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Functional Model and Spectral Analysis of Discrete Singular Hamiltonian System

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Abstract. A space of boundary values is constructed for a minimal symmetric operator, generated by a discrete singular Hamiltonian system, acting in the Hilbert space $\ell^2_{\mathbf{A}}(\mathbb{N}_0; E \oplus E)$ ($\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\dim E = m < \infty$) with maximal deficiency indices (m, m) (in limit-circle case). A description of all maximal dissipative, maximal accumulative, self-adjoint and other extensions of such a symmetric operator is given in terms of boundary conditions at infinity. We construct a self-adjoint dilation of a maximal dissipative operator and its incoming and outgoing spectral representations, which make it possible to determine the scattering matrix of the dilation. We establish a functional model of the dissipative operator and construct its characteristic function in terms of the scattering matrix of the dilation. Finally, we prove the theorem on completeness of the system of eigenvectors and associated vectors (or root vectors) of the dissipative operator.

1. Introduction

The theory of extensions of symmetric operators is one of the main branches in operator theory closely related to various fields of mathematics. In operator theory there exists an abstract scheme of constructing maximal dissipative (self-adjoint) extensions of symmetric operators that are parametrized by contraction (unitary) operators (see [3, 4, 7, 10, 11, 14, 19, 23, 26, 28, 30]). The extension theory developed originally by J. von Neumann [19]. He gives an affirmative answer to the question under which conditions does a symmetric densely defined operator possess self-adjoint extensions and describes all such extensions.

However, regardless of the general scheme, the problem of the description of the maximal dissipative, maximal accumulative, self-adjoint and other extensions of the given symmetric differential and difference operator via the boundary conditions is of considerable interest. This problem is particularly interesting in the case of singular operators, because

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at the singular ends of the interval under consideration the usual boundary conditions in general are meaningless.

As is well known [1–5, 16, 18, 20–22], the most adequate approach in the spectral theory of dissipative (and also contractive) operators is that based on a study of the characteristic function and the corresponding functional model representing an operator unitary equivalent to the original operator and defined in a certain L^2-space of vector-valued functions by a relatively simple formula that is convenient for investigation. According to the well-known series of results of Lax, Phillips, Sz.-Nagy, Foiaş and Pavlov, the computation of the characteristic function is, in turn, preceded by the construction and investigation of a self-adjoint (unitary for contractions) dilation and of the corresponding scattering theory problem, in which this function is realized as the scattering matrix (see [17, 18, 21, 22]). This method (also called the functional model method) has already been used in many investigations, of which we mention only [16–18, 21, 22]. Efficiency of this approach for dissipative discrete Dirac and Hamiltonian operators has been demonstrated in [1–5].

In this paper, we consider the minimal symmetric operator, generated by discrete singular Hamiltonian system, acting in the Hilbert space $\ell^2_{\mathbf{A}}(\mathbb{N}_0; E \oplus E)$ ($\mathbb{N}_0 = \{0, 1, 2, \ldots\}$, $\dim E = m < \infty$) with maximal deficiency indices (m, m) (in limit-circle case). We construct a space of boundary values of minimal symmetric operator and description of all maximal dissipative, maximal accumulative, self-adjoint and others extensions of such a symmetric operator is given in terms of the boundary conditions at infinity. We construct a self-adjoint dilation of the maximal dissipative operator and its incoming and outgoing spectral representations, which makes it possible to determine the scattering matrix of dilation according to the scheme of Lax and Phillips [17]. With the help of the incoming spectral representation, we then construct a functional model of the maximal dissipative operator and define its characteristic function in terms of the scattering matrix of the dilation. Finally, using these results, we prove the theorem on completeness of the system of eigenvectors and associated vectors (or root vectors) of the maximal dissipative discrete Hamiltonian operator.

2. Extensions of symmetric operators generated by discrete Hamiltonian system

Let E be the m-dimensional ($m < \infty$) Euclidean space. For sequences $x^{(1)} = \{x_n^{(1)}\}$, $x^{(2)} = \{x_n^{(2)}\}$ ($n \in \mathbb{N}_0 := \{0, 1, 2, \ldots\}$) of vectors $x_n^{(1)} \in E$, $x_n^{(2)} \in E$, we consider the *discrete Hamiltonian system*

$$(2.1) \qquad (\Lambda_1 x)_n := \begin{cases} -C_n x_{n+1}^{(2)} + B_n x_n^{(2)} + P_n x_n^{(1)} = \lambda(A_n x_n^{(1)} + D_n x_n^{(2)}), \\ B_n^* x_n^{(1)} - C_{n-1} x_{n-1}^{(1)} + Q_n x_n^{(2)} = \lambda(D_n^* x_n^{(2)} + R_n x_n^{(2)}), \end{cases}$$

where λ is a complex spectral parameter, C_{-1}, C_n, B_n, P_n, Q_n, A_n, D_n, R_n, are linear operators (matrices) acting in E, $\det C_l \neq 0$, $(l = -1, 0, 1, 2, \ldots)$, $P_n = P_n^*$, $Q_n = Q_n^*$, and $\mathbf{A}_n := \left(\begin{smallmatrix} A_n & D_n \\ D_n^* & R_n \end{smallmatrix}\right)$ is a positive operator in $E \oplus E$, i.e., $\mathbf{A}_n > 0$, $n \in \mathbb{N}_0$.

System (2.1) is a discrete analog (for $C_{-1} = C_n = \pm I$, $n \in \mathbb{N}_0$, where I is the identity operator on E) of Hamiltonian system shown as

$$\mathbf{J}\frac{dx(t)}{dt} + \mathbf{B}(t)x(t) = \lambda \mathbf{A}(t)x(t), \quad t \in [0, \infty),$$

where

$$\mathbf{J} = \begin{pmatrix} 0 & \mp I \\ \pm I & 0 \end{pmatrix}, \quad x(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}, \quad x_1(t), x_2(t) \in E,$$

$$\mathbf{B}(t) = \begin{pmatrix} P(t) & E(t) \\ E^*(t) & Q(t) \end{pmatrix}, \quad \mathbf{A}(t) = \begin{pmatrix} A(t) & D(t) \\ D^*(t) & R(t) \end{pmatrix},$$

$\mathbf{B}(t) = \mathbf{B}^*(t)$, $\mathbf{A}(t) > 0$ (for almost all $t \in [0, \infty)$), entries of the $2m \times 2m$ matrices $\mathbf{A}(t)$ and $\mathbf{B}(t)$ are Lebesgue measurable and locally integrable functions on $[0, \infty)$ (see, for example, [6, 12, 13, 15]).

For two arbitrary vector-valued sequences

$$x := \{x_n\} := \begin{Bmatrix} x_n^{(1)} \\ x_n^{(2)} \end{Bmatrix} \quad \text{and} \quad y := \{y_n\} := \begin{Bmatrix} y_n^{(1)} \\ y_n^{(2)} \end{Bmatrix},$$

denote by $[x, y]$ the sequence with components $[x, y]_n$ defined by

$$[x, y]_n = (x_n^{(1)}, C_n y_{n+1}^{(2)})_E - (C_n x_{n+1}^{(2)}, y_n^{(1)})_E, \quad n \in \{-1\} \cup \mathbb{N}_0.$$

For all vectors $x = \{x_n\}$, $y = \{y_n\}$, $n \in \{-1\} \cup \mathbb{N}_0$, we have Green's formula

$$\sum_{n=0}^{l} [((\Lambda_1 x)_n, y_n)_E - (x_n, (\Lambda_1 y)_n)_E] = [x, y]_l - [x, y]_{-1}, \quad l \in \mathbb{N}_0. \tag{2.2}$$

For passing from the system (2.1) to operators, we consider the Hilbert space $H := \ell^2_{\mathbf{A}}(\mathbb{N}_0; E \oplus E)$ consisting of all vector-valued sequences $x = \{x_n\}$ $(n \in \mathbb{N}_0)$ such that $\sum_{n=0}^{\infty} (\mathbf{A}_n x_n, x_n)_E < \infty$ with the inner product $(x, y) = \sum_{n=0}^{\infty} (\mathbf{A}_n x_n, y_n)_E$, where $\mathbf{A} := \{\mathbf{A}_n\}$ $(n \in \mathbb{N}_0)$.

Denote by $\Lambda_1 x$ (*resp.* Λx) the vector-valued sequence with components $(\Lambda_1 x)_n$ (resp. $(\Lambda x)_n := \mathbf{A}_n^{-1} (\Lambda_1 x)_n$) for $n \in \mathbb{N}_0$. Next, denote by $D_{\max}$ the linear set of all vectors $x \in H$ such that $\Lambda x \in H$ and $x_{-1}^{(1)} = 0$. We define the *maximal operator* $L_{\max}$ on $D_{\max}$ by the equality $L_{\max} x = \Lambda x$.

