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Breathers of Discrete One-dimensional Nonlinear Schrödinger Equations in Inhomogeneous Media

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Abstract. This paper is concerned with the breathers of discrete one-dimensional nonlinear Schrödinger equations in inhomogeneous media. By using a constrained minimization approach known as the Nehari variational principle or the Nehari manifold approach, we obtain the existence of nontrivial breathers.

1. Introduction

Consider the one-dimensional nonlinear Schrödinger (NLS) equations in inhomogeneous media

$$i\psi_t + (a(x)\psi_x)_x - V(x)\psi + \sigma\gamma(x)f(\psi) = 0, \quad x \in \mathbb{R}, \tag{1.1}$$

where $\sigma = \pm 1$, there exist positive constants M and $\overline{\gamma}$ such that $0 < a(x) \leq M$ and $0 < \gamma(x) \leq \overline{\gamma}$, and the nonlinearity $f(\psi)$ is gauge invariant, i.e., $f(e^{i\omega t}\varphi) = e^{i\omega t}f(\varphi)$.

It is well known that, the NLS equation arises in several areas of Physics, such as Optics or Quantum Mechanics, where it is related to Bose-Einstein condensation or Superfluidity (see [2, 6, 12]). Since the seventies in last century, the classical NLS equation, which corresponds to the case that $a(x) \equiv 1$ in (1.1), has been studied extensively (see [1, 7–10, 14]). Recently, much attention has been paid on the NLS equation in inhomogeneous media because of its important applications, for example, in Optics it naturally corresponds to a variable optical index (see [6] for a survey).

In this paper, we mainly focus on the discretization equation of (1.1), which, by a semi-discretization method (see [3]), has the following form

$$i\dot{\psi}_n + (\mathcal{A}\psi)_n - \upsilon_n\psi_n + \sigma\gamma_n f(\psi_n) = 0, \quad n \in \mathbb{Z}, \tag{1.2}$$

where $\sigma = \pm 1$, $0 < \gamma_n \leq \overline{\gamma}$ and the discrete operator

$$(\mathcal{A}\psi)_n = a_{n+1/2}(\psi_{n+1} - \psi_n) + a_{n-1/2}(\psi_{n-1} - \psi_n).$$

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Over the past decade, the discrete nonlinear Schrödinger (DNLS) equation became a strong focal point, among other reasons due to the increasing visibility of the physical realizations of this deceptively simple-looking mathematical model. The DNLS models were widely used in experimental setups of waveguide arrays (see [5,11]) and Bose-Einstein Condensates trapped in periodic optical lattices. In recent years, the existence of breathers of the DNLS equations has drawn a great deal of interest (see, for example, [14–16]). However, most of the existing literature is devoted to the DNLS equations with constant coefficients. To the best of our knowledge, there are no results on the breathers of discrete nonlinear Schrödinger equation in inhomogeneous media (1.2). Motivated by the previous works, in this paper we investigate the impact of inhomogeneous media and study the existence of nontrivial breathers of (1.2).

The paper is organized as follows. Section 2 contains some preliminaries and basic results on the Nehari manifold. Section 3 gives the main results on the existence and exponential decay properties of breather solutions.

2. Preliminaries and basic results

Denote

$$\ell^p(\mathbb{Z}) = \left\{ \varphi = \{\varphi_n\}_{n\in\mathbb{Z}} : \forall\, n \in \mathbb{Z}, \varphi_n \in \mathbb{R}, \|\varphi\|_{\ell_p} = \left(\sum_{n\in\mathbb{Z}} |\varphi_n|^p\right)^{1/p} < \infty \right\}.$$

It is known that $\ell^q \subset \ell^p$, $\|\varphi\|_{\ell^p} \le \|\varphi\|_{\ell^q}$, $1 \le q \le p \le \infty$.

Now we consider the breather solutions of (1.2) with the form

$$\psi_n = e^{-it\omega}\varphi_n. \tag{2.1}$$

Inserting the ansatz of a breather solution with the form (2.1) into equation (1.2), we obtain the following nonlinear system

$$-(\mathcal{A}\varphi)_n + \upsilon_n\varphi_n - \omega\varphi_n - \sigma\gamma_n f(\varphi_n) = 0, \quad n \in \mathbb{Z}. \tag{2.2}$$

By the definition of $\mathcal{A}$, it is easy to see that

$$(\mathcal{A}\varphi, \upsilon)_{\ell^2} = (\mathcal{A}\upsilon, \varphi)_{\ell^2}, \quad \forall\, \varphi, \upsilon \in \ell^2.$$

Furthermore, by boundedness of $a(x)$, we have

$$0 \le (-\mathcal{A}\varphi, \varphi)_{\ell^2} \le 4M\|\varphi\|_{\ell^2}.$$

Thus, $\mathcal{A}$ is bounded, self-adjoint on ℓ_2 and $\sigma(-\mathcal{A}) \subset [0, 4M]$.

