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A Characterization of Weighted Carleson Measure Spaces

Hsun-Wu Liu* and Kunchuan Wang

Abstract. Using Frazier and Jawerth's φ-transform, we characterize weighted generalized Carleson measure spaces $\dot{C}MO_{p,w}^{\alpha,q}$ for a weight w and show that the definition of this space is well-defined by a Plancherel-Pôlya inequality. Note that $\dot{C}MO_{1,w}^{0,2}$ is the weighted BMO space.

1. Introduction

The general Triebel-Lizorkin spaces $\dot{F}_p^{\alpha,q}$ (homogeneous) and $F_p^{\alpha,q}$ (inhomogeneous), $\alpha \in \mathbb{R}$, $0 < p, q \leq \infty$, include many well-known classical function spaces. For example, $L^p \approx \dot{F}_p^{0,2} \approx F_p^{0,2}$ when $1 < p < \infty$, $\dot{F}_p^{\alpha,2} \approx \dot{L}_p^\alpha$ and $F_p^{\alpha,2} \approx L_p^\alpha$ (Sobolev spaces) when $1 < p < \infty$ and $\alpha > 0$, $H^p \approx \dot{F}_p^{0,2}$ when $0 < p \leq 1$, and $BMO \approx \dot{F}_\infty^{0,2}$. Here the notation "$\approx$" means in a (quasi-)normed vector space V with different norms $\|\cdot\|_a$ and $\|\cdot\|_b$, there exist positive constants c_1 and c_2 such that

$$c_1\|x\|_a \leq \|x\|_b \leq c_2\|x\|_a$$

for all x in V.

We say that a cube $Q \subseteq \mathbb{R}^n$ is *dyadic* if $Q = Q_{j\boldsymbol{k}} = \{x = (x_1, x_2, \ldots, x_n) \in \mathbb{R}^n : 2^{-j}k_i \leq x_i < 2^{-j}(k_i + 1), i = 1, 2, \ldots, n\}$ for some $j \in \mathbb{Z}$ and $\boldsymbol{k} = (k_1, k_2, \ldots, k_n) \in \mathbb{Z}^n$. For any pair of dyadic cubes P and Q, either P and Q are nonoverlapping or one contains the other. Denote by $\ell(Q) = 2^{-j}$ the side length of Q and $x_Q = 2^{-j}\boldsymbol{k}$ the "left lower corner" of Q. In fact $Q = x_Q + [0, 2^{-j})^n$. For $j \in \mathbb{Z}$ let $\mathscr{Q}_j$ be the collection of dyadic cubes with side length 2^{-j} and let $\mathscr{Q}$ be the collection of all dyadic cubes in $\mathbb{R}^n$. Thus $\mathscr{Q} = \bigcup_{j\in\mathbb{Z}} \mathscr{Q}_j$. For a fixed dyadic cube P let $\mathscr{Q}_P$ be the collection of all dyadic cubes in $\mathbb{R}^n$ which are contained in P.

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*Corresponding author.

For any function f defined on $\mathbb{R}^n$, $j \in \mathbb{Z}$ and dyadic cube $Q = Q_{j\boldsymbol{k}}$, set

$$\begin{aligned} f_Q(x) &= |Q|^{-1/2} f((x - x_Q)/\ell(Q)) = 2^{jn/2} f(2^j x - \boldsymbol{k}), \\ f_j(x) &= 2^{jn} f(2^j x), \\ \widetilde{f}(x) &= \overline{f(-x)}. \end{aligned}$$

It is clear that $\widetilde{g}_j * f(x_Q) = |Q|^{-1/2}\langle f, g_Q\rangle$, where $\langle f, g\rangle$ denotes the pairing in the usual sense for g in a Fréchet space X and f in the dual of X.

Next, let us consider more general function spaces. For this purpose, let us recall the φ-transform identity introduced by Frazier and Jawerth [6]. Choose a fixed Schwartz function φ satisfying

$$\text{(1.1)} \qquad \operatorname{supp}(\widehat{\varphi}) \subseteq \{\xi : 1/2 \le |\xi| \le 2\} \quad \text{and} \quad |\widehat{\varphi}(\xi)| \ge c > 0 \quad \text{if } 3/5 \le |\xi| \le 5/3,$$

where $\widehat{f}$ is the Fourier transform of f, i.e.,

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x) e^{-ix\cdot\xi}\, dx.$$

The existence of such a function was provided by Frazier and Jawerth in [6]. Then there exists a function $\psi \in \mathscr{S}$ satisfying the same conditions as φ such that

$$\text{(1.2)} \qquad \sum_{j\in\mathbb{Z}} \overline{\widehat{\varphi}}(2^{-j}\xi)\widehat{\psi}(2^{-j}\xi) = 1 \quad \text{for } \xi \neq 0.$$

Thus the *φ-transform identity* is given by

$$\text{(1.3)} \qquad f = \sum_{Q\in\mathscr{Q}} \langle f, \varphi_Q\rangle \psi_Q,$$

where the identity holds in the sense of $\mathscr{S}'/\mathscr{P}$ (the spaces of all tempered distributions modulo polynomials), $\mathscr{S}_0$ (the subspace of $\mathscr{S}$ that each element has all vanishing moments), $\dot{B}_p^{\alpha,q}$-norm, and $\dot{F}_p^{\alpha,q}$-norm.

Now due to the Littlewood-Paley characterization, define the homogeneous Triebel-Lizorkin spaces as follows. Select a function $\varphi \in \mathscr{S}$ satisfying the conditions above in (1.1). For $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$, and $f \in \mathscr{S}'/\mathscr{P}$, define the *homogeneous Triebel-Lizorkin space* to be the collection of every distribution $f \in \mathscr{S}'/\mathscr{P}$ so that the norm $\|f\|_{\dot{F}_p^{\alpha,q}}$ is finite, where

$$\|f\|_{\dot{F}_p^{\alpha,q}} := \left\| \left[\sum_{\nu\in\mathbb{Z}} (2^{\nu\alpha}|\varphi_\nu * f|)^q \right]^{1/q} \right\|_{L^p}$$

if $0 < p < \infty$ and

$$\|f\|_{\dot{F}_\infty^{\alpha,q}} := \sup_{P\in\mathscr{Q}} \left\{ \frac{1}{|P|} \int_P \sum_{k\ge -\log_2 \ell(P)} (2^{k\alpha}|\varphi_k * f(x)|)^q\, dx \right\}^{1/q}.$$

By an inequality of Plancherel-Plôlya type, the definition of $\dot{F}_p^{\alpha,q}$ is independent of the choice of φ satisfying conditions (1.1).

In 2009 M.-Y. Lee, C.-C. Lin, and Y.-C. Lin [9] characterized the weighted Carleson measure space CMO_w^p for a weight w belonging to the Muckenhoupt class by wavelets. They lifted sequence spaces to prove that CMO_w^p is the dual space of H_w^p. The *weighted Carleson measure space* CMO_p^w is the set of all $g \in L^1_{\mathrm{loc}}$ satisfying

$$\|g\|_{CMO_w^p} := \sup_{J \in \mathscr{D}} \left\{ w(J)^{1-2/p} \sum_{I \in \mathscr{D}_J} \frac{|I|}{w(I)} |\langle g, \psi_I \rangle|^2 \right\}^{1/2} < \infty, \quad 0 < p \le 1,$$

where $w \in A_\infty$ (see Definition 2.2 for the A_p weights), ψ is a certain smooth function so that it is an orthonormal wavelet in $L^2(w)$, $\mathscr{D}$ is the set of all dyadic intervals $I_{j,k}$ with $j, k \in \mathbb{Z}$, and $\mathscr{D}_J$ is the collection of all dyadic intervals contained in J.

In 2012, C.-C. Lin and K. Wang [10] gave another characterization for the dual of $\dot{F}_p^{\alpha,q}$ in terms of Carleson measures for $\alpha \in \mathbb{R}$ and $0 < p \le 1 \le q \le \infty$. The *generalized Carleson measure space* $CMO_p^{\alpha,q}$ is the collection of all $f \in \mathscr{S}'/\mathscr{P}$ satisfying $\|f\|_{CMO_p^{\alpha,q}} < \infty$, where

$$|f\|_{CMO_p^{\alpha,q}} := \sup_{P \in \mathscr{Q}} \left\{ |P|^{-q(1/p-1/q')} \int_P \sum_{Q \in \mathscr{Q}_P} (|Q|^{-\alpha/n-1/2} |\langle f, \varphi \rangle| \chi_Q(x))^q \, dx \right\}^{1/q}$$

for $1 \le q < \infty$, and

$$\|f\|_{CMO_p^{\alpha,\infty}} := \sup_{Q \in \mathscr{Q}} |Q|^{-\alpha/n-1/p+1/2} |\langle f, \varphi_Q \rangle|.$$

Here χ_Q denotes the characteristic function of Q and q' is the conjugate index of q, i.e., $1/q + 1/q' = 1$. Throughout the article, q' is defined as $q' = \infty$ whenever $0 < q \le 1$.

They introduced a new kind of sequence space $c_p^{\alpha,q}$, and then characterized the duals of $\dot{f}_p^{\alpha,q}$ by means of $c_p^{\alpha,q}$. Let us recall the definitions of the sequence spaces $\dot{f}_p^{\alpha,q}$ defined in [6].