Using (2.2) we get that the limit $[x,y]_\infty = \lim_{n\to\infty}[x,y]_n$ exists and is finite for all $x, y \in D_{\max}$. Hence passing to the limit as $l \to \infty$ in (2.2), we get for arbitrary vectors $x, y \in D_{\max}$ that

$$(L_{\max}x, y) - (x, L_{\max}y) = [x,y]_\infty. \tag{2.3}$$

Let D_0' be a dense linear set in H consisting of finite vectors (i.e., vectors having only finite many nonzero components). Denote by L_0' the restriction of the operator $L_{\max}$ to D_0'. It follows from (2.3) that L_0' is symmetric. Consequentially, it admits closure which is denoted by $L_{\min}$. The domain of $L_{\min}$ consists of those vectors $x \in D_{\max}$ satisfying the condition

$$[x,y]_\infty = 0, \quad \forall\, y \in D_{\max}. \tag{2.4}$$

The *minimal operator* $L_{\min}$ is a symmetric operator with deficiency indices (n_-, n_+), where $0 \le n_\mp \le m$, and satisfying $L_{\max} = L_{\min}^*$ [8, 10, 24, 25, 28–30].

In this paper, we assume that the symmetric operator $L_{\min}$ has maximal deficiency indices (m,m), so that the *Weyl limit-circle case* holds for the expression Λ or the symmetric operator $L_{\min}$ (see [8, 24, 25, 28–30]).

The *Wronskian* of the two matrix-valued solutions $Y = \{Y_n\} = \left\{\begin{smallmatrix} Y_n^{(1)} \\ Y_n^{(2)} \end{smallmatrix}\right\}$, $Z = \{Z_n\} = \left\{\begin{smallmatrix} Z_n^{(1)} \\ Z_n^{(2)} \end{smallmatrix}\right\}$ of the system (2.1) is $W_n(Y,Z) := Z_{n+1}^{(2)*}C_n^*Y_n^{(1)} - Z_n^{(1)*}C_nY_{n+1}^{(2)}$, $n \in \mathbb{N}_0$. $W_n(Y,Z)$ is independent of n. The solutions Y and Z of this equation are linearly independent if and only if $W_n(Y,Z)$ is nonzero.

Denote by $P(\lambda) = \{P_n(\lambda)\}$ and $Q(\lambda) = \{Q_n(\lambda)\}$ $(n \in \mathbb{N}_0)$ the matrix-valued solutions of (2.1) satisfying the initial conditions

$$P_{-1}^{(1)} = I, \quad P_0^{(2)} = 0, \quad Q_{-1}^{(1)} = 0, \quad Q_0^{(2)} = C_{-1}^{-1}.$$

We have that $W_n(P(\lambda), Q(\lambda)) = I$, $n \in \mathbb{N}_0$. Consequently $P(\lambda)$ and $Q(\lambda)$ form a fundamental system of solutions of (2.1). Since $L_{\min}$ has deficiency indices (m,m), $P(\lambda)a, Q(\lambda)a \in H$ $(a \in E \oplus E)$ for all $\lambda \in \mathbb{C}$.

We set $\Upsilon = P(0)$, $\Psi = Q(0)$, and

$$x_n^{(i)} := \mathbb{V}_n^{(i)}a := (\Upsilon_n^{(i)}, \Psi_n^{(i)})\begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \Upsilon_n^{(i)}a_1 + \Psi_n^{(i)}a_2, \quad i = 1, 2,$$

where $a_1, a_2 \in E$, and let

$$\mathbb{U}_n = \begin{pmatrix} \Upsilon_n^{(1)} & \Psi_n^{(1)} \\ \Upsilon_{n+1}^{(2)} & \Psi_{n+1}^{(2)} \end{pmatrix}, \quad n \in \mathbb{N}_0.$$

Then one can show that

$$\mathbb{U}_n^{-1} = \begin{pmatrix} \Psi_{n+1}^{(2)*}C_n^* & -\Psi_n^{(1)*}C_n \\ -\Upsilon_{n+1}^{(2)*}C_n^* & \Upsilon_n^{(1)*}C_n \end{pmatrix}, \quad \text{and} \quad \mathbb{U}_n^{-1} = J\mathbb{U}_n^* J \begin{pmatrix} C_n^* & 0 \\ 0 & C_n \end{pmatrix}, \quad n \in \mathbb{N}_0,$$

where $J = i\left(\begin{smallmatrix} 0 & I \\ -I & 0 \end{smallmatrix}\right)$, $J = J^*$, $J^2 = I_{E\oplus E}$, $I_{E\oplus E}$ is the identity operator on $E \oplus E$.

Now we introduce the following notation:

$$(Sx)_n := \begin{pmatrix} (S_1x)_n \\ (S_2x)_n \end{pmatrix} := \mathbb{U}_n^{-1} \begin{pmatrix} x_n^{(1)} \\ x_{n+1}^{(2)} \end{pmatrix} = \begin{pmatrix} \Psi_{n+1}^{(2)*}C_n^*x_n^{(1)} - \Psi_n^{(1)*}C_nx_{n+1}^{(2)} \\ -\Upsilon_{n+1}^{(2)*}C_n^*x_n^{(1)} + \Upsilon_n^{(1)*}C_nx_{n+1}^{(2)} \end{pmatrix}, \quad n \in \mathbb{N}_0.$$

Then we have

Lemma 2.1. *For arbitrary vector $x \in D_{\max}$ the limit $\lim_{n\to\infty}(Sx)_n = (Sx)(\infty)$ exists and is a finite.*

Proof. Let $x, y \in D_{\max}$. Then the Green's formulas (2.2) and (2.3) hold. Further, for $x_n^{(i)} = \mathbb{V}_n^{(i)}a$ $(i = 1, 2)$, $x = \{x_n\} \in D_{\max}$ and $y = \{y_n\} \in D_{\max}$ $(n \in \mathbb{N}_0)$, we have

$$\begin{aligned}
[x, y]_n &= i\left(J\begin{pmatrix} C_n^*x_n^{(1)} \\ C_nx_{n+1}^{(2)} \end{pmatrix}, \begin{pmatrix} y_n^{(1)} \\ y_{n+1}^{(2)} \end{pmatrix}\right)_{E\oplus E} = i\left(J\begin{pmatrix} C_n^*\mathbb{V}_n^{(1)}a \\ C_n\mathbb{V}_{n+1}^{(2)}a \end{pmatrix}, \begin{pmatrix} y_n^{(1)} \\ y_{n+1}^{(2)} \end{pmatrix}\right)_{E\oplus E} \\
&= i\left(J\begin{pmatrix} C_n^* & 0 \\ 0 & C_n \end{pmatrix}\mathbb{U}_na, \begin{pmatrix} y_n^{(1)} \\ y_{n+1}^{(2)} \end{pmatrix}\right)_{E\oplus E} = i\left(a, \mathbb{U}_n^*\begin{pmatrix} C_n & 0 \\ 0 & C_n^* \end{pmatrix} J \begin{pmatrix} y_n^{(1)} \\ y_{n+1}^{(2)} \end{pmatrix}\right)_{E\oplus E} \\
&= i\left(a, J^2\mathbb{U}_n^*J\begin{pmatrix} C_n^* & 0 \\ 0 & C_n \end{pmatrix}\begin{pmatrix} y_n^{(1)} \\ y_{n+1}^{(2)} \end{pmatrix}\right)_{E\oplus E} = i\left(Ja, \mathbb{U}_n^{-1}\begin{pmatrix} y_n^{(1)} \\ y_{n+1}^{(2)} \end{pmatrix}\right)_{E\oplus E} \\
&= i(Ja, (Sy)_n)_{E\oplus E}.
\end{aligned}$$

From this it follows that, for all $y \in D_{\max}$, there exists the limit $\lim_{n\to\infty}(Sy)_n = (Sy)(\infty)$ and is a finite. Lemma 2.1 is proved. □

Lemma 2.2. *For any vectors $\gamma, \delta \in E$, there exists a vector $x \in D_{\max}$ satisfying the conditions*

$$(S_1x)(\infty) = \gamma, \quad (S_2x)(\infty) = \delta.$$

Proof. Consider that g be an arbitrary vector in H satisfying the conditions

$$(g, \Psi\varphi_j) = \gamma_j, \quad (g, \Upsilon\varphi_j) = -\delta_j, \quad j = 1, 2, \ldots, m, \tag{2.5}$$

where $\{\varphi_j\}$ $(j = 1, 2, \ldots, m)$ is an orthonormal basis in E, and $\gamma_j = (\gamma, \varphi_j)_E$, $\delta_j = (\delta, \varphi_j)_E$ $(j = 1, 2, \ldots, m)$. Such an vectors g exists, and even among linear combinations of the vectors $\Upsilon\varphi_j$ and $\Psi\varphi_n$ $(j,n=1,2,\dots,m)$. Since the operator $L_{\min}$ has deficiency indices (m,m), we have $\Upsilon\varphi_j, \Psi\varphi_j \in H$, $j=1,2,\dots,m$. Indeed, if we set $g=\sum_{j=1}^m a_1^{(j)}\Upsilon\varphi_j + \sum_{j=1}^m a_2^{(j)}\Psi\varphi_j$, then condition (2.5) is a system of equations for the constants $a_1^{(j)}$ and $a_2^{(j)}$ $(j=1,2,\dots,m)$ whose determinant is the Gram determinant of the linearly independent vectors $\Upsilon\varphi_j$ and $\Psi\varphi_n$ $(j,n=1,2,\dots,m)$ and is therefore nonzero.

