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# Two Positive Solutions for Kirchhoff Type Problems with Hardy-Sobolev Critical Exponent and Singular Nonlinearities

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Abstract. We consider the following singular Kirchhoff type equation with Hardy-Sobolev critical exponent

$$\begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \dfrac{u^3}{|x|} + \dfrac{\lambda}{|x|^\beta u^\gamma}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $a, b, \lambda > 0$, $0 < \gamma < 1$, and $0 \le \beta < (5+\gamma)/2$. Combining with the variational method and perturbation method, two positive solutions of the equation are obtained.

## 1. Introduction and main result

In this paper, we consider the positive solutions of the Kirchhoff type equation

$$\tag{1.1} \begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \dfrac{u^3}{|x|} + \dfrac{\lambda}{|x|^\beta u^\gamma}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ is a bounded domain with smooth boundary $\partial\Omega$, $0 \in \Omega$, $a, b, \lambda > 0$, $0 < \gamma < 1$ and $0 \le \beta < (5+\gamma)/2$, and 4 is the Hardy-Sobolev critical exponent.

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Liu and Sun [20] considered the following singular Kirchhoff type equation for the first time

$$(1.2)\qquad \begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \lambda g(x)\dfrac{u^p}{|x|^s} + h(x)u^{-\gamma}, & x \in \Omega,\\ u > 0, & x \in \Omega,\\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $3 < p < 5-2s$, $0 \le s < 1$ and $g, h \in C(\overline{\Omega})$ are nontrivial nonnegative functions. By the Nehari method, when $\lambda > 0$ small, they obtained two positive solutions for (1.2). Later, Lei, Liao and Tang studied the critical case of (1.2) with $s = 0$, $p = 5$, $\lambda = g(x) \equiv 1$, and obtained two positive solutions by using the variational method and perturbation method, see [12]. In [21], Liu et al. generalized [12] in dimension four, that is,

$$(1.3)\qquad \begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \mu u^3 + \dfrac{\lambda}{|x|^\beta u^\gamma}, & x \in \Omega,\\ u > 0, & x \in \Omega,\\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^4$ is a bounded smooth domain and $\mu > 0$. For all $\mu > 0$, $\gamma \in (0,1)$ and $0 \le \beta < 3$, they obtained (1.3) has a positive solution. When $\mu > bS^2$, $\gamma \in (0, 1/2)$ and $2 + 2\gamma < \beta < 3$, they proved (1.3) has at least two positive solutions. Moreover, when $s = 0$, $p = 3$, the existence and multiplicity of positive solutions for (1.2) are considered by Liao et al., see [19]. And, Li, Tang and Liao [16] studied (1.2) with $0 \le s < 1$, $p = 3$ and $g \in L^\infty(\Omega)$ may change sign in $\Omega$.

To the best of our knowledge, the first work on the Kirchhoff-type problem with critical Sobolev exponent is from Alves, Corrêa and Figueiredo in [1]. After that, the Kirchhoff type equation with critical exponent has been extensively studied, and some important and interesting results have been obtained, see [4–8, 10, 12–15, 17, 18, 21–24, 27–29].

However, the Kirchhoff type problem with Hardy-Sobolev critical exponent has few been considered. Inspired by [12, 20, 21], we study the existence of positive solutions of (1.1). To the best of our knowledge, most of the Kirchhoff type equation with asymptotically 3-linear are subcritical in $\mathbb{R}^3$. One of the main feature of (1.1) is asymptotically 3-linear and critical, the difficulty is due to the lack of compactness of the embedding $H_0^1(\Omega) \hookrightarrow L^4(\Omega, |x|^{-1}dx)$. And the power of the nonlocal term $b\big(\int_\Omega |\nabla u|^2\,dx\big)\Delta u$ and the critical term is equal. Furthermore, because of the singular term $u^{-\gamma}$, the corresponding energy functional $I$ does not belong to $C^1(H_0^1(\Omega), \mathbb{R})$ which leads to the classic critical point theory for $I$ could not be checked directly. In this article, combining with some analysis techniques and the definition of solution of (1.1), we obtain a positive local minimizer solution of (1.1). While $0 < b < A^{-2}$ and $2+\gamma < \beta < (5+\gamma)/2$, combining with the perturbation method and variational method, we get another positive solution for (1.1).

Let $A$ be the Hardy-Sobolev constant, and $S$ be the best Sobolev constant, namely

$$A = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\left(\int_\Omega \frac{u^4}{|x|}\,dx\right)^{1/2}}, \tag{1.4}$$

$$S = \inf_{u \in H_0^1(\Omega)\setminus\{0\}} \frac{\|u\|^2}{\left(\int_\Omega u^6\,dx\right)^{1/3}}. \tag{1.5}$$

For readers' convenience, we give the definition of the $(C)_c$ condition (see Definition 1.40 in [30]).

**Definition 1.1.** Suppose $\psi \in C^1(H_0^1(\Omega), \mathbb{R})$. For any $c \in \mathbb{R}$, $\{u_n\}$ is called a $(C)_c$ sequence of $\psi$ in $H_0^1(\Omega)$, if $\psi(u_n) \to c$ and $(1 + \|u_n\|)\psi'(u_n) \to 0$ as $n \to \infty$. We say that $\psi$ satisfies the $(C)_c$ condition if every $(C)_c$ sequence of $\psi$ has a converging subsequence in $H_0^1(\Omega)$.

The energy functional of (1.1) is defined by

$$I(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{4}\int_\Omega \frac{(u^+)^4}{|x|}\,dx - \frac{\lambda}{1-\gamma}\int_\Omega \frac{(u^+)^{1-\gamma}}{|x|^\beta}\,dx$$

for all $u \in H_0^1(\Omega)$, where $u^\pm = \max\{\pm u, 0\}$ and $\|u\| = \left(\int_\Omega |\nabla u|^2\,dx\right)^{1/2}$ is the norm of $H_0^1(\Omega)$. Since $0 < \gamma < 1$, the energy functional $I$ is not a $C^1$ functional on $H_0^1(\Omega)$. We say $u$ is a solution of (1.1), if $u \in H_0^1(\Omega)$ with $u > 0$ and for all $\varphi \in H_0^1(\Omega)$ satisfies

$$(a + b\|u\|^2)\int_\Omega (\nabla u, \nabla\varphi)\,dx - \int_\Omega \frac{(u^+)^3\varphi}{|x|}\,dx - \lambda\int_\Omega \frac{(u^+)^{-\gamma}}{|x|^\beta}\varphi\,dx = 0.$$

Our main result is described as follows.

**Theorem 1.2.** *Suppose that $a, b > 0$, $0 < \gamma < 1$, then*

(1) *when $0 \le \beta < (5+\gamma)/2$, there exists $\lambda_* > 0$ such that (1.1) has at least a positive solution for all $0 < \lambda < \lambda_*$;*

(2) *when $0 < b < A^{-2}$ and $2+\gamma < \beta < (5+\gamma)/2$, there exist $\lambda_{**} > 0$ ($\lambda_{**} \le \lambda_*$) such that (1.1) has at least two positive solutions for all $0 < \lambda < \lambda_{**}$.*

*Remark* 1.3. To the best of our knowledge, (1.1) has not been studied up to now. On the one hand, (1.1) is equal to (1.2) with $s = 1$ and $p = 5 - 2s$. In some sense, our result generalizes [20] to the Hardy-Sobolev critical case. Moreover, (1.1) is different from (1.3). Comparing with [21], we consider the Kirchhoff type problem with Hardy-Sobolev critical exponent in dimension three. On the other hand, the Kirchhoff type problem is asymptotically 3-linear and critical, it is worth mentioning that our result demonstrates the relation between the existence of positive solutions and the value range of $b$, $\lambda$. It is worth mentioning that the constraint conditions $0 < b < A^{-2}$ and $2+\gamma < \beta < (5+\gamma)/2$ are ensure the existence of the second positive solution. However, we could not obtain the second solution for (1.1) with $0 \le \beta \le 2+\gamma$.

*Remark* 1.4. The more general problem of (1.1) is

$$(1.6) \qquad \begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\,dx\right)\Delta u = \dfrac{u^{5-2s}}{|x|^s} + \lambda h(x)u^{-\gamma}, & x \in \Omega, \\ u > 0, & x \in \Omega, \\ u = 0, & x \in \partial\Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^3$ and $0 \le s \le 2$. For all $u \in H_0^1(\Omega)$, the energy functional of (1.6) is defined by

$$I_s(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{6-2s}\int_\Omega \frac{(u^+)^{6-2s}}{|x|^s}\,dx - \frac{\lambda}{1-\gamma}\int_\Omega h(x)(u^+)^{1-\gamma}\,dx.$$

According to the first result of Theorem 1.2, for any $0 \le s \le 2$, we can also obtain that (1.6) has at least one positive solution. When $s = 0$, [12] considered (1.6) and obtained two positive solutions. However, we could not obtain the existence of the second positive solution for (1.6) with $s \in (0,1) \cup (1,2]$ by the methods of this paper. Because we could not obtain that $I_{s,\alpha}$ satisfies the local $(C)_c$ condition, where $I_{s,\alpha}$ is the energy functional of the approximating problem of (1.6) which is similar to (2.18). Similar to (2.31), we have $al^2 + bl^4 + bl^2\|u\|^2 = \int_\Omega (w_n^+)^{6-2s}/|x|^s\,dx \le l^{6-2s}/A^{3-s}$. But we could not solve this inequality about $l$ for $s \in (0,1)$. When $s \in (1,2]$, it is difficult to obtain the mountain-pass geometry structure for $I_{s,\alpha}$ in $H_0^1(\Omega)$. Thus, the existence of the second positive solution for (1.6) with $s \in (0,1) \cup (1,2]$ is one of future problems of us.

