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Infinitely Many Solutions for Sublinear Modified Nonlinear Schrödinger Equations Perturbed from Symmetry

Liang Zhang*, Xianhua Tang and Yi Chen

Abstract. In this paper, we consider the existence of infinitely many solutions for the following perturbed modified nonlinear Schrödinger equations

$$\begin{cases} -\Delta u - \Delta(|u|^{\alpha})|u|^{\alpha-2}u = g(x,u) + h(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 1)$ and $\alpha \geq 2$. Under the condition that $g(x,u)$ is sublinear near origin with respect to u, we study the effect of non-odd perturbation term $h(x,u)$ which breaks the symmetry of the associated energy functional. With the help of modified Rabinowitz's perturbation method and the truncation method, we prove that this equation possesses a sequence of small negative energy solutions approaching to zero.

1. Introduction and main results

Consider the following problem

$$\begin{cases} -\Delta u - \Delta(|u|^{\alpha})|u|^{\alpha-2}u = g(x,u) + h(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded smooth domain in $\mathbb{R}^N$ $(N \geq 1)$ and $\alpha \geq 2$. The quasilinear elliptic equation of the form (1.1) is often called modified nonlinear Schrödinger equation, which appears naturally in several physical models such as the superfluid film equation in plasma physics. For more physical motivations and detailed information in applications, we refer readers to [13, 14, 26] and the references therein.

Generally speaking, (1.1) has a variational structure on $H_0^1(\Omega)$, but a major difficulty is that the energy functional of (1.1) is not well defined for all $u \in H_0^1(\Omega)$, which makes the

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*Corresponding author.

study of such a problem quite difficult and interesting. Several methods were developed to overcome this difficulty, such as dual approach [18], the constrained minimization argument [17], the Nehari method [19] and the perturbation method [16]. Along these approaches, there have been a large number of works about existence and multiplicity of solutions for quasilinear elliptic equations, see, e.g., [8, 9, 11, 22, 27].

When $g(x,t)$ is odd in t and $h \equiv 0$, (1.1) possesses a natural $\mathbb{Z}_2$ symmetry and many results of the infinitely many solutions for quasilinear problem in bounded domain or whole space have been established with g satisfying various conditions, see [15, 30, 32, 38] and the references therein. In these works above, the methods rely on the notion of genus for symmetric sets. Therefore, the fact that $g(x,t)$ is odd in t is essential in the application of these techniques. However, if $h \not\equiv 0$ and is not odd in t, such a problem is often called the perturbation from symmetry problem, and the main feature is that the symmetry of the corresponding functional for (1.1) is broken. A long open question is whether the infinite number of solutions persists in the absence of symmetry, and this question is rather complicated. Since the early 1980s, the perturbation from symmetry problem for classical elliptic equations and systems has received increasingly more attention, and there has been much work on this topic, see, e.g., [2–6, 12, 25, 28, 29, 31, 33–37].

However, to the best of our knowledge, few results are known for the perturbation from symmetry problem of modified nonlinear Schrödinger equations. For the special case that $g(x,u) = |u|^{p-2}u$ with $p > 2$ and $h(x,u) \equiv h(x) \in L^2(\Omega)$, Liu and Zhao [20] obtained the existence of infinitely many solutions for a class of more general perturbed quasilinear elliptic equation. Their main approach is mainly based on minimax methods and the perturbation method. Later on, when $g(x,t)$ is indefinite in sign and only locally superlinear with respect to t at origin, the authors [36] studied the existence of infinitely many solutions of (1.1) by using Bolle's perturbation method introduced in [5].

In the sublinear case, i.e., $\lim_{t\to 0} g(x,t)/t = +\infty$ for a.e. $x \in \Omega$, it is natural to ask whether the infinite number of solutions persists for (1.1) with perturbed symmetry. For example, $g(x,t) = a(x)|t|^{-1/2}t\cos|t|^{3/2}$, $(x,t) \in \overline{\Omega} \times \mathbb{R}$, where $a(x)$ is a positive continuous function in $\overline{\Omega}$. The purpose of this paper is to give a positive answer to the perturbation problem of (1.1) in sublinear situation. Our main method is based on a variant of Rabinowitz's perturbation method in [23] for superlinear perturbation problems. Our strategy is to find suitable truncation of the original functional, in order to obtain a modified functional, in which the nonsymmetric part can be estimated, such that the modified functional has almost the same small critical values as the original functional. Next we state our main results as follows.

Theorem 1.1. *Assume that g and h satisfy the following conditions:*

(W1) $g(x,t) = g_1(x,t) + g_2(x,t)$, $g_1 \in C(\overline{\Omega} \times \mathbb{R})$ *and there exist constants* $C_0 > 0$ *and*

$1 < p < 2\alpha$ *such that*

$$|g_1(x,t)| \le C_0|t|^{p-1}, \quad \forall\,(x,t) \in \overline{\Omega} \times \mathbb{R}; \tag{1.2}$$

(W2) *there exist constants* $C_1 > 0$, $1 < \mu < 2$ *and* $2\alpha < \alpha_1 < 2^*\alpha$ *such that*

$$-C_1|t|^{\alpha_1} \le g_1(x,t)t - \mu G_1(x,t) \le 0 \quad \forall\,(x,t) \in \Omega \times \mathbb{R},$$

where $G_1(x,t) := \int_0^t g_1(x,s)\,ds$, $2^* = 2N/(N-2)$ *if* $N \ge 3$ *and* $2^* = \infty$ *if* $N = 1, 2$*;*

(W3) $G_1(x,t) \ge 0$, $(x,t) \in \Omega \times \mathbb{R}$ *and*

$$\lim_{t\to 0} \frac{g_1(x,t)}{t} = +\infty \quad \text{uniformly for } x \in \Omega; \tag{1.3}$$

(W4) $g_1(x,t) = -g_1(x,-t)$, $\forall\,(x,t) \in \Omega \times \mathbb{R}$*;*

(W5) $g_2 \in C(\overline{\Omega} \times \mathbb{R})$ *and there exist constants* $C_2 > 0$, $\delta_0 > 0$ *and* $\alpha_2 > 2\alpha$ *such that*

$$|g_2(x,t)| \le C_2|t|^{\alpha_2 - 1} \quad \text{for } |t| \le \delta_0 \text{ and all } x \in \overline{\Omega};$$

(W6) $g_2(x,t) = -g_2(x,-t)$ *for* $|t| \le \delta_0$ *and all* $x \in \Omega$*;*

(H1) $h \in C(\overline{\Omega} \times \mathbb{R})$ *and there exist constants* $C_3 > 0$, $\delta_1 > 0$ *and* $2\alpha < \sigma \le 2^*\alpha$ *such that*

$$|h(x,t)| \le C_3|t|^{\sigma-1} \quad \text{for } |t| \le \delta_1 \text{ and all } x \in \overline{\Omega};$$

(H2) *the constants* p *and* σ *in* (W1) *and* (H1) *satisfy*

$$\frac{p}{N(2\alpha - p)} > \frac{\alpha}{\sigma - 2\alpha}.$$

Then (1.1) *has a sequence of small negative energy solutions converging to zero.*

Corollary 1.2. *Assume that* g *and* h *satisfy* (W1)–(W6), (H1) *and the following condition:*

(H3) $h(x,t) = -h(x,-t)$ *for* $|t| \le \delta_1$ *and all* $x \in \overline{\Omega}$.

Then (1.1) *possesses a sequence of small negative energy solutions approaching to zero.*

Remark 1.3. Kajikiya [12] considered the perturbation problem for sublinear elliptic equations, but the author only dealt with a special nonlinear term $a|u|^{q-2}u$, where a is a positive constant. It is obvious that the odd nonlinearity $|u|^{q-2}u$ possesses homogeneous property, which is essential in the arguments of [12]. The novelty of our approach is that it allows us to consider some more general nonlinearities without homogeneous property. Moreover, our method can also be applied to solve the perturbation from symmetry problem of elliptic system and Hamiltonian system.

The rest of this paper is organized as follows. In Section 2, we introduce two cut-off functions to define a modified functional φ, and some useful estimates for φ are given. In Section 3, we prove φ satisfies Palais-Smale condition and construct several minimax sequences related to the critical values of φ, then we can obtain a sequence of critical values of φ and show that φ shares the same small critical values as the energy functional of (1.1). At last we give an example to illustrate our result in Section 4.

Notation. Throughout the paper we shall denote C_i various positive constants which may vary from line to line but are not essential to our proofs.

2. Some preliminary lemmas

First we introduce some functional spaces which will be useful in the sequel. As usual, for $1 \le \nu < +\infty$, let

$$\|u\|_\nu = \left(\int_\Omega |u(x)|^\nu \, dx \right)^{1/\nu}, \quad \forall\, u \in L^\nu(\Omega).$$

Throughout this paper, we denote by E the usual Sobolev space $H_0^1(\Omega)$ equipped with the following inner product and induced norm

$$(u,v) = \int_\Omega \nabla u \cdot \nabla v \, dx, \quad \|u\| = (u,u)^{1/2}, \quad \forall\, u,v \in H_0^1(\Omega).$$

It is well known that E is continuously embedded into $L^\nu(\Omega)$ for any $1 \le \nu \le 2^*$, i.e., there exists $\tau_\nu > 0$ such that

$$\|u\|_\nu \le \tau_\nu \|u\|, \quad \forall\, u \in E. \tag{2.1}$$

Moreover, E is compactly embedded into $L^\nu(\Omega)$ only for any $1 \le \nu < 2^*$.

By (W5) and (H1) in Theorem 1.1, the terms g_2 and h are only locally defined, so we can't apply the variational methods directly. To overcome this difficulty, we use cut-off method to modify $g_2(x,t)$ and $h(x,t)$ for t outside a neighbourhood of the origin. In detail, we have the following lemma.