For $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$, the space $\dot{f}_p^{\alpha,q}$ consists of all such sequences $\boldsymbol{s} = \{s_Q\}_{Q \in \mathscr{Q}}$ satisfying $\|\boldsymbol{s}\|_{\dot{f}_p^{\alpha,q}} < \infty$, where

$$\|\boldsymbol{s}\|_{\dot{f}_p^{\alpha,q}} := \left\| \left\{ \sum_{Q \in \mathscr{Q}} (|Q|^{-\alpha/n-1/2} |s_Q| \chi_Q)^q \right\}^{1/q} \right\|_{L^p}$$

if $0 < p < \infty$ and

$$\|\boldsymbol{s}\|_{\dot{f}_\infty^{\alpha,q}} := \sup_{P \in \mathscr{Q}} \left\{ |P|^{-1} \int_P \sum_{Q \in \mathscr{Q}_P} \left(|Q|^{-\alpha/n-1/2} |s_Q| \chi_Q(x) \right)^q \, dx \right\}^{1/q}.$$

As before, the above ℓ^q-norm is modified to the supremum norm for $0 < p < \infty$ and $q = \infty$. For $p = q = \infty$, we adopt the norm

$$\|\boldsymbol{s}\|_{\dot{f}_\infty^{\alpha,\infty}} := \sup_{Q\in\mathscr{Q}} |Q|^{-\alpha/n-1/2}|s_Q|.$$

To study the dual of $\dot{f}_p^{\alpha,q}$, they introduced a *discrete version of Carleson measure spaces* $c_p^{\alpha,q}$. For $\alpha \in \mathbb{R}$ and $0 < p \le 1 \le q \le \infty$, the space $c_p^{\alpha,q}$ is the collection of all sequences $\boldsymbol{t} = \{t_Q\}_{Q\in\mathscr{Q}}$ satisfying $\|\boldsymbol{t}\|_{c_p^{\alpha,q}} < \infty$, where

$$\|\boldsymbol{t}\|_{c_p^{\alpha,q}} := \sup_{P\in\mathscr{Q}} \left\{ |P|^{-q(1/p-1/q')} \int_P \sum_{Q\in\mathscr{Q}_P} \left[|Q|^{-\alpha/n-1/2}|t_Q|\chi_Q(x)\right]^q dx \right\}^{1/q}$$

for $1 \le q < \infty$, and

$$\|\boldsymbol{t}\|_{c_p^{\alpha,\infty}} := \sup_{Q\in\mathscr{Q}} |Q|^{-\alpha/n-1/p+1/2}|t_Q|.$$

For $1 \le q < \infty$ and $f \in CMO_p^{\alpha,q}$, let $S_\varphi(f) := \{\langle f, \varphi_Q\rangle\}_Q = \{s_Q\}_Q = \boldsymbol{s}$. Then the φ-tranform identity shows $f = \sum_Q s_Q\psi_Q$ and $\|f\|_{CMO_p^{\alpha,q}} = \|S_\varphi(f)\|_{c_p^{\alpha,q}} = \|\boldsymbol{s}\|_{c_p^{\alpha,q}}$. In particular, $\|f\|_{CMO_1^{\alpha,q}} = \|S_\varphi(f)\|_{c_1^{\alpha,q}} = \|S_\varphi(f)\|_{\dot{f}_\infty^{\alpha,q}} \approx \|f\|_{\dot{F}_\infty^{\alpha,q}}$. Furthermore, for $\boldsymbol{s} \in c_p^{\alpha,q}$,

$$\|T_\psi(\boldsymbol{s})\|_{CMO_p^{\alpha,q}} = \left\|\sum_P s_P\psi_P\right\|_{CMO_p^{\alpha,q}} = \left\|\left\{\left\langle \sum_P s_P\psi_P, \varphi_Q\right\rangle\right\}_Q\right\|_{c_p^{\alpha,q}} = \|A\boldsymbol{s}\|_{c_p^{\alpha,q}},$$

where $T_\psi(\boldsymbol{s}) := \sum_Q s_Q\psi_Q$ and $A := \{\langle\psi_P, \varphi_Q\rangle\}_{Q,P}$ is $(\alpha + nq(1/p - 1/q'), p, q)$-almost diagonal (cf. [6, Lemma 3.6]). They summarized that $T_\psi \circ S_\varphi|_{CMO_p^{\alpha,q}}$ is also the identity on $CMO_p^{\alpha,q}$.

In [3–5], Bui and Taibleson defined another weighted version of Triebel-Linzorkin spaces. There are some other papers concerning this topic, see [2, 7, 8] for more details.

Given a weight w, let $\mathscr{Q}(w)$ denote the collection of all dyadic cubes $Q \subseteq \mathbb{R}^n$ such that $w(Q) := \int_Q w(Q)\,dx \neq 0$ and for $k \in \mathbb{Z}$, $\mathscr{Q}_k(w)$ denote the subcollections of $\mathscr{Q}(w)$ with side length 2^{-k}. Also, for $P \in \mathscr{Q}(w)$, $\mathscr{Q}_P(w)$ denotes the collection of all dyadic cubes $Q \in \mathscr{Q}(w)$ with $Q \subseteq P$ and $\mathscr{Q}_{P,k}(w)$ denotes the collection of all dyadic cubes satisfying $Q \subseteq P$ and $\ell(Q) = 2^{-k}$. Note that $\mathscr{Q}(w) = \bigcup_{k\in\mathbb{Z}} \mathscr{Q}_k(w)$, and $\mathscr{Q}_P(w) = \bigcup_{k\ge-\log_2\ell(P)} \mathscr{Q}_{P,k}(w)$.

In weighted cases, we adopt similar definitions for S_φ and T_ψ as follows. Define a linear map S_φ from $\mathscr{S}'/\mathscr{P}$ into the family of complex sequences by

$$S_\varphi(f) = \{\langle f, \varphi_Q\rangle\}_{Q\in\mathscr{Q}(w)}, \tag{1.4}$$

and another linear map T_ψ from the family of complex sequences into $\mathscr{S}'/\mathscr{P}$ by

$$T_\psi(\{s_Q\}_{Q\in\mathscr{Q}(w)}) = \sum_{Q\in\mathscr{Q}(w)} s_Q\psi_Q. \tag{1.5}$$

In this article, we study the weighted generalized Carleson measure spaces via the φ-transform identity. To do so, we first need the following definitions.

Definition 1.1. For $\alpha \in \mathbb{R}$, $0 < p, q \le +\infty$, and a weight w, we say that f belongs to the *homogeneous weighted Triebel-Lizorkin space* $\dot{F}^{\alpha,q}_{p,w}$ if $f \in \mathscr{S}'/\mathscr{P}$ satisfies $\|f\|_{\dot{F}^{\alpha,q}_{p,w}} < \infty$, where

$$\|f\|_{\dot{F}^{\alpha,q}_{p,w}} := \begin{cases} \left\| \left\{ \sum_{k\in\mathbb{Z}} (2^{k\alpha}|\varphi_k * f|)^q \right\}^{1/q} \right\|_{L^p(w)} & \text{for } p < \infty, \\ \sup_{P\in\mathscr{Q}(w)} \left\{ |P|^{-1} \int_P \sum_{k=-\log_2 \ell(P)}^{\infty} (2^{k\alpha}|\varphi_k * f(x)|)^q w(x)\,dx \right\}^{1/q} & \text{for } p = \infty. \end{cases}$$

Definition 1.2. Let $\varphi \in \mathscr{S}$ satisfy the conditions in (1.1). For $\alpha \in \mathbb{R}$, $0 < p \le 1$, $0 < q \le \infty$ and a weight w, the *weighted generalized Carleson measure space* $\dot{C}MO^{\alpha,q}_{p,w}$ is the collection of all $f \in \mathscr{S}'/\mathscr{P}$ satisfying $\|f\|_{\dot{C}MO^{\alpha,q}_{p,w}} < \infty$, where

$$\|f\|_{\dot{C}MO^{\alpha,q}_{p,w}} := \sup_{P\in\mathscr{Q}(w)} \left\{ |P|^{-q(1/p-1/q')} \int_P \sum_{Q\in\mathscr{Q}_P(w)} (|Q|^{-\alpha/n-1/2}|\langle f, \varphi_Q\rangle|\chi_Q(x))^q w(x)\,dx \right\}^{1/q}$$

for $0 < q < \infty$, and

$$\|f\|_{\dot{C}MO^{\alpha,\infty}_{p,w}} := \sup_{P\in\mathscr{Q}(w)} |P|^{1-1/p} \sup_{Q\in\mathscr{Q}_P(w)} |Q|^{-\alpha/n-1/2}|\langle f, \varphi_Q\rangle|.$$

In order to prove that the definition of $\dot{C}MO^{\alpha,q}_{p,w}$ is independent of the choice of $\varphi \in \mathscr{S}$ satisfying certain conditions, we need the following Plancherel-Pôlya inequality (for $q = \infty$ and the other case we will give descriptions and prove them in Section 3).

Theorem 1.3 (Plancherel-Pôlya inequality)**.** *Let $\varphi, \phi \in \mathscr{S}$ satisfy* (1.1). *For $\alpha \in \mathbb{R}$, $0 < p \le 1 < q < \infty$ and a weight w with doubling exponent β, if $f \in \mathscr{S}'/\mathscr{P}$ satisfies*

$$\sup_{P\in\mathscr{Q}(w)} \left\{ |P|^{-q(1/p-1/q')} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{Q\in\mathscr{Q}_{P,k}(w)} (2^{k\alpha} \sup_{u\in Q} |\widetilde{\varphi}_k * f(u)|)^q w(Q) \right\}^{1/q} < \infty,$$

then

$$\sup_{P\in\mathscr{Q}(w)} \left\{ |P|^{-q(1/p-1/q')} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{Q\in\mathscr{Q}_{P,k}(w)} (2^{k\alpha} \sup_{u\in Q} |\widetilde{\phi}_k * f(u)|)^q w(Q) \right\}^{1/q}$$
$$\approx \sup_{P\in\mathscr{Q}(w)} \left\{ |P|^{-q(1/p-1/q')} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{Q\in\mathscr{Q}_{P,k}(w)} (2^{k\alpha} \inf_{u\in Q} |\widetilde{\varphi}_k * f(u)|)^q w(Q) \right\}^{1/q}.$$

By the theorem above we can make the following remark.

Remark 1.4. Let $\varphi, \phi \in \mathscr{S}$ satisfy (1.1). For $\alpha \in \mathbb{R}$, $0 < p \le 1 < q < \infty$ and a weight w with doubling exponent β. Denote $\dot{C}MO^{\alpha,q}_{p,w}(\varphi)$ as the collection of all $f \in \mathscr{S}'/\mathscr{P}$ satisfying $\|f\|_{\dot{C}MO^{\alpha,q}_{p,w}(\varphi)} < \infty$ defined in Definition 1.2 with respect to φ. Then, by Theorem 1.3,

$$\begin{aligned}
&\|f\|_{\dot{C}MO^{\alpha,q}_{p,w}(\phi)} \\
&\le \sup_{P \in \mathscr{Q}(w)} \left\{ |P|^{-q(1/p-1/q')} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{Q \in \mathscr{Q}_{P,k}(w)} (2^{k\alpha} \sup_{u \in Q} |\widetilde{\phi}_k * f(u)|)^q w(Q) \right\}^{1/q} \\
&\le C \sup_{P \in \mathscr{Q}(w)} \left\{ |P|^{-q(1/p-1/q')} \sum_{k=-\log_2 \ell(P)}^{\infty} \sum_{Q \in \mathscr{Q}_{P,k}(w)} (2^{k\alpha} \inf_{u \in Q} |\widetilde{\varphi}_k * f(u)|)^q w(Q) \right\}^{1/q} \\
&\le C \|f\|_{\dot{C}MO^{\alpha,q}_{p,w}(\varphi)}.
\end{aligned}$$

Similarly, $\|f\|_{\dot{C}MO^{\alpha,q}_{p,w}(\varphi)} \le C\|f\|_{\dot{C}MO^{\alpha,q}_{p,w}(\phi)}$ by interchanging the roles of φ and ϕ. Hence the definition of $\dot{C}MO^{\alpha,q}_{p,w}(\varphi)$ is independent of the choice of φ and, in short form, is denoted by $\dot{C}MO^{\alpha,q}_{p,w}$.