Let us denote by $x=\{x_n\}$ the solution of the equation $\Lambda x = g$ satisfying the conditions $x_{-1}^{(1)} = x_0^{(2)} = 0$. This solution belongs to H. Applying (2.2) as $l\to\infty$, we obtain

$$\begin{aligned}(g,\Upsilon\varphi_j) &= (\Lambda x,\Upsilon\varphi_j) = [x,\Upsilon\varphi_j]_\infty + (x,\Lambda\Upsilon\varphi_j),\\ (g,\Psi\varphi_j) &= (\Lambda x,\Psi\varphi_j) = [x,\Psi\varphi_j]_\infty + (x,\Lambda\Psi\varphi_j).\end{aligned} \tag{2.6}$$

Taking into account that $\Lambda\Upsilon\varphi_j=0$, $\Lambda\Psi\varphi_j=0$, we have $(x,\Lambda\Upsilon\varphi_j)=0$, $(x,\Lambda\Psi\varphi_j)=0$, $j=1,2,\dots,m$. Then, from relation (2.6) it follows that

$$\begin{aligned}-\delta_j &= [x,\Upsilon\varphi_j]_\infty = -((S_2x)(\infty),\varphi_j)_E,\\ \gamma_j &= [x,\Psi\varphi_j]_\infty = ((S_1x)(\infty),\varphi_j)_E, \quad j=1,2,\dots,m.\end{aligned}$$

From this we have $(S_1x)(\infty)=\gamma$, $(S_2x)(\infty)=\delta$. Lemma 2.2 is proved. □

Lemma 2.3. *For arbitrary vectors $x,y\in D_{\max}$, the identity*

$$[x,y]_n = ((S_1x)_n,(S_2y)_n)_E - ((S_2x)_n,(S_1y)_n)_E, \quad n\in\mathbb{N}_0$$

holds. In particular,

$$[x,y]_\infty = ((S_1x)(\infty),(S_2y)(\infty))_E - ((S_2x)(\infty),(S_1y)(\infty))_E.$$

Proof. For arbitrary $x,y\in D_{\max}$, we have

$$\begin{aligned}&((S_1x)_n,(S_2y)_n)_E - ((S_2x)_n,(S_1y)_n)_E\\ &= i(J(Sx)_n,(Sy)_n)_{E\oplus E} = i\left(J\mathbb{U}_n^{-1}\begin{pmatrix}x_n^{(1)}\\ x_{n+1}^{(2)}\end{pmatrix}, \mathbb{U}_n^{-1}\begin{pmatrix}y_n^{(1)}\\ y_{n+1}^{(2)}\end{pmatrix}\right)_{E\oplus E}\\ &= i\left(J\mathbb{U}_n^{-1}\begin{pmatrix}x_n^{(1)}\\ x_{n+1}^{(2)}\end{pmatrix}, J\mathbb{U}_n^{*}J\begin{pmatrix}C_n^* & 0\\ 0 & C_n\end{pmatrix}\begin{pmatrix}y_n^{(1)}\\ y_{n+1}^{(2)}\end{pmatrix}\right)_{E\oplus E}\\ &= i\left(\begin{pmatrix}C_n & 0\\ 0 & C_n^*\end{pmatrix}J\mathbb{U}_nJ^2\mathbb{U}_n^{-1}\begin{pmatrix}x_n^{(1)}\\ x_{n+1}^{(2)}\end{pmatrix}, \begin{pmatrix}y_n^{(1)}\\ y_{n+1}^{(2)}\end{pmatrix}\right)_{E\oplus E}\\ &= i\left(J\begin{pmatrix}C_n^* & 0\\ 0 & C_n\end{pmatrix}\begin{pmatrix}x_n^{(1)}\\ x_{n+1}^{(2)}\end{pmatrix}, \begin{pmatrix}y_n^{(1)}\\ y_{n+1}^{(2)}\end{pmatrix}\right)_{E\oplus E} = [x,y]_n, \quad n\in\mathbb{N}_0.\end{aligned}$$

Passing to the limit as $n \to \infty$ in the previous equality, we get

$$((S_1x)(\infty), (S_2y)(\infty))_E - ((S_2x)(\infty), (S_1y)(\infty))_E = [x, y]_\infty.$$

Lemma 2.3 is proved. □

Theorem 2.4. *The domain $D_{\min}$ of the operator $L_{\min}$ consists of the vectors $x \in D_{\max}$ satisfying the boundary conditions at infinity:*

$$(S_1x)(\infty) = (S_2x)(\infty) = 0. \tag{2.7}$$

Proof. As noted above, the domain $D_{\min}$ of the operator $L_{\min}$ coincides with the set of all vectors $x \in D_{\max}$ satisfying condition (2.4). Hence using Lemma 2.3, (2.4) is equivalent to the condition

$$((S_1x)(\infty), (S_2y)(\infty))_E - ((S_2x)(\infty), (S_1y)(\infty))_E = 0. \tag{2.8}$$

Since $(S_1y)(\infty)$ and $(S_2y)(\infty)$ ($y \in D_{\max}$) can be arbitrary (see Lemma 2.2), equality (2.8) (for all $y \in D_{\max}$) is possible if and only if conditions (2.7) hold. This proves Theorem 2.4. □

The space of boundary values of the symmetric operator stays in the centre of the extension theory. We shall remind that a triplet $(\mathbb{H}, G_1, G_2)$, where $\mathbb{H}$ is a Hilbert space and G_1 and G_2 are linear mappings of $\mathfrak{D}(\mathbf{T}^*)$ into $\mathbb{H}$, is called (see [10, p. 152]) a *space of boundary values* of a closed symmetric operator $\mathbf{T}$, acting in a Hilbert space $\mathbf{H}$ with equal (finite or infinite) deficiency indices, if

(1) for every $f, g \in D(\mathbf{T}^*)$, $(\mathbf{T}^*f, g)_\mathbf{H} = (f, \mathbf{T}^*g)_\mathbf{H} = (G_1f, G_2f)_\mathbb{H} - (G_1g, G_2g)_\mathbb{H}$;

(2) for every $F_1, F_2 \in \mathbb{H}$, there exists a vector $f \in D(\mathbf{T}^*)$ such that $G_1f = F_1$, $G_2f = F_2$.

We consider the following mappings from $D_{\max}$ into E

$$G_1x = -(S_2x)(\infty), \quad G_2x = (S_1x)(\infty). \tag{2.9}$$

Then we have

Theorem 2.5. *The triple (E, G_1, G_2) defined by (2.9) is the space of boundary values of the minimal operator $L_{\min}$.*

Proof. By Lemma 2.3, for arbitrary $x, y \in D_{\max}$ we have

$$\begin{aligned}(L_{\max}x, y) - (x, L_{\max}y) &= [x, y]_\infty \\ &= ((S_1x)(\infty), (S_2y)(\infty))_E - ((S_2x)(\infty), (S_1y)(\infty))_E \\ &= (G_1x, G_2y)_E - (G_2x, G_1y)_E,\end{aligned}$$

i.e., the first requirement of the definition of the space of boundary values is valid. The second requirement is valid due to Lemma 2.2. The theorem is proved. □

Recall that a linear operator T (with dense domain $D(T)$) acting on some Hilbert space H is called *dissipative (accumulative)* if $\Im(Tf,f) \geq 0$ ($\Im(Tf,f) \leq 0$) for all $f \in D(T)$ and *maximal dissipative (accumulative)* if it does not have a proper dissipative (accumulative) extension (see [10, p. 149]).