Lemma 2.1. *Assume that* (W5), (W6) *and* (H1) *are satisfied. Then there exist two functions* $\widetilde{g}_2(x,t)$ *and* $\widetilde{h}(x,t)$ *possessing the following properties:*

(i) $\widetilde{g}_2 \in C(\overline{\Omega} \times \mathbb{R})$ *and there exists a constant* $2\alpha < \alpha_2' < 2^*\alpha$ *such that* $|\widetilde{g}_2(x,t)| \le C_2 |t|^{\alpha_2' - 1}$, $\forall\, (x,t) \in \overline{\Omega} \times \mathbb{R}$*;*

(ii) *there exists a positive constant* $\delta_0' \le \min\{\delta_0/2, 1/2\}$ *such that*

$$\widetilde{g}_2(x,t) = g_2(x,t) \quad \text{for } |t| \le \delta_0' \text{ and all } x \in \overline{\Omega};$$

(iii) $\widetilde{h} \in C(\overline{\Omega} \times \mathbb{R})$, $|\widetilde{h}(x,t)| \leq C_3|t|^{\sigma-1}$ *and* $|\widetilde{h}(x,t)| \leq C_3|t|^{2\alpha-1}$, $\forall\,(x,t) \in \overline{\Omega} \times \mathbb{R}$, *where the positive constants* C_3 *and* σ *are given in* (H1)*;*

(iv) *there exists a positive constant* $\delta_1' \leq \min\{\delta_1/2, 1/2\}$ *such that*

$$\widetilde{h}(x,t) = h(x,t) \quad \text{for } |t| \leq \delta_1' \text{ and all } x \in \overline{\Omega}.$$

Proof. First we prove (i) and (ii). Choose a constant $\delta_0' = \min\{\delta_0/2, 1/2\}$. Define a cut-off function $\chi_0 \in C^1(\mathbb{R},\mathbb{R})$ such that $\chi_0(t) = 1$ for $t \leq 1$, $\chi_0(t) = 0$ for $t \geq 2$ and $-2 \leq \chi_0'(t) < 0$ for $1 < t < 2$. Set

$$\widetilde{g}_2(x,t) = \chi_0(t^2/{\delta_0'}^2)g_2(x,t), \quad \forall\,(x,t) \in \Omega \times \mathbb{R}. \tag{2.2}$$

By (W5), (W6) and (2.2), it is easy to verify (i) and (ii) hold and

$$\widetilde{g}_2(x,t) = -\widetilde{g}_2(x,-t), \quad \forall\,(x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{2.3}$$

Next we prove (iii) and (iv). By a similar fashion, let $\delta_1' = \min\{\delta_1/2, 1/2\}$, define

$$\widetilde{h}(x,t) = \chi_0(t^2/{\delta_1'}^2)h(x,t), \quad \forall\,(x,t) \in \overline{\Omega} \times \mathbb{R}. \tag{2.4}$$

Both (H1) and (2.4) imply (iii) and (iv). This completes the proof. □

Next we introduce the following modified nonlinear Schrödinger equation

$$\begin{cases} -\Delta u - \Delta(|u|^\alpha)|u|^{\alpha-2}u = \widetilde{g}(x,u) + \widetilde{h}(x,u) & x \in \Omega, \\ u = 0 & x \in \partial\Omega, \end{cases} \tag{2.5}$$

where $\widetilde{g} := g_1 + \widetilde{g}_2$, $\widetilde{g}_2$ and $\widetilde{h}$ are defined by (2.2) and (2.4).

By direct computation, problem (2.5) is the Euler-Lagrange equation associated with the energy functional $J\colon E \to \mathbb{R}$ given by

$$J(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \frac{1}{2\alpha}\int_\Omega |\nabla(|u|^\alpha)|^2\,dx - \int_\Omega \widetilde{G}(x,u)\,dx - \int_\Omega \widetilde{H}(x,u)\,dx, \quad u \in E, \tag{2.6}$$

where $\widetilde{G}(x,t) := \int_0^t \widetilde{g}(x,s)\,ds$ and $\widetilde{H}(x,t) := \int_0^t \widetilde{h}(x,s)\,ds$. It is evident that J is not well defined in E. To overcome this difficulty, we employ a dual approach as in [8, 18]. Precisely speaking, the main idea of the dual approach is that the quasilinear equation can be reduced to a semilinear equation by the use of a suitable function f, then the classical Sobolev space framework can be used as the working space. In the spirit of the transformation introduced in [1], we make the change of variables by $v = f^{-1}(u)$, where the function f can be defined as follows:

$$f'(t) = (1 + \alpha|f(t)|^{2(\alpha-1)})^{-1/2},\ t \in [0,+\infty) \quad \text{and} \quad f(-t) = -f(t),\ t \in (-\infty,0].$$

Next we collect some properties of the function f, which is very useful in the sequel of the paper. The detailed proof can be found in [1].

Lemma 2.2. *The function f and its derivative have the following properties:*

(f1) *f is uniquely defined C^∞ function and invertible;*

(f2) $0 < f'(t) \le 1$ *and* $|f(t)| \le |t|$, $\forall\, t \in \mathbb{R}$;

(f3) $\lim_{t\to 0} |f(t)|/|t| = 1$ *and* $\lim_{t\to\infty} |f(t)|^\alpha/|t| = \sqrt{\alpha}$;

(f4) *there exists a positive constant C such that* $|f(t)|^{\alpha-1} f'(t) \le C$, $\forall\, t \in \mathbb{R}$;

(f5) $f''(t)f(t) = (\alpha-1)(f'(t))^2((f'(t))^2 - 1)$, $\forall\, t \in \mathbb{R}$.

Therefore, by a change of variable and (2.6), we obtain the following functional

$$I(v) := J(f(v)) = \frac{1}{2}\int_\Omega |\nabla v|^2\,dx - \int_\Omega \widetilde{G}(x, f(v))\,dx - \int_\Omega \widetilde{H}(x, f(v))\,dx, \quad \forall\, v \in E.$$

Combining with Lemmas 2.1 and 2.2, we have $I \in C^1(E, \mathbb{R})$ and

$$\langle I'(v), w\rangle = \int_\Omega \nabla v \nabla w\,dx - \int_\Omega \widetilde{g}(x, f(v))f'(v)w\,dx - \int_\Omega \widetilde{h}(x, f(v))f'(v)w\,dx$$

for any $v, w \in E$. It is evident that the critical points of I are the weak solutions of the following problem

$$\begin{cases} -\Delta v = (1 + \alpha|f(v)|^{2(\alpha-1)})^{-1/2}(\widetilde{g}(x, f(v)) + \widetilde{h}(x, f(v))) & x \in \Omega, \\ v = 0 & x \in \partial\Omega. \end{cases} \tag{2.7}$$

Arguing similarly as in the proof of Lemma 2.6 and Remark 2.7 in [1], if $v_0 \in E$ is a critical point of the functional I, then v_0 is a weak solution of (2.7) and $u_0 = f(v_0) \in E$ is a weak solution of (2.5). Next we prove that (2.7) has a sequence of weak solutions $\{v_n\}$ converging to 0. With the aid of elliptic regularity theory and Lemma 2.1, we can show that $u_n = f(v_n)$ are also a sequence of weak solutions of (1.1).

First we introduce a cut-off function $\zeta \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$\begin{cases} \zeta(t) = 1 & \text{if } t \in (-\infty, 1], \\ 0 \le \zeta(t) \le 1 & \text{if } t \in (1, 2), \\ \zeta(t) = 0 & \text{if } t \in [2, \infty), \\ |\zeta'(t)| \le 2 & \text{if } t \in \mathbb{R}. \end{cases} \tag{2.8}$$

With the help of this cut-off function ζ, define

$$k(v) = \zeta\left(\frac{\|v\|^2}{T_0}\right), \quad \forall\, v \in E, \tag{2.9}$$

where T_0 is a small positive constant independent of v determined by (2.20) and (3.17).

Lemma 2.3. *The functional k defined by (2.9) is of $C^1(E,\mathbb{R})$ and*

$$\left|\left\langle k'(v), \frac{f(v)}{f'(v)}\right\rangle\right| \le C_4, \quad \forall\, v \in E, \tag{2.10}$$

where C_4 is a positive constant independent of v.

Proof. By (2.9) and straightforward calculation, we have

$$\langle k'(v), w\rangle = 2\zeta'\left(\frac{\|v\|^2}{T_0}\right)\frac{(v,w)}{T_0}, \quad \forall\, v, w \in E. \tag{2.11}$$

Assume that $v_n \to v_0$ in E. In view of (2.11), for any $w \in E$, we get

$$\begin{aligned}
&|\langle k'(v_n) - k'(v_0), w\rangle| \\
&= 2\left|\zeta'\left(\frac{\|v_n\|^2}{T_0}\right)\frac{(v_n,w)}{T_0} - \zeta'\left(\frac{\|v_0\|^2}{T_0}\right)\frac{(v_0,w)}{T_0}\right| \\
&\le 2{T_0}^{-1}\|w\|\left[\left|\zeta'\left(\frac{\|v_n\|^2}{T_0}\right)\right|\|v_n - v_0\| + \left|\zeta'\left(\frac{\|v_n\|^2}{T_0}\right) - \zeta'\left(\frac{\|v_0\|^2}{T_0}\right)\right|\|v_0\|\right],
\end{aligned}$$

which implies that $\|k'(v_n) - k'(v_0)\|_{E^*} \to 0$, $n \to \infty$. This means that $k \in C^1(E,\mathbb{R})$. By Lemma 2.2(f5) and direct computation, there exists a positive constant C_5 independent of v such that

$$\left\|\frac{f(v)}{f'(v)}\right\| \le C_5\|v\|, \quad \forall\, v \in E. \tag{2.12}$$

In combination with (2.8), (2.11) and (2.12), we see that

$$\left|\left\langle k'(v), \frac{f(v)}{f'(v)}\right\rangle\right| \le 2C_5\left|\zeta'\left(\frac{\|v\|^2}{T_0}\right)\right|\frac{\|v\|^2}{T_0} \le 8C_5, \quad \forall\, v \in E,$$

which implies that (2.10) holds. The proof is completed. □

Next we introduce a new functional $\overline{I}\colon E \to \mathbb{R}$ by

$$\overline{I}(v) = \frac{1}{2}\|v\|^2 - \int_\Omega G_1(x,f(v))\,dx - k(v)\left(\int_\Omega \widetilde{G}_2(x,f(v))\,dx + \int_\Omega \widetilde{H}(x,f(v))\,dx\right) \tag{2.13}$$

for any $v \in E$, where $G_1(x,t) = \int_0^t g_1(x,s)\,ds$ and $\widetilde{G}_2(x,t) = \int_0^t \widetilde{g}_2(x,s)\,ds$. Under assumptions of Theorem 1.1, by Lemmas 2.1 and 2.3, we have $\overline{I} \in C^1(E,\mathbb{R})$ and

$$\begin{aligned}
\langle \overline{I}'(v), w\rangle &= (v,w) - \int_\Omega g_1(x,f(v))f'(v)w\,dx \\
&\quad - k(v)\int_\Omega (\widetilde{g}_2(x,f(v)) + \widetilde{h}(x,f(v)))f'(v)w\,dx \\
&\quad - \langle k'(v), w\rangle\left(\int_\Omega \widetilde{G}_2(x,f(v))\,dx + \int_\Omega \widetilde{H}(x,f(v))\,dx\right), \quad \forall\, v, w \in E.
\end{aligned} \tag{2.14}$$

In order to construct a modified functional, we provide some prior bounds for critical points of $\overline{I}$ in terms of the corresponding critical values in the following lemma.

