_x\|_2^2 + \|3w_x - \psi_x\|_2^2 \right] \\
 & + c_2 (g \circ (3w_x - \psi_x))(t) + N_5 \left(\frac{D\rho}{G} - I_\rho \right) \int_0^1 \varphi_t(3w_{xt} - \psi_{xt}) \, dx.
 \end{aligned}$$

On the other hand, we find that

$$\begin{aligned}
 |\mathcal{L}(t) - NE(t)| & \leq \rho \int_0^1 |\varphi \varphi_t| \, dx + N_2 I_\rho \int_0^1 |(3w - \psi)(3w_t - \psi_t)| \, dx + N_3 I_\rho \int_0^1 |w w_t| \, dx \\
 & + N_4 I_\rho \int_0^1 |3w_t - \psi_t| \int_0^t g(t-s) |(3w - \psi)(t) - (3w - \psi)(s)| \, ds \, dx
 \end{aligned}$$

$$\begin{aligned}
 &+ N_5 \left[\frac{D\rho}{G} \int_0^1 |\varphi_t(3w_x - \psi_x)| \, dx + I_\rho \int_0^1 |(3w_t - \psi_t)(\psi - \varphi_x)| \, dx \right. \\
 &\quad \left. + \frac{\rho}{G} \int_0^1 |\varphi_t| \int_0^t g(t-s)|(3w_x - \psi_x)(s)| \, ds \, dx \right] \\
 &\leq cE(t),
 \end{aligned}$$

where c is a positive constant.

Therefore, we can choose N even large (if needed) so that the proof is completed. \square

4. The case with structural damping under non-equal wave speeds

In this section, we will give an estimate to the decay rate for system with structural damping in the case of non-equal speeds of wave propagation. We give some lemmas that is beneficial to prove the main result.

We introduce the second-order energy as

$$\begin{aligned}
 \tilde{E}(t) = &\frac{1}{2} \left(\rho \|\varphi_{tt}\|_2^2 + I_\rho \|3w_{tt} - \psi_{tt}\|_2^2 + 3I_\rho \|w_{tt}\|_2^2 + G \|\psi_t - \varphi_{xt}\|_2^2 + 3D \|w_{xt}\|_2^2 + 4\gamma \|w_t\|_2^2 \right. \\
 &\left. + \left(D - \int_0^t g(s) \, ds \right) \|3w_{xt} - \psi_{xt}\|_2^2 \right) + \frac{1}{2} (g \circ (3w_{xt} - \psi_{xt}))(t).
 \end{aligned}$$

Then we give the following lemmas.

Lemma 4.1. [13] *Let U be the strong solution of (1.1)–(1.3). Then, the second energy functional satisfies, for all $t \geq 0$,*

$$\begin{aligned}
 \tilde{E}'(t) = &-4\beta \|w_{tt}\|_2^2 - \frac{g(t)}{2} \|3w_{xt} - \psi_{xt}\|_2^2 + \frac{1}{2} (g' \circ (3w_{xt} - \psi_{xt}))(t) \\
 &- g(t) \int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx}) \, dx
 \end{aligned}
 \tag{4.1}$$

and

$$\tilde{E}(t) \leq c(\tilde{E}(0) + \|3w_{0xx} - \psi_{0xx}\|_2^2).
 \tag{4.2}$$

Lemma 4.2. [13] *Let U be the strong solution of (1.1)–(1.3). Then, we have*

$$\begin{aligned}
 &\left(\frac{D\rho}{G} - I_\rho \right) \int_0^1 \varphi_t(3w_{xt} - \psi_{xt}) \, dx \\
 &\leq \varepsilon \|\varphi_t\|_2^2 + \frac{c}{\varepsilon} g(t)E(0) + \frac{c}{\varepsilon} (g \circ (3w_{xt} - \psi_{xt}) - g' \circ (3w_{xt} - \psi_{xt}))(t),
 \end{aligned}$$

where $c > 0$ and $\varepsilon > 0$.

As in [21], we give the following inequality which is important to prove our main result.

Lemma 4.3. *Let U be the strong solution of (1.1)–(1.3). Assume that conditions (A1) and (A2) hold with H being linear. Then*

$$\xi(t)(g \circ (3w_{xt} - \psi_{xt}))(t) \leq c(-\tilde{E}'(t) + C_1g(t)), \quad \forall t \geq 0$$

for some positive constants c and C_1 .