## 2. Proof of Theorem 1.2

We will divide two parts to complete the proof of Theorem 1.2. First, we prove that (1.1) with $0 \le \beta < (5+\gamma)/2$ has a positive local minimizer solution in first part. Secondly, for $0 < b < A^{-2}$ and $2+\gamma < \beta < (5+\gamma)/2$, we study the existence of the second positive solution of (1.1) in second part. In order to overcome the difficulty of the singular term $u^{-\gamma}$, we study an approximating equation of (1.1) and prove that the corresponding approximating equation has at least a positive mountain-pass solution. Finally, we prove the sequence of positive solutions of the approximating equation is convergent in $H_0^1(\Omega)$ and the limit is indeed a positive solution of (1.1).

## 2.1. The existence of the first positive solution

In order to obtain the first positive solution, we give the following important lemmas.

**Lemma 2.1.** *Assume that $a > 0$, $b > 0$, $0 < \gamma < 1$, $0 \le \beta < (5+\gamma)/2$, then there exist $R, \rho > 0$ and $\lambda_* > 0$ such that*

$$I(u)|_{u \in S_R} \ge \rho > 0, \qquad \inf_{u \in \overline{B}_R} I(u) < 0 \tag{2.1}$$

*for every* $0 < \lambda < \lambda_*$, *where* $S_R = \{u \in H_0^1(\Omega) : \|u\| = R\}$, $\overline{B}_R = \{u \in H_0^1(\Omega) : \|u\| \le R\}$.

*Proof.* Let $R_0 > 0$ be a constant such that $\Omega \subset B(0, R_0) = \{x \in \mathbb{R}^3 : |x| < R_0\}$. By Hölder's inequality and (1.5), for all $u \in H_0^1(\Omega)$, since $0 \le \beta < (5+\gamma)/2$, one has

$$\begin{aligned} \int_\Omega \frac{(u^+)^{1-\gamma}}{|x|^\beta}\,dx &\le \int_\Omega \frac{|u|^{1-\gamma}}{|x|^\beta}\,dx \\ &\le \left(\int_\Omega |u|^6\,dx\right)^{(1-\gamma)/6} \left(\int_\Omega \frac{1}{|x|^{6\beta/(5+\gamma)}}\,dx\right)^{(5+\gamma)/6} \\ &\le S^{-(1-\gamma)/2}\|u\|^{1-\gamma}\left(4\pi\int_0^{R_0} r^{\frac{2(5+\gamma)-6\beta}{5+\gamma}}\,dr\right)^{(5+\gamma)/6} \\ &= \left[\frac{4\pi(5+\gamma)}{3(5+\gamma-2\beta)}\right]^{(5+\gamma)/6} R_0^{(5+\gamma-2\beta)/2} S^{-(1-\gamma)/2}\|u\|^{1-\gamma}. \end{aligned} \tag{2.2}$$

By (1.4) and (2.2), we have

(2.3)
$$\begin{aligned} I(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac14\int_\Omega \frac{(u^+)^4}{|x|}\,dx - \frac{\lambda}{1-\gamma}\int_\Omega \frac{(u^+)^{1-\gamma}}{|x|^\beta}\,dx \\ &\ge \frac{a}{2}\|u\|^2 - \frac{1-bA^2}{4A^2}\|u\|^4 - \frac{\lambda R_0^{(5-2\beta+\gamma)/2}}{(1-\gamma)S^{(1-\gamma)/2}}\left[\frac{4\pi(5+\gamma)}{3(5+\gamma-2\beta)}\right]^{(5+\gamma)/6}\|u\|^{1-\gamma} \\ &\ge \|u\|^{1-\gamma}\left\{\frac{a}{2}\|u\|^{1+\gamma} - \frac{1-bA^2}{4A^2}\|u\|^{3+\gamma} - \frac{\lambda R_0^{(5-2\beta+\gamma)/2}}{(1-\gamma)S^{(1-\gamma)/2}}\left[\frac{4\pi(5+\gamma)}{3(5+\gamma-2\beta)}\right]^{(5+\gamma)/6}\right\}. \end{aligned}$$

For all $t \ge 0$, let

$$H(t) = \frac{a}{2}t^{1+\gamma} - \frac{1-bA^2}{4A^2}t^{3+\gamma}.$$

When $0 < b < A^{-2}$, it is easy to obtain a constant $R_1 = \left[\frac{2a(1+\gamma)A^2}{(1-bA^2)(3+\gamma)}\right]^{1/2} > 0$ such that $\max_{t\ge 0} H(t) = H(R_1) > 0$. Letting $\lambda_1 = \frac{(1-\gamma)S^{(1-\gamma)/2}}{2R_0^{(5-2\beta+\gamma)/2}}\left[\frac{3(5+\gamma-2\beta)}{4\pi(5+\gamma)}\right]^{(5+\gamma)/6} H(R_1)$, it follows that there exists a constant $\rho > 0$ such that $I(u)|_{u\in S_{R_1}} \ge \rho$ for every $\lambda \in (0, \lambda_1)$. When $b \ge A^{-2}$, from (2.3) we can see that $I(u) \to +\infty$ as $\|u\| \to +\infty$. Therefore, $I$ is coercive on $H_0^1(\Omega)$. Obviously, we can find an $R_2 > 0$ and a constant $\rho > 0$ such that $I(u)|_{u\in S_{R_2}} \ge \rho$

for every $\lambda \in (0, \lambda_2)$, where $\lambda_2 = \frac{(1-\gamma)S^{(1-\gamma)/2}}{2R_0^{(5-2\beta+\gamma)/2}} \left[\frac{3(5+\gamma-2\beta)}{4\pi(5+\gamma)}\right]^{(5+\gamma)/6} H(R_2)$ and $H(R_2) > 0$. Thus, there exist $R, \lambda_*, \rho > 0$ such that $I(u)|_{u\in S_R} \geq \rho$ for every $\lambda \in (0, \lambda_*)$. Choosing $u \in \overline{B}_R$ with $u^+ \neq 0$, we have

$$\lim_{t\to 0^+} \frac{I(tu)}{t^{1-\gamma}} = -\frac{\lambda}{1-\gamma} \int_\Omega \frac{(u^+)^{1-\gamma}}{|x|^\beta}\, dx < 0,$$

then $I(tu) < 0$ for all $u^+ \neq 0$ and $t$ small enough. Therefore, one has

$$m_0 = \inf_{u\in \overline{B}_R} I(u) < 0. \tag{2.4}$$

Then the proof of Lemma 2.1 is completed. □

**Lemma 2.2.** *Suppose that $a > 0$, $b > 0$, $0 < \gamma < 1$, $0 \leq \beta < (5+\gamma)/2$ and $0 < \lambda < \lambda_*$ ($\lambda_*$ defined in Lemma 2.1), then $I$ attains the local minimizer $m_0$ in $H_0^1(\Omega)$, that is, there exists $u_* \in H_0^1(\Omega)$ such that $I(u_*) = m_0 < 0$.*

*Proof.* First, we prove that there exists $u_* \in \overline{B}_R$ such that $I(u_*) = m_0 < 0$. Actually, by (2.1), we can infer that

$$\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{4}\int_\Omega \frac{(u^+)^4}{|x|}\, dx \geq \rho \quad \text{for } u \in S_R,$$

and

$$\frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{4}\int_\Omega \frac{(u^+)^4}{|x|}\, dx \geq 0 \quad \text{for } u \in \overline{B}_R. \tag{2.5}$$

By the definition of (2.4), there exists a minimizing sequence $\{u_n\} \subset \overline{B}_R$ such that $\lim_{n\to\infty} I(u_n) = m_0 < 0$. Clearly, this minimizing sequence is bounded in $\overline{B}_R$, up to a subsequence, there exists $u_* \in H_0^1(\Omega)$ such that

$$\begin{gathered} u_n \rightharpoonup u_* \text{ weakly in } H_0^1(\Omega), \quad u_n \to u_* \text{ strongly in } L^p(\Omega),\ 1 \leq p < 6, \\ \frac{u_n^4}{|x|} \rightharpoonup \frac{u_*^4}{|x|} \text{ weakly in } L^1(\Omega), \quad u_n(x) \to u_*(x) \text{ a.e. in } \Omega. \end{gathered} \tag{2.6}$$

By (2.2) and (2.6), we have

$$\lim_{n\to\infty} \int_\Omega \frac{(u_n^+)^{1-\gamma}}{|x|^\beta}\, dx = \int_\Omega \frac{(u_*^+)^{1-\gamma}}{|x|^\beta}\, dx + o(1). \tag{2.7}$$

Setting $w_n = u_n - u_*$, we have

$$\|u_n\|^2 = \|w_n\|^2 + \|u_*\|^2 + o(1), \tag{2.8}$$

and

$$\|u_n\|^4 = \|w_n\|^4 + \|u_*\|^4 + 2\|w_n\|^2\|w_0\|^2 + o(1). \tag{2.9}$$

Moreover, by Lemma 4.2 in [9], one has

$$\int_\Omega \frac{(u_n^+)^4}{|x|}\,dx = \int_\Omega \frac{(w_n^+)^4}{|x|}\,dx + \int_\Omega \frac{(u_0^+)^4}{|x|}\,dx + o(1). \tag{2.10}$$

If $u_* = 0$, then $w_n = u_n$, it follows that $w_n \in \overline{B}_R$. If $u_* \neq 0$, from (2.8), we obtain $w_n \in \overline{B}_R$ for $n$ large sufficiently. Hence from (2.5) one has

$$\frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 - \frac{1}{4}\int_\Omega \frac{(w_n^+)^4}{|x|}\,dx \geq 0. \tag{2.11}$$

By (2.7)–(2.11), then we have

$$\begin{aligned}
m_0 &= I(u_n) + o(1) \\
&= I(u_*) + \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u_*\|^2 - \frac{1}{4}\int_\Omega \frac{(w_n^+)^4}{|x|}\,dx + o(1) \\
&\geq I(u_*) + \frac{b}{2}\|w_n\|^2\|u_*\|^2 + o(1) \\
&\geq I(u_*) + o(1),
\end{aligned}$$

which implies that $I(u_*) \leq m_0$. Noting that $\overline{B}_R$ is closed and convex, thus $u_* \in \overline{B}_R$. By (2.4), we have $I(u_*) \geq m_0$. Thus we obtain $I(u_*) = m_0 < 0$, that is, $u_*$ is a local minimizer. Then the proof of Lemma 2.2 is completed. □

Now, we have the following conclusion.