Lemma 2.4. *Under assumptions of* (W2), (W5) *and* (H1), *if v is a critical point of $\overline{I}$, then*

$$\overline{I}(v) \le (4\mu)^{-1}(\mu - 2)\|v\|^2. \tag{2.15}$$

Proof. If v is a critical point of $\overline{I}$ and $\|v\|^2 > 2T_0$, by (2.9) and (2.11), $k(v) = 0$ and $k'(v) = 0$. In view of (W2), Lemma 2.2(f2), (2.13) and (2.14), we obtain

$$\begin{aligned} \overline{I}(v) &= \overline{I}(v) - \mu^{-1}\left\langle \overline{I}'(v), \frac{f(v)}{f'(v)} \right\rangle \\ &\le \frac{\mu - 2\alpha}{2\mu}\|v\|^2 + \frac{\alpha - 1}{\mu}\int_\Omega (f'(v))^2|\nabla v|^2\,dx \\ &\le (2\mu)^{-1}(\mu - 2)\|v\|^2. \end{aligned} \tag{2.16}$$

By Lemma 2.2(f3), there exist positive constants M and C_6 such that

$$|f(t)| \le C_6|t|^{1/\alpha}, \quad |t| \ge M. \tag{2.17}$$

Since $\alpha \ge 2$, in view of Lemma 2.2(f3) and (2.17), there exists a positive constant C_7 independent of t such that

$$|f(t)| \le C_7|t|^{1/\alpha}, \quad t \in \mathbb{R}. \tag{2.18}$$

When v is a critical point of $\overline{I}$ with $\|v\|^2 \le 2T_0$, by Lemma 2.1(i)(iii), (W2), (2.10), (2.13) and (2.14), we have

$$\begin{aligned} \overline{I}(v) &= \overline{I}(v) - \mu^{-1}\left\langle \overline{I}'(v), \frac{f(v)}{f'(v)} \right\rangle \\ &\le (2\mu)^{-1}(\mu - 2)\|v\|^2 + C_8\|v\|^{\alpha_1/\alpha} + C_9\|v\|^{\alpha_2'/\alpha} + C_{10}\|v\|^{\sigma/\alpha}, \end{aligned} \tag{2.19}$$

where $C_8 = C_1 C_7^{\alpha_1}\tau_{\alpha_1/\alpha}^{\alpha_1/\alpha}$, $C_9 = (C_4+1)C_2 C_7^{\alpha_2'}\tau_{\alpha_2'/\alpha}^{\alpha_2'/\alpha}$ and $C_{10} = (C_4+1)C_3 C_7^{\sigma}\tau_{\sigma/\alpha}^{\sigma/\alpha}$. Since $\alpha_1/\alpha > 2$, $\alpha_2'/\alpha > 2$ and $\sigma/\alpha > 2$, we can choose T_0 small enough such that if $\|v\|^2 \le 2T_0$,

$$C_8\|v\|^{\alpha_1/\alpha} + C_9\|v\|^{\alpha_2'/\alpha} + (M_0 + 10C_{10})\|v\|^{\sigma/\alpha} < (4\mu)^{-1}(2 - \mu)\|v\|^2, \tag{2.20}$$

where M_0 is a positive constant independent of v given in (2.41). In view of (2.16), (2.19) and (2.20), (2.15) holds. This completes the proof. □

Next we introduce a cut-off function $\chi \in C^\infty(\mathbb{R}, \mathbb{R})$ satisfying

$$\begin{cases} \chi(t) = 1 & \text{if } t \in (-\infty, A/2], \\ 0 \le \chi(t) \le 1 & \text{if } t \in (A/2, A/4), \\ \chi(t) = 0 & \text{if } t \in [A/4, \infty), \\ |\chi'(t)| \le M_1 & \text{if } t \in \mathbb{R}, \end{cases} \tag{2.21}$$

where $A := (4\mu)^{-1}(\mu - 2) < 0$ and M_1 is a positive constant. By this function χ, set

$$l(v) = \chi(\|v\|^{-2}\overline{I}(v)), \quad \forall\, v \in E \setminus \{0\}. \tag{2.22}$$

By straightforward computation, for $v \in E \setminus \{0\}$ and any $w \in E$, we obtain

$$\langle l'(v), w\rangle = \chi'(\theta(v))\|v\|^{-4}(\|v\|^2\langle \overline{I}'(v), w\rangle - 2\overline{I}(v)(v, w)), \tag{2.23}$$

where $\theta(v) := \|v\|^{-2}\overline{I}(v)$, $\forall\, v \in E \setminus \{0\}$. Under assumptions of Theorem 1.1, it is easy to verify that l is continuously differentiable at any $v \in E \setminus \{0\}$.

Next we introduce a modified functional φ on E as follows:

$$\varphi(v) = \frac{1}{2}\|v\|^2 - \int_\Omega G_1(x, f(v))\,dx - k(v)\int_\Omega \widetilde{G}_2(x, f(v))\,dx - \psi(v), \quad \forall\, v \in E, \tag{2.24}$$

where

$$\psi(v) := \begin{cases} k(v)l(v)P(v) & \text{if } v \in E \setminus \{0\}, \\ 0 & \text{if } v = 0 \end{cases} \tag{2.25}$$

and $P(v) := \int_\Omega \widetilde{H}(x, f(v))\,dx$, $\forall\, v \in E$. In view of Lemma 2.1(iii), (2.1) and (2.18),

$$|P(v)| \le C_3 C_7^\sigma \tau_{\sigma/\alpha}^{\sigma/\alpha}\|v\|^{\sigma/\alpha}, \quad \forall\, v \in E. \tag{2.26}$$

Under assumptions of Theorem 1.1, it is easy to prove that $P \in C^1(E, \mathbb{R})$ and

$$\langle P'(v), w\rangle = \int_\Omega \widetilde{h}(x, f(v))f'(v)w\,dx, \quad \forall\, v, w \in E. \tag{2.27}$$

Remark 2.5. The functional k can assure the coercivity of φ, which allows us to verify the Palais-Smale condition easily, and the functional l gives some important qualitative descriptions for the critical points of functional φ. Under assumptions of Theorem 1.1, we can prove that the modified functional φ shares a sequence of small critical values tending to 0 as the original functional I.

Lemma 2.6. *Suppose that* (W1)–(W6) *and* (H1) *are satisfied. Then*

(i) *the functional* ψ *defined by* (2.25) *is of class* $C^1(E, \mathbb{R})$ *and*

$$\left|\left\langle \psi'(v), \frac{f(v)}{f'(v)}\right\rangle\right| \le (M_0 + C_3C_7^\sigma(C_4 + 1)\tau_{\sigma/\alpha}^{\sigma/\alpha})\|v\|^{\sigma/\alpha}, \quad \forall\, v \in E, \tag{2.28}$$

where M_0 *is a positive constant independent of* v *defined in* (2.41)*;*

(ii) $\varphi \in C^1(E, \mathbb{R})$ *and there exists a constant* C_{11} *independent of* v *such that*

$$|\varphi(v) - \varphi(-v)| \le C_{11}|\varphi(v)|^{\sigma/(2\alpha)}, \quad \forall\, v \in E; \tag{2.29}$$

(iii) *φ has no critical point with positive critical value on E and $K_0 = \{0\}$, where $K_0 := \{v \in E \mid \varphi(v) = 0, \varphi'(v) = 0\}$.*

Proof. For $v = 0$ and any $w \in E$, by Lemma 2.1(iii), (2.9), (2.18), (2.22) and (2.25),

$$|\langle \psi'(0), w\rangle| = \left|\lim_{\lambda\to 0} \frac{\psi(\lambda w) - \psi(0)}{\lambda}\right| \le C_3 C_7^\sigma \int_\Omega |w(x)|^{\sigma/\alpha}\, dx \lim_{\lambda\to 0} |\lambda|^{(\sigma-\alpha)/\alpha} = 0,$$

which implies that $\psi'(0) = 0$. By (2.11), (2.23) and (2.27), for $v \in E \setminus \{0\}$ and $w \in E$,

$$\langle \psi'(v), w\rangle = \langle k'(v), w\rangle l(v) P(v) + k(v)\langle l'(v), w\rangle P(v) + k(v) l(v) \langle P'(v), w\rangle. \tag{2.30}$$

Next we prove $\psi \in C^1(E, \mathbb{R})$. Assume that $v_n \to v_0$. We consider two possibilities.

Case 1*:* $v_0 \ne 0$. By Lemma 2.3, (2.23), (2.27) and (2.30), we obtain $\psi'(v_n) \to \psi'(v_0)$, $n \to \infty$.