Proof. Using the non-increasing property of $\xi(t)$ and H is linear, we obtain

$$\begin{aligned} & \xi(t)(g \circ (3w_{xt} - \psi_{xt}))(t) \\ (4.3) \quad & \leq \int_0^1 \int_0^t \xi(t-s)g(t-s)[(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(s)]^2 ds dx \\ & \leq \int_0^1 \int_0^t g'(t-s)[(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(s)]^2 ds dx \\ & = -(g' \circ (3w_{xt} - \psi_{xt}))(t), \quad \forall t > 0. \end{aligned}$$

From equation (4.1), inequality (4.2) and Young inequality, we have, for any $t \geq 0$,

$$\begin{aligned} (4.4) \quad & -(g' \circ (3w_{xt} - \psi_{xt}))(t) = -2\tilde{E}'(t) - 8\beta\|w_{tt}\|_2^2 - g(t)\|3w_{xt} - \psi_{xt}\|_2^2 \\ & \quad - 2g(t) \int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx}) dx \\ & \leq -2\tilde{E}'(t) - 2g(t) \int_0^1 (3w_{tt} - \psi_{tt})(3w_{0xx} - \psi_{0xx}) dx \\ & \leq -2\tilde{E}'(t) + g(t) [\|3w_{tt} - \psi_{tt}\|_2^2 + \|3w_{0xx} - \psi_{0xx}\|_2^2] \\ & \leq -2\tilde{E}'(t) + g(t) \left(\frac{2\tilde{E}(t)}{I_\rho} + \|3w_{0xx} - \psi_{0xx}\|_2^2 \right) \\ & \leq c(-\tilde{E}'(t) + c_1g(t)), \end{aligned}$$

where c and C_1 are some fixed positive constants.

Combining (4.3) and (4.4), we obtain the desired result. \square

Now, we turn to prove Theorem 2.7.

Proof of Theorem 2.7. Our proof starts with the observation that, for any $t \geq t_0$ and $m > 0$,

$$\begin{aligned} \mathcal{L}'(t) & \leq -c_1 [\|\varphi_t\|_2^2 + \|3w_t - \psi_t\|_2^2 + \|w_t\|_2^2 + \|w_x\|_2^2 + \|w\|_2^2 + \|\psi - \varphi_x\|_2^2 + \|3w_x - \psi_x\|_2^2] \\ & \quad + c_2(g \circ (3w_x - \psi_x))(t) + N_5 \left(\frac{D\rho}{G} - I_\rho \right) \int_0^1 \varphi_t(3w_{xt} - \psi_{xt}) dx \\ & \leq -mE(t) + c_2(g \circ (3w_x - \psi_x))(t) + N_5 \left(\frac{D\rho}{G} - I_\rho \right) \int_0^1 \varphi_t(3w_{xt} - \psi_{xt}) dx \\ & \leq -(m - \varepsilon)E(t) + c_2(g \circ (3w_x - \psi_x))(t) \\ & \quad + \frac{c}{\varepsilon}g(t)E(0) + \frac{c}{\varepsilon}(g \circ (3w_{xt} - \psi_{xt})) - (g' \circ (3w_x - \psi_x))(t), \end{aligned}$$

which are derived from (3.6) and Lemma 4.2. After fixing ε small enough, we arrive at

$$\mathcal{L}'(t) \leq -m_1 E(t) + c(g \circ (3w_x - \psi_x) + g \circ (3w_{xt} - \psi_{xt}))(t) + cg(t)E(0) - cE'(t),$$

where m_1 is a fixed positive constant. Taking $\mathcal{F}(t) := \mathcal{L}(t) + cE(t)$, which is obviously equivalent to $E(t)$, to get, for any $t \geq t_0$,

$$(4.5) \quad \mathcal{F}'(t) \leq -m_1 E(t) + c(g \circ (3w_x - \psi_x) + g \circ (3w_{xt} - \psi_{xt}))(t) + cg(t).$$

We consider the following two cases relying on the ideas present in [22].

Case 1: H is linear. We multiply (4.5) by $\xi(t)$, then on account of assumption (A2), Lemma 4.3 and estimate (4.2), we obtain, for any $t \geq t_0$,

$$\xi(t)\mathcal{F}'(t) \leq -m_1 \xi(t)E(t) - cE'(t) + c(-\tilde{E}'(t) + C_1 g(t)) + c\xi(0)g(t).$$

As $\xi(t)$ is non-increasing, we have, for some fixed positive constant c_3 ,

$$(\xi\mathcal{F} + cE + c\tilde{E})'(t) \leq -m_1 \xi(t)E(t) + c_3 g(t), \quad \forall t \geq t_0.$$

It follows immediately that

$$m_1 \xi(t)E(t) \leq -(\xi\mathcal{F} + cE + c\tilde{E})'(t) + c_3 g(t), \quad \forall t \geq t_0.$$

