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On Partial Galois Algebras

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Abstract. We generalize, in the context of partial group action, the Kanzaki commutator theorem for Galois extensions and the structure theorem for Galois algebras given by Szeto and Xue.

1. Introduction

The notion of Galois extension for commutative ring extensions was firstly introduced by Auslander and Goldman [2], and Chase, Harrison and Rosenberg continued the study, giving characterizations and the fundamental theorem for a commutative Galois algebra in the celebrated paper [5]. DeMeyer and Kanzaki respectively generalized the notion of commutative Galois extension to the case of noncommutative ring (see [6, 7, 15, 17]). Since then more investigation have been done by several authors (see [1, 8, 11, 13, 14, 16, 21, 24]). Particularly, Kanzaki showed the following important commutator theorem (see [16, Proposition 1]): if R is a Galois extension of R^G with Galois group G and C is the center of R, then the commutator subring of R^G in R is a direct sum of certain C-submodules of R, namely, $J_g := \{x \in R \mid xr = g(r)x \text{ for all } r \in R\}$, where $g \in G$. By investigating further these J_g and applying the Kanzaki commutator theorem, Szeto and Xue derived a structure theorem for Galois algebras (see [25, Theorem 3.8]), which we will describe in the next section.

The notion of partial Galois extension was recently introduced by Dokuchaev, Ferrero and Paques in [10], where the authors developed the partial Galois theory of rings, generalizing the results on Galois theory of commutative rings given in [5]. More properties were obtained for partial Galois extensions in [4], using the theory of Galois corings, and for partial Galois Azumaya extensions in [12, 22, 23], generalizing the results in [1]. In the series of papers [18–20], the authors, among other things, characterized (partial) Galois extensions generated by central idempotents in a partial Galois extension, which we will apply later in Section 4.

The purpose of the present paper is to generalize, in the context of partial group action, the Kanzaki commutator theorem for Galois extensions and the structure theorem

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for Galois algebras given by Szeto and Xue. Let R be a ring with a partial action α of a finite group G and C denote its center. For each $g \in G$, let $I_g = \{x \in R \mid xr = \alpha_g(r1_{g^{-1}})x$ for all $r \in R\}$. In Section 3, we generalize the Kanzaki commutator theorem for Galois extensions by showing that if R is an α-partial Galois extension of R^α, then the commutator subring of R^α in R is a direct sum of these C-submodules I_g of R (see Theorem 3.3). In Section 4, we firstly derive several properties of these I_g, associated to which a Boolean semigroup is then introduced. By investigating certain minimal elements of this Boolean semigroup and applying the generalized commutator theorem for partial Galois extensions, we extend the structure theorem for Galois algebras given by Szeto and Xue to a structure theorem for partial Galois algebras (see Theorem 4.11). It is worth mentioning that this structure theorem can be applied straightforwardly without going through the process of the globalization of R as defined in [9]. In the next section, we recall some notions and results which will be used later.

2. Preliminary

Let R be a ring with 1. Suppose that G is a finite automorphism group of R and let $R^G = \{r \in R \mid g(r) = r$ for each $g \in G\}$. If there exist elements a_i, b_i in R, $i = 1, 2, \ldots, m$ for some integer m, such that $\sum_{i=1}^m a_i g(b_i) = \delta_{1,g} 1_R$ for each $g \in G$, then R is called a Galois extension of R^G with Galois group G, and the set $\{a_i, b_i \mid i = 1, 2, \ldots, m\}$ is called a G-Galois system for R. Furthermore, let C denote the center of R; if $R^G \subseteq C$, then the Galois extension R is called a Galois algebra with Galois group G or simply a G-Galois algebra, and a central G-Galois algebra when $R^G = C$.

For any ring Y and any non-empty subset X of Y, let $C_Y(X)$ denote the centralizer (or commutator subring) of X in Y. Below we recall the Kanzaki commutator theorem for Galois extensions.

Theorem 2.1. [16, Proposition 1] *Suppose R is a Galois extension of R^G with Galois group G. Let $J_g = \{x \in R \mid xr = g(r)x$ for all $r \in R\}$ for each $g \in G$. Then*

$$C_R(R^G) = \bigoplus_{g \in G} J_g \quad \text{as } C\text{-modules.}$$

More properties of J_g, $g \in G$, were derived in [25], followed by a structure theorem for Galois algebras as stated below.