By [7], [10, p. 156, Theorem 1.6] and [14], Theorem 3.1 implies the following

Theorem 2.6. *For any contraction R in E, the restriction of the operator $L_{\max}$ to the set of vectors $x \in D_{\max}$ satisfying the boundary condition*

$$(R-I)G_1x + i(R+I)G_2x = 0 \tag{2.10}$$

or

$$(R-I)G_1x - i(R+I)G_2x = 0 \tag{2.11}$$

is respectively the maximal dissipative and maximal accumulative extension of the operator $L_{\min}$. Conversely, every maximal dissipative (resp. accumulative) extension of the operator $L_{\min}$ is the restriction of the operator $L_{\max}$ to the set of vectors $x \in D_{\max}$ satisfying (2.10) *(resp.* (2.11)*), and the contraction R is uniquely determined by the extension. These conditions give a self-adjoint extensions if R is unitary. In this case* (2.10), (2.11) *are equivalent to the condition $(\cos T)G_1x - (\sin T)G_2x = 0$, where T is a self-adjoint operator in E.*

The general form of dissipative and accumulative extensions of an operator $L_{\min}$ is given by the conditions

$$R(G_1x + iG_2x) = G_1x - iG_2x, \quad G_1x + iG_2x \in D(R), \tag{2.12}$$

$$R(G_1x - iG_2x) = G_1x + iG_2x, \quad G_1x - iG_2x \in D(R) \tag{2.13}$$

respectively, where R is a linear operator with $\|Rf\| \leq \|f\|$, $f \in D(R)$. The general form of symmetric extensions is given by (2.12) *and* (2.13)*, where R is an isometric operator in E.*

In this paper, we consider the maximal dissipative operator L_R, where R is the strict contraction in E, i.e., $\|R\| < 1$, generated by the difference expression Λ and the boundary condition (2.10). Since R is a strict contraction, the operator $R + I$ must be invertible, and the boundary condition (2.10) is equivalent to the condition

$$G_1x - KG_2x = 0, \tag{2.14}$$

where $K = -i(R-I)^{-1}(R+I)$, $\Im K > 0$ and R is the Cayley transform of the dissipative operator K. We denote by $\widetilde{L}_K$ $(= L_R)$ the maximal dissipative operator generated by expression Λ and the boundary condition (2.14).

3. Self-adjoint dilation of the maximal dissipative operator

We add the 'incoming' and 'outgoing' channels $\mathcal{L}^2((-\infty,0);E)$ and $\mathcal{L}^2((0,\infty);E)$ to the space H and then form the *main Hilbert space of the dilation* $\mathcal{H} := \mathcal{L}^2((-\infty,0);E)\oplus H\oplus \mathcal{L}^2((0,\infty);E)$. In $\mathcal{H}$ we consider the operator $\mathcal{L}_K$ generated by the expression

$$\mathcal{L}\langle \phi^-, u, \phi^+\rangle = \left\langle i\frac{d\phi^-}{d\xi}, \Lambda u, i\frac{d\phi^+}{d\varsigma}\right\rangle$$

on the set of vectors $D(\mathcal{L}_K)$ satisfying the conditions: $\phi^- \in W_2^1((-\infty,0);E)$, $\phi^+ \in W_2^1((0,\infty);E)$, $u \in D_{\max}$,

$$G_1u - KG_2u = T\phi^-(0), \quad G_1u - K^*G_2u = T\phi^+(0), \tag{3.1}$$

where W_2^1 is the Sobolev space, and $T^2 := 2\Im K$, $T > 0$. Then we have

Theorem 3.1. *The operator $\mathcal{L}_K$ is self-adjoint in $\mathcal{H}$ and is a self-adjoint dilation of the dissipative operator $\widetilde{L}_K$ (L_R).*

Proof. For $U_1 = \langle \phi_1^-, u_1, \phi_1^+\rangle \in D(\mathcal{L}_K)$ and $U_2 = \langle \phi_2^-, u_2, \phi_2^+\rangle \in D(\mathcal{L}_K)$, we have $(\mathcal{L}_KU_1, U_2)_{\mathcal{H}} - (U_1, \mathcal{L}_KU_2)_{\mathcal{H}} = 0$. Hence $\mathcal{L}_K$ is a symmetric operator. One can show that operators $\mathcal{L}_K$ and $\mathcal{L}_K^*$ can be written by the same differential expression $\mathcal{L}$. Moreover, if we suppose that $U_2 \in D(\mathcal{L}_K^*)$ and if we accept that $\phi_2^- \in W_2^1((-\infty,0);E)$, $\phi_2^+ \in W_2^1((0,\infty);E)$, $u_2 \in D_{\max}$ then the following conditions are satisfied: $G_1u_2 - KG_2u_2 = T\phi_2^-(0)$, $G_1u_2 - K^*G_2u_2 = T\phi_2^+(0)$. Hence, $\mathcal{L}_K^* \subseteq \mathcal{L}_K$ or $\mathcal{L}_K^* = \mathcal{L}_K$.

It is known that the self-adjoint operator $\mathcal{L}_K$ generates the unitary group $\mathcal{X}(s) = \exp(i\mathcal{L}_Ks)$ ($s \in \mathbb{R} := (-\infty,\infty)$) on $\mathcal{H}$. Denote by $P\colon \mathcal{H} \to H$ and $P_1\colon H \to \mathcal{H}$ the mappings acting according to the formulae $P\colon \langle \phi^-, u, \phi^+\rangle \to u$ and $P_1\colon u \to \langle 0, u, 0\rangle$, respectively. Let $Z(s) = P\mathcal{X}(s)P_1$, $s \geq 0$. The operator family $\{Z(s)\}$ ($s \geq 0$) of operators is a strongly continuous semigroup of completely nonunitary contractions on H. Let us denote by T the generator of this semigroup $\{Z(s)\}$: $Tx = \lim_{s\to+0}(is)^{-1}(Z(s)x - x)$. The operator T is a maximal dissipative and operator $\mathcal{L}_K$ is called the *self-adjoint dilation* of T. We shall show that $T = \widetilde{L}_K$. So we will have shown that $\mathcal{L}_K$ is a self-adjoint dilation of $\widetilde{L}_K$. We want to verify the following equality

$$P(\mathcal{L}_K - \lambda I)^{-1}P_1x = (\widetilde{L}_K - \lambda I)^{-1}x, \quad x \in H,\ \Im\lambda < 0. \tag{3.2}$$

For this purpose, we set $(\mathcal{L}_K - \lambda I)^{-1}P_1x = g = \langle \psi^-, y, \psi^+\rangle$. Hence we get that $(\mathcal{L}_K - \lambda I)g = P_1x$, and $L_{\max}y - \lambda y = x$, $\psi^-(\xi) = \psi^-(0)e^{-i\lambda\xi}$, $\psi^+(\varsigma) = \psi^+(0)e^{-i\lambda\varsigma}$. Since $g \in D(\mathcal{L}_K)$, hence $\psi^- \in L^2((-\infty,0);E)$; it follows that $\psi^-(0) = 0$, and consequently, y satisfies the boundary condition $G_1y - KG_2y = 0$. Therefore, $y \in D(\widetilde{L}_K)$, and since a point λ with $\Im\lambda < 0$ cannot be an eigenvalue of a dissipative operator, it follows that $y = (\widetilde{L}_K - \lambda I)^{-1}x$. Thus for $x \in H$ and $\Im\lambda < 0$ we have

$$(\mathcal{L}_K - \lambda I)^{-1}P_1x = \left\langle 0, (\widetilde{L}_K - \lambda I)^{-1}x, T^{-1}(G_1x - K^*G_2x)e^{-i\lambda\zeta} \right\rangle .$$

Applying the mapping P to this equality, we get (3.2).

It is now easy to show that $T = \widetilde{L}_K$. Indeed, by (3.2)

$$\begin{aligned}(\widetilde{L}_K - \lambda I)^{-1} &= P(\mathcal{L}_K - \lambda I)^{-1}P_1 = -iP\int_0^\infty \mathcal{X}(s)e^{-i\lambda s}\,ds\,P_1 \\ &= -i\int_0^\infty Z(s)e^{-i\lambda s}\,ds = (T-\lambda I)^{-1}.\end{aligned}$$

This implies that $\widetilde{L}_K = T$ and completes the proof of Theorem 3.1. □

4. Scattering theory of dilation and functional model of dissipative operator

The unitary group $\{\mathcal{X}(s)\}$ has an important property which allows us to apply the Lax-Phillips scheme [17]. Namely, it has the *incoming* and *outgoing subspaces* $\mathcal{D}^- = \langle L^2((-\infty, 0); E), 0, 0\rangle$ and $\mathcal{D}^+ = \langle 0, 0, L^2((0, -\infty); E)\rangle$ with the following properties:

(1) $\mathcal{X}(s)\mathcal{D}^- \subset \mathcal{D}^-$, $s \le 0$; $\mathcal{X}(s)\mathcal{D}^+ \subset \mathcal{D}^+$, $s \ge 0$;

(2) $\bigcap_{s\le 0}\mathcal{X}(s)\mathcal{D}^- = \bigcap_{s\le 0}\mathcal{X}(s)\mathcal{D}^+ = \{0\}$;

(3) $\overline{\bigcup_{s\ge 0}\mathcal{X}(s)\mathcal{D}^-} = \overline{\bigcup_{s\le 0}\mathcal{X}(s)\mathcal{D}^+} = \mathcal{H}$;

(4) $\mathcal{D}^- \perp \mathcal{D}^+$.