According to the non-increasing property of $E(t)$ and estimate (4.2), we may now integrate over (t, t_0) to conclude that, for any $t > t_0$,

$$\begin{aligned} m_1 E(t) \int_{t_0}^t \xi(s) \, ds &\leq -(\xi\mathcal{F} + cE + c\tilde{E})(t) + (\xi\mathcal{F} + cE + c\tilde{E})(t_0) + c_3 \int_0^t g(s) \, ds \\ &\leq (\xi\mathcal{F} + cE + c\tilde{E})(0) + c \int_0^1 (3w_{0xx} - \psi_{0xx})^2 \, dx + c_3(D - l). \end{aligned}$$

Thus, we have, for some fixed positive constant c ,

$$E(t) \leq \frac{c}{\int_{t_0}^t \xi(s) \, ds}, \quad \forall t > t_0.$$

Case 2: H is nonlinear. Taking account of (2.1), Lemma 2.5 and estimate (4.4), we get, for any $t \geq t_0$ and $c_4 > 0$,

$$\begin{aligned} &\int_0^1 \int_0^{t_0} g(s)[(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t - s)]^2 \, ds \, dx \\ &+ \int_0^1 \int_0^{t_0} g(s)[(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t - s)]^2 \, ds \, dx \\ &\leq \frac{1}{\xi(t_0)} \int_0^1 \int_0^{t_0} \xi(s)g(s)[(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t - s)]^2 \, ds \, dx \end{aligned}$$

$$\begin{aligned}
 & + \frac{1}{\xi(t_0)} \int_0^1 \int_0^{t_0} \xi(s)g(s)[(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)]^2 dsdx \\
 \leq & -\frac{g(0)}{a\xi(t_0)} \int_0^1 \int_0^{t_0} g'(s)[(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)]^2 dsdx \\
 & + \frac{g(0)}{a\xi(t_0)} \int_0^1 \int_0^{t_0} g'(s)[(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)]^2 dsdx \\
 \leq & -c(E'(t) + \tilde{E}'(t)) + c_4g(t).
 \end{aligned}$$

Inserting this estimate into (4.5), we obtain, for any $t \geq t_0$ and $c_5 > 0$,

$$\begin{aligned}
 \mathcal{F}'(t) \leq & -m_1E(t) - c(E'(t) + \tilde{E}'(t)) + c_5g(t) \\
 & + c \int_{t_0}^t g(s)\|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|_2^2 ds \\
 & + c \int_{t_0}^t g(s)\|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|_2^2 ds.
 \end{aligned}
 \tag{4.6}$$

Now, we define the functional $\theta(t)$ by

$$\begin{aligned}
 \theta(t) := & - \int_{t_0}^t g'(s)(\|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|_2^2 \\
 & + \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|_2^2) ds.
 \end{aligned}$$

Clearly, we have

$$\begin{aligned}
 \theta(t) \leq & - \int_0^t g'(s)(\|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|_2^2 \\
 & + \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|_2^2) ds \\
 \leq & -c(E'(t) + \tilde{E}'(t)) + c_6g(t), \quad \forall t > t_0,
 \end{aligned}
 \tag{4.7}$$

where $c_6 > 0$ is a fixed constant.

After that, we define another functional $\eta(t)$ by, for any $t > t_0$,

$$\begin{aligned}
 \eta(t) := & \frac{\gamma}{t-t_0} \int_{t_0}^t (\|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|_2^2 \\
 & + \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|_2^2) ds.
 \end{aligned}$$

Now, the following inequality holds under energy functional $E(t)$, second energy functional $\tilde{E}(t)$, (4.1), Lemmas 2.5 and 4.1 that

$$\begin{aligned}
 & \frac{1}{t-t_0} \int_{t_0}^t (\|(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s)\|_2^2 \\
 & + \|(3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s)\|_2^2) ds
 \end{aligned}$$

$$\begin{aligned} &\leq \frac{4}{l(t-t_0)} \int_{t_0}^t (E(t) + E(t-s) + \tilde{E}(t) + \tilde{E}(t-s)) \, ds \\ &\leq \frac{8}{l(t-t_0)} \int_{t_0}^t \left[E(0) + c \left(\tilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx}) \, dx \right) \right] \, ds \\ &\leq \frac{8}{l} \left[E(0) + c \left(\tilde{E}(0) + \int_0^1 (3w_{0xx} - \psi_{0xx}) \, dx \right) \right] < +\infty, \quad \forall t > t_0. \end{aligned}$$