In order to obtain a norm equivalence, we need to define a *discrete version of weighted Carleson measure spaces* $\dot{c}^{\alpha,q}_{p,w}$. Before giving the definition of these spaces, let us recall the weighted homogeneous Triebel-Lizorkin sequence spaces.

Definition 1.5. For $\alpha \in \mathbb{R}$, $0 < p, q \le \infty$ and a weight w, the space $\dot{f}^{\alpha,q}_{p,w}$ consists of all such sequences $\boldsymbol{s} = \{s_Q\}_{Q \in \mathscr{Q}(w)}$ satisfying $\|\boldsymbol{s}\|_{\dot{f}^{\alpha,q}_{p,w}} < \infty$, where

$$\|\boldsymbol{s}\|_{\dot{f}^{\alpha,q}_{p,w}} := \left\| \left\{ \sum_{Q \in \mathscr{Q}} (|Q|^{-\alpha/n-1/2} |s_Q| \chi_Q)^q \right\}^{1/q} \right\|_{L^p(w)}$$

if $0 < p < \infty$ and

$$\|\boldsymbol{s}\|_{\dot{f}^{\alpha,q}_{\infty,w}} := \sup_{P \in \mathscr{Q}(w)} \left\{ |P|^{-1} \int_P \sum_{Q \in \mathscr{Q}_P(w)} \left(|Q|^{-\alpha/n-1/2} |s_Q| \chi_Q(x) \right)^q w(x)\, dx \right\}^{1/q}.$$

As before, the above ℓ^q-norm is modified to the supremum norm for $0 < p < \infty$ and $q = \infty$. For $p = q = \infty$, we adopt the norm

$$\|\boldsymbol{s}\|_{\dot{f}^{\alpha,\infty}_{\infty,w}} := \sup_{Q \in \mathscr{Q}(w)} |Q|^{-\alpha/n-1/2} |s_Q|.$$

Next, let us define the weighted generalized Carleson measure sequence spaces.

Definition 1.6. Let w be a weight. For $\alpha \in \mathbb{R}$, $0 < p \le 1$ and $0 < q \le \infty$, the space $\dot{c}^{\alpha,q}_{p,w}$ is the collection of all sequences $\boldsymbol{t} = \{t_Q\}_{Q \in \mathscr{Q}(w)}$ satisfying $\|\boldsymbol{t}\|_{\dot{c}^{\alpha,q}_{p,w}} < \infty$, where

$$\|\boldsymbol{t}\|_{\dot{c}^{\alpha,q}_{p,w}} := \sup_{P \in \mathscr{Q}(w)} \left\{ |P|^{-q(1/p-1/q')} \int_P \sum_{Q \in \mathscr{Q}_P(w)} \left[|Q|^{-\alpha/n-1/2} |t_Q| \chi_Q(x) \right]^q w(x)\,dx \right\}^{1/q}$$

for $0 < q < \infty$, and

$$\|\boldsymbol{t}\|_{\dot{c}^{\alpha,\infty}_{p,w}} := \sup_{P \in \mathscr{Q}(w)} |P|^{1-1/p} \sup_{Q \in \mathscr{Q}_P(w)} |Q|^{-\alpha/n-1/2} |t_Q|.$$

As a consequence of Plancherel-Pôlya inequalities, we have a result concerning the norm equivalence between generalized function spaces and corresponding sequence spaces.

Theorem 1.7. *Suppose $\alpha \in \mathbb{R}$, $0 < p \le 1$, $0 < q \le \infty$, $w \in A_\infty$ and φ, ψ in $\mathscr{S}$ satisfy* (1.1) *and* (1.2). *The linear operators $S_\varphi \colon \dot{C}MO^{\alpha,q}_{p,w} \mapsto \dot{c}^{\alpha,q}_{p,w}$ and $T_\psi \colon \dot{c}^{\alpha,q}_{p,w} \mapsto \dot{C}MO^{\alpha,q}_{p,w}$ defined by* (1.4) *and* (1.5), *respectively, are bounded. Furthermore $T_\psi \circ S_\varphi$ is the identity on $\dot{C}MO^{\alpha,q}_{p,w}$. In particular, $\|f\|_{\dot{C}MO^{\alpha,q}_{1,w}} = \|S_\varphi(f)\|_{\dot{c}^{\alpha,q}_{1,w}} = \|S_\varphi(f)\|_{\dot{f}^{\alpha,q}_{\infty,w}} \approx \|f\|_{\dot{F}^{\alpha,q}_{\infty,w}}$.*

The organization of this article is as follows. We recall weights and some preliminary results in Section 2. In Section 3, we show the Plancherel-Pôlya inequalities that give us the independence of the choice of φ for the definition of weighted generalized Carleson measure spaces. In Section 4, we show a norm equivalence between $\dot{C}MO^{\alpha,q}_{p,w}$ and $\dot{c}^{\alpha,q}_{p,w}$. Through the article, we use $j \wedge k$ to denote the minimum of j and k, use $j \vee k$ to denote the maximum of j and k, and use C to denote a positive constant independent of the main variables, which may vary from line to line.

2. Weights

We say that w is a *weight* if w is a non-negative measurable function on $\mathbb{R}^n$. At the beginning of this section, let us recall the definition of "doubling condition".

Definition 2.1. A weight w is called a *doubling measure*, if there exists a constant $C = C_n$ such that for any $\delta > 0$ and any $z \in \mathbb{R}^n$,

$$\int_{B_{2\delta}(z)} w(t)\,dt \le C \int_{B_\delta(z)} w(t)\,dt, \tag{2.1}$$

where $B_\delta(z)$ is an open ball in $\mathbb{R}^n$ centered at z with radius δ. If $C = 2^\beta$ is the smallest constant such that the inequality (2.1) holds, then β is called the *doubling exponent* of w.

Here, we recall the definition of A_p weights. For $s > 0$ and a weight w, define w^{-s} by

$$w^{-s}(x) = \begin{cases} [w(x)]^{-s} & \text{if } w(x) \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Definition 2.2 (A_p weights)**.** Let w be a non-negative and locally integrable function on $\mathbb{R}^n$. We say that $w \in A_p$ if $\|w\|_{A_p}$ is finite, where $\|w\|_{A_p}$ is defined by

$$\|w\|_{A_p} = \begin{cases} \sup_Q \operatorname{ess\,sup}_{y\in Q} w^{-1}(y) \frac{1}{|Q|} \int_Q w(t)\,dt & \text{if } p = 1, \\ \sup_Q \left(\frac{1}{|Q|}\int_Q w(x)\,dx\right)\left(\frac{1}{|Q|}\int_Q w^{1-p'}(x)\,dx\right)^{p-1} & \text{if } 1 < p < \infty, \end{cases}$$

where the suprema are taken over all cubes (with sides parallel to the coordinate axes) in $\mathbb{R}^n$. We also set $A_\infty = \bigcup_{1\le p<\infty} A_p$.

Note that we can replace any cubes by any balls in $\mathbb{R}^n$ in the last definition. Also note that when $w \in A_p$ for $1 \le p \le \infty$, $w(x)\,dx$ is a doubling measure (see [13, Theorem 2.1, p. 226]).

The following is a characterization of Muckenhoupt A_p weights.

Lemma 2.3. *Let $1 < q < \infty$ and $w \in A_q$. Then, for every cube $Q \subseteq \mathbb{R}^n$*

$$|Q| \le [w(Q)]^{1/q}[w^{1-q'}(Q)]^{(q-1)/q} \le C|Q|,$$

where C is dependent only on the constant of A_q condition. Moreover,

$$\left(\frac{|Q|}{w(Q)}\right)^{q'} w(Q) \approx w^{1-q'}(Q).$$

Proof. By Hölder's inequality and A_q condition, we have, for a cube Q in $\mathbb{R}^n$

$$\begin{aligned} |Q| &\le \left(\int_Q w(x)\,dx\right)^{1/q}\left(\int_Q w^{-q'/q}(x)\,dx\right)^{-1/q'} \\ &= \left\{[w(Q)]\left[w^{-(q'-1)}(Q)\right]^{q-1}\right\}^{1/q} \le C|Q|. \end{aligned}$$

That implies

$$|Q|^q \le w(Q)[w^{1-q'}(Q)]^{q-1} \le C|Q|^q,$$

then

$$1 \le \frac{w(Q)}{|Q|}\left(\frac{w^{1-q'}(Q)}{|Q|}\right)^{q-1} \le C.$$

Therefore, we have

$$\begin{aligned} \left(\frac{|Q|}{w(Q)}\right)^{q'} w(Q) &= \left(\frac{|Q|}{w(Q)}\right)^{q'-1} |Q| \\ &\approx \left(\frac{w^{1-q'}(Q)}{|Q|}\right)^{(q-1)(q'-1)} |Q| = w^{1-q'}(Q), \end{aligned}$$

and the proof is finished. □

Also there is a weighted version of Fefferman-Stein vector-valued maximal inequality for Hardy-Littlewood maximal function M, which is given below.

Proposition 2.4. [1] *Let $1 < p, q < \infty$ and $w \in A_p$. There is a constant $C = C_p$ independent of $\{f_i\}_i$ such that*

$$\left\| \left(\sum_i |Mf_i|^q \right)^{1/q} \right\|_{L^p(w)} \le C \left\| \left(\sum_i |f_i|^q \right)^{1/q} \right\|_{L^p(w)}$$

for any $\{f_i\}_i \in L^p(w)(\ell^q)$.

3. Plancherel-Pôlya inequalities

In this section, we show Plancherel-Pôlya inequalities that give us the independence of the choice of φ for the definition of weighted Carleson measure spaces. Before proving those, let us recall a basic estimate of Roudenko [12].

Lemma 3.1. [12] *Let w be a weight with doubling exponent β. If $L > \beta$, then for $r \ge \ell(Q)$,*

$$\int_{\mathbb{R}^n} w(x) \left(1 + \frac{|x - x_Q|}{r} \right)^{-L} dx \le c_\beta \left[\frac{r}{\ell(Q)} \right]^\beta \int_Q w(x)\, dx.$$

Now we can prove the following Plancherel-Pôlya inequalities.