Assume that the potential $V = \{\upsilon_n\}_{n\in\mathbb{Z}}$ is positive, bounded from below and satisfies:

$$\lim_{|n|\to\infty} \upsilon_n = \infty. \tag{2.3}$$

Without loss of generality we assume that $V \geq 1$, and define the operator

$$L = -\mathcal{A} + V,$$

and Hilbert space

$$H = \{\varphi \in \ell^2(\mathbb{Z}) : L^{1/2}\varphi \in \ell^2(\mathbb{Z})\}, \quad \|\varphi\|_H = \|L^{1/2}\varphi\|_{\ell^2}.$$

It is obvious that L is a self-adjoint operator on $\ell^2(\mathbb{Z})$. Furthermore, according to the definition of equivalent norms, we can obtain the relation in H:

$$\|\varphi\|_H \sim \|V^{1/2}\varphi\|_{\ell^2}.$$

Lemma 2.1. [4] *Let $\Theta = \{\theta_n\}_{n\in\mathbb{Z}}$ be a multiplication operator from ℓ^2 to ℓ^2 defined by $(\Theta\varphi)_n = \theta_n\varphi_n$ for any $n \in \mathbb{Z}$. If $\lim_{|n|\to\infty} \theta_n = 0$, then the operator Θ is compact.*

Lemma 2.2. [14] *If V satisfies the condition* (2.3)*, then*

(1) *the embedding map from H into ℓ^p is compact for any $2 \leq p \leq \infty$;*

(2) *the spectrum $\sigma(L)$ is discrete, that is, the eigenvalues of the operator L have finite multiplicities.*

Now we consider the system (2.2), which can be rewritten as

$$L\varphi_n - \omega\varphi_n - \sigma\gamma_n f(\varphi_n) = 0. \tag{2.4}$$

We first make the following hypotheses:

(H1) The function $f \in C^1(\mathbb{R})$ is an odd function satisfying $f(0) = f'(0) = 0$, and there exist $C_1 > 0$ and $2 < p < \infty$ such that

$$|f(\varphi) - f(\psi)| \leq C_1(1 + |\varphi|^{p-2} + |\psi|^{p-2})|\varphi - \psi|. \tag{2.5}$$

(H2) There exists $2 < q < \infty$ such that

$$0 < (q-1)f(\varphi)\varphi \leq f'(\varphi)\varphi^2, \quad \forall\, \varphi \neq 0 \tag{2.6}$$

and

$$\lim_{\varphi\to 0} \frac{f(\varphi)}{|\varphi|^{q-2}\varphi} = C_0 > 0. \tag{2.7}$$

Remark 2.3. We would like to remark that, a typical example which satisfies (H1) and (H2) is $f(u) = C|u|^{p-2}u$ with $C \neq 0$ and $p > 2$.

By (H1), it is easy to obtain that

$$|f(\varphi)| \leq C_2(1 + |\varphi|^{p-1}) \tag{2.8}$$

holds for some constant $C_2 > 0$. In fact, by taking $\psi = 0$ in (2.5), we have

$$|f(\varphi)| \leq C_1(1 + |\varphi|^{p-2})|\varphi| = C_1(|\varphi| + |\varphi|^{p-1}). \tag{2.9}$$

For $|\varphi| \leq 1$, it is obvious that

$$|f(\varphi)| \leq C_1(|\varphi| + |\varphi|^{p-1}) \leq C_1(1 + |\varphi|^{p-1}).$$

On the other hand, for $|\varphi| > 1$, we know

$$|\varphi| \leq |\varphi|^{p-1}$$

since $p > 2$. Thus, for $|\varphi| > 1$, we have

$$|f(\varphi)| \leq C_1(|\varphi| + |\varphi|^{p-1}) \leq 2C_1|\varphi|^{p-1}.$$

Therefore, for any $\varphi \in \mathbb{R}$, we have

$$|f(\varphi)| \leq 2C_1(1 + |\varphi|^{p-1}),$$

which shows (2.8) holds for $C_2 = 2C_1$.