Property (4) follows from the inner product in $\mathcal{H}$. We prove property (1) and (2) for $\mathcal{D}^+$. The proof for $\mathcal{D}^-$ can be done similarly. Let $\mathcal{R}_\lambda = (\mathcal{L}_K - \lambda I)^{-1}$. For all λ with $\Im\lambda < 0$ and for all $f = \langle 0, 0, \phi\rangle \in \mathcal{D}^+$, we have

$$\mathcal{R}_\lambda f = \left\langle 0, 0, -ie^{-i\lambda\xi}\int_0^\xi e^{i\lambda s}\phi(s)\,ds \right\rangle .$$

From this we get that $\mathcal{R}_\lambda f \in \mathcal{D}^+$ and if $g \perp \mathcal{D}^+$, then one obtains that

$$0 = (\mathcal{R}_\lambda f, g)_\mathcal{H} = -i\int_0^\infty e^{-i\lambda s}(\mathcal{X}(s)f, g)_\mathcal{H}\,ds, \quad \Im\lambda < 0,$$

which implies that$(\mathcal{X}(s)f, g)_\mathcal{H} = 0$ for all $s \ge 0$. Consequently, $\mathcal{X}(s)\mathcal{D}^+ \subset \mathcal{D}^+$ for $s \ge 0$, and property (1) is proved.

To prove property (2), let $P_+ \colon \mathcal{H} \to H$ and $\mathcal{P}_+ \colon H \to \mathcal{D}^+$ be the mappings acting according to the formulae $P_+ \colon \langle\phi^-, x, \phi^+\rangle \to \phi^+$ and $\mathcal{P}_+ \colon \phi \to \langle 0, 0, \phi\rangle$, and we observe that the semigroup of isometries $U^+(s) = P_+\mathcal{X}(s)\mathcal{P}_+$ $(s \geq 0)$ is the one-sided shift in H. Indeed, the differentiation operator $i\left(\frac{d}{d\xi}\right)$ with boundary condition $\phi(0) = 0$ is the generator of the one-sided shift semigroup $V(s)$ on H $(V(s)\phi(\xi) = \phi(\xi - s)$ for $\xi > s$, and $V(s)\phi(\xi) = 0$ for $0 \leq \xi \leq s)$. On the other hand, the generator $\mathbb{S}$ of the semigroup of isometries $U^+(s)$, $s \geq 0$, is given by $\mathbb{S}\phi = P_+\mathcal{L}_K\mathcal{P}_+\phi = P_+\mathcal{L}_K\langle 0, 0, \phi\rangle = P_+\langle 0, 0, i\frac{d\phi}{d\xi}\rangle = i\frac{d\phi}{d\xi}$, where $\phi \in W_2^1((0,\infty); E)$ and $\phi(0) = 0$. But since a semigroup is uniquely determined by its generator, $U^+(s) = V(s)$, hence

$$\bigcap_{s\geq 0} \mathcal{X}(s)\mathcal{D}^+ = \left\langle 0, 0, \bigcap_{s\geq 0} V(s)L^2((0,\infty); E)\right\rangle = \{0\},$$

i.e., property (2) is proved.

We shall remind that the linear operator $\mathbf{T}$ (with domain $D(\mathbf{T})$) acting in the Hilbert space $\mathbf{H}$ is called *completely non-self-adjoint* (or *simple*) if there is no invariant subspace $M \subseteq D(\mathbf{T})$ $(M \neq \{0\})$ of the operator $\mathbf{T}$ on which the restriction $\mathbf{T}$ to M is self-adjoint.

In order to prove property (3), let us prove the following

Lemma 4.1. *The operator $\widetilde{L}_K$ is totally non-self-adjoint (simple).*

Proof. Let $M \subset H$ be the subspace on which $\widetilde{L}_K$ induces a self-adjoint operator $\widetilde{L}'_K$ (i.e., the subspace M is invariant with respect to the semigroup of isometries $V_s = \exp(i\widetilde{L}'_K s)$, $V_s^* = \exp(-i\widetilde{L}'^*_K s)$, $V_s^{*-1} = V_s$, $s > 0)$. If $f \in M \cap (\widetilde{L}_K)$, then $f \in D(\widetilde{L}^*_K)$ and $G_1 f - KG_2 f = 0$, $G_1 f - K^* G_2 f = 0$, i.e., $G_1 f = G_2 f = 0$. From this for eigenvectors u_λ of operator $\widetilde{L}_K$ that lie in K and are eigenvectors of $\widetilde{L}'_K$ we have $(S_1 u_\lambda)(\infty) = 0$, $(S_2 u_\lambda)(-\infty) = 0$. Now we shall prove that $u_\lambda \equiv 0$. Suppose that it is not correct, i.e., $u_\lambda \neq 0$. The eigenvectors u_λ of operator $\widetilde{L}_K$ is expressed by the matrix solution $Q(\lambda)$: $u_\lambda = Q(\lambda)a$, $a \neq 0$. It follows from Lemma 2.3 that $[P(\lambda)a, Q(\lambda)a]_\infty = 0$, and from the other hand, we know that $[P(\lambda)a, Q(\lambda)a]_\infty = a \neq 0$. The contradiction we have obtained gives $u_\lambda \equiv 0$. Since all solutions of $\Lambda x = \lambda x$ belong to H, from this it can be concluded that the resolvent $R_\lambda(\widetilde{L}_K)$ of the operator $\widetilde{L}_K$ is compact operator, and the spectrum of $\widetilde{L}_K$ is purely discrete. Hence, by theorem on expansion in eigenvectors of the self-adjoint operator $\widetilde{L}'_K$, we have $M = \{0\}$, i.e., the operator L_K is simple. The lemma is proved. □

To prove property (3), we let

$$\mathcal{H}_- = \overline{\bigcup_{s\geq 0} \mathcal{X}(s)\mathcal{D}^-}, \quad \mathcal{H}_+ = \overline{\bigcup_{s\leq 0} \mathcal{X}(s)\mathcal{D}^+}$$

and first prove

Lemma 4.2. *The equality $\mathcal{H}_- + \mathcal{H}_+ = \mathcal{H}$ holds.*

Proof. Indeed, by taking into account property (1) of the subspace $\mathcal{D}^{\pm}$, it is easy to see that the subspace $\mathcal{H}' = \mathcal{H} \ominus (\mathcal{H}_- + \mathcal{H}_+)$ is invariant under to the group $\{\mathcal{X}(s)\}$ and has the form $\mathcal{H}' = \langle 0, H', 0\rangle$, where H' is subspace of H. hence in the case that the subspace $\mathcal{H}'$ (and also H') is nontrivial, then the unitary group $\{U'(s)\}$ restricted of $\mathcal{X}(s)$ to H' will be a self-adjoint operator in H'. But this is impossible from Lemma 4.1. Hence the lemma is proved. □

According to the Lax-Phillips scattering theory one can construct the scattering matrix with the help of the spectral representations. Now we shall construct the spectral representations. We now proceed to their construction, and on this path we also prove property (3).

The Weyl matrix-valued function $M_\infty(\lambda)$ of the self-adjoint operator L_∞, generated by the boundary condition $(S_1x)(\infty) = 0$, is uniquely determined from the condition $(S_1(P + M_\infty(\lambda)Q)(\infty) = 0$. In view of this we have

$$M(\lambda) := M_\infty(\lambda) = -(S_1P)(\infty)[(S_1Q)(\infty)]^{-1}.$$

Now let us define $F(\lambda)$ and $\Phi(\lambda)$ by $F(\lambda) = -(S_2Q)(\infty)[(S_1P)(\infty)]^{-1}$ and $\Phi(\lambda) = F(\lambda)M(\lambda)$, respectively. It is easy to show that the matrix-valued function $\Phi(\lambda)$ is a meromorphic in $\mathbb{C}$ with all its poles on real axis $\mathbb{R}$, and that it has the following properties:

(a) $\Im\Phi(\lambda) \leq 0$ for $\Im\lambda > 0$, and $\Im\Phi(\lambda) \geq 0$ for $\Im\lambda < 0$;

(b) $\Phi^*(\lambda) = \Phi(\overline{\lambda})$ for all $\lambda \in \mathbb{C}$ except at the poles of $\Phi(\lambda)$.