Proof of Theorem 1.3. Without loss of generality, we may assume $\alpha = 0$. By the φ-transform identity, (1.3), we rewrite $\widetilde{\phi}_j * f(u)$ as

$$\begin{aligned} \widetilde{\phi}_j * f(u) &= \sum_{Q \in \mathscr{Q}} \langle f, \varphi_Q \rangle \int \widetilde{\phi}_j(u - x) \psi_Q(x)\, dx \\ &= \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathscr{Q}_k} |Q| \langle f, \varphi_k(\,\cdot\, - x_Q) \rangle \int \widetilde{\phi}_j(u - x) \psi_k(x - x_Q)\, dx. \end{aligned}$$

Using the inequality (B.5) in [6, p. 151],

$$\left| \int \widetilde{\phi}_j(u - x) \psi_k(x - x_Q)\, dx \right| \le C 2^{-K|j-k|} \frac{2^{-(j \wedge k)(J-n)}}{(2^{-(j \wedge k)} + |u - x_Q|)^J},$$

where $j \wedge k = \min\{j, k\}$, $J > \beta + n$ and $K > (\beta - n) \vee (J - \beta - n)$, we obtain

$$|\widetilde{\phi}_j * f(u)| \le C \sum_{k \in \mathbb{Z}} \sum_{Q \in \mathscr{Q}_k} 2^{-K|j-k|} |Q| \frac{2^{-(j \wedge k)(J-n)}}{(2^{-(j \wedge k)} + |u - x_Q|)^J} |\widetilde{\varphi}_j * f(x_Q)|.$$

Thus, for $\ell(Q')=2^{-j}$,

$$\left(\sup_{u\in Q'}|\widetilde{\phi}_j*f(u)|\right)^q\le C\left(\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathscr{Q}_k}2^{-K|j-k|}|Q|\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|\widetilde{\varphi}_k*f(x_Q)|\right)^q$$
$$\le C\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathscr{Q}_k}2^{-K|j-k|}|Q|\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|\widetilde{\varphi}_k*f(x_Q)|^q,$$

where the last inequality is followed by Hölder's inequality and

$$\sum_{Q\in\mathscr{Q}_k}|Q|\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}\le C.$$

Denote T_Q by

$$T_Q:=\inf_{u\in Q}|\widetilde{\varphi}_k*f(u)|^q. \tag{3.1}$$

Since x_Q can be replaced by any point in Q in the last inequality,

$$\left(\sup_{u\in Q'}|\widetilde{\phi}_j*f(u)|\right)^q\le C\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathscr{Q}_k}2^{-K|j-k|}|Q|\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}T_Q.$$

Given a dyadic cube P with $\ell(P)=2^{-k_0}$, the above estimates yield

$$\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\left(\sup_{u\in Q'}|\widetilde{\phi}_j*f(u)|\right)^q w(Q')$$
$$\le C\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathscr{Q}_k}2^{-K|j-k|}\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|Q|\frac{w(Q')}{w(Q)}T_Qw(Q)$$
$$:=CE_1+CE_2,$$

where

$$E_1=\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k\ge j}\sum_{Q\in\mathscr{Q}_k}2^{-K|j-k|}\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|Q|\frac{w(Q')}{w(Q)}T_Qw(Q)$$

and

$$E_2=\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k<j}\sum_{Q\in\mathscr{Q}_k}2^{-K|j-k|}\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|Q|\frac{w(Q')}{w(Q)}T_Qw(Q).$$

Then E_1 can be further decomposed as

$$E_1=\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k\ge j}\sum_{Q\in\mathscr{Q}_{3P,k}}2^{-K|j-k|}\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|Q|\frac{w(Q')}{w(Q)}T_Qw(Q)$$
$$+\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k\ge j}\sum_{\substack{Q\cap 3P=\emptyset\\ Q\in\mathscr{Q}_k}}2^{-K|j-k|}\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|Q|\frac{w(Q')}{w(Q)}T_Qw(Q)$$
$$:=E_{11}+E_{12}.$$

There are 3^n dyadic cubes in $3P$ with the same side length as P, so if $P' \subseteq 3P$ then $|P'| = |P|$ and

$$\sum_{Q\in\mathcal{Q}_{3P,k}} T_Q w(Q) \le 3^n \sup_{\substack{P'\subseteq 3P\\ \ell(P')=\ell(P)}} \sum_{Q\in\mathcal{Q}_{P',k}} T_Q w(Q).$$

Let $J > \beta$, $k \ge j$ and w be a weight with doubling exponent β. By Lemma 3.1, we have

$$\int_{\mathbb{R}^n} \left(1+\frac{|x-x_Q|}{2^{-j}}\right)^{-J} w(x)\,dx \le c_\beta 2^{(k-j)\beta} w(Q),$$

and so

$$\begin{aligned}
&\sum_{j=k_0}^{\infty} \sum_{Q'\in\mathcal{Q}_{P,j}} 2^{-K|j-k|} \frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J} |Q| w(Q') \\
&\le C \sum_{j=k_0}^{\infty} 2^{-K(k-j)} 2^{jn} 2^{-kn} \int_{\mathbb{R}^n} \left(1+\frac{|x-x_Q|}{2^{-j}}\right)^{-J} w(x)\,dx \\
&\le C w(Q).
\end{aligned}$$

Hence

$$\begin{aligned}
|P|^{-q(1/p-1/q')} E_{11} &\le C |P|^{-q(1/p-1/q')} \sum_{j=k_0}^{\infty} \sum_{Q'\in\mathcal{Q}_{P,j}} \sum_{k\ge j} \sum_{Q\in\mathcal{Q}_{3P,k}} 2^{-K|j-k|} \\
&\qquad \times \left(\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_P|)^J} |Q| \frac{w(Q')}{w(Q)} T_Q w(Q) \right) \\
&\le C \sup_{P'\in\mathcal{Q}(w)} |P'|^{-q(1/p-1/q')} \sum_{k=-\log_2 \ell(P')}^{\infty} \sum_{Q\in\mathcal{Q}_{P',k}} \inf_{u\in Q} |\widetilde{\varphi}_k * f(u)|^q w(Q).
\end{aligned}$$

Next, we decompose the set of dyadic cubes $\{Q : Q \cap 3P = \emptyset, \ell(Q) = \ell(P)\}$ into $\{B_i\}_{i\in\mathbb{N}}$ according to the distance between each Q and P. Namely, for each $i \in \mathbb{N}$,

$$B_i := \{P' \in \mathcal{Q} : P' \cap 3P = \emptyset, \ell(P) = \ell(P'), 2^{i-k_0} \le \|y_{P'} - y_P\| < 2^{i-k_0+1}\}, \tag{3.2}$$

where y_Q denotes the center of Q. Then we obtain

$$\begin{aligned}
|P|^{-q(1/p-1/q')} E_{12} &\le C \sum_{i=1}^{\infty} \sum_{P'\in B_i} |P'|^{-q(1/p-1/q')} \sum_{j=k_0}^{\infty} \sum_{Q'\in\mathcal{Q}_{P,j}} \sum_{k\ge j} \sum_{\substack{Q\in\mathcal{Q}_{P',k}\\ P'\in B_i}} 2^{-K|j-k|} \\
&\qquad \times \left(\frac{2^{-j(J-n)}}{(2^{-j}+|x_{Q'}-x_P|)^J} |Q| \frac{w(Q')}{w(Q)} T_Q w(Q) \right).
\end{aligned}$$

Since $w(Q')/w(Q) \le C2^{\beta(k-k_0+i)}$ and $\|x_{P'} - x_P\| \approx 2^{i-k_0}$ for $P' \in B_i$, the right-hand side of the inequality above is dominated by

$$C\sum_{i=1}^{\infty}\sum_{P'\in B_i}\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k\ge j}\sum_{\substack{Q\in\mathscr{Q}_{P',k}\\ P'\in B_i}} 2^{\beta(k-k_0+i)}2^{-K(k-j)}2^{-(i-k_0)J}2^{-j(J-n)}|Q| \times\left(\sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{k\ge k_0}\sum_{Q\in\mathscr{Q}_{P',k}}T_Qw(Q)\right).$$

Because there are at most $2^{(i+2)n}$ dyadic cubes in B_i, $J > \beta + n$, $K + n > \beta$ and $|Q| = |Q'|\frac{|Q|}{|Q'|}$,

$$\begin{aligned}
&|P|^{-q(1/p-1/q')}E_{12}\\
&\le C\left\{\sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{k\ge k_0}\sum_{Q\in\mathscr{Q}_{P',k}}T_Qw(Q)\right\}\\
&\quad\times\left[\sum_{i=1}^{\infty}\sum_{j=k_0}^{\infty}\sum_{k\ge j}2^{\beta(k-k_0+i)}2^{-K(k-j)}2^{-(i-k_0)J}2^{-(J-n)j}2^{-k_0n}2^{(j-k)n}2^{in}\right]\\
&\le C\sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{k=-\log_2\ell(P')}^{\infty}\sum_{Q\in\mathscr{Q}_{P',k}}\inf_{u\in Q}|\widetilde{\varphi}_k * f(u)|^q w(Q).
\end{aligned}$$

To estimate E_2, for $i \in \mathbb{N}$ and $k < k_0$, set

$$G_{i,k} := \{Q : \ell(Q) = 2^{-k} \text{ and } x_Q \in 2^{i+1}P \setminus 2^iP\}. \tag{3.3}$$

Then $|x_Q - x_P| \approx 2^{i-k_0}$ for $Q \in G_{i,k}$ and

$$E_2 = \sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k<j}\sum_{i=1}^{\infty}\sum_{Q\in G_{i,k}}\frac{2^{-K|j-k|}}{|Q|^{-q(1/p-1/q')}}\frac{2^{-k(J-n)}}{(2^{-k}+|x_{Q'}-x_Q|)^J}\frac{w(Q')}{w(Q)}|Q| \times|Q|^{-q(1/p-1/q')}T_Qw(Q).$$

Since $J > \beta + n$ and $K > J - n - \beta$, there are at most $2^{(i+k-k_0)n}$ dyadic cubes contained in $G_{i,k}$ and

$$|Q|^{-q(1/p-1/q')}T_Qw(Q) \le \sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{m\ge-\log_2\ell(P')}\sum_{R\in\mathscr{Q}_{P',m}}T_Rw(R),$$

$$
\begin{aligned}
|Q|^{-q(1/p-1/q')}E_2 &\leq C\left\{\sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{m\geq-\log_2\ell(P')}\sum_{Q\in\mathscr{Q}_{P',m}}T_Q w(Q)\right\}\\
&\quad\times\left[\sum_{j=k_0}^{\infty}\sum_{k<j}\sum_{i=1}^{\infty}2^{-K(j-k)}2^{-(i-k_0)J}2^{-k(J-n)}2^{\beta(k-k_0+i)}2^{-kn}2^{(i+k-k_0)n}\right]\\
&\leq C\sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{m\geq-\log_2\ell(P')}\sum_{Q\in\mathscr{Q}_{P',m}}\inf_{u\in Q}|\widetilde{\varphi}_m*f(u)|^q w(Q).
\end{aligned}
$$

According to the estimates of E_1 and E_2, we complete the proof. □

For $q=\infty$ and $q\leq 1$ we have following results.