Moveover, by (H1), we can also conclude that for any given $\varepsilon > 0$, there is $A(\varepsilon) > 0$, such that

$$f(\varphi)\varphi \leq \varepsilon|\varphi|^2 + A(\varepsilon)|\varphi|^p. \tag{2.10}$$

Note that $f'(0) = 0$, then f is superlinear near 0, i.e.,

$$\lim_{\varphi \to 0} \frac{f(\varphi)}{\varphi} = 0.$$

Of course, this assertion can also be obtained directly from (2.6). Thus, for any given $\varepsilon > 0$, there is $\delta > 0$, such that

$$|f(\varphi)| \leq \varepsilon|\varphi|$$

holds for any $0 < |\varphi| \leq \delta$. Since f is an odd function with $f(0) = 0$, it is known that

$$f(\varphi)\varphi \leq \varepsilon|\varphi|^2$$

holds for any $|\varphi| \le \delta$. Furthermore, for $|\varphi| > \delta$, it is obvious that

$$\frac{|\varphi|^2 + |\varphi|^p}{|\varphi|^p} = 1 + \frac{1}{|\varphi|^{p-2}} \le 1 + \frac{1}{\delta^{p-2}},$$

i.e.,

$$|\varphi|^2 + |\varphi|^p \le \left(1 + \frac{1}{\delta^{p-2}}\right) |\varphi|^p.$$

Thus, by (2.9), we have

$$f(\varphi)\varphi \le C_1(|\varphi|^2 + |\varphi|^p) \le C_1 \left(1 + \frac{1}{\delta^{p-2}}\right) |\varphi|^p$$

holds for $|\varphi| > \delta$. Therefore, for any $\varphi \in \mathbb{R}$, we have

$$f(\varphi)\varphi \le \varepsilon|\varphi|^2 + C_1 \left(1 + \frac{1}{\delta^{p-2}}\right) |\varphi|^p,$$

which shows (2.10) holds for $A(\varepsilon) = C_1(1 + 1/\delta^{p-2})$.

Finally, since f is an odd function, (2.6) implies that

$$0 < qF(\varphi) \le f(\varphi)\varphi, \quad \forall\, \varphi \ne 0, \tag{2.11}$$

where $F(\varphi) = \int_0^\varphi f(t)\,dt$ is an even function. By combining (2.8) and (2.11), we can conclude

$$q \le p.$$

Furthermore, by (2.11), we have

$$\frac{F'(\varphi)}{F(\varphi)} \ge \frac{q}{\varphi}, \quad \forall\, \varphi > 0,$$

which implies that

$$(\ln F(\varphi))' \ge q(\ln \varphi)', \quad \forall\, \varphi > 0,$$

i.e.,

$$\left(\ln \frac{F(\varphi)}{\varphi^q}\right)' \ge 0, \quad \forall\, \varphi > 0.$$

This implies that $\ln(F(\varphi)/\varphi^q)$ is nondecreasing in $\varphi \in (0, +\infty)$, so $F(\varphi)/\varphi^q$ is also nondecreasing in $\varphi \in (0, +\infty)$. Moreover, by L'Hôpital's rule, (2.7) gives that

$$\lim_{\varphi \to 0^+} \frac{F(\varphi)}{\varphi^q} = \lim_{\varphi \to 0^+} \frac{f(\varphi)}{q\varphi^{q-1}} = \frac{C_0}{q} > 0.$$

Denote $C_3 = C_0/q$. Then we have

$$F(\varphi) \ge C_3\varphi^q, \quad \forall\, \varphi \ge 0.$$

Note that $F(\varphi) = \int_0^\varphi f(t)\,dt$ is an even function, therefore we have

$$F(\varphi) \geq C_3|\varphi|^q, \quad \forall\, \varphi \in \mathbb{R}. \tag{2.12}$$

Define

$$J(\varphi) = \frac{1}{2}((L-\omega)\varphi, \varphi) - \sigma \sum_{n\in\mathbb{Z}} \gamma_n F(\varphi_n).$$

It is obvious that $J(\varphi) \in C^1(H, \mathbb{R})$, and

$$(J'(\varphi), \psi) = ((L-\omega)\varphi, \psi) - \sigma \sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n)\psi_n.$$

We denote

$$I(\varphi) = (J'(\varphi), \varphi) = ((L-\omega)\varphi, \varphi) - \sigma \sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n)\varphi_n,$$

and the Nehari manifold

$$\mathcal{N} = \{\varphi \in H : I(\varphi) = 0, \varphi \neq 0\}.$$

Define $\lambda_1 = \inf \sigma(L)$, which is the smallest eigenvalue of L. Now we list some lemmas on the Nehari manifold.

Lemma 2.4. *If $\sigma = 1$, $\omega \in \mathbb{R}$ or $\sigma = -1$, $\omega > \lambda_1$, then the Nehari manifold $\mathcal{N}$ is a nonempty closed C^1 manifold in H.*

Proof. From the definition of $J(\varphi)$ and $I(\varphi)$, we can rewrite

$$J(\varphi) = \frac{1}{2}(\|\varphi\|_H^2 - \omega\|\varphi\|_{l^2}^2) - \sigma \sum_{n\in\mathbb{Z}} \gamma_n F(\varphi_n)$$

and

$$I(\varphi) = \|\varphi\|_H^2 - \omega\|\varphi\|_{l^2}^2 - \sigma \sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n)\varphi_n.$$

We distinguish two cases to prove $\mathcal{N}$ is nonempty.