Let

$$\begin{aligned}\mathcal{V}_{\lambda j}^{-}(\xi, \varsigma) = \big\langle e^{-i\lambda\xi}\varphi_j, -Q(\lambda)[(S_1Q)(\infty)]^{-1}[\Phi(\lambda) + K]^{-1}T\varphi_j,\\ T^{-1}(\Phi(\lambda) + K^*)(\Phi(\lambda) + K)^{-1}Te^{-i\lambda\varsigma}\varphi_j\big\rangle, \quad j = 1, 2, \dots, m,\end{aligned}$$

where $\varphi_1, \varphi_2, \dots, \varphi_m$ are an orthonormal basis for E. It must be noted that vectors $\mathcal{V}_{\lambda j}^{-}$ $(j = 1, 2, \dots, m)$ for all $\lambda \in \mathbb{R}$ do not belong to $\mathcal{H}$. However, $\mathcal{V}_{\lambda j}^{-}$ $(j = 1, 2, \dots, m)$ satisfy the equation $\mathcal{L}\mathcal{V} = \lambda\mathcal{V}$ and the boundary conditions (3.1) for the operator $\mathcal{L}_K$.

With the help of $\mathcal{V}_{\lambda j}^{-}$ $(j = 1, 2, \dots, m)$, we define the transformation $\mathcal{F}_- \colon f \to \widetilde{f}_-(\lambda)$ by

$$(\mathcal{F}_-f)(\lambda) := \widetilde{f}_-(\lambda) := \sum_{j=1}^{m} f_j^-(\lambda)\varphi_j := \frac{1}{\sqrt{2\pi}}\sum_{j=1}^{m}(f, \mathcal{V}_{\lambda j}^{-})_{\mathcal{H}}\varphi_j$$

on the vectors $f = \langle\phi^-, u, \phi^+\rangle$ in which ϕ^-, ϕ^+ are smooth, compactly supported vector-valued functions, and $u = \{u_n\}$ $(n \in \mathbb{N}_0)$ is a finite vector sequence.

Lemma 4.3. *The transformation $\mathcal{F}_-$ maps $\mathcal{H}_-$ isometrically onto $L^2(\mathbb{R}; E)$, and for all $f, g \in \mathcal{H}_-$ the Parseval equality and the inversion formula are valid:*

$$(f,g)_{\mathcal{H}} = (\widetilde{f}_-, \widetilde{g}_-)_{L^2} = \int_{-\infty}^{\infty} \sum_{j=1}^{m} \widetilde{f}_j^-(\lambda)\overline{\widetilde{g}_j^-(\lambda)}\, d\lambda,$$

$$f = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \sum_{j=1}^{m} \mathcal{V}_{\lambda j}^- \widetilde{f}_j^-(\lambda)\, d\lambda,$$

where $\widetilde{f}_-(\lambda) = (\mathcal{F}_- f)(\lambda)$ and $\widetilde{g}_-(\lambda) = (\mathcal{F}_- g)(\lambda)$.

Proof. For arbitrary $f, g \in \mathcal{D}^-$ such that $f = \langle \phi^-, 0, 0\rangle$, $g = \langle \psi^-, 0, 0\rangle$, we have

$$\widetilde{f}_-(\lambda) = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{m} (f, \mathcal{V}_{\lambda j}^-)_{\mathcal{H}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} \phi^-(\xi) e^{i\lambda\xi}\, d\xi \in H_-^2(E).$$

Using the Parseval equality for Fourier integrals we get that

$$\begin{aligned}(f,g)_{\mathcal{H}} &= \int_{-\infty}^{0} (\phi^-(\xi), \psi^-(\xi))_E\, d\xi \\ &= \int_{-\infty}^{\infty} (\widetilde{f}_-(\lambda), \widetilde{g}_-(\lambda))_E\, d\lambda = (\mathcal{F}_- f, \mathcal{F}_- g)_{L^2}.\end{aligned}$$

Here and below, $H_\pm^2(E)$ denotes the Hardy classes in $L^2(\mathbb{R}; E)$ consisting of the vector-valued functions analytically extendible to the upper and lower half-planes, respectively.

To extend the Parseval equality to whole of $\mathcal{H}_-$, we shall consider the dense set $\mathcal{H}'_-$ in $\mathcal{H}_-$ consisting of the vectors obtained as follows from the smooth, compactly supported vector-valued functions $\mathcal{D}^-$: $f \in \mathcal{H}'_-$ if $f = \mathcal{X}(s) f_0$, $f_0 = \langle \phi^-, 0, 0\rangle$, $\phi^- \in C_0^\infty((-\infty, 0); E)$, where s_f is a non-negative number (depending on f). In this case, if $f, g \in \mathcal{H}_-$, then for $s > s_f$ and $s > s_g$ we have $U(-s)f, U(-s)g \in \mathcal{D}^-$ and, moreover, the first components of these vectors belong to $C_0^\infty((-\infty, 0); E)$. Therefore, since the operators $\mathcal{X}(s)$ $(s \in \mathbb{R})$ are unitary, by the equality

$$\mathcal{F}_- \mathcal{X}(s) f = \frac{1}{\sqrt{2\pi}} \sum_{j=1}^{m} (\mathcal{X}(s) f, \mathcal{V}_{\lambda j}^-)_{\mathcal{H}} \varphi_j = \frac{1}{\sqrt{2\pi}} e^{i\lambda s} \sum_{j=1}^{m} (f, \mathcal{V}_{\lambda j}^-)_{\mathcal{H}} \varphi_j = e^{i\lambda s} \mathcal{F}_- f,$$

we have that

$$\begin{aligned}(f,g)_{\mathcal{H}} &= (U(-s)f, U(-s)g)_{\mathcal{H}} = (\mathcal{F}_- U(-s) f, \mathcal{F} U(-s) g)_{L^2} \\ &= (e^{-i\lambda s} \mathcal{F}_- f, e^{-i\lambda s} \mathcal{F}_- g)_{L^2} = (\mathcal{F}_- f, \mathcal{F}_- g)_{L^2}.\end{aligned} \tag{4.1}$$

Hence taking closure in (4.1), we obtain the Parseval equality for the whole of $\mathcal{H}_-$. The inversion formula follows from the Parseval equality if all the integrals in it are understood as limits in the mean of integrals over finite intervals. Finally we get that

$$\mathcal{F}_- \mathcal{H}_- = \overline{\bigcup_{s \geq 0} \mathcal{X}(s) \mathcal{D}^-} = \overline{\bigcup_{s \geq 0} e^{-i\lambda s} H_-^2(E)} = L^2(\mathbb{R}; E),$$

i.e., $\mathcal{F}_-$ maps $\mathcal{H}_-$ onto the whole of $L^2(\mathbb{R};E)$. Lemma 4.3 is proved. $\square$

We set

$$\begin{aligned}&\mathcal{V}^+_{\lambda j}(\xi,\varsigma)\\&=\langle \mathcal{S}_K e^{-i\lambda\xi}\varphi_j, -Q(\lambda)[(S_1Q)(\infty)]^{-1}(\Phi(\lambda)+K^*)^{-1}T\varphi_j, e^{-i\lambda\varsigma}\varphi_j\rangle, \quad j=1,2,\dots,m,\end{aligned}$$

where

$$\mathcal{S}_K(\lambda) := T^{-1}(\Phi(\lambda)+K)(\Phi(\lambda)+K^*)^{-1}T. \tag{4.2}$$

It must be noted that vectors $\mathcal{V}^-_{\lambda j}$ $(j=1,2,\dots,m)$ for all $\lambda\in\mathbb{R}$ do not belong to $\mathcal{H}$. However, $\mathcal{V}^+_{\lambda j}$ $(j=1,2,\dots,m)$ satisfy the equation $\mathcal{L}\mathcal{V}=\lambda\mathcal{V}$ and the boundary conditions (3.1) for the operator $\mathcal{L}_K$. With the help of $\mathcal{V}^+_{\lambda j}$ $(j=1,2,\dots,m)$ we define the transformation $\mathcal{F}_+\colon f\to \widetilde{f}_+(\lambda)$ by

$$(\mathcal{F}_+f)(\lambda)=\widetilde{f}_+(\lambda)=\sum_{j=1}^m \widetilde{f}^+_j(\lambda)\varphi_j=\frac{1}{\sqrt{2\pi}}\sum_{j=1}^m (f,\mathcal{V}^+_{\lambda j})_{\mathcal{H}}\varphi_j$$

on the vectors $f=\langle\phi^-,u,\phi^+\rangle$ in which ϕ^-, ϕ^+ are smooth, compactly supported vector-valued functions, and $u=\{u_n\}$ $(n\in\mathbb{N}_0)$ is a finite vector sequence. The proof of the next result is analogous to that of Lemma 4.3.

Lemma 4.4. *The transformation $\mathcal{F}_+$ maps $\mathcal{H}_+$ isometrically onto $L^2(\mathbb{R};E)$, and for all $f,g\in\mathcal{H}_+$ the Parseval equality and the inversion formula are valid:*

$$(f,g)_{\mathcal{H}}=(\widetilde{f}_+,\widetilde{g}_+)_{L^2}=\int_{-\infty}^{\infty}\sum_{j=1}^m \widetilde{f}^+_j(\lambda)\overline{\widetilde{g}^+_j(\lambda)}\,d\lambda,$$

$$f=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\sum_{j=1}^m \mathcal{V}^+_{\lambda j}\widetilde{f}^+_j(\lambda)\,d\lambda,$$

where $\widetilde{f}_+(\lambda)=(\mathcal{F}_+f)(\lambda)$ and $\widetilde{g}_+(\lambda)=(\mathcal{F}_+g)(\lambda)$.