**Theorem 2.3.** *Assume that $a > 0$, $b > 0$, $0 < \gamma < 1$, $0 \leq \beta < (5+\gamma)/2$, then* (1.1) *has at least a positive solution for $0 < \lambda < \lambda_*$ ($\lambda_*$ defined in Lemma* 2.1*).*

*Proof.* By Lemma 2.2, there exists $u_* \in \overline{B}_R \subset H_0^1(\Omega)$ such that $I(u_*) = m_0 < 0$, we only need prove that $u_*$ is a positive solution of (1.1). Then for any $\varphi \in H_0^1(\Omega)$, $\varphi \geq 0$, letting $t > 0$ small enough, such that $u_* + t\varphi \in \overline{B}_R$, we have

$$\begin{aligned}
0 &\leq \liminf_{t\to 0^+} \int_\Omega \frac{I(u_* + t\varphi) - I(u_*)}{t}\,dx \\
&= (a + b\|u_*\|^2)\int_\Omega (\nabla u_*, \nabla\varphi)\,dx - \int_\Omega \frac{(u_*^+)^3\varphi}{|x|}\,dx \\
&\quad - \frac{\lambda}{1-\gamma}\limsup_{t\to 0^+}\int_\Omega \frac{(u_*^+ + t\varphi)^{1-\gamma} - (u_*^+)^{1-\gamma}}{|x|^\beta t}\,dx.
\end{aligned} \tag{2.12}$$

By the mean value theorem and Fatou lemma, there exists $\theta > 0$ such that

$$\begin{aligned}
\limsup_{t\to 0^+}\int_\Omega \frac{(u_*^+ + t\varphi)^{1-\gamma} - (u_*^+)^{1-\gamma}}{(1-\gamma)|x|^\beta t}\,dx &\geq \liminf_{t\to 0^+}\int_\Omega \frac{(u_*^+ + t\varphi)^{1-\gamma} - (u_*^+)^{1-\gamma}}{(1-\gamma)|x|^\beta t}\,dx \\
&= \liminf_{t\to 0^+}\int_\Omega \frac{(u_*^+ + \theta t\varphi)^{-\gamma}\varphi}{|x|^\beta t}\,dx \\
&\geq \lambda\int_\Omega \frac{(u_*^+)^{-\gamma}\varphi}{|x|^\beta}\,dx,
\end{aligned}$$

where $\theta \to 0$ and $(u_*^+ + \theta t\varphi)^{-\gamma}\varphi \to (u_*^+)^{-\gamma}\varphi$ as $t \to 0^+$, and $(u_*^+ + \theta t\varphi)^{-\gamma}\varphi \geq 0$. Consequently, it follows from (2.12) that

$$(a + b\|u_*\|^2) \int_\Omega (\nabla u_*, \nabla\varphi)\, dx - \int_\Omega \frac{(u_*^+)^3\varphi}{|x|}\, dx - \lambda \int_\Omega \frac{(u_*^+)^{-\gamma}\varphi}{|x|^\beta}\, dx \geq 0, \quad \varphi \geq 0. \tag{2.13}$$

Now, we will prove that (2.13) holds for any $\varphi \in H_0^1(\Omega)$. By Lemma 2.2, we know that $I(u_*) < 0$. Combining with (2.1), one has $u_* \notin S_R$, that is, $\|u_*\| < R$. For $u_*$, there exists $\sigma \in (0,1)$ such that $(1+t)u_* \in \overline{B}_R$ for $|t| \leq \sigma$. Define $\tau : [-\sigma, \sigma]$ by $\tau(t) = I((1+t)u_*)$. Clearly, $\tau(t)$ achieves its minimum at $t = 0$, namely

$$\tau'(t)|_{t=0} = a\|u_*\|^2 + b\|u_*\|^4 - \int_\Omega \frac{(u_*^+)^4}{|x|}\, dx - \lambda \int_\Omega \frac{(u_*^+)^{1-\gamma}}{|x|^\beta}\, dx = 0. \tag{2.14}$$

For any $\varphi \in H_0^1(\Omega)$ and $\varepsilon > 0$, we define $\Phi \in H_0^1(\Omega)$ by

$$\Phi = (u_*^+ + \varepsilon\varphi)^+.$$

Then it follows from (2.13) and (2.14) that

$$\begin{aligned}
0 &\leq \int_\Omega (a + b\|u_*\|^2)(\nabla u_*, \nabla\Phi)\, dx - \lambda \int_\Omega \frac{(u_*^+)^{-\gamma}}{|x|^\beta}\Phi\, dx - \int_\Omega \frac{(u_*^+)^3}{|x|}\Phi\, dx \\
&= \int_{\{u_*^+ + \varepsilon\varphi > 0\}} (a + b\|u_*\|^2)(\nabla u_*, \nabla(u_*^+ + \varepsilon\varphi))\, dx \\
&\quad - \int_{\{u_*^+ + \varepsilon\varphi > 0\}} \left[ \frac{(u_*^+)^3(u_*^+ + \varepsilon\varphi)}{|x|} + \lambda \frac{(u_*^+)^{-\gamma}}{|x|^\beta}(u_*^+ + \varepsilon\varphi) \right] dx \\
&= \left( \int_\Omega - \int_{\{u_*^+ + \varepsilon\varphi \leq 0\}} \right) \Bigg[ (a + b\|u_*\|^2)(\nabla u_*, \nabla(u_*^+ + \varepsilon\varphi)) \\
&\qquad\qquad - \frac{(u_*^+)^3(u_*^+ + \varepsilon\varphi)}{|x|} - \lambda \frac{(u_*^+)^{-\gamma}}{|x|^\beta}(u_*^+ + \varepsilon\varphi) \Bigg] dx \\
&\leq a\|u_*\|^2 + b\|u_*\|^4 - \lambda \int_\Omega \frac{(u_*^+)^{1-\gamma}}{|x|^\beta}\, dx - \int_\Omega \frac{(u_*^+)^4}{|x|}\, dx \\
&\quad + \varepsilon \int_\Omega \left[ (a + b\|u_*\|^2)(\nabla u_*, \nabla\varphi) - \lambda \frac{(u_*^+)^{-\gamma}}{|x|^\beta}\varphi - \frac{(u_*^+)^3\varphi}{|x|} \right] dx \\
&\quad - \int_{\{u_*^+ + \varepsilon\varphi \leq 0\}} (a + b\|u_*\|^2)(\nabla u_*, \nabla(u_*^+ + \varepsilon\varphi))\, dx \\
&\quad + \int_{\{u_*^+ + \varepsilon\varphi \leq 0\}} \left[ \lambda \frac{(u_*^+)^{-\gamma}}{|x|^\beta}(u_*^+ + \varepsilon\varphi) + \frac{(u_*^+)^3(u_*^+ + \varepsilon\varphi)}{|x|} \right] dx \\
&\leq \varepsilon \int_\Omega \left[ (a + b\|u_*\|^2)(\nabla u_*, \nabla\varphi) - \lambda \frac{(u_*^+)^{-\gamma}}{|x|^\beta}\theta - \frac{(u_*^+)^3\varphi}{|x|} \right] dx \\
&\quad - \varepsilon \int_{\{u_*^+ + \varepsilon\varphi \leq 0\}} (a + b\|u_*\|^2)(\nabla u_*, \nabla\varphi)\, dx.
\end{aligned} \tag{2.15}$$

Since $\operatorname{meas}(\{u_*^+ + \varepsilon\varphi \leq 0\}) \to 0$ as $\varepsilon \to 0^+$, it follows that

$$\lim_{\varepsilon \to 0^+} \int_{\{u_*^+ + \varepsilon\varphi \leq 0\}} (\nabla u_*, \nabla\varphi)\, dx = 0.$$

Therefore, dividing by $\varepsilon$ and letting $\varepsilon \to 0^+$ in (2.15), we deduce that

$$(2.16) \qquad (a + b\|u_*\|^2) \int_\Omega (\nabla u_*, \nabla\varphi)\, dx - \lambda \int_\Omega \frac{(u_*^+)^{-\gamma}}{|x|^\beta} \varphi\, dx - \int_\Omega \frac{(u_*^+)^3 \varphi}{|x|}\, dx \geq 0.$$