Case 1*: $\sigma = 1$, $\omega \in \mathbb{R}$.* Note that, for any $\omega \in \mathbb{R}$, there exists $N \in \mathbb{N}$, such that

$$v_n \geq \omega + 1, \quad \forall\, |n| \geq N.$$

Thus we can construct a nonzero element $\varphi \in H$ such that

$$\varphi_n = 0, \quad \forall\, |n| \leq N$$

and

$$\|\varphi\|_H^2 - \omega\|\varphi\|_{\ell^2}^2 \geq \|\varphi\|_{\ell^2}^2.$$

Then, by (2.11) and (2.12), we have

$$I(\varphi) \le \|\varphi\|_H^2 - \omega\|\varphi\|_{\ell^2}^2 - q\sum_{n\in\mathbb{Z}} \gamma_n F(\varphi_n) \le \|\varphi\|_H^2 - \omega\|\varphi\|_{\ell^2}^2 - qC_3\sum_{n\in\mathbb{Z}} \gamma_n |\varphi_n|^q.$$

Define

$$I_1(\varphi) = \|\varphi\|_H^2 - \omega\|\varphi\|_{\ell^2}^2 - qC_3\sum_{n\in\mathbb{Z}} \gamma_n |\varphi_n|^q.$$

It is obvious that

$$\lim_{t\to\infty} I_1(t\varphi) = -\infty,$$

which implies that

$$\lim_{t\to\infty} I(t\varphi) = -\infty.$$

On the other hand, since $\varphi \in H$, there exists $M_0 > 0$ such that $|\varphi_n| < M_0$ for any $n \in \mathbb{N}$. Furthermore, note that f is superlinear near 0, then we can choose $\varepsilon > 0$ satisfying $\varepsilon\overline{\gamma} < 1/2$, such that

$$|f(t\varphi_n)| < \varepsilon|t\varphi_n|, \quad n \in \mathbb{N}$$

holds for $t > 0$ small enough. Consequently, if $t > 0$ is small enough, we have

$$I(t\varphi) > t^2\|\varphi\|_{\ell^2}^2 - t\sum_{n\in\mathbb{Z}} \overline{\gamma} f(t\varphi_n)\varphi_n > t^2\left(\|\varphi\|_{\ell^2}^2 - \sum_{n\in\mathbb{Z}} \overline{\gamma}\varepsilon\|\varphi_n\|^2\right) > \frac{t^2}{2}\|\varphi\|_{\ell^2}^2 > 0.$$

By the property of continuous function, there exists a $t^* > 0$, such that $I(t^*\varphi) = 0$. Therefore, $\mathcal{N}$ is nonempty.

Furthermore, notice that $I(\varphi) = 0$, implies

$$\|\varphi\|_H^2 - \omega\|\varphi\|_{\ell^2}^2 = \sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n)\varphi_n,$$

thus, by (2.6), we have

$$\begin{aligned}(I'(\varphi), \varphi) &= 2(\|\varphi\|_H^2 - \omega\|\varphi\|_{\ell^2}^2) - \sum_{n\in\mathbb{Z}} \gamma_n (f'(\varphi_n)\varphi_n^2 + f(\varphi_n)\varphi_n) \\ &= \sum_{n\in\mathbb{Z}} \gamma_n (f(\varphi_n)\varphi_n - f'(\varphi_n)\varphi_n^2) \\ &\le (2-q)\sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n)\varphi_n < 0.\end{aligned}$$

The implicit function theorem shows that $\mathcal{N}$ is a closed C^1 manifold in H.

Case 2: $\sigma = -1$, $\omega > \lambda_1$. Let u satisfy $Lu = \lambda_1 u$, then

$$I(u) = -(\omega - \lambda_1)\|u\|_{\ell^2}^2 + \sum_{n\in\mathbb{Z}} \gamma_n f(u_n)u_n.$$

Since f is superlinear near 0, then for $t > 0$ small enough, we have $I(tu) < 0$. Meanwhile,

$$I(u) = -(\omega - \lambda_1)\|u\|_{\ell^2}^2 + \sum_{n\in\mathbb{Z}} \gamma_n f(u_n)u_n \geq -(\omega - \lambda_1)\|u\|_{\ell^2}^2 + qC_3 \sum_{n\in\mathbb{Z}} \gamma_n |u_n|^q.$$

Denote

$$I_2(u) = -(\omega - \lambda_1)\|u\|_{\ell^2}^2 + qC_3 \sum_{n\in\mathbb{Z}} \gamma_n |u_n|^q.$$

It is obvious that

$$\lim_{t\to\infty} I_2(tu) = \infty,$$

which implies

$$\lim_{t\to\infty} I(tu) = \infty.$$

Thus there exists $t_1 > 0$ such that $t_1 u \in \mathcal{N}$. Just like Case 1, we can prove $\mathcal{N}$ is a closed C^1 manifold in H. □

Similar to Lemma 4.2 in [14], we also have the following result.