It is clear that $\mathcal{S}_K(\lambda)$ is unitary for all $\lambda\in\mathbb{R}$. So using the explicit expression for the vectors $\mathcal{V}^+_{\lambda j}$ and $\mathcal{V}^-_{\lambda j}$ $(j=1,2,\dots,m)$ we get that

$$\mathcal{V}^+_{\lambda j}=\sum_{n=1}^m \mathcal{S}_{jn}(\lambda)\mathcal{V}^-_{\lambda n}, \quad j=1,2,\dots,m,$$

where $\mathcal{S}_{jn}$ $(j,n=1,2,\dots,m)$ are entries of the matrix $\mathcal{S}_K(\lambda)$. Therefore, from Lemmas 4.3 and 4.4 one obtains that $\mathcal{H}_-=\mathcal{H}_+$. Together with Lemma 4.2, this shows that $\mathcal{H}=\mathcal{H}_-=\mathcal{H}_+$, and property (3) for $\mathcal{X}(s)$ above has been established for incoming and outgoings subspaces.

Thus, the transformation $\mathcal{F}_-$ maps $\mathcal{H}$ isometrically onto $L^2(\mathbb{R};E)$; the subspace $\mathcal{D}^-$ is mapped onto $H^2_-(E)$, while the operator $\mathcal{X}(s)$ passing into operators of multiplication by $e^{i\lambda s}$. These results shows that $\mathcal{F}_-$ and $\mathcal{F}_+$ are the incoming and respectively outgoing spectral representations of the unitary group $\{\mathcal{X}(s)\}$. From the explicit formulas for $\mathcal{V}^-_{\lambda j}$ and $\mathcal{V}^+_{\lambda j}$ $(j=1,2,\dots,m)$, it follows that the passage from the $\mathcal{F}_-$-representation of a vector $f\in\mathcal{H}$ to its $\mathcal{F}_+$-representation is accomplished as follows: $\widetilde{f}_+(\lambda)=\mathcal{S}_K^{-1}(\lambda)\widetilde{f}_-(\lambda)$. According to [17], we have thus proved the following theorem.

Theorem 4.5. *The matrix $\mathcal{S}_K^{-1}(\lambda)$ is the scattering matrix of the unitary group $\{\mathcal{X}(s)\}$ (or of the self-adjoint operator $\mathcal{L}_K$).*

Let $S(\lambda)$ be an arbitrary non-constant inner matrix-valued function on the upper half-plane (the analytic matrix-valued function $S(\lambda)$ on the upper half-plane $\mathbb{C}_+$ is called *inner function* on $\mathbb{C}_+$ if $\|S(\lambda)\|\le 1$ for $\lambda\in\mathbb{C}_+$ and $S(\lambda)$ is a unitary matrix for almost all $\lambda\in\mathbb{R}$). Define $\mathbf{M}=H^2_+\ominus SH^2_+$. Then $\mathbf{M}\neq\{0\}$ is a subspace of the Hilbert space H^2_+. We consider the semigroup of the operators $\mathbf{V}(s)$ $(s\ge 0)$ acting in $\mathbf{M}$ according to the formula $\mathbf{V}(s)\phi=\mathbf{P}[e^{i\lambda s}\phi]$, $\phi:=\phi(\lambda)\in\mathbf{M}$, where $\mathbf{P}$ is the orthogonal projection from H^2_+ onto $\mathbf{M}$. The generator of the semigroup $\{\mathbf{V}(s)\}$ is denoted by $\mathbf{B}$: $\mathbf{B}\phi=\lim_{s\to+0}(is)^{-1}(\mathbf{V}(s)\phi-\phi)$. $\mathbf{B}$ is a maximal dissipative operator acting in $\mathbf{M}$ and its domain $D(\mathbf{B})$ consists of all vectors $\phi\in\mathbf{M}$ for which the above limit exists. The operator $\mathbf{B}$ is called a *model dissipative operator* (we remark that this model dissipative operator, which is associated with the names of Lax and Phillips [17], is a special case of a more general model dissipative operator constructed by Sz.-Nagy and Foiaş [18]). We claim that $S(\lambda)$ is the *characteristic function* of the dissipative operator $\mathbf{B}$.

We have proved that under the unitary transformation $\mathcal{F}_-$ we get the following mappings:

$$\mathcal{H}\to L^2(\mathbb{R};E),\quad f\to\widetilde{f}_-(\lambda)=(\mathcal{F}_-f)(\lambda),\quad \mathcal{D}^-\to H^2_-(E),$$
$$\mathcal{D}^+\to\mathcal{S}_KH^2_+(E),\quad \mathcal{H}\ominus(\mathcal{D}^-\oplus\mathcal{D}^+)\to H^2_+(E)\ominus\mathcal{S}_KH^2_+(E),$$
$$\mathcal{X}(s)f\to(\mathcal{F}_-\mathcal{X}(s)\mathcal{F}_-^{-1}\widetilde{f}_-)(\lambda)=e^{i\lambda s}\widetilde{f}_-(\lambda).$$

These formulas also imply that the operator $\widetilde{L}_K$ (L_R) is unitary equivalent to the model dissipative operator with characteristic function $\mathcal{S}_K(\lambda)$. Hence we have proved the following theorem.

Theorem 4.6. *The characteristic function of the maximal dissipative operator $\widetilde{L}_K$ (L_R) coincides with the matrix-valued function $\mathcal{S}_K(\lambda)$ determined by formula* (4.2)*. The matrix-valued function $\mathcal{S}_K(\lambda)$ is meromorphic in the complex plane $\mathbb{C}$ and is an inner function in the upper half-plane.*

5. The spectral analysis of a dissipative operators

The complex number λ_0 is called an *eigenvalue* of the operator $\mathbf{T}$, which is the linear operator acting in the Hilbert space $\mathbf{H}$ with the domain $D(\mathbf{T})$, if there exists a nonzero vector $f_0 \in D(\mathbf{T})$ such that $\mathbf{T}f_0 = \lambda_0 f_0$. This vector f_0 is called the *eigenvector* of the operator $\mathbf{T}$ corresponding to the eigenvalue λ_0. The eigenvector for λ_0 spans a subspace of $D(\mathbf{T})$, called the *eigenspace* for λ_0 and the *geometric multiplicity* of λ_0 is the dimension of its eigenspace. If the vectors $f_1, f_2, \ldots, f_k$ belong to $D(\mathbf{T})$ and satisfy the equalities $\mathbf{T}f_j = \lambda_0 f_j + f_{j-1}$, $j = 1, 2, \ldots, k$, then they are called the *associated vectors* of the eigenvector f_0. If all powers of $\mathbf{T}$ such that $(\mathbf{T} - \lambda_0 I)^n f = 0$ (for some integer n) are defined on the vector $f \in D(\mathbf{T})$, $f \neq 0$, then it is called a *root vector* of the operator $\mathbf{T}$ corresponding to the eigenvalue λ_0. The set of all root vectors of $\mathbf{T}$ corresponding to the same eigenvalue λ_0 with the vector $f = 0$ forms a linear set $\mathbf{N}_{\lambda_0}$ and is called the root lineal. The dimension of the lineal $\mathbf{N}_{\lambda_0}$ is called the *algebraic multiplicity* of the eigenvalue λ_0. The root lineal $\mathbf{N}_{\lambda_0}$ coincides with the linear span of all eigenvectors and associated vectors of $\mathbf{T}$ corresponding to the eigenvalue λ_0. Consequently, the completeness of the system of all eigenvectors and associated vectors of $\mathbf{L}$ is equivalent to the completeness of the system of all root vectors of this operator.

Questions of the spectral analysis of the dissipative operator L_R $(\widetilde{L}_K)$ can be solved in terms of characteristic function. Thus, for example, the absence of the singular factor $s(\lambda)$ in the factorization $\det \mathcal{S}_K(\lambda) = s(\lambda)\mathcal{B}(\lambda)$ ($\mathcal{B}(\lambda)$ is the Blaschke product) ensures the completeness of the system of eigenvectors and associated vectors (or root vectors) of the operator L_R $(\widetilde{L}_K)$ in the space H (see [3, 4, 9, 16, 18, 20]).