Case 2*:* $v_0 = 0$. Without loss of generality, we can assume $\|v_n\|^2 < T_0$. In view of (2.8), (2.9) and (2.11), $k'(v_n) = 0$ and $k(v_n) = 1$. By (2.30), we have

$$\langle \psi'(v_n), w\rangle = \langle l'(v_n), w\rangle P(v_n) + l(v_n)\langle P'(v_n), w\rangle, \quad \forall\, w \in E. \tag{2.31}$$

In view of (2.23), we can divide $\langle l'(v_n), w\rangle P(v_n)$ into two parts as follows:

$$\langle l'(v_n), w\rangle P(v_n) = P_1(v_n, w) - P_2(v_n, w), \tag{2.32}$$

where

$$P_1(v_n, w) := \chi'(\theta(v_n))\|v_n\|^{-2}\langle \overline{I}'(v_n), w\rangle P(v_n), \quad \forall\, w \in E \tag{2.33}$$

and

$$P_2(v_n, w) := 2\chi'(\theta(v_n))\theta(v_n)\|v_n\|^{-2} P(v_n)(v_n, w), \quad \forall\, w \in E. \tag{2.34}$$

It follows from Lemma 2.1(iii), (2.21), (2.26), (2.33) and (2.34) that

$$|P_1(v_n, w)| \le C_{12}\|\overline{I}'(v_n)\|_{E^*}\|v_n\|^{(\sigma-2\alpha)/\alpha}\|w\| \tag{2.35}$$

and

$$|P_2(v_n, w)| \le C_{13}\|v_n\|^{(\sigma-\alpha)/\alpha}\|w\|. \tag{2.36}$$

Since $k'(v_n) = 0$, $k(v_n) = 1$ and $v_n \to 0$, $n \to \infty$, by Lemma 2.1(ii)(iii), (W1), (2.14) and (2.27), we conclude that

$$\|\overline{I}'(v_n)\|_{E^*} \to 0 \quad \text{and} \quad \|P'(v_n)\|_{E^*} \to 0 \quad \text{as } n \to \infty. \tag{2.37}$$

Combining with (2.31), (2.32), (2.35)–(2.37), we obtain

$$\|\psi'(v_n) - \psi'(0)\|_{E^*} = \sup_{\|w\| \le 1} |\langle l'(v_n), w\rangle P(v_n) + l(v_n)\langle P'(v_n), w\rangle| \to 0 \quad \text{as } n \to \infty,$$

which implies the continuity of ψ'. So we have $\psi \in C^1(E, \mathbb{R})$.

When $\|v\|^2 > 2T_0$ or $v = 0$, by (2.8), (2.9), (2.11) and (2.30), $\langle \psi'(v), v\rangle = 0$. Otherwise, $\|v\|^2 \le 2T_0$ and $v \neq 0$. Arguing similarly as in (2.19), we have

$$\left|\overline{I}(v) - \mu^{-1}\left\langle \overline{I}'(v), \frac{f(v)}{f'(v)}\right\rangle\right| \le 2|A|\|v\|^2 + C_8\|v\|^{\alpha_1/\alpha} + C_9\|v\|^{\alpha_2'/\alpha} + C_{10}\|v\|^{\sigma/\alpha}. \tag{2.38}$$

When $\|v\|^2 \le 2T_0$, by (2.20) and (2.38), we get

$$\left|\left\langle \overline{I}'(v), \frac{f(v)}{f'(v)}\right\rangle\right| \le \mu(3|A|\|v\|^2 + |\overline{I}(v)|). \tag{2.39}$$

In combination with (2.21) and (2.23), if $\theta(v) \notin [A/2, A/4]$, we have $l'(v) = 0$. Otherwise, $A/2 \le \theta(v) \le A/4$, then the definition of θ implies that

$$|\overline{I}(v)| \le |A|\|v\|^2. \tag{2.40}$$

By Lemma 2.1(iii), (2.12), (2.23), (2.26), (2.39) and (2.40), if $\|v\|^2 \le 2T_0$ and $v \neq 0$,

$$\begin{aligned} \left|k(v)\left\langle l'(v), \frac{f(v)}{f'(v)}\right\rangle P(v)\right| &\le 2M_1\|v\|^{-2}\left(C_5|\overline{I}(v)| + \left|\left\langle \overline{I}'(v), \frac{f(v)}{f'(v)}\right\rangle\right|\right)|P(v)| \\ &\le M_0\|v\|^{\sigma/\alpha}, \end{aligned} \tag{2.41}$$

where M_0 is a positive constant independent of v. In view of Lemma 2.1(iii), (2.10), (2.18), (2.22), (2.26) and (2.27), for any $v \in E \setminus \{0\}$, we have

$$\left|\left\langle k'(v), \frac{f(v)}{f'(v)}\right\rangle l(v)P(v) + k(v)l(v)\left\langle P'(v), \frac{f(v)}{f'(v)}\right\rangle\right| \le C_3C_7^{\sigma}(C_4+1)\tau_{\sigma/\alpha}^{\sigma/\alpha}\|v\|^{\sigma/\alpha}. \tag{2.42}$$

In combination with (2.30), (2.41) and (2.42), we obtain (2.28).

To prove (ii), by Lemmas 2.1, 2.3, 2.6(i) and (2.24), $\varphi \in C^1(E, \mathbb{R})$ and

$$\begin{aligned} \langle \varphi'(v), w\rangle = (v, w) &- \int_\Omega g_1(x, f(v))f'(v)w\,dx - k(v)\int_\Omega \widetilde{g}_2(x, f(v))f'(v)w\,dx \\ &- \langle k'(v), w\rangle \int_\Omega \widetilde{G}_2(x, f(v))\,dx - \langle \psi'(v), w\rangle, \quad \forall\, v, w \in E. \end{aligned} \tag{2.43}$$

If $\|v\|^2 > 2T_0$ or $\theta(v) > A/4$, by (2.8), (2.9) or (2.21), (2.22) and (2.25), we have $\psi(v) = 0$. It follows from (W4), (2.3) and (2.24) that (2.29) holds. So we can assume $\|v\|^2 \le 2T_0$ and $\theta(v) \le A/4$. When $\theta(v) \le A/4$, by the definition of θ, we obtain

$$|\overline{I}(v)| \ge \frac{|A|}{4}\|v\|^2. \tag{2.44}$$

By Lemma 2.1(iii), (W4), (2.3), (2.9), (2.13), (2.20), (2.22), (2.24)–(2.26) and (2.44), if $\|v\|^2 \le 2T_0$ and $\theta(v) \le A/4$, we get

$$|\varphi(v)| \ge |\bar{I}(v)| - 2|P(v)| \ge \frac{|A|}{4}\|v\|^2 - 2C_3C_7^\sigma \tau_{\sigma/\alpha}^{\sigma/\alpha}\|v\|^{\sigma/\alpha} \ge \frac{|A|}{20}\|v\|^2. \tag{2.45}$$

Combining with Lemma 2.1(iii), (W4), (2.3), (2.9), (2.22), (2.24)–(2.26), we have

$$|\varphi(v) - \varphi(-v)| \le 2C_3C_7^\sigma \tau_{\sigma/\alpha}^{\sigma/\alpha}\|v\|^{\sigma/\alpha}, \quad \forall\, v \in E. \tag{2.46}$$

In view of (2.45) and (2.46), we conclude that (2.29) holds.

Next we prove (iii) by contradiction. If v_0 is a critical point of φ with $\varphi(v_0) > 0$, by Lemma 2.1(i)(iii), (W1), (2.24) and (2.25), $v_0 \ne 0$. Without loss of generality, we can assume $\|v_0\|^2 \le 2T_0$. Otherwise, by (2.9), (2.11) and (2.30), $k(v_0) = 0$, $k'(v_0) = 0$ and $\psi'(v_0) = 0$. Then it follows from (W2), (2.24) and (2.43) that

$$0 < \varphi(v_0) = \varphi(v_0) - \mu^{-1}\left\langle \varphi'(v_0), \frac{f(v_0)}{f'(v_0)} \right\rangle \le 2A\|v_0\|^2 < 0, \tag{2.47}$$

which yields a contradiction, so $\|v_0\|^2 \le 2T_0$. In view of Lemma 2.1(i)(iii), (W2), (2.20), (2.24), (2.28) and (2.43),

$$\begin{aligned} 0 < \varphi(v_0) &= \varphi(v_0) - \mu^{-1}\left\langle \varphi'(v_0), \frac{f(v_0)}{f'(v_0)} \right\rangle \\ &\le 2A\|v_0\|^2 + C_8\|v_0\|^{\alpha_1/\alpha} + C_9\|v_0\|^{\alpha_2'/\alpha} + (M_0 + 10C_{10})\|v_0\|^{\sigma/\alpha} < 0, \end{aligned}$$

which is a contradiction. Next we prove $K_0 = \{0\}$. By Lemma 2.1(i)(iii), (W1), (2.24) and (2.25), we have $0 \in K_0$. If $v_0 \ne 0$ and $v_0 \in K_0$, by a similar estimate as in (2.47), we obtain $\|v_0\|^2 \le 2T_0$. Then it follows from Lemma 2.1(i)(iii), (W2), (2.20), (2.24), (2.28) and (2.43) that

$$0 = \varphi(v_0) = \varphi(v_0) - \mu^{-1}\left\langle \varphi'(v_0), \frac{f(v_0)}{f'(v_0)} \right\rangle \le 2A\|v_0\|^2 < 0,$$

which is impossible. So we have $K_0 = \{0\}$. The proof is completed. □

3. Proofs of main results

Lemma 3.1. *Under assumptions* (W1), (W5) *and* (H1)*, the functional φ satisfies the Palais-Smale condition.*

Proof. First we show that φ is bounded from below. In combination with (1.2), (2.8), (2.9), (2.18), (2.24) and (2.25), when $\|v\|^2 > 2T_0$, we have

$$\varphi(v) \ge \frac{1}{2}\|v\|^2 - C_{14}(\|v\|^{p/\alpha} + 1). \tag{3.1}$$

Since $1 < p < 2\alpha$, (3.1) implies that $\varphi(v) \to +\infty$ as $\|v\| \to +\infty$.