Lemma 2.5. *If $\sigma = 1$, $\omega \leq \lambda_1$, then there exist two positive constants α_0 and β_0 such that*

$$\|\varphi\|_H \geq \alpha_0, \quad J(\varphi) \geq \beta_0$$

hold for all $\varphi \in \mathcal{N}$.

Lemma 2.6. *If φ satisfies $J(\varphi) = \min_{u\in\mathcal{N}} J(u)$, then φ is a weak solution of system* (2.4).

Proof. By Lagrange multiplier method, it is known that φ is the critical point of functional $G = J(u) + \Lambda I(u)$. Thus

$$I(\varphi) = 0, \tag{2.13}$$

and

$$(G'(\varphi), u) = (J'(\varphi), u) + \Lambda(I'(\varphi), u) = 0, \quad \forall\, u \in H. \tag{2.14}$$

In particular, when $u = \varphi$, we have

$$I(\varphi) + \Lambda(I'(\varphi), \varphi) = 0. \tag{2.15}$$

Furthermore, by the proof of Lemma 2.4, we know that

$$(I'(\varphi), \varphi) \neq 0. \tag{2.16}$$

Thus, by combining (2.13), (2.15) and (2.16), we have $\Lambda = 0$. Therefore, by (2.14), we can conclude that $(J'(\varphi), u) = 0$ for any $u \in H$, which means that φ is a weak solution of (2.4). □

Lemma 2.7. *Suppose that there exists a positive constant $\underline{\gamma}$ such that $\underline{\gamma} \le \gamma_n$. Let*

$$d = \min_{\varphi \in \mathcal{N}} J(\varphi).$$

If $\{\varphi^{(k)}\}_{k\in\mathbb{N}}$ is a sequence in $\mathcal{N}$ such that

$$\lim_{k\to\infty} J(\varphi^{(k)}) = d,$$

then $\{\varphi^{(k)}\}_{k\in\mathbb{N}}$ is bounded in H. Furthermore, if $\sigma = 1$ and $\omega < \lambda_1$, then the same conclusion can be obtained without the positive lower bound $\underline{\gamma}$, which means, in this special case it is enough to assume $\gamma_n > 0$.

Proof. We distinguish two cases to show that $\{\varphi^{(k)}\}_{k\in\mathbb{N}}$ is bounded in H.

Case 1*:* $\sigma = -1$. Note that, $\varphi^{(k)} \in \mathcal{N}$ for each $k \in \mathbb{N}$, we know that $I(\varphi^{(k)}) = 0$. Thus, by choosing $\beta > 1/2$ and taking into consideration of (2.11), we have

$$\begin{aligned} J(\varphi^{(k)}) &= J(\varphi^{(k)}) - \beta I(\varphi^{(k)}) \\ &= \left(\frac{1}{2} - \beta\right)((L-\omega)\varphi^{(k)}, \varphi^{(k)}) + \sum_{n\in\mathbb{Z}} \gamma_n (F(\varphi^{(k)}) - \beta f(\varphi^{(k)})\varphi^{(k)}) \\ &\le \left(\frac{1}{2} - \beta\right)\|\varphi^{(k)}\|_H^2 - \left(\frac{1}{2} - \beta\right)\omega\|\varphi^{(k)}\|_{\ell^2}^2 + \sum_{n\in\mathbb{Z}} \gamma_n (1 - q\beta) F(\varphi^{(k)}) \\ &\le \left(\frac{1}{2} - \beta\right)\|\varphi^{(k)}\|_H^2 - \left(\frac{1}{2} - \beta\right)|\omega|\|\varphi^{(k)}\|_{\ell^2}^2 + \sum_{n\in\mathbb{Z}} \gamma_n (1 - q\beta) F(\varphi^{(k)}), \end{aligned}$$

which is equivalent to

$$-J(\varphi^{(k)}) \ge \left(\beta - \frac{1}{2}\right)\|\varphi^{(k)}\|_H^2 - \left(\beta - \frac{1}{2}\right)|\omega|\|\varphi^{(k)}\|_{\ell^2}^2 + \sum_{n\in\mathbb{Z}} \gamma_n (q\beta - 1) F(\varphi^{(k)}). \tag{2.17}$$

By (2.3), there must exist $N \in \mathbb{N}$ such that

$$\upsilon_n \ge 2|\omega|, \quad \forall\, |n| > N,$$

which implies

$$\omega\|\varphi^{(k)}\|_{\ell^2}^2 \le |\omega| \sum_{|n|<N} (\varphi_n^{(k)})^2 + \frac{1}{2}\|\varphi^{(k)}\|_H^2. \tag{2.18}$$