Then the above inequality allows for a constant $0 < \gamma < 1$ chosen so that, for all $t > t_0$,

$$0 < \eta(t) < 1;$$

otherwise we get the following decay rate from (4.6)

$$E(t) \leq \frac{c}{t-t_0}, \quad \forall t > t_0.$$

Moreover, recalling that H is strict convex on $(0, r]$ and $H(0) = 0$, then

$$H(s\tau) \leq sH(\tau) \quad \text{for } 0 \leq s \leq 1 \text{ and } \tau \in (0, r].$$

From assumption (A2), Lemma 2.4 and (4.6), it follows that, for any $t > t_0$,

$$\begin{aligned} \theta(t) &= - \int_{t_0}^t g'(s) (\| (3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s) \|_2^2 \\ &\quad + \| (3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s) \|_2^2) \, ds \\ &= - \frac{1}{\eta(t)} \int_{t_0}^t \eta(t) g'(s) (\| (3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s) \|_2^2 \\ &\quad + \| (3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s) \|_2^2) \, ds \\ &\geq \frac{1}{\eta(t)} \int_{t_0}^t \eta(t) \xi(s) H(g(s)) (\| (3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s) \|_2^2 \\ &\quad + \| (3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s) \|_2^2) \, ds \\ &\geq \frac{\xi(t)(t-t_0)}{\gamma \eta(t)} \int_{t_0}^t H(\eta(t)g(s)) \frac{\gamma}{t-t_0} (\| (3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s) \|_2^2 \\ &\quad + \| (3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s) \|_2^2) \, ds \\ &\geq \frac{\xi(t)(t-t_0)}{\gamma} H \left(\frac{1}{\eta(t)} \int_{t_0}^t \eta(t)g(s) \frac{\gamma}{t-t_0} (\| (3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s) \|_2^2 \right. \\ &\quad \left. + \| (3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s) \|_2^2) \, ds \right) \\ &= \frac{\xi(t)(t-t_0)}{\gamma} H \left(\frac{\gamma}{t-t_0} \int_{t_0}^t g(s) (\| (3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s) \|_2^2 \right. \\ &\quad \left. + \| (3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s) \|_2^2) \, ds \right). \end{aligned}$$

Since \overline{H} is an extension of H (see Remark 2.1(2)), we also have

$$\theta(t) \geq \frac{\xi(t)(t-t_0)}{\gamma} \overline{H} \left(\frac{\gamma}{t-t_0} \int_{t_0}^t g(s) (\| (3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s) \|_2^2 + \| (3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s) \|_2^2) ds \right).$$

In this way,

$$\begin{aligned} & \int_{t_0}^t g(s) (\| (3w_x - \psi_x)(t) - (3w_x - \psi_x)(t-s) \|_2^2 + \| (3w_{xt} - \psi_{xt})(t) - (3w_{xt} - \psi_{xt})(t-s) \|_2^2) ds \\ & \leq \frac{t-t_0}{\gamma} \overline{H}^{-1} \left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)} \right), \quad \forall t > t_0. \end{aligned}$$

Then, taking $\mathcal{F}_1(t) := \mathcal{F}(t) + cE(t) + c\tilde{E}(t)$, (4.6) becomes

$$(4.8) \quad \begin{aligned} \mathcal{F}'_1(t) & := \mathcal{F}'(t) + cE'(t) + c\tilde{E}'(t) \\ & \leq -m_1E(t) + \frac{c}{\gamma}(t-t_0)\overline{H}^{-1} \left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)} \right) + c_5g(t), \quad \forall t > t_0. \end{aligned}$$

Let $0 < \varepsilon_1 < r$, we define the functional $\mathcal{F}_2(t)$ by, for any $t > t_0$,

$$\mathcal{F}_2(t) := \overline{H}' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) \mathcal{F}_1(t).$$

Then, recalling that $E'(t) \leq 0$, $\overline{H}' > 0$ and $\overline{H}'' > 0$ as well as making use of estimate (4.8), we have

$$(4.9) \quad \begin{aligned} \mathcal{F}'_2(t) & = \left(-\frac{\varepsilon_1}{(t-t_0)^2} \frac{E(t)}{E(0)} + \frac{\varepsilon_1}{t-t_0} \frac{E'(t)}{E(0)} \right) \overline{H}'' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) \mathcal{F}_1(t) \\ & \quad + \overline{H}' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) \mathcal{F}'_1(t) \\ & \leq \overline{H}' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) \left(-m_1E(t) + \frac{c}{\gamma}(t-t_0)\overline{H}^{-1} \left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)} \right) + c_5g(t) \right) \\ & = -m_1E(t)\overline{H}' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) + c_5\overline{H}' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) g(t) \\ & \quad + \frac{c}{\gamma}(t-t_0)\overline{H}' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) \overline{H}^{-1} \left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)} \right), \quad \forall t > t_0. \end{aligned}$$