Next we show that φ satisfies the Palais-Smale condition. Assume that $\{v_n\}_{n\in\mathbb{N}} \subset E$ is a (PS) sequence, i.e., $\{\varphi(v_n)\}_{n\in\mathbb{N}}$ is bounded and $\varphi'(v_n) \to 0$ as $n \to +\infty$. We need to prove that $\{v_n\}$ has a convergent subsequence. Since φ is coercive, then $\{v_n\}$ is bounded, passing to subsequence, also denoted by $\{v_n\}$, it can be assumed that $v_n \rightharpoonup v_0$, $n \to \infty$. Since $v_n \rightharpoonup v_0$, by Lemma 2.1(i), Lemma 2.2(f2), (W1) and (2.18), we get

$$\int_\Omega g_1(x, f(v_n))f'(v_n)(v_n - v_0)\,dx \to 0, \quad n \to \infty \tag{3.2}$$

and

$$\int_\Omega \widetilde{g}_2(x, f(v_n))f'(v_n)(v_n - v_0)\,dx \to 0, \quad n \to \infty. \tag{3.3}$$

Similarly, in view of Lemma 2.1(iii), Lemma 2.2(f2) and (2.18), we also obtain

$$\int_\Omega \widetilde{h}(x, f(v_n))f'(v_n)(v_n - v_0)\,dx \to 0, \quad n \to \infty. \tag{3.4}$$

If $\|v_n\|^2 > 2T_0$ or $v_n = 0$, by (2.8), (2.9), (2.11) and (2.30), $k'(v_n) = 0$ and $\psi'(v_n) = 0$. In view of (2.43), (3.2) and (3.3), we obtain

$$|\langle \varphi'(v_n), v_n - v_0\rangle| \geq \|v_n - v_0\|^2 + o_n(1). \tag{3.5}$$

When $\|v_n\|^2 \leq 2T_0$ and $v_n \neq 0$, by Lemma 2.1(iii), (2.11), (2.18) and (2.26), we have

$$|\langle k'(v_n), v_n - v_0\rangle P(v_n)| \leq 2^{(\sigma+2\alpha)/(2\alpha)} C_3 C_7^\sigma \tau_{\sigma/\alpha}^{\sigma/\alpha} T_0^{(\sigma-2\alpha)/(2\alpha)} \|v_n - v_0\|^2 + o_n(1) \tag{3.6}$$

and

$$|\langle k'(v_n), v_n - v_0\rangle l(v_n)P(v_n)| \leq 2^{(\sigma+2\alpha)/(2\alpha)} C_3 C_7^\sigma \tau_{\sigma/\alpha}^{\sigma/\alpha} T_0^{(\sigma-2\alpha)/(2\alpha)} \|v_n - v_0\|^2 + o_n(1). \tag{3.7}$$

Similarly, in view of Lemma 2.1(i), (2.11) and (2.18), we obtain

$$\begin{aligned} &\left| \langle k'(v_n), v_n - v_0\rangle \int_\Omega \widetilde{G}_2(x, f(v_n))\,dx \right| \\ &\quad \leq 2^{(\alpha_2'+2\alpha)/(2\alpha)} C_2 C_7^{\alpha_2'} \tau_{\alpha_2'/\alpha}^{\alpha_2'/\alpha} T_0^{(\alpha_2'-2\alpha)/(2\alpha)} \|v_n - v_0\|^2 + o_n(1). \end{aligned} \tag{3.8}$$

By (2.9), (2.32), (2.33) and (2.34), we have

$$|k(v_n)\langle l'(v_n), v_n - v_0\rangle P(v_n)| \leq |P_1(v_n, v_n - v_0)| + |P_2(v_n, v_n - v_0)|. \tag{3.9}$$

It follows from (2.21), (2.26) and (2.33) that

$$|P_1(v_n, v_n - v_0)| \leq M_1 C_3 C_7^\sigma \tau_{\sigma/\alpha}^{\sigma/\alpha} (2T_0)^{(\sigma-2\alpha)/(2\alpha)} |\langle \overline{I}'(v_n), v_n - v_0\rangle|. \tag{3.10}$$

In view of (2.14), (3.2)–(3.4), (3.6) and (3.8), we have

$$|\langle \overline{I}'(v_n), v_n - v_0\rangle| \le (C_{15}+1)\|v_n - v_0\|^2 + o_n(1), \tag{3.11}$$

where $C_{15} := 2^{(\sigma+2\alpha)/(2\alpha)} C_3 C_7^{\sigma} \tau_{\sigma/\alpha}^{\sigma/\alpha} T_0^{(\sigma-2\alpha)/(2\alpha)} + 2^{(\alpha_2'+2\alpha)/(2\alpha)} C_2 C_7^{\alpha_2'} \tau_{\alpha_2'/\alpha}^{\alpha_2'/\alpha} T_0^{(\alpha_2'-2\alpha)/(2\alpha)}$. By (3.10) and (3.11),

$$|P_1(v_n, v_n - v_0)| \le C_{16}\|v_n - v_0\|^2 + o_n(1), \tag{3.12}$$

where $C_{16} := (C_{15}+1) M_1 C_3 C_7^{\sigma} \tau_{\sigma/\alpha}^{\sigma/\alpha} (2T_0)^{(\sigma-2\alpha)/(2\alpha)}$. Combining (2.21), (2.26) and (2.34), we get

$$|P_2(v_n, v_n - v_0)| \le 4M_1 C_3 C_7^{\sigma} \tau_{\sigma/\alpha}^{\sigma/\alpha} (2T_0)^{(\sigma-2\alpha)/(2\alpha)} \|v_n - v_0\|^2 + o_n(1). \tag{3.13}$$

It follows from (3.9), (3.12) and (3.13) that

$$|k(v_n)\langle l'(v_n), v_n - v_0\rangle P(v_n)| \le (C_{16} + 4M_1 C_{15})\|v_n - v_0\|^2 + o_n(1). \tag{3.14}$$

Combining with Lemma 2.1(iii), Lemma 2.2(f2), (2.9), (2.22) and (2.27), we conclude that

$$|k(v_n) l(v_n)\langle P'(v_n), v_n - v_0\rangle| \le o_n(1). \tag{3.15}$$

By (2.30), (3.7), (3.14) and (3.15), we have

$$|\langle \psi'(v_n), v_n - v_0\rangle| \le (C_{16} + (4M_1+1)C_{15})\|v_n - v_0\|^2 + o_n(1). \tag{3.16}$$

Since $\alpha_2' > 2$ and $\sigma > 2$, we can choose T_0 small enough such that

$$C_{16} + (4M_1+2)C_{15} < 2^{-1}. \tag{3.17}$$

It follows from (2.43), (3.2), (3.3), (3.8), (3.16) and (3.17) that

$$|\langle \varphi'(v_n), v_n - v_0\rangle| \ge 2^{-1}\|v_n - v_0\|^2 + o_n(1). \tag{3.18}$$

In combination with (3.5) and (3.18), we have $v_n \to v_0$, $n \to \infty$. This completes the proof. □

It is well known that the eigenvalue problem for the following equation

$$\begin{cases} -\Delta u = \lambda u & x \in \Omega, \\ u = 0 & x \in \partial\Omega \end{cases}$$

has a sequence of eigenvalues λ_n (counted with multiplicity) and $0 < \lambda_1 < \lambda_2 < \cdots < \lambda_n < \cdots \to \infty$. The corresponding system of normalized eigenfunctions $\{e_n \mid n \in \mathbb{N}\}$

forming an orthogonal basis in E. By this normalized orthogonal sequence $\{e_n\}_{n=1}^{\infty}$, we can define some subspaces as follows:

$$E_n = \operatorname{span}\{e_1, e_2, \ldots, e_n\}, \quad B^n = \{v \in E_n \mid \|v\| \le 1\}, \quad S^n := \{v \in E_n \mid \|v\| = 1\}$$

and

$$S_+^{n+1} := \{v = w + te_{n+1} \mid \|v\| = 1, w \in B^n, 0 \le t \le 1\}.$$

With the help of these subspaces, we can introduce some continuous maps and minimax sequences of φ as follows:

$$\Lambda_n = \{\gamma \in C(S^n, E)) \mid \gamma \text{ is odd}\}, \quad \Gamma_n = \{\gamma \in C(S_+^{n+1}, E) \mid \gamma|_{S^n} \in \Lambda_n\} \tag{3.19}$$

and

$$b_n = \inf_{\gamma \in \Lambda_n} \max_{v \in S^n} \varphi(\gamma(v)), \quad c_n = \inf_{\gamma \in \Gamma_n} \max_{v \in S_+^{n+1}} \varphi(\gamma(v)). \tag{3.20}$$

For any $\delta > 0$, set

$$\Gamma_n(\delta) = \{\gamma \in \Gamma_n \mid \varphi(\gamma(v)) \le b_n + \delta, v \in S^n\} \tag{3.21}$$

and

$$c_n(\delta) = \inf_{\gamma \in \Gamma_n(\delta)} \max_{v \in S_+^{n+1}} \varphi(\gamma(v)). \tag{3.22}$$

In combination with (3.19)–(3.22), we have $b_n \le c_n \le c_n(\delta)$, $n \in \mathbb{N}$. Next we give some useful estimates for minimax values b_n and $c_n(\delta)$.

Lemma 3.2. *Assume that* (W3), (W5) *and* (H1) *hold. Then for any $n \in \mathbb{N}$, $b_n < 0$.*

Proof. Since E_n is a finite dimensional space, there exists $\varrho_n > 0$ such that

$$\|v\| \le \varrho_n \|v\|_2, \quad \forall\, v \in E_n. \tag{3.23}$$

Since $f(0) = 0$, by Lagrange mean value theorem, there exists a positive constant C_{17} independent of n such that

$$|f(t)| \ge C_{17}|t|, \quad |t| \le 1. \tag{3.24}$$

In view of (1.3), we can choose $0 < r_0 \le 1$ such that

$$g_1(x,t) \ge 8\varrho_n^2 C_{17}^{-2} t \tag{3.25}$$

for all $x \in \Omega$ and $0 \le t \le r_0$. By (3.25) and direct computation, we have

$$G_1(x,t) \ge 4\varrho_n^2 C_{17}^{-2} t^2 \tag{3.26}$$

for all $x \in \Omega$ and $0 \leq t \leq r_0$. In view of (W4), $G_1(x,t)$ is an even function in t. In combination Lemma 2.2(f2), (3.24) and (3.26), we see that

$$G_1(x, f(t)) \geq 4\varrho_n^2 C_{17}^{-2} f^2(t) \geq 4\varrho_n^2 t^2, \quad x \in \Omega \text{ and } |t| \leq r_0. \tag{3.27}$$

Since E_n is finite dimensional, we claim that there exists a constant $\kappa > 0$ such that

$$\frac{1}{2}\int_\Omega |v(x)|^2\,dx \geq \int_{|v|>r_0} |v(x)|^2\,dx, \quad \forall\, v \in E_n \text{ with } \|v\| \leq \kappa. \tag{3.28}$$

If (3.28) is not true, there exists a sequence of $\{v_k\} \subset E_n \setminus \{0\}$ such that $v_k \to 0$ in E_n and

$$\frac{1}{2}\int_\Omega |v_k(x)|^2\,dx < \int_{|v_k|>r_0} |v_k(x)|^2\,dx, \quad \forall\, k \in \mathbb{N}. \tag{3.29}$$

Set $u_k = \|v_k\|_2^{-1} v_k$, $k \in \mathbb{N}$. By (3.23) and (3.29), $\{u_k\}_{k\in\mathbb{N}}$ is bounded and

$$\frac{1}{2} < \int_{|v_k|>r_0} |u_k(x)|^2\,dx, \quad \forall\, k \in \mathbb{N}. \tag{3.30}$$

On the other hand, since E_n is a finite dimensional space, we can assume that $u_k \to u_0$ in E_n. So $u_k \to u_0$ in $L^2(\Omega)$. Moreover, in view of $v_k \to 0$ in E_n, we have

$$\text{meas}\{x \in \Omega \mid |v_k(x)| > r_0\} \to 0 \quad \text{as } k \to \infty. \tag{3.31}$$

Therefore, it follows from (3.31) that

$$\int_{|v_k|>r_0} |u_k|^2\,dx \leq 2\int_\Omega |u_k - u_0|^2\,dx + 2\int_{|v_k|>r_0} |u_0|^2\,dx \to 0, \quad k \to \infty,$$

which contradicts (3.30). So (3.28) holds.