For any $\varepsilon > 0$, $q > 2$, let

$$D(\varepsilon) = \max_{x\in\mathbb{R}} (x^2 - \varepsilon |x|^q).$$

It is easy to see that

$$0 < D(\varepsilon) < \infty.$$

By $\varphi^{(k)} \in \ell^2$ and Hölder inequality, thus $\varphi^{(k)} \in \ell^q$ and there must exist a positive constant C such that

$$\sum_{|n|<N} (\varphi_n^{(k)})^2 \leq C\|\varphi^{(k)}\|_{\ell^q}^2 \leq CD(\varepsilon) + C\varepsilon\|\varphi^{(k)}\|_{\ell^q}^q. \tag{2.19}$$

Thus, by the assumption $\gamma_n \geq \underline{\gamma}$ and combining (2.17), (2.18) and (2.19), we can obtain

$$-J(\varphi^{(k)}) \geq \frac{2\beta-1}{4}\|\varphi^{(k)}\|_H^2 - \frac{2\beta-1}{2}C|\omega|D(\varepsilon) + \left[(q\beta-1)C_3\underline{\gamma} - \frac{2\beta-1}{2}C|\omega|\varepsilon\right]\|\varphi^{(k)}\|_{\ell^q}^q,$$

which can be rewritten as

$$-J(\varphi^{(k)}) \geq \frac{2\beta-1}{4}\|\varphi^{(k)}\|_H^2 + B_1\|\varphi^{(k)}\|_{\ell^q}^q - B_2$$

with

$$B_1 = (q\beta-1)C_3\underline{\gamma} - \frac{2\beta-1}{2}C|\omega|\varepsilon, \quad B_2 = \frac{2\beta-1}{2}C|\omega|D(\varepsilon).$$

It is easy to see that both B_1 and B_2 are positive constants for $\varepsilon > 0$ small enough. Therefore, we have

$$-J(\varphi^{(k)}) \geq \frac{2\beta-1}{4}\|\varphi^{(k)}\|_H^2 - B_2,$$

which implies $\{\varphi^{(k)}\}$ is bounded.

Similarly, we can prove the case $\sigma = 1$ by taking $1/q < \beta < 1/2$. Here we omit the details.

Finally, we pay more attention on the special case $\sigma = 1$ with $\omega < \lambda_1$. By Lemma 2.5, if $\sigma = 1$ and $\omega < \lambda_1$, then $J(\varphi) \geq \beta_0 > 0$ for any $\varphi \in \mathcal{N}$, which shows that

$$d = \min_{\varphi\in\mathcal{N}} J(\varphi) > 0, \quad \max_k J(\varphi^{(k)}) > 0.$$

Then, since $I(\varphi^{(k)}) = 0$, we have

$$\|\varphi^{(k)}\|_H^2 - \omega\|\varphi^{(k)}\|_{\ell^2}^2 = \sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n^{(k)})\varphi_n^{(k)}.$$

Thus, by (2.11), we have

$$\begin{aligned} J(\varphi^{(k)}) &= \frac{1}{2}(\|\varphi^{(k)}\|_H^2 - \omega\|\varphi^{(k)}\|_{\ell^2}^2) - \sum_{n\in\mathbb{Z}} \gamma_n F(\varphi_n^{(k)}) \\ &\geq \frac{1}{2}(\|\varphi^{(k)}\|_H^2 - \omega\|\varphi^{(k)}\|_{\ell^2}^2) - \frac{1}{q}\sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n^{(k)})\varphi_n^{(k)} \\ &= \left(\frac{1}{2} - \frac{1}{q}\right)(\|\varphi^{(k)}\|_H^2 - \omega\|\varphi^{(k)}\|_{\ell^2}^2). \end{aligned}$$

Thus, if $\omega \leq 0$, we have

$$J(\varphi^{(k)}) \geq \left(\frac{1}{2} - \frac{1}{q}\right) \|\varphi^{(k)}\|_H^2,$$

which implies

$$\|\varphi^{(k)}\|_H \leq \sqrt{\left(\frac{1}{2} - \frac{1}{q}\right)^{-1} \Delta}, \tag{2.20}$$

where $\Delta = \max_k J(\varphi^{(k)})$. If $0 < \omega < \lambda_1$, it is easy to show that

$$J(\varphi^{(k)}) \geq \left(\frac{1}{2} - \frac{1}{q}\right)\left(1 - \frac{\omega}{\lambda_1}\right) \|\varphi^{(k)}\|_H^2,$$

which implies

$$\|\varphi^{(k)}\|_H \leq \sqrt{\left[\left(\frac{1}{2} - \frac{1}{q}\right)\left(1 - \frac{\omega}{\lambda_1}\right)\right]^{-1} \Delta}, \tag{2.21}$$

where Δ is same as in (2.20). The inequalities (2.20) and (2.21) show that $\{\varphi^{(k)}\}$ is bounded in H. $\square$

3. Breather solution

Theorem 3.1. *Let $0 < \gamma_n \leq \overline{\gamma}$, the nonlinearity f satisfy* (H1) *and* (H2), *and* (2.3) *hold in system* (2.4). *Then*

(1) *If $\sigma = -1$, $\omega \leq \lambda_1$, there is no nontrivial solution for system* (2.4).