Theorem 3.2 (Plancherel-Pôlya inequality for $q=\infty$)**.** *Let $\varphi,\phi\in\mathscr{S}$ satisfy* (1.1)*. For $\alpha\in\mathbb{R}$, $0<p\leq 1$, $q=\infty$ and a weight w with doubling exponent β, if $f\in\mathscr{S}'/\mathscr{P}$ satisfies*

$$
\sup_{P\in\mathscr{Q}(w)}|P|^{1-1/p}\sup_{\substack{Q\in\mathscr{Q}_{P,j}\\ j\geq-\log_2\ell(P)}}2^{j\alpha}\sup_{u\in Q}|\widetilde{\varphi}_j*f(u)|<\infty,
$$

then

$$
\begin{aligned}
&\sup_{P\in\mathscr{Q}(w)}|P|^{1-1/p}\sup_{\substack{Q\in\mathscr{Q}_{P,j}\\ j\geq-\log_2\ell(P)}}2^{j\alpha}\sup_{u\in Q}|\widetilde{\varphi}_j*f(u)|\\
\approx&\sup_{P\in\mathscr{Q}(w)}|P|^{1-1/p}\sup_{\substack{Q\in\mathscr{Q}_{P,j}\\ j\geq-\log_2\ell(P)}}2^{j\alpha}\inf_{u\in Q}|\widetilde{\phi}_j*f(u)|.
\end{aligned}
$$

Proof. Without loss of generality, we may assume $\alpha=0$. By a similar argument as the proof of Theorem 1.3, we have

$$
\sup_{u\in Q'}|\widetilde{\phi}_j*f(u)|\leq C\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathscr{Q}_k}2^{-K|j-k|}|Q|\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}T_Q,
$$

where $J>n$, $K>J-n$ and

$$
T_Q:=\inf_{u\in Q}|\widetilde{\varphi}_k*f(u)|.
$$

Given a dyadic cube P with $\ell(P)=2^{-k_0}$, the above estimates yield

$$
\begin{aligned}
&\sup_{\substack{Q'\in\mathscr{Q}_{P,j}\\ j\geq-\log_2\ell(P)}}\sup_{u\in Q'}|\widetilde{\phi}_j*f(u)|\\
&\leq C\sup_{\substack{Q'\in\mathscr{Q}_{P,j}\\ j\geq-\log_2\ell(P)}}\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathscr{Q}_k}2^{-K|j-k|}|Q|\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}T_Q\\
&:=CE_1+CE_2,
\end{aligned}
$$

where

$$E_1 = \sup_{\substack{Q' \in \mathscr{Q}_{P,j} \\ j \geq -\log_2 \ell(P)}} \sum_{k \geq j} \sum_{Q \in \mathscr{Q}_k} 2^{-K|j-k|} |Q| \frac{2^{-(j \wedge k)(J-n)}}{(2^{-(j \wedge k)} + |x_{Q'} - x_Q|)^J} T_Q$$

and

$$E_2 = \sup_{\substack{Q' \in \mathscr{Q}_{P,j} \\ j \geq -\log_2 \ell(P)}} \sum_{k < j} \sum_{Q \in \mathscr{Q}_k} 2^{-K|j-k|} |Q| \frac{2^{-(j \wedge k)(J-n)}}{(2^{-(j \wedge k)} + |x_{Q'} - x_Q|)^J} T_Q.$$

Then E_1 can be further decomposed as

$$\begin{aligned} E_1 = & \sup_{\substack{Q' \in \mathscr{Q}_{P,j} \\ j \geq -\log_2 \ell(P)}} \sum_{k \geq j} \sum_{Q \in \mathscr{Q}_{3P,k}} 2^{-K|j-k|} |Q| \frac{2^{-(j \wedge k)(J-n)}}{(2^{-(j \wedge k)} + |x_{Q'} - x_Q|)^J} T_Q \\ & + \sup_{\substack{Q' \in \mathscr{Q}_{P,j} \\ j \geq -\log_2 \ell(P)}} \sum_{k \geq j} \sum_{\substack{Q \cap 3P = \emptyset \\ Q \in \mathscr{Q}_k}} 2^{-K|j-k|} |Q| \frac{2^{-(j \wedge k)(J-n)}}{(2^{-(j \wedge k)} + |x_{Q'} - x_Q|)^J} T_Q \\ := & E_{11} + E_{12}. \end{aligned}$$

There are 3^n dyadic cubes in $3P$ with the same side length as P, so if $P' \subseteq 3P$ then $|P'| = |P|$. Thus

$$\begin{aligned} |P|^{1-1/p} E_{11} \leq & C|P|^{1-1/p} \sup_{\substack{Q' \in \mathscr{Q}_{P,j} \\ j \geq -\log_2 \ell(P)}} \sum_{k \geq j} \sum_{\substack{P' \in \mathscr{Q}_{3P} \\ \ell(P') = \ell(P)}} 2^{-K|j-k|} \\ & \times \left(\sum_{Q \in \mathscr{Q}_{P',k}} |Q| \frac{2^{-j(J-n)}}{(2^{-j} + |x_{Q'} - x_P|)^J} \sup_{\substack{Q \in \mathscr{Q}_{P',k} \\ k \geq -\log_2 \ell(P')}} T_Q \right) \\ \leq & C|P|^{1-1/p} \sum_{k \geq j} 2^{-K(k-j)} \sup_{\substack{Q \in \mathscr{Q}_{P',k} \\ k \geq -\log_2 \ell(P')}} T_Q \\ \leq & C \sup_{P' \in \mathscr{Q}(w)} |P'|^{1-1/p} \sup_{\substack{Q \in \mathscr{Q}_{P',k} \\ k \geq -\log_2 \ell(P')}} \inf_{u \in Q} |\widetilde{\varphi}_k * f(u)|, \end{aligned}$$

since $\sum_{Q \in \mathscr{Q}_k} |Q| \frac{2^{-j(J-n)}}{(2^{-j} + |x_{Q'} - x_P|)^J}$ is independent of $k \in \mathbb{N}$.

Next, we decompose the set of dyadic cubes $\{Q : Q \cap 3P = \emptyset, \ell(Q) = \ell(P)\}$ into $\{B_i\}_{i \in \mathbb{N}}$ as (3.2). Then we obtain

$$\begin{aligned} & |P|^{1-1/p} E_{12} \\ & \leq C \sum_{i=1}^{\infty} \sum_{P' \in B_i} |P'|^{1-1/p} \sup_{\substack{Q' \in \mathscr{Q}_{P,j} \\ j \geq -\log_2 \ell(P)}} \sum_{k \geq j} \sum_{\substack{Q \in \mathscr{Q}_{P',k} \\ P' \in B_i}} 2^{-K|j-k|} |Q| \frac{2^{-j(J-n)}}{(2^{-j} + |x_{Q'} - x_P|)^J} T_Q. \end{aligned}$$

There are at most $2^{(i+2)n}$ dyadic cubes in B_i with length $\ell(P)$. Since $|x_{P'}-x_P|\approx 2^{i-k_0}$ for $P'\in B_i$, the right-hand side of the inequality is dominated by

$$C\sum_{i=1}^{\infty}\sum_{P'\in B_i}\sum_{k\geq j}2^{-K(k-j)}2^{-kn}2^{-(i-k_0)J}2^{-j(J-n)}\times\left(|P'|^{1-1/p}\sup_{\substack{Q\in\mathscr{Q}_{P',k}\\ k\geq -\log_2\ell(P')}}T_Q\right).$$

Applying this with $K+n>J$, $J>n$, and $|Q|=|Q'|\frac{|Q|}{|Q'|}$ to the last inequality, it follows that

$$\begin{aligned}
&|P|^{1-1/p}E_{12}\\
&\leq C\left\{\sup_{P'\in\mathscr{Q}(w)}|P'|^{1-1/p}\sup_{\substack{Q\in\mathscr{Q}_{P',k}\\ k\geq -\log_2\ell(P')}}T_Q\left[\sum_{i=1}^{\infty}\sum_{k\geq j}2^{-K(k-j)}2^{-(i-k_0)J}2^{-j(J-n)}2^{-jn}2^{-(k-j)n}2^{in}\right]\right\}\\
&\leq C\sup_{P'\in\mathscr{Q}(w)}|P'|^{1-1/p}\sup_{\substack{Q\in\mathscr{Q}_{P',k}\\ k\geq -\log_2\ell(P')}}\inf_{u\in Q}|\widetilde{\varphi}_k*f(u)|.
\end{aligned}$$

To estimate E_2, for $i\in\mathbb{N}$ and $k<j$, set $G_{i,k}$ as (3.3). Then $|x_Q-x_P|\approx 2^{i-k_0}$ for $Q\in G_{i,k}$ and

$$E_2=\sup_{\substack{Q'\in\mathscr{Q}_{P,j}\\ j\geq -\log_2\ell(P)}}\sum_{k<j}\sum_{i=1}^{\infty}\sum_{Q\in G_{i,k}}\frac{2^{-K|j-k|}}{|Q|^{1-1/p}}\frac{2^{-k(J-n)}}{(2^{-k}+|x_{Q'}-x_Q|)^J}|Q||Q|^{1-1/p}T_Q.$$

Since $J>n$ and $K+n>J$, there are at most $2^{(i+k-k_0)n}$ dyadic cubes contained in $G_{i,k}$ and

$$|Q|^{1-1/p}T_Q\leq\sup_{P'\in\mathscr{Q}(w)}|P'|^{1-1/p}\sup_{\substack{R\in\mathscr{Q}_{P',m}\\ m\geq -\log_2\ell(P')}}T_R,$$

$$\begin{aligned}
&|Q|^{1-1/p}E_2\\
&\leq C\left\{\sup_{P'\in\mathscr{Q}(w)}|P'|^{1-1/p}\sup_{\substack{Q\in\mathscr{Q}_{P',m}\\ m\geq -\log_2\ell(P')}}T_Q\left[\sum_{k<j}\sum_{i=1}^{\infty}2^{-K(j-k)}2^{-(i-k_0)J}2^{-k(J-n)}2^{-kn}2^{(i+k-k_0)n}\right]\right\}\\
&\leq C\sup_{P'\in\mathscr{Q}(w)}|P'|^{1-1/p}\sup_{\substack{Q\in\mathscr{Q}_{P',m}\\ m\geq -\log_2\ell(P')}}\inf_{u\in Q}|\widetilde{\varphi}_m*f(u)|.
\end{aligned}$$