By the arbitrariness of $\varphi$, this inequality also holds for $-\varphi$, i.e.,

$$(2.17) \qquad (a + b\|u_*\|^2) \int_\Omega (\nabla u_*, \nabla\varphi)\, dx - \lambda \int_\Omega \frac{(u_*^+)^{-\gamma}}{|x|^\beta} \varphi\, dx - \int_\Omega \frac{(u_*^+)^3 \varphi}{|x|}\, dx = 0.$$

On the one hand, taking the test function $\varphi = u_*^-$ in (2.17), one has $\|u_*^-\| = 0$, which implies that $u_* \geq 0$. Hence, $u_*$ is a nonzero solution of (1.1). On the other hand, from (2.16), one has

$$-\Delta u_* \geq 0 \quad \text{in } \Omega.$$

Recalling that $u_* \geq 0$ and $u_* \not\equiv 0$, by using the maximum principle of the weak solution (see Theorem 3 in [3]), one has $u_* > 0$ in $\Omega$. Therefore, $u_*$ is a positive solution of (1.1) with $I(u_*) = m_0 < 0$. This completes the proof of Theorem 2.3. $\square$

### 2.2. The existence of the second positive solution

In the part, we will prove that (1.1) has the second positive solution. It is well known that the singular term leads to the non-differentiability of the functional $I$ on $H_0^1(\Omega)$. In order to overcome the difficulty caused by the singular term and get the second positive solution of (1.1), we consider the following approximating equation

$$(2.18) \qquad \begin{cases} -\left(a + b\displaystyle\int_\Omega |\nabla u|^2\, dx\right) \Delta u = \dfrac{u^3}{|x|} + \dfrac{\lambda}{|x|^\beta (u+\alpha)^\gamma}, & x \in \Omega, \\ u = 0, & x \in \partial\Omega \end{cases}$$

for any $\alpha > 0$. The energy functional of (2.18) $I_\alpha$ is defined by

$$I_\alpha(u) = \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{1}{4}\int_\Omega \frac{(u^+)^4}{|x|}\, dx - \frac{\lambda}{1-\gamma} \int_\Omega \frac{(u^+ + \alpha)^{1-\gamma} - \alpha^{1-\gamma}}{|x|^\beta}\, dx.$$

Obviously, $I_\alpha$ is a $C^1$-function on $H_0^1(\Omega)$. As well known, there exists a one to one correspondence between the nonnegative solutions of (2.18) and the critical points of $I_\alpha$ on $H_0^1(\Omega)$. More precisely, we say that $u \in H_0^1(\Omega)$ is a solution of (2.18), if $u$ satisfies

$$(2.19) \qquad (a + b\|u\|^2) \int_\Omega (\nabla u, \nabla\varphi)\, dx - \int_\Omega \frac{(u^+)^3 \varphi}{|x|}\, dx - \lambda \int_\Omega \frac{\varphi}{(u^+ + \alpha)^\gamma |x|^\beta}\, dx = 0$$

for any $\varphi \in H_0^1(\Omega)$.

Now, for any $\alpha > 0$, we will prove that (2.18) has a mountain-pass solution. First, we show that $I_\alpha$ satisfies the local $(C)_c$ condition.

**Lemma 2.4.** *Suppose that $a > 0$, $0 < b < A^{-2}$, $0 < \gamma < 1$, $0 \le \beta < (5+\gamma)/2$, then $I_\alpha$ satisfies the $(C)_c$ condition on $H_0^1(\Omega)$ with $c \in \big(0, \frac{a^2A^2}{4(1-bA^2)} - D\lambda^{2/(1+\gamma)}\big)$, where*

$$D = \frac{a(1+\gamma)}{4(1-\gamma)} \left( \frac{2R_0^{(5-2\beta+\gamma)/2}}{aS^{-(1-\gamma)/2}} \right)^{2/(1+\gamma)} \left[ \frac{4\pi(5+\gamma)}{3(5+\gamma-2\beta)} \right]^{(5+\gamma)/[3(1+\gamma)]}$$

*and $R_0$ is defined in* (2.2).

*Proof.* Suppose that $\{u_n\}$ is a $(C)_c$ sequence, for $c \in \big(0, \frac{(aA)^2}{4(1-bA^2)} - D\lambda^{2/(1+\gamma)}\big)$, i.e.,

$$I_\alpha(u_n) \to c, \quad (1 + \|u_n\|) I'_\alpha(u_n) \to 0 \quad \text{as } n \to \infty. \tag{2.20}$$

First, we prove that $\{u_n\}$ is a bounded sequence. By (2.20), one has $\lim_{n\to\infty}\langle I'_\alpha(u_n), u_n^-\rangle = 0$, that is,

$$\lim_{n\to\infty} \left[ -(a + b\|u_n\|^2)\|u_n^-\|^2 - \lambda \int_\Omega \frac{u_n^-}{\alpha^\gamma |x|^\beta}\, dx \right] = 0,$$

which implies that

$$\lim_{n\to\infty} \int_\Omega \frac{u_n^-}{\alpha^\gamma |x|^\beta}\, dx = 0. \tag{2.21}$$

Since $0 < \gamma < 1$, it follows from the subadditivity that

$$(u^+ + \alpha)^{1-\gamma} - \alpha^{1-\gamma} \le (u^+)^{1-\gamma}, \quad \forall\, u \in H_0^1(\Omega). \tag{2.22}$$

By (2.2), (2.20)–(2.22), we have

$$\begin{aligned}
c + 1 &\ge I_\alpha(u_n) - \frac{1}{4}\langle I'_\alpha(u_n), u_n\rangle + o(1) \\
&= \frac{a}{4}\|u_n\|^2 + \frac{\lambda}{4}\int_\Omega \frac{u_n}{|x|^\beta (u_n^+ + \alpha)^\gamma}\, dx - \frac{\lambda}{1-\gamma}\int_\Omega \frac{(u_n^+ + \alpha)^{1-\gamma} - \alpha^{1-\gamma}}{|x|^\beta}\, dx + o(1) \\
&\ge \frac{a}{4}\|u_n\|^2 - \frac{\lambda}{4}\int_\Omega \frac{u_n^-}{\alpha^\gamma |x|^\beta}\, dx - \frac{\lambda}{1-\gamma}\int_\Omega \frac{(u_n^+)^{1-\gamma}}{|x|^\beta}\, dx + o(1) \\
&\ge \frac{a}{4}\|u_n\|^2 - \frac{\lambda}{1-\gamma}\left[\frac{4\pi(5+\gamma)}{3(5+\gamma-2\beta)}\right]^{(5+\gamma)/6} R_0^{(5+\gamma-2\beta)/2} S^{-(1-\gamma)/2}\|u_n\|^{1-\gamma} + o(1),
\end{aligned}$$

since $0 < 1-\gamma < 1$, which implies that $\{u_n\}$ is bounded in $H_0^1(\Omega)$. Going if necessary to a subsequence, one can get $u_n \rightharpoonup u$ in $H_0^1(\Omega)$. Up to a subsequence, there exists $u \in H_0^1(\Omega)$ such that

$$\begin{gathered}
u_n \rightharpoonup u \text{ weakly in } H_0^1(\Omega), \quad u_n \to u \text{ strongly in } L^p(\Omega),\ 1 \le p < 6, \\
\frac{u_n^4}{|x|} \rightharpoonup \frac{u^4}{|x|} \text{ weakly in } L^1(\Omega), \quad u_n(x) \to u(x) \text{ a.e. in } \Omega, \\
\text{there exists } \phi \in L^p(\Omega)\ (1 \le p < 6) \text{ such that } |u_n(x)|, |u(x)| \le \phi(x), \text{ a.e. in } \Omega,
\end{gathered} \tag{2.23}$$

where the last conclusion is from Lemma A.1 in [26]. From (2.23), we obtain

$$\left|\frac{u_n}{|x|^\beta(u_n^+ + \alpha)^\gamma}\right| \le \frac{|u_n|}{\alpha^\gamma |x|^\beta} \le \frac{\phi(x)}{\alpha^\gamma |x|^\beta}.$$

Since $1 < (5+4\beta)/(5-\beta) < 6$, we choose $\phi \in L^{(5+4\beta)/(5-\beta)}(\Omega)$, we have

$$\begin{aligned}
&\int_\Omega \frac{\phi(x)}{|x|^\beta \alpha^\gamma}\,dx \\
&\le \frac{1}{\alpha^\gamma}\left(\int_\Omega |\phi(x)|^{(5+4\beta)/(5-\beta)}\,dx\right)^{(5-\beta)/(5+4\beta)} \left(\int_\Omega |x|^{-(5+4\beta)/5}\,dx\right)^{5\beta/(5+4\beta)} \\
&\le \frac{1}{\alpha^\gamma}|\phi|_{(5+4\beta)/(5-\beta)} \left(\int_{B_{(0,R_0)}} |x|^{-(5+4\beta)/5}\,dx\right)^{5\beta/(5+4\beta)} \\
&\le \frac{C}{\alpha^\gamma}(R_0)^{2\beta(5-2\beta)/(5+4\beta)}|\phi|_{(5+4\beta)/(5-\beta)}.
\end{aligned} \tag{2.24}$$

From (2.24), we know $\phi(x)/(\alpha^\gamma|x|^\beta) \in L^1(\Omega)$. Thus, applying the dominated convergence theorem, one has

$$\lim_{n\to\infty}\int_\Omega \frac{u_n}{|x|^\beta(u_n^+ + \alpha)^\gamma}\,dx = \int_\Omega \frac{u}{|x|^\beta(u^+ + \alpha)^\gamma}\,dx.$$