(2) *If $\sigma = 1$, $\omega < \lambda_1$, there exists a pair of nontrivial solution $\pm\varphi \in \ell^2$ for system* (2.4).

Proof. (1) If $\sigma = -1$, $\omega \leq \lambda_1$, then it is easy to prove that $I(\varphi) = 0$ if and only if $\varphi = 0$. Therefore, in this case the system (2.4) has no nontrivial solution.

(2) If $\sigma = 1$, $\omega < \lambda_1$, then, by Lemma 2.5, $J(\varphi) \geq \beta_0 > 0$ for any $\varphi \in \mathcal{N}$. Thus, $d = \min_{\varphi \in \mathcal{N}} J(\varphi) > 0$. Furthermore, there exists a sequence $\{\varphi^{(k)}\} \subset \mathcal{N}$ such that

$$d = \lim_{k \to \infty} J(\varphi^{(k)}).$$

By Lemma 2.7, the sequence $\{\varphi^{(k)}\}$ is bounded. Consequently, there exists a subsequence, also denoted by $\varphi^{(k)}$, such that $\varphi^{(k)}$ weakly converges to $\varphi \in H$. By Lemma 2.2, it is obvious that

$$\varphi^{(k)} \to \varphi$$

in the space ℓ^q for any $2 \leq q \leq \infty$. Moreover, by (2.10), we have

$$\begin{aligned}
&\left|\sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n^{(k)})\varphi_n^{(k)} - \sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n)\varphi_n\right| \\
&\leq \sum_{n\in\mathbb{Z}} \gamma_n |f(\varphi_n)||\varphi_n^{(k)} - \varphi_n| + \sum_{n\in\mathbb{Z}} \gamma_n |f(\varphi_n^{(k)}) - f(\varphi_n)||\varphi_n^{(k)}| \\
&\leq \overline{\gamma}|\varphi_n^{(k)} - \varphi_n| \sum_{n\in\mathbb{Z}^m} \left[\varepsilon|\varphi_n| + A|\varphi_n|^{p-1} + C_1(2 + |\varphi_n|^{p-2} + |\varphi_n^{(k)}|^{p-2})|\varphi_n^{(k)}|\right],
\end{aligned}$$

which, combining with the Hölder inequality, yields that

$$\lim_{k\to\infty} \sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n^{(k)})\varphi_n^{(k)} = \sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n)\varphi_n. \tag{3.1}$$

Meanwhile, in view of (2.10) and by the mean value theorem, there exists $t^* \in (0,1)$ such that

$$\begin{aligned}
&\left|\sum_{n\in\mathbb{Z}} \gamma_n [F(\varphi_n^{(k)}) - F(\varphi_n)]\right| \\
&= \left|\sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n) + t^*(\varphi_n^{(k)} - \varphi_n)(\varphi_n^{(k)} - \varphi_n)\right| \\
&\leq \sum_{n\in\mathbb{Z}} \gamma_n |f(\varphi_n) + t^*(\varphi_n^{(k)} - \varphi_n)||\varphi_n^{(k)} - \varphi_n| \\
&\leq \overline{\gamma} \sum_{n\in\mathbb{Z}} \left[\varepsilon|\varphi_n| + A|\varphi_n|^{p-1} + \varepsilon|\varphi_n^{(k)} - \varphi_n| + A|\varphi_n^{(k)} - \varphi_n|^{p-1}\right] |\varphi_n^{(k)} - \varphi_n|,
\end{aligned}$$

which implies

$$\lim_{k\to\infty} \sum_{n} \gamma_n F(\varphi_n^{(k)}) = \sum_{n} \gamma_n F(\varphi_n). \tag{3.2}$$

Thus,

$$\begin{aligned}
J(\varphi^{(k)}) &= \left(\frac{1}{2} - \frac{1}{q}\right)(\|\varphi^{(k)}\|_H^2 - \omega\|\varphi^{(k)}\|_{\ell^2}^2) - \sum_{n\in\mathbb{Z}} \gamma_n F(\varphi_n^{(k)}) \\
&= \frac{1}{2}\sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n^{(k)})\varphi_n^{(k)} - \sum_{n\in\mathbb{Z}} \gamma_n F(\varphi_n^{(k)}).
\end{aligned} \tag{3.3}$$