By (W3), Lemma 2.1(i)(iii), (2.24) and (2.26), there exists a constant $\kappa' > 0$ such that

$$\varphi(v) \leq \|v\|^2 - \int_{\Omega_{r_0}} G_1(x, f(v))\,dx, \quad \forall\, v \in E_n \text{ with } \|v\| \leq \kappa', \tag{3.32}$$

where $\Omega_{r_0} := \{x \in \Omega \mid |v(x)| \leq r_0\}$. Combining with (3.27), (3.28) and (3.32), if $v \in E_n$ with $\|v\| \leq \min\{\kappa, \kappa'\}$, we have

$$\begin{aligned}
\varphi(v) &\leq \|v\|^2 - \int_{\Omega_{r_0}} G_1(x, f(v))\,dx \\
&\leq \|v\|^2 - 4\varrho_n^2 \int_{\Omega_{r_0}} |v(x)|^2\,dx \\
&= \|v\|^2 - 4\varrho_n^2 \left(\int_\Omega |v(x)|^2\,dx - \int_{\Omega\setminus\Omega_{r_0}} |v(x)|^2\,dx\right) \\
&\leq -\|v\|^2.
\end{aligned} \tag{3.33}$$

Choose $0 < \rho_0 < \min\{\kappa, \kappa'\}$, let $\gamma(v) = \rho_0 v$, $v \in S^n$. In view of (3.33), we conclude that $b_n < 0$. The proof is completed. □

Lemma 3.3. *Assume that* (W1)–(W3), (W5) *and* (H1) *are satisfied. Then for any* $n \in \mathbb{N}$ *and any* $\delta > 0$, $c_n(\delta) < 0$.

Proof. Combining with (3.21) and (3.22), for fixed $n \in \mathbb{N}$, when $0 < \delta < \delta'$, we have $\Gamma_n(\delta) \subset \Gamma_n(\delta')$ and $c_n(\delta) \geq c_n(\delta')$. So we only need to prove $c_n(\delta) < 0$ for any $\delta \in (0, |b_n|)$. For any $\delta \in (0, |b_n|)$, by the definition of b_n in (3.20), there exists $\gamma_0 \in \Lambda_n$ such that $\max_{v \in S^n} \varphi(\gamma_0(v)) \leq b_n + \delta/2$. Since $\gamma_0(S^n)$ is a compact set in E, there exists a positive integer m_0 such that

$$\max_{v \in S^n} \varphi((P_{m_0} \circ \gamma_0)v) \leq b_n + \delta, \tag{3.34}$$

where P_{m_0} denotes the orthogonal projective operator from E to E_{m_0}.

For any $c \in \mathbb{R}$, let $\varphi^c = \{v \in E \mid \varphi(v) \leq c\}$. Choose $\overline{\varepsilon} = -(b_n + \delta)/2 > 0$. Arguing as in Lemma 3.2, there exists $\rho_{m_0+1} > 0$ such that if $v \in \overline{B}(0, \rho_0) \cap E_{m_0+1}$, $\varphi(v) \leq 0$, where $B(x_0, \rho)$ denotes the open ball of radius ρ centred at x_0 in E and $\overline{B}(x_0, \rho)$ denotes the closure of $B(x_0, \rho)$ in E. In view of $\varphi \in C^1(E, \mathbb{R})$ and $\varphi(0) = 0$, we have $\operatorname{dist}(0, \varphi^{-\overline{\varepsilon}}) > 0$. Define $\rho_0' = \min\{\rho_{m_0+1}, \operatorname{dist}(0, \varphi^{-\overline{\varepsilon}})\}$, then $\rho_0' > 0$. By Deformation Theorem (see Theorem A.4 in [24]), there exists $\varepsilon \in (0, \overline{\varepsilon})$ and a continuous map $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1, v) = v \quad \text{if } \varphi(v) \notin [-\overline{\varepsilon}, \overline{\varepsilon}] \tag{3.35}$$

and

$$\eta(1, \varphi^{\varepsilon} \setminus B(0, \rho_0')) \subset \varphi^{-\varepsilon}, \tag{3.36}$$

where $B(0, \rho_0')$ is a neighbourhood of K_0 given by Lemma 2.6(iii).

By (3.19), $P_{m_0} \circ \gamma_0 \in C(S^n, E_{m_0})$. Since E_{n+1} is a metric space with the norm $\|\cdot\|$ and S^n is a closed subset in E_{n+1}, there exists an extension $\widetilde{P_{m_0} \circ \gamma_0} \colon E_{n+1} \to E_{m_0}$ of $P_{m_0} \circ \gamma_0$ by Dugundji extension theorem (see Theorem 4.1 in [10]); furthermore,

$$((\widetilde{P_{m_0} \circ \gamma_0})E_{n+1}) \subset \operatorname{co}((P_{m_0} \circ \gamma_0)S^n), \tag{3.37}$$

where co denotes the convex hull. Since $(P_{m_0} \circ \gamma_0)S^n$ is a compact set in E_{m_0}, by the definition of convex hull, $\operatorname{co}((P_{m_0} \circ \gamma_0)S^n)$ is a bounded set in E_{m_0}. Then there exists a constant ν such that $\varphi(v) \leq \nu$, $\forall\, v \in \operatorname{co}((P_{m_0} \circ \gamma_0)S^n)$. It follows from (3.37) that

$$\varphi((\widetilde{P_{m_0} \circ \gamma_0})v) \leq \nu, \quad \forall\, v \in E_{n+1}. \tag{3.38}$$

Next we consider two possible cases.

Case 1*:* $\nu \leq \varepsilon$. Since $\widetilde{P_{m_0} \circ \gamma_0} \in C(E_{n+1}, E_{m_0})$, by (3.38), we have

$$(\widetilde{P_{m_0} \circ \gamma_0})v \in \varphi^{\varepsilon}_{m_0}, \quad \forall\, v \in E_{n+1}, \tag{3.39}$$

where $\varphi_{m_0}^{\varepsilon} := \{v \in E_{m_0} \mid \varphi(v) \leq \varepsilon\}$. Define a map T as follows:

$$T(v) = \begin{cases} v & \text{if } v \notin \overline{B}(0,\rho_0') \cap E_{m_0}, \\ v + (\rho_0'^2 - \|v\|^2)^{1/2} e_{m_0+1} & \text{if } v \in \overline{B}(0,\rho_0') \cap E_{m_0}. \end{cases} \tag{3.40}$$

By (3.40), we have $T \in C(E_{m_0}, E_{m_0+1})$ and

$$(T \circ (\widetilde{P_{m_0} \circ \gamma_0}))v \notin B(0,\rho_0'), \quad \forall\, v \in E_{n+1}. \tag{3.41}$$

When $v \in E_{n+1}$ and $\|(\widetilde{P_{m_0} \circ \gamma_0})v\| > \rho_0'$, by (3.39) and (3.40), we obtain

$$(T \circ (\widetilde{P_{m_0} \circ \gamma_0}))v = (\widetilde{P_{m_0} \circ \gamma_0})v \in \varphi_{m_0}^{\varepsilon}. \tag{3.42}$$

Otherwise, if $v \in E_{n+1}$ and $\|(\widetilde{P_{m_0} \circ \gamma_0})v\| \leq \rho_0'$, in view of (3.40), $\|(T \circ (\widetilde{P_{m_0} \circ \gamma_0}))v\| = \rho_0'$. By the definition of ρ_0' and (3.42), we have

$$(T \circ (\widetilde{P_{m_0} \circ \gamma_0}))v \subset \varphi^{\varepsilon}, \quad \forall\, v \in E_{n+1}. \tag{3.43}$$

Define a map $H_0 \colon E_{n+1} \to E$ as follows:

$$H_0(\cdot) = \eta(1, (T \circ (\widetilde{P_{m_0} \circ \gamma_0}))(\cdot)). \tag{3.44}$$

We need to prove $H_0 \in \Gamma_n(\delta)$ and $\max_{v \in S_+^{n+1}} \varphi(H_0(v)) < 0$. First, it is obvious that $H_0 \in C(S_+^{n+1}, E)$. Next we prove $H_0|_{S^n} \in \Lambda_n$. By Dugundji extension theorem,

$$(\widetilde{P_{m_0} \circ \gamma_0})v = (P_{m_0} \circ \gamma_0)v, \quad \forall\, v \in S^n. \tag{3.45}$$

By (3.34), $(P_{m_0} \circ \gamma_0)v \in \varphi^{-2\overline{\varepsilon}}$, $v \in S^n$. By the definition of ρ_0' and $\varphi^{-2\overline{\varepsilon}} \subset \varphi^{-\overline{\varepsilon}}$, we have

$$\|(P_{m_0} \circ \gamma_0)v\| \geq \rho_0', \quad \forall\, v \in S^n. \tag{3.46}$$

It follows from (3.40), (3.45) and (3.46) that

$$(T \circ (\widetilde{P_{m_0} \circ \gamma_0}))v = T \circ ((P_{m_0} \circ \gamma_0)v) = (P_{m_0} \circ \gamma_0)v, \quad \forall\, v \in S^n. \tag{3.47}$$

Since $(P_{m_0} \circ \gamma_0)v \in \varphi^{-2\overline{\varepsilon}}$, $v \in S^n$, by (3.35), (3.44) and (3.47), we get

$$H_0(v) = \eta(1, (T \circ (\widetilde{P_{m_0} \circ \gamma_0}))v) = (P_{m_0} \circ \gamma_0)v, \quad \forall\, v \in S^n, \tag{3.48}$$

which implies that $H_0|_{S^n} \in \Lambda_n$. Moreover, in view of (3.34) and (3.48), we have $H_0 \in \Gamma_n(\delta)$. Since $S^{n+1} \subset E_{n+1}$, by (3.41) and (3.43), $(T \circ (\widetilde{P_{m_0} \circ \gamma_0}))v \notin B(0,\rho_0')$, $\forall\, v \in S_+^{n+1}$ and $(T \circ (\widetilde{P_{m_0} \circ \gamma_0}))v \in \varphi^{\varepsilon}$, $\forall\, v \in S_+^{n+1}$. It follows from (3.36) and (3.44) that $\max_{v \in S_+^{n+1}} \varphi(H_0(v)) \leq -\varepsilon < 0$, which implies that $c_n(\delta) < 0$.