For given $\alpha > 0$ and $|u|/[|x|^\beta(u_n^+ + \alpha)^\gamma] \le |u(x)|/(\alpha^\gamma|x|^\beta)$, by the dominated convergence theorem and (2.23), we can obtain

$$\lim_{n\to\infty}\int_\Omega \frac{u}{|x|^\beta(u_n^+ + \alpha)^\gamma}\,dx = \int_\Omega \frac{u}{|x|^\beta(u^+ + \alpha)^\gamma}\,dx. \tag{2.25}$$

Let $w_n = u_n - u$, we claim that $\|w_n\| \to 0$ as $n \to \infty$. Otherwise, there exists a subsequence (still denoted by $\|w_n\|$) such that

$$\lim_{n\to\infty}\|w_n\| = l > 0.$$

By $I'_\alpha(u_n) \to 0$ in $(H_0^1(\Omega))^*$, we can deduce that

$$a\|u_n\|^2 + b\|u_n\|^4 - \lambda\int_\Omega \frac{(u_n^+ + \alpha)^{-\gamma}u_n}{|x|^\beta}\,dx - \int_\Omega \frac{(u_n^+)^4}{|x|}\,dx = o(1).$$

By Brézis-Lieb's Lemma (see [2]) and (2.25), we obtain

$$\begin{aligned}
o(1) &= a\|w_n\|^2 + a\|u\|^2 + b\|w_n\|^4 + b\|u\|^4 + 2b\|w_n\|^2\|u\|^2 \\
&\quad - \lambda\int_\Omega \frac{(u^+ + \alpha)^{-\gamma}}{|x|^\beta}u\,dx - \left(\int_\Omega \frac{(u^+)^4}{|x|}\,dx + \int_\Omega \frac{(w_n^+)^4}{|x|}\,dx\right).
\end{aligned} \tag{2.26}$$

It also follows from (2.20) that

$$\begin{aligned}0 &= \lim_{n\to\infty}\langle I'_\alpha(u_n), u\rangle \\ &= a\|u\|^2 + b\left(\lim_{n\to\infty}\int_\Omega |\nabla w_n|^2\,dx + 2\lim_{n\to\infty}\int_\Omega (\nabla w_n,\nabla u)\,dx + \int_\Omega |\nabla u|^2\,dx\right)\|u\|^2 \\ &\quad - \lambda\int_\Omega \frac{(u^+ + \alpha)^{-\gamma}}{|x|^\beta}u\,dx - \int_\Omega \frac{(u^+)^4}{|x|}\,dx,\end{aligned}$$

which implies that

$$a\|u\|^2 + bl^2\|u\|^2 + b\|u\|^4 - \lambda\int_\Omega \frac{(u^+ + \alpha)^{-\gamma}}{|x|^\beta}u\,dx - \int_\Omega \frac{(u^+)^4}{|x|}\,dx = 0. \tag{2.27}$$

On the one hand, by (2.2), (2.22) and (2.27), one has

$$\begin{aligned}I_\alpha(u) &= \frac{a}{2}\|u\|^2 + \frac{b}{4}\|u\|^4 - \frac{\lambda}{1-\gamma}\int_\Omega \frac{(u^+ + \alpha)^{1-\gamma} - \alpha^{1-\gamma}}{|x|^\beta}\,dx - \frac{1}{4}\int_\Omega \frac{(u^+)^4}{|x|}\,dx \\ &\geq \frac{a}{4}\|u\|^2 - \frac{1}{4}bl^2\|u\|^2 - \frac{\lambda}{1-\gamma}\int_\Omega \frac{(u^+ + \alpha)^{1-\gamma} - \alpha^{1-\gamma}}{|x|^\beta}\,dx \\ &\quad + \frac{\lambda}{4}\int_\Omega \frac{(u^+ + \alpha)^{-\gamma}}{|x|^\beta}u\,dx \\ &\geq \frac{a}{4}\|u\|^2 - \frac{\lambda}{1-\gamma}\int_\Omega \frac{|u|^{1-\gamma}}{|x|^\beta}\,dx - \frac{1}{4}bl^2\|u\|^2 \\ &\geq \frac{a}{4}\|u\|^2 - \frac{\lambda}{1-\gamma}\left[\frac{4\pi(5+\gamma)}{3(5+\gamma-2\beta)}\right]^{(5+\gamma)/6}\frac{R_0^{(5-2\beta+\gamma)/2}}{S^{(1-\gamma)/2}}\|u\|^{1-\gamma} - \frac{1}{4}bl^2\|u\|^2 \\ &\geq -\frac{a(1+\gamma)}{4(1-\gamma)}\left(\frac{2R_0^{(5-2\beta+\gamma)/2}}{aS^{-(1-\gamma)/2}}\right)^{2/(1+\gamma)}\left[\frac{4\pi(5+\gamma)}{3(5+\gamma-2\beta)}\right]^{(5+\gamma)/[3(1+\gamma)]}\lambda^{2/(1+\gamma)} \\ &\quad - \frac{1}{4}bl^2\|u\|^2 \\ &= -D\lambda^{2/(1+\gamma)} - \frac{1}{4}bl^2\|u\|^2,\end{aligned} \tag{2.28}$$

where the last inequality is from the Young inequality. On the other hand, it follows from (2.26) and (2.27) that

$$I_\alpha(u_n) = I_\alpha(u) + \frac{a}{2}\|w_n\|^2 + \frac{b}{4}\|w_n\|^4 + \frac{b}{2}\|w_n\|^2\|u\|^2 - \frac{1}{4}\int_\Omega \frac{(w_n^+)^4}{|x|}\,dx + o(1) \tag{2.29}$$

and

$$a\|w_n\|^2 + b\|w_n\|^4 + b\|w_n\|^2\|u\|^2 - \int_\Omega \frac{(w_n^+)^4}{|x|}\,dx = o(1). \tag{2.30}$$

By (2.30) and (1.4), we obtain

$$al^2 + bl^4 + bl^2\|u\|^2 = \int_\Omega \frac{(w_n^+)^4}{|x|}\,dx \leq \frac{l^4}{A^2}. \tag{2.31}$$

Consequently, for $0 < b < A^{-2}$, by (2.31), one has

$$l^2 \geq \frac{(a + b\|u\|^2)A^2}{1 - bA^2} \geq \frac{aA^2}{1 - bA^2}.$$

It follows from (2.29) and (2.30) that

$$I_\alpha(u) = I_\alpha(u_n) - \frac{a}{4}\|w_n\|^2 - \frac{b}{4}\|w_n\|^2\|u\|^2 + o(1).$$

Consequently, for $c < (aA)^2/[4(1 - bA^2)] - D\lambda^{2/(1+\gamma)}$, letting $n \to +\infty$, we deduce that

$$\begin{aligned} I_\alpha(u) &= c - \frac{a}{4}l^2 - \frac{1}{4}bl^2\|u\|^2 \\ &\leq c - \frac{a^2A^2}{4(1 - bA^2)} - \frac{1}{4}bl^2\|u\|^2 \\ &< -D\lambda^{2/(1+\gamma)} - \frac{1}{4}bl^2\|u\|^2, \end{aligned}$$

which contradicts to (2.28). Then, $u_n \to u$ in $H_0^1(\Omega)$ as $n \to \infty$. Therefore, $I_\alpha$ satisfies the $(C)_c$ condition with $0 < c < (aA)^2/[4(1-bA^2)]-D\lambda^{2/(1+\gamma)}$. Thus, the proof of Lemma 2.4 is completed. □

By Lemma 2.2 in [11], we know that $A$ is attained when $\Omega = \mathbb{R}^3$ by the functions

$$y_\varepsilon(x) = \frac{(2\varepsilon)^{1/2}}{\varepsilon + |x|}$$

for all $\varepsilon > 0$. Moreover, the functions $y_\varepsilon(x)$ solve the equation

$$-\Delta u = \frac{u^3}{|x|}, \quad x \in \mathbb{R}^3 \setminus \{0\}.$$

Let

$$c_\varepsilon = (2\varepsilon)^{1/2}, \quad U_\varepsilon(x) = \frac{y_\varepsilon(x)}{c_\varepsilon}.$$

Define a cut-off function $\varphi \in C_0^\infty(\Omega)$ such that $\varphi(x) = 1$ for $|x| \leq R$, $\varphi(x) = 0$ for $|x| \geq 2R$, $0 \leq \varphi(x) \leq 1$, where $B_{2R}(0) \subset \Omega$, set $u_\varepsilon(x) = \varphi(x)U_\varepsilon(x)$, $v_\varepsilon = \frac{u_\varepsilon(x)}{\left(\int_\Omega u_\varepsilon^4/|x|\,dx\right)^{1/4}}$, so that $\int_\Omega v_\varepsilon^4/|x|\,dx = 1$. According to Lemma 11.1 in [9], one has

$$\|v_\varepsilon\|^2 = A + o(\varepsilon), \tag{2.32}$$

and

$$\|v_\varepsilon\|^4 = A^2 + o(\varepsilon). \tag{2.33}$$

Next, we prove that the energy functional $I_\alpha$ satisfies the mountain-pass geometry structure on $H_0^1(\Omega)$.