We give the following generalized Young inequality

$$(4.10) \quad AB \leq \overline{H}^*(A) + \overline{H}(B),$$

where \overline{H}^* is the convex conjugate of \overline{H} in the sense of Young [2], i.e.,

$$\overline{H}^*(s) = s(\overline{H}')^{-1}(s) - \overline{H}[(\overline{H}')^{-1}(s)].$$

Choosing $A = \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right)$, $B = \overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)}\right)$ and combining (4.9) and (4.10), we obtain

$$\begin{aligned}
 & \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) \overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)}\right) \\
 (4.11) \quad & \leq \overline{H}^*\left(\overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right)\right) + \overline{H}\left(\overline{H}^{-1}\left(\frac{\gamma\theta(t)}{\xi(t)(t-t_0)}\right)\right) \\
 & = \overline{H}^*\left(\overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right)\right) + \frac{\gamma\theta(t)}{\xi(t)(t-t_0)}
 \end{aligned}$$

and

$$\begin{aligned}
 (4.12) \quad \overline{H}^*\left(\overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right)\right) & = \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) (\overline{H}')^{-1}\left(\overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right)\right) \\
 & \quad - \overline{H}\left[(\overline{H}')^{-1}\left(\overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right)\right)\right] \\
 & = \frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) - \overline{H}\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) \\
 & \leq \frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right).
 \end{aligned}$$

So, combining (4.9), (4.11) and (4.12), we obtain

$$\mathcal{F}'_2(t) \leq -(m_1 E(0) - c\varepsilon_1) \frac{E(t)}{E(0)} \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) + c_5 \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) g(t) + c \frac{\theta(t)}{\xi(t)}.$$

Then, if we fix ε_1 much smaller (if needed), we will arrive at, for all $t > t_0$ and $m_2 > 0$,

$$(4.13) \quad \mathcal{F}'_2(t) \leq -m_2 \frac{E(t)}{E(0)} \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) + c_5 \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) g(t) + c \frac{\theta(t)}{\xi(t)}.$$

From this, we multiply (4.13) by $\xi(t)$ to get

$$\xi(t) \mathcal{F}'_2(t) \leq -m_2 \xi(t) \frac{E(t)}{E(0)} \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) + c_5 \xi(t) \overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) g(t) + c \theta(t).$$

Then, using the fact that, as $\varepsilon_1 \frac{E(t)}{E(0)} < r$, $\overline{H}'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) = H'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right)$ and inequality (4.7) to get, for $c_7 > 0$,

$$\begin{aligned}
 \xi(t) \mathcal{F}'_2(t) & \leq -m_2 \xi(t) \frac{E(t)}{E(0)} H'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) + c_5 \xi(t) H'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) g(t) \\
 & \quad - c(E'(t) + \tilde{E}'(t)) + c_7 g(t).
 \end{aligned}$$

Since $1/(t-t_0) \rightarrow 0$ when $t \rightarrow \infty$, there exists $t_1 > t_0$ such that $1/(t-t_0) < 1$, whenever $t > t_1$. According to the strictly increasing property of H' , non-increasing properties of E and ξ , we conclude that

$$\xi(t) H'\left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)}\right) \leq H'(\varepsilon_1) \xi(0), \quad \forall t > t_1.$$

Therefore, it is sufficient to prove that, for some positive constant c_8 ,

$$\begin{aligned} \xi(t)\mathcal{F}'_2(t) + c(E'(t) + \tilde{E}'(t)) &\leq -m_2\xi(t)\frac{E(t)}{E(0)}H' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) + c_5H'(\varepsilon_1)\xi(0) + c_7g(t) \\ &\leq -m_2\xi(t)\frac{E(t)}{E(0)}H' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) + c_8g(t). \end{aligned}$$

After that, defining $\mathcal{F}_3(t) = \xi\mathcal{F}_2(t) + c(E(t) + \tilde{E}(t))$ and using the non-increasing property of $\xi(t)$, we arrive at, for any $t > t_1$,

$$(4.14) \quad m_2\xi(t)\frac{E(t)}{E(0)}H' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) \leq -\mathcal{F}'_3(t) + c_8g(t).$$