Case 2: $\nu > \varepsilon$. Let $\varphi|_{E_{m_0}}$ denote the restriction of φ on E_{m_0}. By a similar proof as in Lemmas 2.6 and 3.1, we can prove that $\varphi|_{E_{m_0}} \in C^1(E_{m_0}, \mathbb{R})$ and satisfies Palais-Smale condition. Moreover, $\varphi|_{E_{m_0}}$ has no critical point with positive critical values on E_{m_0}. By Noncritical interval theorem (see Theorem 5.1.6 in [7]), $\varphi^{\varepsilon}_{m_0}$ is a strong deformation retraction of $\varphi^{\nu}_{m_0}$. So there exists a map ς such that $\varsigma \in C(\varphi^{\nu}_{m_0}, \varphi^{\varepsilon}_{m_0})$ and $\varsigma(v) = v$, if $v \in \varphi^{\varepsilon}_{m_0}$. Define a map from $E_{n+1} \to E$ as follows:

$$\overline{H}_0(\cdot) = \eta(1, (T \circ (\varsigma \circ (\widetilde{P_{m_0} \circ \gamma_0})))(\cdot)).$$

By a similar proof as in Case 1, $\overline{H}_0 \in \Gamma_n(\delta)$ and $\max_{v \in S^{n+1}_+} \varphi(\overline{H_0}(v)) \leq -\varepsilon < 0$. In view of (3.22), we have $c_n(\delta) < 0$. This completes the proof. □

Lemma 3.4. *Suppose that* (W1), (W5) *and* (H1) *are satisfied. Then there exists a positive constant C_{18} independent of n such that for all n large enough*

$$b_n \geq -C_{18} n^{-2p/(N(2\alpha-p))}. \tag{3.49}$$

Proof. For any $\gamma \in \Lambda_n$ $(n \geq 2)$, if $0 \notin \gamma(S^n)$, then the genus $\vartheta(\gamma(S^n))$ is well defined and $\vartheta(\gamma(S^n)) \geq \vartheta(S^n) = n$. By Proposition 7.8 in [24], $\gamma(S^n) \cap E^{\perp}_{n-1} \neq \emptyset$. Otherwise, when $0 \in \gamma(S^n)$, then $0 \in \gamma(S^n) \cap E^{\perp}_{n-1}$. So for any $\gamma \in \Lambda_n$ $(n \geq 2)$, $\gamma(S^n) \cap E^{\perp}_{n-1} \neq \emptyset$. Therefore, for any $\gamma \in \Lambda_n$ $(n \geq 2)$, we have

$$\max_{v \in S^n} \varphi(\gamma(v)) \geq \inf_{v \in E^{\perp}_{n-1}} \varphi(v). \tag{3.50}$$

In view of Lemma 2.1(i)(iii), (W1), (2.9), (2.18), (2.20) and (2.24)–(2.26), we get

$$\varphi(v) \geq \frac{1}{4}\|v\|^2 - C_{19}\|v\|_2^{p/\alpha}, \quad \forall\, v \in E. \tag{3.51}$$

When $v \in E^{\perp}_{n-1}$, $\lambda_n \|v\|_2^2 \leq \|v\|^2$. If $v \in E^{\perp}_{n-1}$, by (3.51), we conclude that

$$\varphi(v) \geq \frac{1}{4}\|v\|^2 - C_{19}\lambda_n^{-p/(2\alpha)}\|v\|^{p/\alpha}. \tag{3.52}$$

In combination with (3.20), (3.50) and (3.52), for $n \geq 2$, we have

$$b_n \geq \inf_{t \geq 0}\left\{\frac{1}{4}t^2 - C_{19}\lambda_n^{-p/(2\alpha)} t^{p/\alpha}\right\} = -C_{20}\lambda_n^{-p/(2\alpha-p)}, \tag{3.53}$$

where C_{20} is a positive constant independent of n and λ_n. When n is large enough, it is well known that $\lambda_n \geq C_{21} n^{2/N}$. By (3.53), (3.49) holds. The proof is completed. □

Lemma 3.5. *If $c_n = b_n$ for all $n \geq n_0$, where $n_0 \in \mathbb{N}$, then there exists a positive integer n_1 such that*

$$|b_n| \geq C_{22} n^{2\alpha/(2\alpha-\sigma)}, \quad n \geq n_1, \tag{3.54}$$

where C_{22} is a positive constant independent of n.

Proof. For any $n \geq n_0$ and any $\varepsilon \in (0, |b_n|)$, by Lemma 3.2 and (3.20), there exists a map $\gamma_1 \in \Gamma_n$ such that

$$\max_{v \in S_+^{n+1}} \varphi(\gamma_1(v)) < c_n + \varepsilon = b_n + \varepsilon < 0. \tag{3.55}$$

Since $S^{n+1} = S_+^{n+1} \cup (-S_+^{n+1})$, so γ_1 can be continuously extended to S^{n+1} as an odd function, also denoted by γ_1, so $\gamma_1 \in \Lambda_{n+1}$. Therefore in view of (3.20), we have

$$b_{n+1} \leq \max_{v \in S^{n+1}} \varphi(\gamma_1(v)) = \varphi(\gamma_1(v_0)) \tag{3.56}$$

for some $v_0 \in S^{n+1}$. If $v_0 \in S_+^{n+1}$, by (3.55) and (3.56), $b_{n+1} \leq \varphi(\gamma_1(v_0)) < b_n + \varepsilon$. So for any $\varepsilon \in (0, |b_n|)$,

$$b_{n+1} < b_n + \varepsilon + C_{11}|b_{n+1}|^{\sigma/(2\alpha)}, \tag{3.57}$$

where C_{11} is given in (2.29). Otherwise, $v_0 \in -S_+^{n+1}$. By (2.29) and (3.55), we see that

$$\begin{aligned} \varphi(\gamma_1(v_0)) &\leq \varphi(\gamma_1(-v_0)) + C_{11}|\varphi(\gamma_1(v_0))|^{\sigma/(2\alpha)} \\ &\leq b_n + \varepsilon + C_{11}|\varphi(\gamma_1(v_0))|^{\sigma/(2\alpha)}. \end{aligned} \tag{3.58}$$

Next we consider two possible cases.

Case 1*:* $\varphi(\gamma_1(v_0)) \leq |b_{n+1}|$. In view of (3.56) and (3.58), for any $\varepsilon \in (0, |b_n|)$, we have

$$b_{n+1} \leq b_n + \varepsilon + C_{11}|b_{n+1}|^{\sigma/(2\alpha)}. \tag{3.59}$$

Case 2*:* $\varphi(\gamma_1(v_0)) > |b_{n+1}|$. By (3.55), there exists $v_1 \in S_+^{n+1}$ such that

$$\varphi(\gamma_1(v_1)) < b_n + \varepsilon < 0. \tag{3.60}$$

By the assumption in Case 2 and (3.60), $\varphi(\gamma_1(v_0)) > |b_{n+1}|$ and $\varphi(\gamma_1(v_1)) < 0$. Since $(\varphi \circ \gamma_1) \in C(S^{n+1}, \mathbb{R})$ and S^{n+1} is a connected space, by Intermediate Value Theorem (see Theorem 24.3 in [21]), there exists $v_2 \in S^{n+1}$ such that

$$\varphi(\gamma_1(v_2)) = \frac{|b_{n+1}|}{2}. \tag{3.61}$$

By (3.55), we have $v_2 \in -S_+^{n+1}$. It follows from (2.29), (3.55) and (3.61) that

$$\begin{aligned} b_{n+1} < \varphi(\gamma_1(v_2)) &\leq \varphi(\gamma_1(-v_2)) + C_{11}|\varphi(\gamma_1(v_2))|^{\sigma/(2\alpha)} \\ &< b_n + \varepsilon + C_{11}|\varphi(\gamma_1(v_2))|^{\sigma/(2\alpha)} \\ &< b_n + \varepsilon + C_{11}|b_{n+1}|^{\sigma/(2\alpha)} \end{aligned} \tag{3.62}$$

for any $\varepsilon \in (0, |b_n|)$. By Lemma 3.2, $b_n < 0$ for any $n \in \mathbb{N}$. In combination with (3.57), (3.59) and (3.62), we get

$$|b_n| \leq |b_{n+1}| + C_{11}|b_{n+1}|^{\sigma/(2\alpha)}, \quad n \geq n_0. \tag{3.63}$$

Next we show that (3.63) implies (3.54). The proof will be done by induction. Next we introduce a useful inequality as follows:

$$(1+x)^\beta \geq 1+\frac{\beta x}{2}, \quad x \in [0,\delta], \tag{3.64}$$

where β, δ are positive constants and δ depends on β. Set $\beta = 2\alpha(\sigma-2\alpha)^{-1}$. Then $\beta > 0$ by (H1). In view of (3.64), there exists $\overline{n}_0 \in \mathbb{N}$ such that

$$\left(1+\frac{1}{n}\right)^{2\alpha/(\sigma-2\alpha)} \geq 1+\frac{\alpha}{(\sigma-2\alpha)n}, \quad n \geq \overline{n}_0. \tag{3.65}$$