**Lemma 2.5.** *Assume that $a > 0$, $0 < b < A^{-2}$, $0 < \gamma < 1$ and $0 \leq \beta < (5+\gamma)/2$, satisfying $0 < \alpha < 1$ and $R, \rho > 0$, $0 < \lambda < \lambda_*$ (where $\lambda_*$, $R$ and $\rho$ are defined in Lemma 2.1). Then the functional $I_\alpha$ satisfies the following conditions:*

(a) *$I_\alpha(u) \geq \rho > 0$ for all $u \in H_0^1(\Omega)$,*

(b) *there exists a function $u_0 \in H_0^1(\Omega)$ such that $\|u_0\| > R$ and $I_\alpha(u_0) < \rho$.*

*Proof.* (a) From (2.22), we deduce that

$$I_\alpha(u) \geq I(u), \quad \forall\, u \in H_0^1(\Omega).$$

Therefore, from (2.1), we obtain (a).

(b) For every $v_\varepsilon \in H_0^1(\Omega)$, $v_\varepsilon \neq 0$ and $t > 0$. Using (2.32) and (2.33), we have

$$\begin{aligned} I_\alpha(tv_\varepsilon) &= \frac{at^2}{2}\|v_\varepsilon\|^2 + \frac{bt^4}{4}\|v_\varepsilon\|^4 - \frac{t^4}{4}\int_\Omega \frac{v_\varepsilon^4}{|x|}\,dx \\ &\quad - \frac{\lambda}{1-\gamma}\int_\Omega \frac{(t^{1-\gamma}v_\varepsilon + \alpha)^{1-\gamma} - \alpha^{1-\gamma}}{|x|^\beta}\,dx \\ &\leq \frac{at^2}{2}\|v_\varepsilon\|^2 + \frac{bt^4}{4}\|v_\varepsilon\|^4 - \frac{t^4}{4}\int_\Omega \frac{v_\varepsilon^4}{|x|}\,dx \\ &= \frac{at^2}{2}[A + o(\varepsilon)] - \frac{bt^4}{4}[1 - bA^2 + o(\varepsilon)], \end{aligned}$$

since $b < A^{-2}$, which implies that $\lim_{t\to+\infty} I_\alpha(tv_\varepsilon) = -\infty$. Thus, let $u_0 = t_0 v_\varepsilon$ choosing $t_0 > 0$ sufficiently large such that $\|u_0\| > R$ and $I_\alpha(u_0) < \rho$. This completes the proof of Lemma 2.5. □

Finally, we estimate the level value of the mountain-pass and obtain the following conclusion.

**Lemma 2.6.** *Assume that $a > 0$, $0 < b < A^{-2}$, $0 < \gamma < 1$ and $2 + \gamma < \beta < (5+\gamma)/2$, then there exists $\lambda_0 > 0$, for all $0 < \lambda < \lambda_0$, such that*

$$\sup_{t\geq 0} I_\alpha(tv_\varepsilon) < \frac{a^2A^2}{4(1-bA^2)} - D\lambda^{2/(1+\gamma)}, \tag{2.34}$$

*where $D$ is defined in Lemma 2.4.*

*Proof.* Let $\lambda < \left(\frac{a^2A^2}{4(1-bS^2)D}\right)^{(1+\gamma)/2}$, we have $a^2A^2/[4(1-bS^2)] - D\lambda^{2/(1+\gamma)} > 0$. Since $I_\alpha(0) = 0$ and $\lim_{t\to\infty} I_\alpha(tv_\varepsilon) = -\infty$, by Lemma 2.5, there exists $t_\varepsilon > 0$ such that $I_\alpha(t_\varepsilon v_\varepsilon) = \max_{t>0} I_\alpha(tv_\varepsilon) \geq \rho > 0$. Moreover, by the continuity of $I_\alpha$, there exist positive constants $t_1$ and $t_2$ such that $0 < t_1 \leq t_\varepsilon \leq t_2$. Set $I_\alpha(t_\varepsilon v_\varepsilon) = g(t_\varepsilon v_\varepsilon) - \lambda h(t_\varepsilon v_\varepsilon)$, where $g$ and $h$ are defined by

$$g(t_\varepsilon v_\varepsilon) = \frac{at_\varepsilon^2}{2}\|v_\varepsilon\|^2 + \frac{bt_\varepsilon^4}{4}\|v_\varepsilon\|^4 - \frac{t_\varepsilon^4}{4}$$

and

$$h(t_\varepsilon v_\varepsilon) = \frac{1}{1-\gamma}\int_\Omega \frac{(t_\varepsilon v_\varepsilon+\alpha)^{1-\gamma}-\alpha^{1-\gamma}}{|x|^\beta}\,dx.$$

First, we claim that there exists a constant $C_1 > 0$ (independent of $\lambda$, $\varepsilon$) such that

$$g(t_\varepsilon v_\varepsilon) \le \frac{a^2A^2}{4(1-bA^2)} + C_1\varepsilon.$$

Indeed, by (2.32) and (2.33), it holds that

$$\begin{aligned}
g(t_\varepsilon v_\varepsilon) &= \frac{at_\varepsilon^2}{2}\|v_\varepsilon\|^2 + \frac{bt_\varepsilon^4}{4}\|v_\varepsilon\|^4 - \frac{t_\varepsilon^4}{4}\\
&\le \frac{a(A+o(\varepsilon))}{2}t_\varepsilon^2 - \frac{1-bA^2+o(\varepsilon)}{4}t_\varepsilon^4\\
&\le \frac{a^2A^2+o(\varepsilon)}{4[1-b(A^2+o(\varepsilon))]} + o(\varepsilon)\\
&\le \frac{a^2A^2}{4(1-bA^2)} + C_1\varepsilon.
\end{aligned} \tag{2.35}$$

Next, we prove that there exists a constant $C_2 > 0$ (independent of $\lambda$, $\varepsilon$) such that

$$h(t_\varepsilon v_\varepsilon) \ge C_2\varepsilon^{(7+\gamma-3\beta)/4}.$$

In fact, for $\beta > 2+\gamma$, we have

$$\begin{aligned}
h(t_\varepsilon v_\varepsilon) &\ge \frac{1}{1-\gamma}\int_{|x|\le\varepsilon^{2/3}} \frac{(t_\varepsilon v_\varepsilon+\alpha)^{1-\gamma}-\alpha^{1-\gamma}}{|x|^\beta}\,dx\\
&\ge \frac{1}{1-\gamma}\int_{|x|\le\varepsilon^{2/3}} \frac{(t_\varepsilon v_\varepsilon)^{1-\gamma}-\alpha^{1-\gamma}}{|x|^\beta}\,dx\\
&\ge C\int_{|x|\le\varepsilon^{2/3}} \frac{\varepsilon^{(1-\gamma)/2}}{(\varepsilon+|x|)^{1-\gamma}|x|^\beta}\,dx - C\int_{|x|\le\varepsilon^{2/3}}\frac{1}{|x|^\beta}\,dx\\
&\ge C\int_{|x|\le\varepsilon^{2/3}} \frac{\varepsilon^{(1-\gamma)/2}}{(\varepsilon+\varepsilon^{2/3})^{1-\gamma}\varepsilon^{2\beta/3}}\,dx - C\int_0^{\varepsilon^{2/3}} r^{2-\beta}\,dr\\
&\ge C\int_{|x|\le\varepsilon^{2/3}} \frac{\varepsilon^{(1-\gamma)/2}}{\varepsilon^{2(1-\gamma)/3}\varepsilon^{2\beta/3}}\,dx - C\int_0^{\varepsilon^{2/3}} r^{2-\beta}\,dr\\
&\ge C\varepsilon^{\frac{1-\gamma}{2}-\frac{2}{3}(1-\gamma)+\frac{6-2\beta}{3}} - C\varepsilon^{(6-2\beta)/3}\\
&\ge C\varepsilon^{(11+\gamma-4\beta)/6} - C\varepsilon^{(6-2\beta)/3}\\
&\ge C_2\varepsilon^{(11+\gamma-4\beta)/6},
\end{aligned} \tag{2.36}$$

where $C$ and $C_2$ are positive constants and independent of $\lambda$ and $\varepsilon$. Therefore, combining (2.35) and (2.36), we have

$$I_\alpha(t_\varepsilon v_\varepsilon) = g(t_\varepsilon v_\varepsilon) - \lambda h(t_\varepsilon v_\varepsilon) \le \frac{a^2A^2}{4(1-bA^2)} + C_1\varepsilon - C_2\lambda\varepsilon^{(11+\gamma-4\beta)/6}.$$

Since $2+\gamma<\beta<(5+\gamma)/2$, let $\varepsilon=\lambda^{2/(1+\gamma)}$, $\lambda<\left(\frac{C_2}{C_1+D}\right)^{3(1+\gamma)/[4(\beta-\gamma-2)]}$, it holds that

$$\begin{aligned}C_1\varepsilon-C_2\lambda\varepsilon^{(11+\gamma-4\beta)/6}&=C_1\lambda^{2/(1+\gamma)}-C_2\lambda^{(14+4\gamma-4\beta)/[3(1+\gamma)]}\\&=\lambda^{2/(1+\gamma)}\left[C_1-C_2\lambda^{4(2+\gamma-\beta)/[3(1+\gamma)]}\right]\\&<-D\lambda^{2/(1+\gamma)}.\end{aligned}$$

Thus, we conclude that

$$\begin{aligned}I_\alpha(t_\varepsilon v_\varepsilon)&=g(t_\varepsilon v_\varepsilon)-\lambda h(t_\varepsilon v_\varepsilon)\\&\le\frac{a^2A^2}{4(1-bA^2)}+C_1\varepsilon-C_2\lambda\varepsilon^{(11+\gamma-4\beta)/6}\\&\le\frac{a^2A^2}{4(1-bA^2)}-D\lambda^{2/(1+\gamma)},\end{aligned}$$

which implies that (2.34) holds provided that $0<\lambda<\lambda_0$ where

$$\lambda_0=\min\left\{\left(\frac{C_2}{C_1+D}\right)^{3(1+\gamma)/[4(\beta-\gamma-2)]},\left[\frac{a^2A^2}{4(1-bA^2)D}\right]^{(1+\gamma)/2}\right\}.$$

Then, the proof of Lemma 2.6 is completed. □

Thus, $I_\alpha$ satisfies $(C)_c$ condition on $H_0^1(\Omega)$ provided that $0<c<\frac{(aA)^2}{4(1-bA^2)}-D\lambda^{2/(1+\gamma)}$. We have the following result.