Therefore,

$$d = \sum_{n\in\mathbb{Z}} \frac{1}{2}\gamma_n [f(\varphi_n)(\varphi_n) - 2F(\varphi_n)].$$

Combining (3.1), (3.2), (3.3) and by the Fatou Lemma, we have

$$\begin{aligned}\|\varphi\|_H^2 = \|\text{weak} - \lim \varphi^{(k)}\|_H^2 &\le \liminf \|\varphi^{(k)}\|_H^2 \\ &= \liminf \left(\sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n^{(k)})\varphi_n^{(k)} + \omega\|\varphi^{(k)}\|_{\ell^2}^2\right) \\ &= \sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n)\varphi_n + \omega\|\varphi\|_{\ell^2}^2,\end{aligned}$$

which implies

$$I(\varphi) = \|\varphi\|_H^2 - \omega\|\varphi\|_{\ell^2}^2 - \sum_{n\in\mathbb{Z}} \gamma_n f(\varphi_n)\varphi_n \le 0.$$

Through a similar demonstration in the proof of Lemma 2.4, we know that there exists a positive constant $0 < t_0 \le 1$ such that $I(t_0\varphi) = 0$. Thus we can conclude that $t_0\varphi \in \mathcal{N}$. Indeed, in what follows we can show that $t_0 = 1$.

Define

$$P(t) = \sum_{n\in\mathbb{Z}} \gamma_n [f(t\varphi_n)(t\varphi_n) - 2F(t\varphi_n)].$$

Thus, $J(t_0\varphi) = \frac{1}{2}P(t_0)$ and $P(1) = 2d$. Notice that, for any $0 < t < \infty$,

$$P'(t) = \sum_{n\in\mathbb{Z}} \gamma_n [tf'(t\varphi_n)\varphi_n^2 - f(t\varphi_n)\varphi_n] \ge \frac{q-2}{t}\sum_{n\in\mathbb{Z}} \gamma_n f(t\varphi_n)(t\varphi_n) > 0,$$

which implies that $P(t)$ is strictly increasing on $(0, \infty)$. Moreover,

$$d \le J(t_0\varphi) = \frac{1}{2}P(t_0) \le \frac{1}{2}P(1) = d.$$

Therefore, $t_0 = 1$, $J(\varphi) = d$ and $\varphi \in \mathcal{N}$. Thus, by Lemma 2.6, φ is a weak solution of system (2.4). Since the functional $J(u)$ is even, then $-\varphi$ is also a weak solution. □

Now we will state our results about the exponential decay property of the breathers.

Theorem 3.2. *Let φ be a solution of system* (2.4) *and satisfy*

$$\lim_{|n|\to\infty} \gamma_n f(\varphi_n) = 0.$$

Then there exist two positive constants K and α such that

$$|\varphi_n| \le Ke^{-\alpha|n|}, \quad n \in \mathbb{Z}.$$

Proof. Define the operators

$$\Theta = \{\theta_n = -\sigma\gamma_n f(\varphi_n)\}_{n\in\mathbb{Z}}$$

and

$$P\varphi = (L\varphi)_n + \theta_n.$$

Then system (2.4) is equivalent to

$$P\varphi = \omega\varphi.$$

By Lemma 2.1, we know that Θ is a compact operator in ℓ^2. Hence

$$\sigma_{\text{ess}}(P) = \sigma_{\text{ess}}(L).$$

Then by the definition of essential spectrum, we have

$$\sigma_{\text{ess}}(P) = \sigma_{\text{ess}}(L) = \emptyset.$$

By Lemma 2.5 in [13], there exist two positive constants K and α such that $|\varphi_n| \leq Ke^{-\alpha|n|}$. $\square$

Moreover, Theorem 3.2 could be generalized to the following case.

Theorem 3.3. *If $\varphi \in \ell^2(\mathbb{Z})$ is a solution of system* (2.4)*, and the sequence $\{\gamma_n f(\varphi_n)\}_{n\in\mathbb{Z}}$ is bounded, then there exist two positive constants K and α such that*

$$|\varphi_n| \leq Ke^{-\alpha|n|}, \quad n \in \mathbb{Z}.$$

Proof. Let P, Θ be the same as defined in the proof of Theorem 3.2. By Lemma 2.2, we know that L^{-1} is compact in ℓ^2. Since Θ is bounded, then ΘL^{-1} is compact. Therefore,

$$\sigma_{\text{ess}}(P) = \sigma_{\text{ess}}(L) = \emptyset.$$

We can accomplish the proof through a similar argument in Theorem 3.2. $\square$

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