Moreover, it suffices to show that the map $t \mapsto E(t)H' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right)$ is non-increasing by the non-increasing property of $E(t)$ and $H'' > 0$. Consequently, we can take an integration of (4.14) over (t_1, t) yields

$$m_2 \int_{t_1}^t \xi(s) \, ds \frac{E(t)}{E(0)}H' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) \leq \mathcal{F}_3(t_1) + c_8(D-l).$$

If we multiply the above inequality by $1/(t-t_0) > 0$, for any $t > t_1$, we will obtain

$$m_2 \frac{1}{t-t_0} \int_{t_1}^t \xi(s) \, ds \frac{E(t)}{E(0)}H' \left(\frac{\varepsilon_1}{t-t_0} \frac{E(t)}{E(0)} \right) \leq \frac{\mathcal{F}_3(t_1) + c_8(D-l)}{t-t_0}.$$

Finally, define

$$H_2(t) = tH'(t).$$

It is immediate that $H'_2(t), H_2(t) > 0$ on $(0, r]$ by the strict convexity of H on $(0, r]$. Therefore, we obtain, for two positive constants k_1 and k_2 ,

$$E(t) \leq k_2(t-t_0)H_2^{-1} \left(\frac{k_1}{(t-t_0) \int_{t_1}^t \xi(s) \, ds} \right), \quad \forall t > t_1.$$

This completes the proof. □

5. The case without structural damping under equal wave speeds

In this section, we will take account of the general decay for system (1.1)–(1.3) without structural damping ($\beta = 0$) in the case of equal speeds of wave propagation.

To overcome the absence of structural damping ($\beta = 0$), we need to construct a new perturbed functional to estimate $\|w_t\|_2^2$. As in [18], we consider the functional

$$J(t) = -I_\rho \int_0^1 w_t(\psi - \varphi_x) \, dx + I_\rho \int_0^1 \varphi_t w_x \, dx.$$

Then the following result holds:

Lemma 5.1. *Let U be the solution of (1.1)–(1.3). The functional $J(t)$ satisfies the estimate*

$$J'(t) \leq -\frac{3I_\rho}{2}\|w_t\|_2^2 + \frac{2I_\rho}{3}\|(3w - \psi)_t\|_2^2 + \frac{4\gamma\varepsilon_6}{3}\|w\|_2^2 + \left(G + \frac{4\gamma}{3\varepsilon_6}\right)\|\psi - \varphi_x\|_2^2,$$

where $\varepsilon_6 > 0$.

Proof. Differentiating $J(t)$ with respect to t , using (1.1) and integrating by parts, we obtain

$$\begin{aligned} J'(t) &= -I_\rho \int_0^1 w_{tt}(\psi - \varphi_x) \, dx - I_\rho \int_0^1 w_t(\psi_t - \varphi_{xt}) \, dx + I_\rho \int_0^1 \varphi_{tt}w_x \, dx + I_\rho \int_0^1 \varphi_t w_{xt} \, dx \\ &= G\|\psi - \varphi_x\|_2^2 + \frac{4\gamma}{3} \int_0^1 w(\psi - \varphi_x) \, dx + \left(D - \frac{I_\rho G}{\rho}\right) \int_0^1 w_x(\psi - \varphi_x) \, dx - I_\rho \int_0^1 w_t \psi_t \, dx. \end{aligned}$$

Using $\psi_t = 3w_t - (3w - \psi)_t$, Young’s inequality and $G/\rho = D/I_\rho$, we know that, for some $\varepsilon_6 > 0$,

$$\begin{aligned} J'(t) &= -3I_\rho\|w_t\|_2^2 + I_\rho \int_0^1 w_t(3w - \psi)_t \, dx + G\|\psi - \varphi_x\|_2^2 + \frac{4\gamma}{3} \int_0^1 w(\psi - \varphi_x) \, dx \\ &\quad + \left(D - \frac{I_\rho G}{\rho}\right) \int_0^1 w_x(\psi - \varphi_x) \, dx \\ &\leq -3I_\rho\|w_t\|_2^2 + \frac{3I_\rho}{2}\|w_t\|_2^2 + \frac{2I_\rho}{3}\|(3w - \psi)_t\|_2^2 + G\|\psi - \varphi_x\|_2^2 \\ &\quad + \frac{4\gamma}{3}\varepsilon_6\|w\|_2^2 + \frac{4\gamma}{3\varepsilon_6}\|\psi - \varphi_x\|_2^2 \\ &= -\frac{3I_\rho}{2}\|w_t\|_2^2 + \frac{2I_\rho}{3}\|(3w - \psi)_t\|_2^2 + \frac{4\gamma\varepsilon_6}{3}\|w\|_2^2 + \left(G + \frac{4\gamma}{3\varepsilon_6}\right)\|\psi - \varphi_x\|_2^2. \end{aligned}$$