Combining the estimates of E_1 and E_2, we prove the theorem. □

Theorem 3.3 (Plancherel-Pôlya inequality for $q\leq 1$)**.** *Let $\varphi,\phi\in\mathscr{S}$ satisfy* (1.1)*. For $\alpha\in\mathbb{R}$, $0<p\leq 1$, $q\leq 1$ and a weight w with doubling exponent β, if $f\in\mathscr{S}'/\mathscr{P}$ satisfies*

$$\sup_{P\in\mathscr{Q}(w)}\left\{|P|^{-q(1/p-1/q')}\sum_{k=-\log_2\ell(P)}^{\infty}\sum_{Q\in\mathscr{Q}_{P,k}}\left(2^{k\alpha}\sup_{u\in Q}|\widetilde{\varphi}_k*f(u)|\right)^q w(Q)\right\}^{1/q}<\infty,$$

then

$$\sup_{P\in\mathscr{Q}(w)}\left\{|P|^{-q(1/p-1/q')}\sum_{k=-\log_2\ell(P)}^{\infty}\sum_{Q\in\mathscr{Q}_{P,k}}\left(2^{k\alpha}\sup_{u\in Q}|\widetilde{\phi}_k*f(u)|\right)^q w(Q)\right\}^{1/q}$$
$$\approx\sup_{P\in\mathscr{Q}(w)}\left\{|P|^{-q(1/p-1/q')}\sum_{k=-\log_2\ell(P)}^{\infty}\sum_{Q\in\mathscr{Q}_{P,k}}\left(2^{k\alpha}\inf_{u\in Q}|\widetilde{\varphi}_k*f(u)|\right)^q w(Q)\right\}^{1/q}.$$

Proof. Without loss of generality, we may assume $\alpha=0$. By a similar argument as the proofs of Theorems 1.3 and 3.2 we have

$$\sup_{u\in Q'}|\widetilde{\phi}_j*f(u)|\le C\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathscr{Q}_k}\left(2^{-K|j-k|}|Q|\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)(J-n)}+|x_{Q'}-x_Q|)^J}\right)^q T_Q,$$

where $J>\beta/q+n/q$, $K>(\beta/q-n)\vee(J-n/q-\beta/q)$ and T_Q is as set out in (3.1).

Given a dyadic cube P with $\ell(P)=2^{-k_0}$, the above estimates yield

$$\begin{aligned}&\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\left(\sup_{u\in Q'}|\widetilde{\phi}_j*f(u)|\right)^q w(Q')\\&\le C\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k\in\mathbb{Z}}\sum_{Q\in\mathscr{Q}_k}\left(2^{-K|j-k|}\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|Q|\right)^q\frac{w(Q')}{w(Q)}T_Qw(Q)\\&:=CE_1+CE_2,\end{aligned}$$

where

$$E_1=\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k\ge j}\sum_{Q\in\mathscr{Q}_k}\left(2^{-K|j-k|}\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|Q|\right)^q\frac{w(Q')}{w(Q)}T_Qw(Q)$$

and

$$E_2=\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k<j}\sum_{Q\in\mathscr{Q}_k}\left(2^{-K|j-k|}\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|Q|\right)^q\frac{w(Q')}{w(Q)}T_Qw(Q).$$

Then E_1 can be further decomposed as

$$\begin{aligned}E_1=&\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k\ge j}\sum_{Q\in\mathscr{Q}_{3P,k}}\left(2^{-K|j-k|}\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|Q|\right)^q\frac{w(Q')}{w(Q)}T_Qw(Q)\\&+\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k\ge j}\sum_{\substack{Q\cap 3P=\emptyset\\Q\in\mathscr{Q}_k}}\left(2^{-K|j-k|}\frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)}+|x_{Q'}-x_Q|)^J}|Q|\right)^q\frac{w(Q')}{w(Q)}T_Qw(Q)\\&:=E_{11}+E_{12}.\end{aligned}$$

There are 3^n dyadic cubes in $3P$ with the same side length as P, so if $P' \subseteq 3P$ then $|P'| = |P|$ and

$$\sum_{Q\in\mathscr{Q}_{3P,k}} T_Q w(Q) \le 3^n \sup_{\substack{P'\subseteq 3P\\ \ell(P')=\ell(P)}} \sum_{Q\in\mathscr{Q}_{P',k}} T_Q w(Q).$$

Let $J > \beta/q$, $k \ge j$ and w be a weight with doubling exponent β. By Lemma 3.1 we have

$$\int_{\mathbb{R}^n} \left(1 + \frac{|x - x_Q|}{2^{-j}}\right)^{-J} w(x)\,dx \le c_\beta 2^{(k-j)\beta} w(Q),$$

and so

$$\begin{aligned}
&\sum_{j=k_0}^{\infty} \sum_{Q'\in\mathscr{Q}_{P,j}} \left(2^{-K|j-k|} \frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)} + |x_{Q'} - x_Q|)^J} |Q|\right)^q w(Q') \\
&\le C \sum_{j=k_0}^{\infty} 2^{-Kq(k-j)} 2^{jnq} 2^{-knq} \int_{\mathbb{R}^n} \left(1 + \frac{|x - x_Q|}{2^{-j}}\right)^{-Jq} w(x)\,dx \\
&\le C w(Q).
\end{aligned}$$

Hence

$$\begin{aligned}
&|P|^{-q(1/p-1/q')} E_{11} \\
&\le C|P|^{-q(1/p-1/q')} \sum_{j=k_0}^{\infty} \sum_{Q'\in\mathscr{Q}_{P,j}} \sum_{k\ge j} \sum_{Q\in\mathscr{Q}_{3P,k}} \left(2^{-K|j-k|} \frac{2^{-(j\wedge k)(J-n)}}{(2^{-(j\wedge k)} + |x_{Q'} - x_P|)^J} |Q|\right)^q \\
&\qquad \times \frac{w(Q')}{w(Q)} T_Q w(Q) \\
&\le C \sup_{P'\in\mathscr{Q}(w)} |P'|^{-q(1/p-1/q')} \sum_{k=-\log_2 \ell(P')}^{\infty} \sum_{Q\in\mathscr{Q}_{P',k}} \inf_{u\in Q} |\widetilde{\varphi}_k * f(u)|^q w(Q).
\end{aligned}$$

Next, we decompose the set of dyadic cubes $\{Q : Q \cap 3P = \emptyset, \ell(Q) = \ell(P)\}$ into $\{B_i\}_{i\in\mathbb{N}}$ as (3.2). Then we obtain

$$\begin{aligned}
&|P|^{-q(1/p-1/q')} E_{12} \\
&\le C \sum_{i=1}^{\infty} \sum_{P'\in B_i} |P'|^{-q(1/p-1/q')} \\
&\quad \times \sum_{j=k_0}^{\infty} \sum_{Q'\in\mathscr{Q}_{P,j}} \sum_{k\ge j} \sum_{\substack{Q\in\mathscr{Q}_{P',k}\\ P'\in B_i}} \left(2^{-K|j-k|} \frac{2^{-j(J-n)}}{(2^{-j} + |x_{Q'} - x_P|)^J} |Q|\right)^q \frac{w(Q')}{w(Q)} T_Q w(Q).
\end{aligned}$$

Since $w(Q')/w(Q) \le C 2^{\beta(k-k_0+i)}$ and $\|x_{P'} - x_P\| \approx 2^{i-k_0}$ for $P' \in B_i$, the right-hand side of the inequality is dominated by

$$C\sum_{i=1}^{\infty}\sum_{P'\in B_i}\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k\geq j}\sum_{\substack{Q\in\mathscr{Q}_{P',k}\\ P'\in B_i}} 2^{\beta(k-k_0+i)}2^{-Kq(k-j)}2^{-q(i-k_0)J}2^{-j(J-n)q}|Q|^q$$
$$\times\left(\sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{k\geq k_0}\sum_{Q\in\mathscr{Q}_{P',k}}T_Qw(Q)\right).$$

Because there are at most $2^{(i+2)n}$ dyadic cubes in B_i, $J>\beta/q+n$, $K+n>\beta q$ and $|Q|^q=|Q'|^q\frac{|Q|^q}{|Q'|^q}$,

$$\begin{aligned}
&|P|^{-q(1/p-1/q')}E_{12}\\
&\leq C\left\{\sup_{P'\in\mathscr{Q}}|P'|^{-q(1/p-1/q')}\sum_{k\geq k_0}\sum_{Q\in\mathscr{Q}_{P',k}}T_Qw(Q)\right\}\\
&\quad\times\left[\sum_{i=1}^{\infty}\sum_{j=k_0}^{\infty}\sum_{k\geq j}2^{\beta(k-k_0+i)}2^{-Kq(k-j)}2^{-q(i-k_0)J}2^{-q(J-n)j}2^{-k_0nq}2^{(j-k)nq}2^{in}\right]\\
&\leq C\sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{k=-\log_2\ell(P')}^{\infty}\sum_{Q\in\mathscr{Q}_{P',k}}\inf_{u\in Q}|\widetilde{\varphi}_k*f(u)|^qw(Q).
\end{aligned}$$

To estimate E_2, for $i\in\mathbb{N}$ and $k<k_0$, set $G_{i,k}$ as (3.3). Then $|x_Q-x_P|\approx 2^{i-k_0}$ for $Q\in G_{i,k}$ and

$$E_2=\sum_{j=k_0}^{\infty}\sum_{Q'\in\mathscr{Q}_{P,j}}\sum_{k<j}\sum_{i=1}^{\infty}\sum_{Q\in G_{i,k}}\frac{2^{-K|j-k|q}}{|Q|^{-q(1/p-1/q')}}\left(\frac{2^{-k(J-n)}}{(2^{-k}+|x_{Q'}-x_Q|)^J}\right)^q\frac{w(Q')}{w(Q)}|Q|^q$$
$$\times|Q|^{-q(1/p-1/q')}T_Qw(Q).$$

Since $J>\beta/q+n/q$ and $K>J-n/q-\beta/q$, there are at most $2^{(i+k-k_0)n}$ dyadic cubes contained in $G_{i,k}$ and

$$|Q|^{-q(1/p-1/q')}T_Qw(Q)\leq\sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{m\geq-\log_2\ell(P')}\sum_{R\in\mathscr{Q}_{P',m}}T_Rw(R),$$

$$\begin{aligned}
&|Q|^{-q(1/p-1/q')}E_2\\
&\leq C\left\{\sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{m\geq-\log_2\ell(P')}\sum_{Q\in\mathscr{Q}_{P',m}}T_Qw(Q)\right\}\\
&\quad\times\left[\sum_{j=k_0}^{\infty}\sum_{k<j}\sum_{i=1}^{\infty}2^{-Kq(j-k)}2^{-q(i-k_0)J}2^{-kq(J-n)}2^{\beta(k-k_0+i)}2^{-knq}2^{(i+k-k_0)n}\right]\\
&\leq C\sup_{P'\in\mathscr{Q}(w)}|P'|^{-q(1/p-1/q')}\sum_{m\geq-\log_2\ell(P')}\sum_{Q\in\mathscr{Q}_{P',m}}\inf_{u\in Q}|\widetilde{\varphi}_m*f(u)|^qw(Q).
\end{aligned}$$

By those estimates above, we have the desired result. □

Remark 3.4. The classical Plancherel-Pôlya inequality [14] concludes that if $\{x_k\}$ is an appropriate set of points in $\mathbb{R}^n$, e.g., lattice points, where the length of the mesh is sufficiently small, then

$$\left(\sum_{k=1}^{\infty}|f(x_k)|^p\right)^{1/p} \approx \|f\|_p$$

for all $0 < p \le \infty$, with a modification if $p = \infty$.