Define

$$C_{22} = \min\left\{ n_1^{2\alpha/(\sigma-2\alpha)}|b_{n_1}|, \left(\frac{\alpha}{C_{11}(\sigma-2\alpha)}\right)^{2\alpha/(\sigma-2\alpha)} \right\}, \tag{3.66}$$

where $n_1 := \max\{n_0, \overline{n}_0\}$. Then we claim (3.54) holds. In view of (3.66), it is obvious that $|b_{n_1}| \geq C_{22} n_1^{2\alpha/(2\alpha-\sigma)}$. Suppose that (3.54) holds for $j \geq n_1$. Then we only need to prove (3.54) also holds for $j+1$. If not, we have

$$|b_{j+1}| < C_{22}(j+1)^{2\alpha/(2\alpha-\sigma)}. \tag{3.67}$$

Since (3.54) holds for j, by (3.63) and (3.67), we get

$$\begin{aligned} C_{22} j^{2\alpha/(2\alpha-\sigma)} &\leq |b_j| \leq |b_{j+1}| + C_{11}|b_{j+1}|^{\sigma/(2\alpha)} \\ &< C_{22}(j+1)^{2\alpha/(2\alpha-\sigma)} + C_{11} C_{22}^{\sigma/(2\alpha)} (j+1)^{\sigma/(2\alpha-\sigma)}. \end{aligned} \tag{3.68}$$

When we divide (3.68) by $C_{22}(j+1)^{2\alpha/(2\alpha-\sigma)}$ on both sides, by (3.66), we have

$$\left(1+\frac{1}{j}\right)^{2\alpha/(\sigma-2\alpha)} < 1 + C_{11} C_{22}^{(\sigma-2\alpha)/(2\alpha)} \frac{1}{j+1} < 1 + C_{11} C_{22}^{(\sigma-2\alpha)/(2\alpha)} \frac{1}{j} \leq 1 + \frac{\alpha}{(\sigma-2\alpha)j},$$

which contradict (3.65). So (3.54) holds. This completes the proof. □

Combining (H2), (3.49) and (3.54), we conclude that it is impossible that $c_n = b_n$ for all large n. Next we can construct critical values of φ as follows.

Lemma 3.6. *Suppose that $c_n > b_n$. Then for any $\delta \in (0, c_n - b_n)$, $c_n(\delta)$ defined by* (3.22) *is a critical value of φ.*

Proof. We prove this lemma by contradiction. For any $\delta \in (0, c_n - b_n)$, if $c_n(\delta)$ is not a critical value for the functional φ, define $\overline{\varepsilon} = (c_n - b_n - \delta)/2$, by Deformation Theorem, there exist $\varepsilon \in (0, \overline{\varepsilon})$ and $\eta \in C([0,1] \times E, E)$ such that

$$\eta(1,v) = v \quad \text{if } \varphi(v) \notin [c_n(\delta) - \overline{\varepsilon}, c_n(\delta) + \overline{\varepsilon}] \tag{3.69}$$

and

$$\eta(1, \varphi^{c_n(\delta)+\varepsilon}) \subset \varphi^{c_n(\delta)-\varepsilon}. \tag{3.70}$$

By (3.22), there exists $\gamma_2 \in \Gamma_n(\delta)$ such that

$$\max_{v \in S_+^{n+1}} \varphi(\gamma_2(v)) < c_n(\delta) + \varepsilon. \tag{3.71}$$

Define

$$\overline{\gamma}_2(v) = \eta(1, \gamma_2(v)), \quad v \in S_+^{n+1}. \tag{3.72}$$

It is obvious that $\overline{\gamma}_2 \in C(S_+^{n+1}, E)$. Since $\gamma_2 \in \Gamma_n(\delta)$, in view of (3.21), we obtain

$$\varphi(\gamma_2(v)) \le b_n + \delta = c_n - 2\overline{\varepsilon} \le c_n(\delta) - 2\overline{\varepsilon}, \quad v \in S^n. \tag{3.73}$$

It follows from (3.69), (3.72) and (3.73) that $\overline{\gamma}_2(v) = \gamma_2(v)$, $v \in S^n$, which yields

$$\overline{\gamma}_2|_{S_n} \in \Lambda_n \quad \text{and} \quad \varphi(\overline{\gamma}_2(v)) = \varphi(\gamma_2(v)) \le b_n + \delta, \quad v \in S^n. \tag{3.74}$$

By (3.74), we have $\overline{\gamma}_2 \in \Gamma_n(\delta)$. In combination with (3.70)–(3.72), we get

$$\max_{v \in S_+^{n+1}} \varphi(\overline{\gamma}_2(v)) = \max_{v \in S_+^{n+1}} \varphi(\eta(1, \gamma_2(v))) \le c_n(\delta) - \varepsilon,$$

which contradicts (3.22). The proof is completed. □

Proof of Theorem 1.1. By (H2), (3.49) and (3.54), it is impossible that $c_n = b_n$ for all large n. Then we can choose a subsequence $\{n_j\} \subset \mathbb{N}$ such that $c_{n_j} > b_{n_j}$. It follows from Lemmas 3.4 and 3.6 that there exists a sequence of critical points $\{v_{n_j}\}_{j=1}^{\infty}$ of φ such that

$$-C_{18} n_j^{-2p/(N(2\alpha-p))} \le b_{n_j} < c_{n_j} \le c_{n_j}(\delta_j) = \varphi(v_{n_j}) < 0, \tag{3.75}$$

where $\delta_j \in (0, c_{n_j} - b_{n_j})$. By (2.24) and the fact $\varphi(v_{n_j}) < 0$, we have $v_{n_j} \ne 0$, $j \in \mathbb{N}$. Next we consider two possibilities.

Case 1*:* $\|v_{n_j}\|^2 > 2T_0$. By (2.8), (2.9) and (2.30), we obtain $k(v_{n_j}) = 0$, $k'(v_{n_j}) = 0$ and $\psi'(v_{n_j}) = 0$. Combining with (W2), (2.13) and (2.43), we get

$$\overline{I}(v_{n_j}) = \overline{I}(v_{n_j}) - \mu^{-1} \left\langle \varphi'(v_{n_j}), \frac{f(v_{n_j})}{f'(v_{n_j})} \right\rangle \le 2A\|v_{n_j}\|^2 < A\|v_{n_j}\|^2. \tag{3.76}$$

Case 2*:* $\|v_{n_j}\|^2 \le 2T_0$. By (W2), (2.10), (2.13), (2.20), (2.26), (2.28) and (2.43), we have

$$\begin{aligned} \overline{I}(v_{n_j}) &= \overline{I}(v_{n_j}) - \mu^{-1} \left\langle \varphi'(v_{n_j}), \frac{f(v_{n_j})}{f'(v_{n_j})} \right\rangle \\ &\le 2A\|v_{n_j}\|^2 + C_8\|v_{n_j}\|^{\alpha_1/\alpha} + C_9\|v_{n_j}\|^{\alpha_2'/\alpha} + (M_0 + 10C_{10})\|v_{n_j}\|^{\sigma/\alpha} \\ &\le A\|v_{n_j}\|^2. \end{aligned} \tag{3.77}$$

In view of (2.21)–(2.23), (3.76) or (3.77), $l(v_{n_j}) = 1$ and $l'(v_{n_j}) = 0$. Then it follows from (2.13), (2.24) and (2.25) that $\varphi(v_{n_j}) = \overline{I}(v_{n_j}) \le A\|v_{n_j}\|^2 < 0$. Moreover, by (3.75), we have $\|v_{n_j}\| \to 0$, $j \to \infty$. So there exists $j_0 \in \mathbb{N}$ such that $\|v_{n_j}\|^2 < T_0$, $j \ge j_0$. In view of (2.9) and (2.11), we get $k(v_{n_j}) = 1$ and $k'(v_{n_j}) = 0$ for all $j \ge j_0$. Combining with (2.9), (2.30) and (2.43), when $j \ge j_0$, v_{n_j} are also critical points of $\overline{I}$ and weak solutions of (2.7). Moreover, by elliptic regularity theory and $\|v_{n_j}\| \to 0$, there exists $j_1 \in \mathbb{N}$ such that $\|v_{n_j}\|_\infty < \min\{\delta_0', \delta_1'\}$ for all $j \ge j_1$, where δ_0' and δ_1' are given in Lemma 2.1(ii)(iv). It follows from Lemma 2.2(f2) that $\|f(v_{n_j})\|_\infty < \min\{\delta_0', \delta_1'\}$ for all $j \ge j_1$. Set $j_2 = \max\{j_0, j_1\}$. Combining with Lemma 2.1(ii)(iv), $u_{n_j} = f(v_{n_j})$ are also a sequence of weak solutions of (1.1) for all $j \ge j_2$. This completes the proof. □

4. Example

Example 4.1. In (1.1), let Ω be a bounded smooth domain in $\mathbb{R}^3$ and $\alpha = 2$. Define $g(x,t) = a(x)|t|^{-1/7} t \arctan(1 + t^4)$ and $h(x,t) = t^{10}$, $(x,t) \in \overline{\Omega} \times \mathbb{R}$, where $a(x)$ is a positive continuous function in $\overline{\Omega}$ with $\inf_{x \in \overline{\Omega}} a(x) > 0$. Set

$$g_1(x,t) = \frac{\pi}{4} a(x)|t|^{-1/7} t, \quad g_2(x,t) = a(x)|t|^{-1/7} t(\arctan(1 + t^4) - \pi/4).$$

It is obvious that $g = g_1 + g_2$. By Lagrange mean value theorem, $|g_2(x,t)| \le M'|t|^{34/7}$, $(x,t) \in \overline{\Omega} \times \mathbb{R}$, where M' is a positive constant. Choose $\mu = p = 13/7$, $\alpha_1 = 5$, $\alpha_2 = 41/7$ and $\sigma = 11$, so all the conditions of Theorem 1.1 are satisfied. By Theorem 1.1, problem (1.1) has a sequence of weak solutions approaching to 0. Since $h(x,t)$ is not odd in t, the results in the reference cannot be applied to this case.

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Liang Zhang
School of Mathematical Sciences, University of Jinan, Jinan, Shandong 250022,
P. R. China
E-mail address: `mathspaper2012@163.com`

Xianhua Tang
School of Mathematics and Statistics, Central South University, Changsha, Hunan
410083, P. R. China
E-mail address: `tangxh@mail.csu.edu.cn`

Yi Chen
Department of Mathematics, China University of Mining and Technology, Xuzhou,
221116, P. R. China
E-mail address: `chenyi@cumt.edu.cn`