We first use the following

Lemma 5.1. *The characteristic function $\widetilde{\mathcal{S}}_R(\lambda)$ of the operator L_R has the form*

$$\widetilde{\mathcal{S}}_R(\lambda) := \mathcal{S}_K(\lambda) = Y_1(I - RR^*)^{-1}(\Theta(\xi) - R)(I - R^*\Theta(\xi))^{-1}(I - R^*R)^{1/2}Y_2,$$

where R is the Cayley transformation of the dissipative operator K and $\Theta(\xi)$ is the Cayley transformation of the matrix-valued function $M_\infty(\lambda)$, $\xi = (\lambda - i)(\lambda + i)^{-1}$ and $Y_1 := (\Im K)^{-1/2}(I - R)^{-1}(I - RR^*)^{1/2}$, $Y_2 := (I - R^*R)^{-1/2}(I - R^*)(\Im K)^{1/2}$, $|\det Y_1| = |\det Y_2| = 1$.

Proof. Using Theorem 4.6, we get that

$$\mathcal{S}_K(\lambda) = (\Im K)^{-1/2}(\Phi(\lambda) + K)(\Phi(\lambda) + K^*)^{-1}(\Im K)^{1/2}.$$

Further obtains that

$$\Im K = \frac{1}{2i}(K - K^*) = \frac{1}{2}[(I - R)^{-1}(I + R) + (I + R^*)(I - R^*)^{-1}]$$

$$
\begin{aligned}
&= \frac{1}{2}[(I-R)^{-1} + (I-R)^{-1}R + (I-R^*)^{-1} + R^*(I-R^*)^{-1}] \\
&= \frac{1}{2}[(I-R)^{-1} + (I-R)^{-1} - I + (I-R^*)^{-1} + (I-R^*)^{-1} - I] \\
&= (I-R)^{-1} + (I-R^*)^{-1} - I \\
&= (I-R)^{-1}[I-R^* + I - R - (I-R)(I-R^*)](I-R^*)^{-1} \\
&= (I-R)^{-1}(I-RR^*)(I-R^*)^{-1}.
\end{aligned} \tag{5.1}
$$

Similarly,

$$
\Im K = (I-R^*)^{-1}(I-R^*R)(I-R)^{-1}. \tag{5.2}
$$

Let us denote the Cayley transformation of the accumulative operator $\Phi(\lambda)$ for $\Im\lambda > 0$ by $\Theta_1(\lambda)$. Then we have $\Phi(\lambda) = -i(I-\Theta_1(\lambda))^{-1}(I+\Theta_1(\lambda))$. Hence we get that

$$
\begin{aligned}
\Phi(\lambda) + K &= -i[(I-\Theta_1(\lambda))^{-1}(I+\Theta_1(\lambda)) - (I-R)^{-1}(I+R)] \\
&= -i[-(I-\Theta_1(\lambda))^{-1}(I-\Theta_1(\lambda)) - 2I + (I-R)^{-1}(I-R-2I)] \\
&= -i[-I + 2(I-\Theta_1(\lambda))^{-1} + I - 2(I-R)^{-1}] \\
&= -2i[(I-\Theta_1(\lambda))^{-1} - (I-R)^{-1}] \\
&= -2i(I-R)^{-1}(\Theta_1(\lambda) - R)(I-\Theta_1(\lambda))^{-1}.
\end{aligned} \tag{5.3}
$$

Similarly, $\Phi(\lambda) + K^* = -2i(I-R^*)^{-1}(I-R^*\Theta_1(\lambda))(I-\Theta_1(\lambda))^{-1}$ and

$$
(\Phi(\lambda) + K^*)^{-1} = -\frac{1}{2i}(I-\Theta_1(\lambda))(I-R^*\Theta_1(\lambda))^{-1}(I-R^*). \tag{5.4}
$$

Using (5.1), (5.2), (5.3) and (5.4), we have

$$
\widetilde{\mathcal{S}}_R(\lambda) = \mathcal{S}_K(\lambda) = Y_1(I-RR^*)^{-1/2}(\Theta(\xi)-R)(I-R^*\Theta(\xi))(I-R^*R)^{1/2}Y_2,
$$

where $\Theta(\xi) = \Theta_1[-i(\xi+1)(\xi-1)^{-1}], Y_1 = (\Im K)^{-1/2}(I-R)(I-RR^*)^{1/2}$, $Y_2 = (I-R^*R)^{-1}(I-R^*)(\Im K)^{1/2}$. It is evident that $|\det Y_1| = |\det Y_2| = 1$. Hence, Lemma 5.1 is proved. □

We shall remind that the inner matrix-valued function $\widetilde{\mathcal{S}}_R(\lambda)$ is a Blaschke-Potapov product if and only if $\det \widetilde{\mathcal{S}}_R(\lambda)$ is a Blaschke product [3, 4, 9, 16, 18, 20]. Hence one gets from Lemma 5.1 that the characteristic function $\widetilde{\mathcal{S}}_R(\lambda)$ is a Blaschke-Potapov product if and only if the matrix-valued function

$$
Y_R(\xi) = (I-RR^*)^{-1/2}(\Theta(\xi)-R)(I-R^*\Theta(\xi))^{-1}(I-R^*R)^{1/2}
$$

is a Blaschke-Potapov product in a unit disk.

In order to state the completeness theorem, we will first define a suitable form for the Γ-capacity [9, 27].

Let $\mathbf{E}$ be an N-dimensional ($N < +\infty$) Euclidean space. In $\mathbf{E}$, we fix an orthonormal basis $\varphi_1, \varphi_2, \ldots, \varphi_N$ and denote by $\mathbf{E}_k$ ($k = 1, 2, \ldots, N$) the linear span vectors $\varphi_1, \varphi_2, \ldots, \varphi_k$. If $\mathbf{M} \subset \mathbf{E}_k$, then the set of $v \in \mathbf{E}_{k-1}$ with the property $\operatorname{Cap}\{\mu : \mu \in \mathbb{C}, (v + \mu\varphi_k) \in \mathbf{M}\} > 0$ will be denoted by $\Gamma_{k-1}\mathbf{M}$. ($\operatorname{Cap} G$ is the inner logarithmic capacity of the set $G \subset \mathbb{C}$.) The Γ-*capacity of the set* $\mathbf{M} \subset \mathbf{E}$ is a number

$$\Gamma - \operatorname{Cap}\mathbf{M} := \sup \operatorname{Cap}\{\mu : \mu \in \mathbb{C}, \mu\varphi_1 \subset \Gamma_1\Gamma_2 \cdots \Gamma_{N-1}\mathbf{M}\},$$

where the sup is taken with respect to all orthonormal bases in $\mathbf{E}$. It is known [17, 30] that every set $M \subset \mathbf{E}$ of zero Γ-capacity has zero $2N$-dimensional Lebesgue measure (in the decomplexified space $\mathbf{E}$), however, the converse is false.

Denote by $\mathcal{L}(E)$ the set of all linear operators acting in E. To convert $\mathcal{L}(E)$ into an m^2-dimensional Hilbert space, we introduce the inner product $\langle T, S\rangle = \operatorname{tr} S^*T$ for $T, S \in \mathcal{L}(E)$ ($\operatorname{tr} S^*T$ is the trace of the operator S^*T). Hence, we may introduce the Γ-capacity of a set in $\mathcal{L}(E)$.

We will utilize the following important result of [9].

Lemma 5.2. *Let $Y(\xi)$ ($|\xi| < 1$) be a holomorphic function with the values to be contractive operators in $\mathcal{L}(E)$ ($\|Y(\xi)\| \le 1$). Then for Γ-quasi every strictly contractive operators (i.e., for all strictly contractive $R \in \mathcal{L}(E)$ possible with the exception of a set of Γ of zero capacity) the inner part of the contractive function*

$$Y_R(\xi) := (I - RR^*)^{-1/2}(Y(\xi) - R)(I - R^*Y(\xi))^{-1}(I - R^*R)^{1/2}$$

is a Blaschke-Potapov product.

Hence considering all the obtained results for the dissipative operator L_R ($\widetilde{L}_K$), we have proved the following theorem.

Theorem 5.3. *For Γ-quasi-every strictly contractive $R \in \mathcal{L}(E)$ the characteristic function $\widetilde{\mathcal{S}}_R(\lambda)$ of the maximal dissipative operator L_R is a Blaschke-Potapov product, and spectrum of L_R is purely discrete and belongs to the open upper half-plane. For Γ-quasi-every strictly contractive $R \in \mathcal{L}(E)$, the operator L_R has a countable number of isolated eigenvalues with finite multiplicity and limit points at infinity, and the system of all eigenvectors and associated vectors (or all root vectors) of this operator is complete in $\ell^2_{\mathbf{A}}(\mathbb{N}_0; E \oplus E)$.*

It should be noted that all results obtained for maximal dissipative operators can be immediately transferred to maximal accumulative operators, because a linear operator $\mathbf{S}$ acting in a Hilbert space $\mathbf{H}$ is maximal accumulative if and only if $-\mathbf{S}$ is maximal dissipative.

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