4. Norm equivalence

In this section, we study the norm equivalence between $\dot{C}MO^{\alpha,q}_{p,w}$ and $\dot{c}^{\alpha,q}_{p,w}$. Suppose that $w \in A_\infty$ and let $r_0 = \inf\{r : w \in A_r\}$. For $\alpha \in \mathbb{R}$ and $0 < p, q \le \infty$, let $H = \max\{n, nr_0/p, n/q\}$. We say that a matrix $A = \{a_{QP}\}_{Q,P}$ is (α, p, q)-almost diagonal, denoted by $A \in \mathrm{ad}^{\alpha,q}_p(w)$ if there exists $\varepsilon > 0$ such that

$$\sup_{Q,P}\frac{|a_{QP}|}{\omega_{QP(\varepsilon)}} < \infty, \tag{4.1}$$

where

$$\begin{aligned}\omega_{QP}(\varepsilon) = &\left(\frac{\ell(Q)}{\ell(P)}\right)^{\alpha}\left(1+\frac{|x_Q-x_P|}{\max(\ell(P),\ell(Q))}\right)^{-H-\varepsilon}\\ &\times\min\left\{\left(\frac{\ell(Q)}{\ell(P)}\right)^{(n+\varepsilon)/2},\left(\frac{\ell(P)}{\ell(Q)}\right)^{(n+\varepsilon)/2+H-n}\right\}.\end{aligned}\tag{4.2}$$

Lemma 4.1. *Let $\alpha \in \mathbb{R}$, $0 < p < \infty$, $0 < q \le \infty$ and $w \in A_\infty$. If A is (α, p, q)-almost diagonal, then A is bounded on $\dot{f}^{\alpha,q}_{p,w}$.*

Proof. We may assume $\alpha = 0$, since the case implies the general case, and set $r = \min(p/r_0, q)$. We shall consider the case $r > 1$ first. Let A be an $(0, p, q)$ almost diagonal operator on $\dot{f}^{0,q}_{p,w}$ with matrix $\{a_{Q,P}\}_{Q,P}$. We decompose the matrix operator A as the sum of $A = A_u + A_l$, namely for $Q \in \mathscr{Q}(w)$

$$(A_u\boldsymbol{s})_Q = \sum_{\substack{P\in\mathscr{Q}(w)\\ \ell(P)\ge\ell(Q)}} a_{QP}s_P \quad\text{and}\quad (A_l\boldsymbol{s})_Q = \sum_{\substack{P\in\mathscr{Q}(w)\\ \ell(P)<\ell(Q)}} a_{QP}s_P$$

for $\boldsymbol{s} = \{s_P\} \in \dot{f}^{0,q}_{p,w}$. According to Lemma A.2 in [6], with $\lambda = H + \varepsilon$ and $a = r = 1$,

$$\begin{aligned}|(A_u\boldsymbol{s})_Q| &\le C\sum_{\substack{P\in\mathscr{Q}(w)\\ \ell(P)\ge\ell(Q)}}\left(1+\frac{|x_Q-x_P|}{\ell(P)}\right)^{-H-\varepsilon}\left(\frac{\ell(Q)}{\ell(P)}\right)^{(n+\varepsilon)/2}|s_P|\\ &\le C\sum_{j\le k}2^{(j-k)(n+\varepsilon)/2}M\left(\sum_{P\in\mathscr{Q}_j(w)}|s_P|\chi_P\right)(x)\quad\text{for } x\in Q,\end{aligned}$$

when $\ell(Q) = 2^{-k}$.

Hence, since $|Q|^{-1/2} = 2^{(k-j)n/2}|P|^{-1/2}$ if $\ell(P) = 2^{-j}$,

$$\begin{aligned}\|A_u\boldsymbol{s}\|_{\dot{f}^{0,q}_{p,w}} &\le C\left\|\left(\sum_{k\in\mathbb{Z}}\left(\sum_{j\le k}2^{(j-k)(n+\varepsilon)/2}M\left(\sum_{P\in\mathscr{Q}_j(w)}|P|^{-1/2}|s_P|\chi_P\right)(x)\right)^q\right)^{1/q}\right\|_{L^p(w)}\\ &\le C\left\|\left(\sum_{j\in\mathbb{Z}}\left(M\left(\sum_{P\in\mathscr{Q}_j(w)}|P|^{-1/2}|s_P|\chi_P\right)(x)\right)^q\right)^{1/q}\right\|_{L^p(w)}\end{aligned}$$

by Minkowski's inequality. Applying the weighted version of Fefferman-Stein vector-valued maximal inequality which was characterized by Andersen and John [1], we find that

$$\|A_u\boldsymbol{s}\|_{\dot{f}^{0,q}_{p,w}} \le C\|\boldsymbol{s}\|_{\dot{f}^{0,q}_{p,w}}$$

since $q > 1$, $p > r_0$ and $w \in A_p$.

Next, consider the case for the matrix operator A_l. Observe that if $\ell(P) \le \ell(Q)$, then

$$\frac{\ell(P)}{\ell(Q)}\left(1+\frac{|x_Q-x_P|}{\ell(Q)}\right) \le 1+\frac{|x_Q-x_P|}{\ell(Q)}.$$

Thus

$$\begin{aligned}\left(1+\frac{|x_Q-x_P|}{\ell(Q)}\right)^{-H-\varepsilon} &\le \left(1+\frac{|x_Q-x_P|}{\ell(Q)}\right)^{-H-\varepsilon/4}\\ &\le \left(1+\frac{|x_Q-x_P|}{\ell(Q)}\right)^{-H-\varepsilon/4}\left(\frac{\ell(P)}{\ell(Q)}\right)^{-H-\varepsilon/4},\end{aligned}$$

and, for $Q \in \mathscr{Q}_k(w)$,

$$\begin{aligned}|(A_l\boldsymbol{s})_Q| &\le C\sum_{\substack{P\in\mathscr{Q}(w)\\ \ell(P)<\ell(Q)}}\left(1+\frac{|x_Q-x_P|}{\ell(Q)}\right)^{-H-\varepsilon}\left(\frac{\ell(P)}{\ell(Q)}\right)^{(n+\varepsilon)/2+H-n}|s_P|\\ &\le C\sum_{\substack{P\in\mathscr{Q}(w)\\ \ell(P)<\ell(Q)}}\left(1+\frac{|x_Q-x_P|}{\ell(Q)}\right)^{-H-\varepsilon/4}\left(\frac{\ell(P)}{\ell(Q)}\right)^{\varepsilon/4-n/2}|s_P|\\ &\le C\sum_{j>k}2^{(k-j)(-n/2+\varepsilon/4)}M\left(\sum_{P\in\mathscr{Q}_j(w)}|s_P|\chi_P\right)(x)\quad\text{for } x\in Q.\end{aligned}$$

Hence, since $|Q|^{-1/2} = 2^{(k-j)n/2}|P|^{-1/2}$ if $\ell(P) = 2^{-j}$,

$$\|A_l \boldsymbol{s}\| \le C \left\| \left(\sum_{k\in\mathbb{Z}} \left(\sum_{j>k} 2^{(k-j)\varepsilon/4} M \left(\sum_{P\in\mathscr{Q}_j(w)} |P|^{-1/2}|s_P|\chi_P \right)(x) \right)^q \right)^{1/q} \right\|_{L^p(w)}$$
$$\le C \left\| \left(\sum_{j\in\mathbb{Z}} \left(M \left(\sum_{P\in\mathscr{Q}_j(w)} |P|^{-1/2}|s_P|\chi_P \right)(x) \right)^q \right)^{1/q} \right\|_{L^p(w)}$$

by Minkowski's inequality. Applying Proposition 2.4, we find that

$$\|A_l \boldsymbol{s}\|_{\dot f^{0,q}_{p,w}} \le C\|\boldsymbol{s}\|_{\dot f^{0,q}_{p,w}}$$

since $p, q > 1$, $p > r_0$ and $w \in A_p$.

The case $r \le 1$ and $q < \infty$ is in fact a consequence of the case $r > 1$. We pick an $\widetilde{r} < r$ so close to r that (4.1) is still satisfied with $r = \min(p/r_0, q)$ replaced by $\widetilde{r}$. This means that $p/\widetilde{r} > 1$ and $q/\widetilde{r} > 1$, and that the matrix $\widetilde{A} = \{\widetilde{a}_{QP}\} := \{|a_{QP}|^{\widetilde{r}}(|Q|/|P|)^{1/2-\widetilde{r}/2}\}$ satisfies (4.2) for a different value of ε. Define $\boldsymbol{t} = \{t_Q\}_Q$ by $t_Q = |Q|^{1/2-\widetilde{r}/2}|s_Q|^{\widetilde{r}}$. Then $\|\boldsymbol{t}\|^{1/\widetilde{r}}_{\dot f^{0,q/\widetilde{r}}_{p/\widetilde{r},w}} = \|\boldsymbol{s}\|_{\dot f^{0,q}_{p,w}}$. By the $\widetilde{r}$-triangle inequality, we have

$$|(A\boldsymbol{s})_Q| \le \left(\sum_{P\in\mathscr{Q}(w)} |a_{QP}|^{\widetilde{r}}|s_P|^{\widetilde{r}} \right)^{1/\widetilde{r}}.$$

Hence, $\|A\boldsymbol{s}\|_{\dot f^{0,q}_{p,w}} \le \|\widetilde{A}\boldsymbol{t}\|^{1/\widetilde{r}}_{\dot f^{0,q/\widetilde{r}}_{p/\widetilde{r},w}}$. Therefore the boundedness of A on $\dot f^{0,q}_{p,w}$ follows from the boundedness of $\widetilde{A}$ on $\dot f^{0,q/\widetilde{r}}_{p/\widetilde{r},w}$. By duality, the case $q = \infty$ and $p > 1$ follows from the result of $q = 1$ which we have just obtained. Finally, for $p \le 1$ and $q = \infty$, we reduce to the case $p > 1$ as before. □

Next let us consider the boundedness of almost diagonal operators acting on weighted Carleson measure sequence spaces $\dot c^{\alpha,q}_{p,w}$. In particular, consider the boundedness of almost diagonal operators acting on $\dot f^{\alpha,q}_{\infty,w}$. Under this situation, we always assume $H = \max\{n, nr_0/p\}$ in the definition of almost diagonality.