**Theorem 2.7.** *Suppose that $a>0$, $0<b<A^{-2}$, $0<\gamma,\alpha<1$ and $2+\gamma<\beta<(5+\gamma)/2$, then there exists $\lambda_{**}>0$ such that (2.18) has at least a positive $u_\alpha\in H_0^1(\Omega)$ with $I_\alpha(u_\alpha)>\rho$ ($\rho$ is defined in Lemma 2.1) for all $0<\lambda<\lambda_{**}$.*

*Proof.* Let $\lambda_{**}=\min\{\lambda_*,\lambda_0\}$, Lemmas 2.4–2.6 hold for $0<\lambda<\lambda_{**}$. Now, we define

$$\Gamma:=\{\eta\in C([0,1],H_0)^1(\Omega)\mid\eta(0)=0,\eta(1)=u_0\},$$
$$c_\alpha=\inf_{\eta\in\Gamma}\max_{t\in[0,1]}I_\alpha(\eta(t)),$$

where $u_0=t_0v_\varepsilon$ is defined in Lemma 2.5. By Lemma 2.4 and Theorem 2.1 in [25], there exists a sequence $\{u_n\}\subset H_0^1(\Omega)$, such that

$$I_\alpha(u_n)\to c_\alpha>\rho\quad\text{and}\quad(1+\|u_n\|)I'_\alpha(u_n)\to 0.$$

Moreover, from Lemmas 2.5 and 2.6, we obtain

$$\begin{aligned}\rho<c_\alpha&\le\max_{t\in[0,1]}I_\alpha(tu_0)=\max_{t\in[0,1]}I_\alpha(tt_0v_\varepsilon)\\&\le\sup_{t\ge0}I_\alpha(tt_0v_\varepsilon)<\frac{(aA)^2}{4(1-bA^2)}-D\lambda^{2/(1+\gamma)}.\end{aligned}\tag{2.37}$$

According to Lemma 2.4, we obtain $\{u_n\} \subset H_0^1(\Omega)$ has a convergent subsequence, still denoted by $\{u_n\}$. Assume that $\{u_n\}$ converges to $u_\alpha \in H_0^1(\Omega)$. Thus, we have $I_\alpha(u_\alpha) = c_\alpha > 0$ and $I'_\alpha(u_\alpha) = 0$, that is, $u_\alpha$ is a nonzero solution of (2.18). Consequently, $u_\alpha$ satisfies (2.19). Choosing $u = u_\alpha$ and $\varphi = u_\alpha^-$ in (2.19), we obtain $(a + b\|u_\alpha\|^2)\|u_\alpha^-\|^2 = 0$, which implies that $u_\alpha^- = 0$. Thus, $u_\alpha \geq 0$ and $u_\alpha \not\equiv 0$. By the maximum principle of the weak solution (see Theorem 3 in [3]), we obtain that $u_\alpha$ is a positive solution of (2.18). Therefore, the proof of Theorem 2.7 is completed. □

According to Theorem 2.7, for every $\alpha \in (0,1)$, (2.18) has at least a positive mountain-pass solution $\{u_\alpha\}$ with $I_\alpha(u_\alpha) > \rho > 0$. Thus, there exist $\{\alpha_n\} \subset (0,1)$ with $\alpha_n \to 0$ as $n \to \infty$, such that $\{u_{\alpha_n}\}$ is a sequence positive mountain-pass solutions of (2.18) with $I_{\alpha_n}(u_{\alpha_n}) > \rho > 0$. Now, we shall prove that the limit point of the sequence of positive solutions $\{u_{\alpha_n}\}$ of problem (2.18) is the second positive solution of (1.1) with $0 < b < A^{-2}$ and $2 + \gamma < \beta < (5+\gamma)/2$.

**Theorem 2.8.** *Suppose that $a > 0$, $0 < b < A^{-2}$, $0 < \gamma < 1$ and $2 + \gamma < \beta < (5+\gamma)/2$, then for any $0 < \lambda < \lambda_{**}$ ($\lambda_{**}$ is defined in Theorem 2.7), (1.1) has a positive solution $u_{**}$ satisfying $I(u_{**}) > 0$.*

*Proof.* Noting that $\{u_{\alpha_n}\}$ is a sequence of positive solutions of (2.18), we have

$$-(a + b\|u_{\alpha_n}\|^2)\Delta u_{\alpha_n} = \frac{u_{\alpha_n}^3}{|x|} + \frac{\lambda}{|x|^\beta (u_{\alpha_n} + \alpha_n)^\gamma} \geq \min\left\{1, \frac{\lambda}{R_0^\beta 2^\gamma}\right\}.$$

Consequently, we obtain

$$-\Delta u_{\alpha_n} \geq \frac{1}{a + b\|u_{\alpha_n}\|^2} \min\left\{1, \frac{\lambda}{R_0^\beta 2^\gamma}\right\}.$$

Let $e$ be the positive solution of the following problem

$$\begin{cases} -\Delta u = 1 & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

then $e(x) > 0$ in $\Omega$. Therefore, by the comparison principle, one has

$$u_{\alpha_n} \geq \frac{1}{a + b\|u_{\alpha_n}\|^2} \min\left\{1, \frac{\lambda}{R_0^\beta 2^\gamma}\right\} e > 0. \tag{2.38}$$

Furthermore, from (2.2), (2.22) and (2.37), we deduce that

$$\begin{aligned}
&\frac{(aA)^2}{4(1-bA^2)} - D\lambda^{2/(1+\gamma)} \\
&> I_{\alpha_n}(u_{\alpha_n}) - \frac{1}{4}\langle I'_{\alpha_n}(u_{\alpha_n}), u_{\alpha_n}\rangle \\
&= \frac{a}{4}\|u_{\alpha_n}\|^2 - \frac{\lambda}{1-\gamma}\int_\Omega \frac{(u_{\alpha_n}+\alpha_n)^{1-\gamma} - \alpha_n^{1-\gamma}}{|x|^\beta}\,dx \\
&\quad + \frac{\lambda}{4}\int_\Omega \frac{u_{\alpha_n}}{|x|^\beta(u_{\alpha_n}+\alpha_n)^\gamma}\,dx \\
&\ge \frac{a}{4}\|u_{\alpha_n}\|^2 - \frac{\lambda}{1-\gamma}\left[\frac{4\pi(5+\gamma)}{3(5+\gamma-2\beta)}\right]^{(5+\gamma)/6} R_0^{(5-2\beta+\gamma)/2} S^{-(1-\gamma)/2}\|u_{\alpha_n}\|^{1-\gamma},
\end{aligned}$$

which implies that $\{u_{\alpha_n}\}$ is bounded in $H_0^1(\Omega)$. Up to a subsequence, there exists $u_{**} \ge 0$ with $u_{**} \in H_0^1(\Omega)$ such that

$$\begin{aligned}
&u_{\alpha_n} \rightharpoonup u_{**} \text{ weakly in } H_0^1(\Omega), \quad u_{\alpha_n} \to u_{**} \text{ strongly in } L^p(\Omega),\ 1 \le p < 6, \\
&\frac{u_{\alpha_n}^4}{|x|} \rightharpoonup \frac{u_{**}^4}{|x|} \text{ weakly in } L^1(\Omega), \quad u_{\alpha_n}(x) \to u_{**}(x) \text{ a.e. in } \Omega.
\end{aligned} \tag{2.39}$$

Now, we prove that $u_{\alpha_n} \to u_{**}$ in $H_0^1(\Omega)$ as $n \to \infty$. As usual, let $w_{\alpha_n} = u_{\alpha_n} - u_{**}$, we claim that $\|w_{\alpha_n}\| \to 0$ as $n \to \infty$. By contradiction, assume that $\|w_{\alpha_n}\| \not\to 0$, then there exists a subsequence (still denoted by $\|w_{\alpha_n}\|$) such that $\lim_{n\to\infty}\|w_{\alpha_n}\| = l > 0$. Since

$$\frac{u_{\alpha_n}}{|x|^\beta(u_{\alpha_n}+\alpha_n)^\gamma} \le \frac{u_{\alpha_n}^{1-\gamma}}{|x|^\beta},$$

by the dominated convergence theorem and (2.39), one gets

$$\lim_{n\to\infty}\int_\Omega \frac{u_{\alpha_n}}{|x|^\beta(u_{\alpha_n}+\alpha_n)^\gamma}\,dx = \int_\Omega \frac{u_{**}^{1-\gamma}}{|x|^\beta}\,dx.$$

From $I'_{\alpha_n}(u_{\alpha_n}) \to 0$ in $(H_0^1(\Omega))^*$, we obtain

$$a\|u_{\alpha_n}\|^2 + b\|u_{\alpha_n}\|^4 - \lambda\int_\Omega \frac{u_{\alpha_n}}{|x|^\beta(u_{\alpha_n}+\alpha_n)^\gamma}\,dx - \int_\Omega \frac{u_{\alpha_n}^4}{|x|}\,dx = o(1).$$