This proof is now completed. □

We define a new Lyapunov function $\mathcal{L}_1(t)$:

$$\mathcal{L}_1(t) := NE(t) + F_1(t) + N_2F_2(t) + N_3F_3(t) + N_4F_4(t) + N_5F_5(t) + N_6J(t),$$

where $N, N_2, N_3, N_4, N_5, N_6 \geq 0$ and $F_i(t)$, $i = 1, 2, 3, 4, 5$ remain as defined in Lemmas 3.1–3.5 with the derivatives of $F_i(t)$, $i = 1, 2, 4, 5$ remain the same while derivatives of $E(t)$ and $F_3(t)$ are given as follows:

$$E'(t) = -\frac{g(t)}{2}\|3w_x - \psi_x\|_2^2 + \frac{1}{2}(g' \circ (3w_x - \psi_x))(t) \leq 0, \quad \forall t \geq 0$$

and

$$\begin{aligned} F_3'(t) &\leq I_\rho\|w_t\|_2^2 + G\varepsilon_3\|w\|_2^2 + \frac{G}{4\varepsilon_3}\|\psi - \varphi_x\|_2^2 - \frac{4\gamma}{3}\|w\|_2^2 - D\|w_x\|_2^2 \\ &= -\left(\frac{4}{3}\gamma - G\varepsilon_3\right)\|w\|_2^2 - D\|w_x\|_2^2 + I_\rho\|w_t\|_2^2 + \frac{G}{4\varepsilon_3}\|\psi - \varphi_x\|_2^2. \end{aligned}$$

Now, we are ready to prove Theorem 2.10.

Proof of Theorem 2.10. Taking derivative about $\mathcal{L}_1(t)$, using lemmas in Section 3 and $G/\rho = D/I_\rho$, we have

$$\begin{aligned} \mathcal{L}'_1(t) \leq & -\left(\rho - \frac{N_5\varepsilon_5\rho}{G}(1 + g(0))\right) \|\varphi_t\|_2^2 - \left(\frac{3N_6I_\rho}{2} - N_3I_\rho - 18N_5\varepsilon_5I_\rho\right) \|w_t\|_2^2 \\ & - I_\rho \left(\frac{g_1N_4}{2} - N_2 - N_5\left(2\varepsilon_5 + \frac{1}{4\varepsilon_5}\right) - \frac{2N_6}{3}\right) \|3w_t - \psi_t\|_2^2 \\ & - \left(N_3\left(\frac{4\gamma}{3} - G\varepsilon_3\right) - \frac{4N_6\gamma\varepsilon_6}{3}\right) \|w\|_2^2 - \left(N_3D - \frac{9G}{2\varepsilon_1}\right) \|w_x\|_2^2 \\ & - \left(N_5G - \varepsilon_1G - \frac{GN_2}{4\varepsilon_2} - \frac{GN_3}{4\varepsilon_3} - \frac{N_4\varepsilon_4G}{2} - N_6\left(G + \frac{4\gamma}{3\varepsilon_6}\right)\right) \|\psi - \varphi_x\|_2^2 \\ & - \left(g(t)\left(\frac{N}{2} - \frac{N_5\rho}{4\varepsilon_5G}\right) + N_2(l - \varepsilon_2(G + 1)) - \frac{G}{2\varepsilon_1} - N_4\varepsilon_4\frac{D}{2}\right) \|3w_x - \psi_x\|_2^2 \\ & + (D - l) \left(\frac{N_2}{4\varepsilon_2} + N_4\left(\frac{G\lambda_0}{2\varepsilon_4} + \frac{D}{2\varepsilon_4}\right)\right) (g \circ (3w_x - \psi_x))(t) \\ & + \left(\frac{N}{2} - \frac{N_4I_\rho g(0)\lambda_0}{2g_0(t)} - \frac{N_5\rho g(0)}{4\varepsilon_5G}\right) (g' \circ (3w_x - \psi_x))(t). \end{aligned}$$