Lemma 4.2. *For $\alpha \in \mathbb{R}$, $0 < p \le 1$, $0 < q \le \infty$ and $w \in A_\infty$, an $(\alpha + nq(1/p - 1/q'), p, q)$-almost diagonal matrix is bounded on $\dot c^{\alpha,q}_{p,w}$.*

Proof. We may assume $\alpha = 0$, since the case implies the general case. Let $A = \{a_{QP}\}_{Q,P}$ be an $(nq(1/p - 1/q'), p, q)$-almost diagonal matrix. Write $A = A_u + A_l$ with

$$(A_u\boldsymbol{s})_Q = \sum_{\substack{P\in\mathscr{Q}(w)\\ \ell(P)\ge\ell(Q)}} a_{QP}s_P \quad \text{and} \quad (A_l\boldsymbol{s})_Q = \sum_{\substack{P\in\mathscr{Q}(w)\\ \ell(P)<\ell(Q)}} a_{QP}s_P$$

for $\boldsymbol{s} = \{s_P\} \in \dot{c}^{0,q}_{p,w}$. Set $r = nq(1/p - 1/q')$. If $\ell(Q) = 2^{-k}$, then

$$\begin{aligned}
|(A_u\boldsymbol{s})_Q| &\le C \sum_{\substack{P\in\mathcal{Q}(w)\\ \ell(P)\ge\ell(Q)}} \left(\frac{\ell(Q)}{\ell(P)}\right)^r \left(1 + \frac{|x_Q - x_P|}{\max(\ell(P),\ell(Q))}\right)^{-H-\varepsilon} \left(\frac{\ell(Q)}{\ell(P)}\right)^{(n+\varepsilon)/2} |s_P| \\
&\le C \sum_{j=-\infty}^{k} 2^{(j-k)[r+(n+\varepsilon)/2]} M\left(\sum_{P\in\mathcal{Q}_j(w)} |s_P|\chi_P\right)(x) \quad \text{for } x \in Q.
\end{aligned}$$

Hence, since $|Q|^{-1/2} = 2^{(k-j)n/2}|P|^{-1/2}$ if $\ell(P) = 2^{-j}$,

$$|Q|^{-1/2}|(A_u\boldsymbol{s})_Q|\chi_Q(x) \le C \sum_{j=-\infty}^{k} 2^{(j-k)(r+\varepsilon/2)} M\left(\sum_{P\in\mathcal{Q}_j(w)} |P|^{-1/2}|s_P|\chi_P\right)(x)\chi_Q(x)$$

and, by Hölder's inequality,

$$\begin{aligned}
\|A_u\boldsymbol{s}\|_{\dot{c}^{0,q}_{p,w}} &\le C \sup_{R\in\mathcal{Q}(w)} \Bigg\{|R|^{-r/n} \int_R \sum_{k\ge -\log_2 \ell(R)} \sum_{Q\in\mathcal{Q}_{R,k}(w)} \sum_{j=-\infty}^{k} 2^{(j-k)(r+\varepsilon/2)} \\
&\qquad \times \left[M\Bigg(\sum_{P\in\mathcal{Q}_j(w)} |P|^{-1/2}|s_P|\chi_P\Bigg)(x)\right]^q \chi_Q(x)w(x)\,dx\Bigg\}^{1/q} \\
&\le C \sup_{R\in\mathcal{Q}(w)} \Bigg\{|R|^{-r/n} \int_{\mathbb{R}^n} \sum_{k\ge -\log_2 \ell(R)} \sum_{j=-\infty}^{k} 2^{(j-k)(r+\varepsilon/2)} \\
&\qquad \times \left[M\Bigg(\sum_{P\in\mathcal{Q}_j(w)} |P|^{-1/2}|s_P|\chi_P\Bigg)(x)\right]^q w(x)\,dx\Bigg\}^{1/q}.
\end{aligned}$$

Note that for given R with $\ell(R) = 2^{-\delta}$,

$$\begin{aligned}
&\sum_{k\ge -\log_2 \ell(R)} \sum_{j=-\infty}^{k} 2^{(j-k)(r+\varepsilon/2)} \left[M\left(\sum_{P\in\mathcal{Q}_j(w)} |P|^{-1/2}|s_P|\chi_P\right)(x)\right]^q \\
&= \sum_{j=-\infty}^{\delta-1} \sum_{k=\delta}^{\infty} 2^{(j-\delta)(r+\varepsilon/2)} 2^{(\delta-k)(r+\varepsilon/2)} \left[M\left(\sum_{P\in\mathcal{Q}_j(w)} |P|^{-1/2}|s_P|\chi_P\right)(x)\right]^q \\
&\quad + \sum_{j=\delta}^{\infty} \sum_{k=j}^{\infty} 2^{(j-k)(r+\varepsilon/2)} \left[M\left(\sum_{P\in\mathcal{Q}_j(w)} |P|^{-1/2}|s_P|\chi_P\right)(x)\right]^q \\
&\le C \sum_{j=-\infty}^{\delta-1} 2^{(j-\delta)(r+\varepsilon/2)} \left[M\left(\sum_{P\in\mathcal{Q}_j(w)} |P|^{-1/2}|s_P|\chi_P\right)(x)\right]^q \\
&\quad + C \sum_{j=\delta}^{\infty} \left[M\left(\sum_{P\in\mathcal{Q}_j(w)} |P|^{-1/2}|s_P|\chi_P\right)(x)\right]^q.
\end{aligned}$$

Applying Proposition 2.4, if P_0 is the cube containing R with $\ell(P_0) = 2^{-j}$, then we get

$$\begin{aligned}
&|R|^{-r/n} \int_R \sum_{k \geq -\log_2 \ell(R)} \sum_{j=-\infty}^{k} 2^{(j-k)(r+\varepsilon/2)} \left[M \left(\sum_{P \in \mathscr{Q}_j(w)} |P|^{-1/2} |s_P| \chi_P \right)(x) \right]^q w(x)\, dx \\
&\leq C \sum_{j=-\infty}^{\delta-1} 2^{(j-\delta)(r+\varepsilon/2)} \left(\frac{|R|}{|P_0|} \right)^{-r/n} |P_0|^{-r/n} \int_{P_0} \left(|P_0|^{-1/2} |s_{P_0}| \chi_{P_0}(x) \right)^q w(x)\, dx \\
&\quad + C|R|^{-r/n} \int_R \sum_{j \geq -\log_2 \ell(R)} \left(\sum_{P \in \mathscr{Q}_j(w)} |P|^{-1/2} |s_P| \chi_P(x) \right)^q w(x)\, dx \\
&\leq C \sum_{j=-\infty}^{\delta-1} 2^{(j-\delta)(r+\varepsilon/2)} 2^{(j-\delta)n(-r/n)} |P_0|^{-r/n} \int_{P_0} \left(|P_0|^{-1/2} |s_{P_0}| \chi_{P_0}(x) \right)^q w(x)\, dx \\
&\quad + C|R|^{-r/n} \int_R \sum_{j \geq -\log_2 \ell(R)} \left(\sum_{P \in \mathscr{Q}_j(w)} |P|^{-1/2} |s_P| \chi_P(x) \right)^q w(x)\, dx,
\end{aligned}$$

because $|R|/|P_0| = 2^{(j-\delta)n}$. Hence

$$\|A_u \boldsymbol{s}\|_{\dot{c}^{0,q}_{p,w}} \leq C \|\boldsymbol{s}\|_{\dot{c}^{0,q}_{p,w}}.$$

A similar argument for A_l yields $\|A_l \boldsymbol{s}\|_{\dot{c}^{0,q}_{p,w}} \leq C \|\boldsymbol{s}\|_{\dot{c}^{0,q}_{p,w}}$. □

Finally, we can give a proof for Theorem 1.7.

Proof of Theorem 1.7. For $0 < q < \infty$ and $f \in \dot{C}MO^{\alpha,q}_{p,w}$, let $\boldsymbol{s} := \{s_Q\}_Q = S_\varphi(f)$. Then the φ-transform identity shows $f = \sum_Q s_Q \psi_Q$ and $\|f\|_{\dot{C}MO^{\alpha,q}_{p,w}} = \|S_\varphi(f)\|_{\dot{c}^{\alpha,q}_{p,w}} = \|\boldsymbol{s}\|_{\dot{c}^{\alpha,q}_{p,w}}$. In particular, $\|f\|_{\dot{C}MO^{\alpha,q}_{1,w}} = \|S_\varphi(f)\|_{\dot{c}^{\alpha,q}_{1,w}} = \|S_\varphi(f)\|_{\dot{f}^{\alpha,q}_{\infty,w}} \approx \|f\|_{\dot{F}^{\alpha,q}_{\infty,w}}$. Furthermore, for $\boldsymbol{s} \in \dot{c}^{\alpha,q}_{p,w}$

$$\|T_\psi(\boldsymbol{s})\|_{\dot{C}MO^{\alpha,q}_{p,w}} = \left\| \sum_P s_P \psi_P \right\|_{\dot{C}MO^{\alpha,q}_{p,w}} = \left\| \left\{ \left\langle \sum_P s_P \psi_P, \varphi_Q \right\rangle \right\}_Q \right\|_{\dot{c}^{\alpha,q}_{p,w}} = \|A\boldsymbol{s}\|_{\dot{c}^{\alpha,q}_{p,w}},$$

where $A := \{\langle \psi_P, \varphi_Q \rangle\}_{Q,P}$ is $(\alpha + nq(1/p - 1/q'), p, q)$-almost diagonal (cf. Lemma 3.6 in [6]) and hence A is bounded on $\dot{c}^{\alpha,q}_{p,w}$ by Lemma 4.2. Therefore, S_φ is bounded from $\dot{C}MO^{\alpha,q}_{p,w}$ to $\dot{c}^{\alpha,q}_{p,w}$ and T_φ is bounded from $\dot{c}^{\alpha,q}_{p,w}$ to $\dot{C}MO^{\alpha,q}_{p,w}$. We summarize that $T_\psi \circ S_\varphi|_{\dot{C}MO^{\alpha,q}_{p,w}}$ is also the identity on $\dot{C}MO^{\alpha,q}_{p,w}$. □

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Hsun-Wu Liu and Kunchuan Wang
Department of Applied Mathematics, National Dong Hwa University, Hualien, Taiwan
E-mail address: takeshi.liu@livemail.tw, kcwang@gms.ndhu.edu.tw