Consequently, by Brézis-Lieb's Lemma, we deduce that

$$\begin{aligned}
o(1) &= a\|w_{\alpha_n}\|^2 + a\|u_{**}\|^2 + b\|w_{\alpha_n}\|^4 + b\|u_{**}\|^4 + 2b\|w_{\alpha_n}\|^2\|u_{**}\|^2 \\
&\quad - \lambda\int_\Omega \frac{u_{\alpha_n}^{1-\gamma}}{|x|^\beta}\,dx - \left(\int_\Omega \frac{u_{**}^4}{|x|}\,dx + \int_\Omega \frac{w_{\alpha_n}^4}{|x|}\,dx\right).
\end{aligned} \tag{2.40}$$

From (2.38), let $n \to \infty$, we have $u_{**} > 0$. Since $u_{\alpha_n}$ satisfies (2.19), choosing $u = u_{\alpha_n}$ and taking the test function $\varphi = \phi \in H_0^1(\Omega) \cap C_0(\Omega)$ ($C_0(\Omega)$ is the subset of $C(\Omega)$ consisting of functions with compact support in $\Omega$), let $n \to \infty$, we obtain

$$(a + bl^2 + b\|u_{**}\|^2)\int_\Omega (\nabla u_{**}, \nabla\phi)\,dx = \lambda\int_\Omega \frac{u_{**}^{-\gamma}}{|x|^\beta}\phi\,dx + \int_\Omega \frac{u_{**}^3}{|x|}\phi\,dx. \tag{2.41}$$

We will show that (2.41) holds for any $\phi \in H_0^1(\Omega)$. Indeed, since $H_0^1(\Omega) \cap C_0(\Omega)$ is dense in $H_0^1(\Omega)$, for any $\phi \in H_0^1(\Omega)$ there exists a sequence $\{\phi_n\} \subset H_0^1(\Omega) \cap C_0(\Omega)$ such that $\phi_n \to \phi$ in $H_0^1(\Omega)$ as $n \to \infty$. For $n, m \in N^+$ large enough, replacing $\phi$ with $\phi_n - \phi_m$ in (2.41), we obtain

$$\begin{aligned}(a + bl^2 + b\|u_{**}\|^2) \int_\Omega (\nabla u_{**}, \nabla|\phi_n - \phi_m|)\, dx &= \lambda \int_\Omega \frac{u_{**}^{-\gamma}}{|x|^\beta} |\phi_n - \phi_m|\, dx \\ &\quad + \int_\Omega \frac{u_{**}^3}{|x|} |\phi_n - \phi_m|\, dx.\end{aligned} \tag{2.42}$$

On one hand, since $\phi_n \to \phi$, from (2.42) we can infer that $\{\phi_n/(|x|^\beta u_{**}^\gamma)\}$ is a Cauchy sequence in $L^1(\Omega)$. Hence, there exists $\psi \in L^1(\Omega)$ satisfying $\phi_n/(|x|^\beta u_{**}^\gamma) \to \psi$ in $L^1(\Omega)$, which means that $\phi_n/(|x|^\beta u_{**}^\gamma) \to \psi(x)$ in measure. By Riesz's Theorem, $\{\phi_n/(|x|^\beta u_{**}^\gamma)\}$ has a subsequence, still denoted by $\{\phi_n/(|x|^\beta u_{**}^\gamma)\}$, such that

$$\frac{\phi_n}{|x|^\beta u_{**}^\gamma} \to \psi(x) \quad \text{a.e. } x \in \Omega. \tag{2.43}$$

On the other hand, since $\phi_n/(|x|^\beta u_{**}^\gamma) \to \phi/(|x|^\beta u_{**}^\gamma)$ a.e. in $\Omega$, from (2.43), one has $\phi/(|x|^\beta u_{**}^\gamma) = \psi$. Thus,

$$\int_\Omega \frac{\phi_n}{|x|^\beta u_{**}^\gamma}\, dx \to \int_\Omega \frac{\phi}{|x|^\beta u_{**}^\gamma}\, dx$$

as $n \to \infty$. Then, taking the test function $\phi = \phi_n$ in (2.41) and passing to the limit as $n \to \infty$, we deduce that (2.41) holds for any $\phi \in H_0^1(\Omega)$.

In particular, choosing $\phi = u_{**}$ in (2.41), we have

$$a\|u_{**}\|^2 + b\|u_{**}\|^4 + bl^2\|u_{**}\|^2 - \int_\Omega \frac{u_{**}^4}{|x|}\, dx - \lambda \int_\Omega \frac{u_{**}^{1-\gamma}}{|x|^\beta} = 0. \tag{2.44}$$

From (2.40) and (2.44), we can deduce that

$$a\|w_{\alpha_n}\|^2 + b\|w_{\alpha_n}\|^4 + b\|w_{\alpha_n}\|^2\|u_{**}\|^2 - \int_\Omega \frac{w_{\alpha_n}^4}{|x|}\, dx = o(1). \tag{2.45}$$

By (1.4) and let $n \to \infty$, it follows from (2.45) that

$$al^2 + bl^4 + bl^2\|u_{**}\|^2 = \int_\Omega \frac{w_{\alpha_n}^4}{|x|}\, dx \le \frac{l^4}{A^2}.$$

Since $0 < b < A^{-2}$, one has

$$l^2 \ge \frac{(a + b\|u_{**}\|^2)A^2}{1 - bA^2} > \frac{aA^2}{1 - bA^2}. \tag{2.46}$$

It follows from (2.45) and Brézis-Lieb's lemma that

$$I(u_{**}) = I_{\alpha_n}(u_{\alpha_n}) - \frac{a}{4}l^2 - \frac{b}{4}l^2\|u_{**}\|^2 + o(1). \tag{2.47}$$

On the one hand, combining (2.2) and (2.44), similar to obtain (2.28), one obtains

$$
\begin{aligned}
& I(u_{**}) \\
& \geq \frac{a}{2}\|u_{**}\|^2 - \left(\frac{1}{1-\gamma} - \frac{1}{4}\right)\lambda \int_\Omega \frac{u_{**}^{1-\gamma}}{|x|^\beta}\,dx - \frac{1}{4}bl^2\|u_{**}\|^2 \\
& \geq \frac{a}{4}\|u_{**}\|^2 - \frac{\lambda}{1-\gamma}\int_\Omega \frac{u_{**}^{1-\gamma}}{|x|^\beta}\,dx - \frac{1}{4}bl^2\|u_{**}\|^2 \\
& \geq \frac{a}{4}\|u_{**}\|^2 - \frac{\lambda}{1-\gamma}\left[\frac{4\pi(5+\gamma)}{3(5+\gamma-2\beta)}\right]^{(5+\gamma)/6} \frac{R_0^{(5-2\beta+\gamma)/2}}{S^{(1-\gamma)/2}}\|u_{**}\|^{1-\gamma} - \frac{1}{4}bl^2\|u_{**}\|^2 \qquad (2.48) \\
& \geq -\frac{a(1+\gamma)}{4(1-\gamma)}\left(\frac{2R_0^{(5-2\beta+\gamma)/2}}{aS^{-(1-\gamma)/2}}\right)^{2/(1+\gamma)} \left[\frac{4\pi(5+\gamma)}{3(5+\gamma-2\beta)}\right]^{4\pi(5+\gamma)/[3(1+\gamma)]} \lambda^{2/(1+\gamma)} \\
& \quad - \frac{1}{4}bl^2\|u_{**}\|^2 \\
& = -D\lambda^{2/(1+\gamma)} - \frac{1}{4}bl^2\|u_{**}\|^2.
\end{aligned}
$$

On the other hand, since $I_{\alpha_n}(u_{\alpha_n}) < \frac{a^2A^2}{4(1-bS^2)} - D\lambda^{2/(1+\gamma)}$, it follows from (2.46) and (2.47) that

$$
\begin{aligned}
I(u_{**}) &= \frac{(aA)^2}{4(1-bA^2)} - D\lambda^{2/(1+\gamma)} - \frac{a}{4}l^2 - \frac{1}{4}bl^2\|u_{**}\|^2 \\
&< \frac{(aA)^2}{4(1-bA^2)} - D\lambda^{2/(1+\gamma)} - \frac{(aA)^2}{4(1-bA^2)} - \frac{1}{4}bl^2\|u_{**}\|^2 \\
&= -D\lambda^{2/(1+\gamma)} - \frac{1}{4}bl^2\|u_{**}\|^2,
\end{aligned}
$$

which contradicts to (2.48). Thus, $l = 0$ and our claim is true. That is, $u_{\alpha_n} \to u_{**}$ in $H_0^1(\Omega)$ as $n \to \infty$. By (2.41), one has

$$
(a + b\|u_{**}\|^2)\int_\Omega (\nabla u_{**}, \nabla\phi)\,dx = \int_\Omega \frac{u_{**}^3}{|x|}\phi\,dx + \lambda\int_\Omega \frac{\phi}{|x|^\beta u_{**}^\gamma}\,dx
$$

for any $\phi \in H_0^1(\Omega)$. Consequently, $u_{**}$ is a positive solution of (1.1). Moreover, one has $I(u_{**}) = \lim_{n\to\infty} I_{\alpha_n}(u_{\alpha_n}) > \rho > 0$. This completes the proof of Theorem 2.8. $\square$

Therefore, according to Theorems 2.3 and 2.8, Theorem 1.2 is proved.


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