At this point, we need to choose our constants very carefully. First, we choose

$$\varepsilon_1 = G, \quad \varepsilon_2 = \frac{1}{N_2}, \quad \varepsilon_3 = \frac{1}{N_3}, \quad \varepsilon_4 = \frac{1}{N_4}, \quad \varepsilon_5 = \frac{G}{2N_5(1 + g(0))}, \quad \varepsilon_6 = \frac{1}{N_6},$$

so that

(5.1)

$$\begin{aligned} \mathcal{L}'_1(t) \leq & -\frac{\rho}{2}\|\varphi_t\|_2^2 - I_\rho \left(\frac{N_4g_1}{2} - N_2 - \frac{N_5^2(1 + g(0))}{2G} - \frac{G}{1 + g(0)} - \frac{2N_6}{3}\right) \|3w_t - \psi_t\|_2^2 \\ & - \left(N_3D - \frac{9}{2}\right) \|w_x\|_2^2 - \left(\frac{3N_6I_\rho}{2} - N_3I_\rho - \frac{9I_\rho G}{1 + g(0)}\right) \|w_t\|_2^2 \\ & - \left(\frac{4N_3\gamma}{3} - \left(G + \frac{4\gamma}{3}\right)\right) \|w\|_2^2 \\ & - \left(N_5G - \frac{GN_2^2}{4} - \frac{GN_3^2}{4} - \frac{G}{2} - G^2 - N_6G - \frac{4\gamma N_6^2}{3}\right) \|\psi - \varphi_x\|_2^2 \\ & - \left(g(t)\left(\frac{N}{2} - \frac{N_5^2\rho(1 + g(0))}{2G^2}\right) + N_2l - \left(\frac{3}{2} + G + \frac{D}{2}\right)\right) \|3w_x - \psi_x\|_2^2 \\ & + (D - l) \left(\frac{N_2^2}{4} + N_4\left(\frac{G\lambda_0}{2} + \frac{D}{2}\right)\right) (g \circ (3w_x - \psi_x))(t) \\ & + \left(\frac{N}{2} - \frac{N_4I_\rho g(0)\lambda_0}{2g_0(t)} - \frac{N_5^2\rho(1 + g(0))}{2G^2}\right) (g' \circ (3w_x - \psi_x))(t). \end{aligned}$$

Next, we select N_2 large enough so that

$$N_2l - \left(\frac{3}{2} + G + \frac{D}{2}\right) > 0.$$

Then, we choose N_3 large enough so that

$$N_3 D - \frac{9}{2} > 0 \quad \text{and} \quad \frac{4N_3\gamma}{3} - \left(G + \frac{4\gamma}{3}\right) > 0.$$

Furthermore, we select N_6 large enough so that

$$\frac{3N_6 I_\rho}{2} - N_3 I_\rho - \frac{9I_\rho G}{1 + g(0)} > 0.$$

After that, we select N_5 large enough so that

$$N_5 G - \frac{GN_2^2}{4} - \frac{GN_3^2}{4} - \frac{G}{2} - G^2 - N_6 G - \frac{4\gamma N_6^2}{3} > 0.$$

Then, we select N_4 large enough so that

$$I_\rho \left(\frac{N_4 g_1}{2} - N_2 - \frac{N_5^2(1 + g(0))}{2G} - \frac{G}{1 + g(0)} - \frac{2N_6}{3} \right) > 0.$$

Finally, we choose N large enough so that

$$\frac{N}{2} - \frac{N_5^2 \rho(1 + g(0))}{2G^2} > 0 \quad \text{and} \quad \frac{N}{2} - \frac{N_4 I_\rho g(0) \lambda_0}{2g_0(t)} - \frac{N_5^2 \rho(1 + g(0))}{2G^2} > 0.$$

Combining (5.1) and the above, we deduce that (5.1) becomes, for positive constants C_i , $i = 2, 3, 4$,

$$\begin{aligned} \mathcal{L}'_1(t) &\leq -C_2 [\|\varphi_t\|_2^2 + \|3w_t - \psi_t\|_2^2 + \|w_x\|_2^2 + \|w_t\|_2^2 + \|w\|_2^2 + \|\psi - \varphi_x\|_2^2 + \|3w_x - \psi_x\|_2^2] \\ &\quad + C_3(g \circ (3w_x - \psi_x))(t) + C_4(g' \circ (3w_x - \psi_x))(t) \\ &\leq -C_2 [\|\varphi_t\|_2^2 + \|3w_t - \psi_t\|_2^2 + \|w_x\|_2^2 + \|w_t\|_2^2 + \|w\|_2^2 + \|\psi - \varphi_x\|_2^2 + \|3w_x - \psi_x\|_2^2] \\ &\quad + C_3(g \circ (3w_x - \psi_x))(t) \\ &\leq -mE(t) - cE'(t) + c \int_0^1 \int_{t_0}^t g(s)[(3w_x - \psi_x)(t) - (3w_x - \psi_x)(t - s)]^2 ds dx. \end{aligned}$$

The remainder of the proof is similar to that in the proof of Theorem 2.7 (see [22, 23]). This completes the proof of Theorem 2.10. □

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