Theorem 2.2. [25, Theorem 3.8] *Suppose R is a Galois algebra with Galois group G. Then there exist orthogonal central idempotents $e_1, e_2, \ldots, e_m$ and subgroups $H_1, H_2, \ldots, H_m$ of G such that each Re_i is a central Galois algebra with Galois group H_i and $R = \bigoplus_{i=1}^m Re_i$ or $R = \bigoplus_{i=0}^m Re_i$, where $e_0 = 1 - \sum_{i=1}^m e_i$ and $Re_0 = Ce_0$ is a commutative Galois algebra with Galois group G.*

The main purpose of this paper is to generalize these two results in the context of partial group action. To do this, we recall the notions of partial group action and partial Galois extension and some derived properties we shall use later.

Let R be a ring with a partial action α of a finite group G. This means, as defined in [9], there exist a collection $\{R_g \mid g \in G\}$ of ideals of R and isomorphisms of (non-necessarily unital) rings $\alpha_g \colon R_{g^{-1}} \to R_g$ such that

(i) $R_1 = R$ and α_1 is the identity automorphism of R;

(ii) $\alpha_g(R_{g^{-1}} \cap R_h) = R_g \cap R_{gh}$ for all $g, h \in G$;

(iii) $(\alpha_g \circ \alpha_h)(r) = \alpha_{gh}(r)$ for every $r \in R_{h^{-1}} \cap R_{(gh)^{-1}}$ and $g, h \in G$.

In this paper, we assume that for each $g \in G$, R_g has an identity 1_g which is a central idempotent of R. Under this assumption, α has a globalization (see [9, Theorem 4.5]). This means that there exist a ring T and a (global) action β of G on T by automorphisms of T such that R can be considered as an ideal of T generated by a central idempotent 1_R of T and the following conditions hold:

(i) $T = \sum_{g\in G} \beta_g(R)$;

(ii) $R_g = R \cap \beta_g(R)$ for every $g \in G$;

(iii) $\alpha_g = \beta_g|_{R_{g^{-1}}}$ for every $g \in G$.

We have the following properties (see [10, p. 79]):

(F1) $1_g = 1_R\beta_g(1_R)$ for every $g \in G$;

(F2) $\alpha_g(r1_{g-1}) = \beta_g(r)1_R$ for every $r \in R$ and $g \in G$; in particular,

(F3) $\alpha_g(1_h1_{g-1}) = 1_{gh}1_g$ for all $g, h \in G$.

As defined in [10], the subring of the invariant elements of R under α is defined to be $R^\alpha = \{r \in R \mid \alpha_g(r1_{g^{-1}}) = r1_g \text{ for all } g \in G\}$, and R is called an α-partial Galois extension of R^α if there exist elements x_i, y_i in R, $i = 1, 2, \ldots, m$ for some integer m, such that $\sum_{i=1}^m x_i\alpha_g(y_i1_{g^{-1}}) = \delta_{1,g}1_R$ for each $g \in G$; in this case, $\{x_i, y_i \mid i = 1, 2, \ldots, m\}$ is called an α-partial Galois system for R. Here, if $R^\alpha \subseteq C$, then we call the α-partial Galois extension R an α-partial Galois algebra, and a central α-partial Galois algebra when $R^\alpha = C$.

For a ring R with a partial action α of a finite group G and its globalization T with an action β of G, we list some known results we shall apply later without special mention.

(i) (see [3, Theorem 1.4]) $R^\alpha = T^\beta 1_R$.

(ii) (see [22, Lemma 2.1]) $C_R(R^\alpha) = C_T(T^\beta)1_R$ and $C_R(R) = C_T(T)1_R$.

(iii) (see [10, Theorem 3.3]) R is an α-partial Galois extension of R^α if and only if T is a Galois extension of T^β with Galois group G.

(iv) (see [10, Theorem 4.2]) If R is an α-partial Galois algebra, then it is separable over R^α.

3. The generalized Kanzaki commutator theorem

Throughout the rest of this paper, let R denote a ring with a partial action α of a finite group G and T its globalization with an action β of G; let C and Z denote the center of R and that of T, respectively. For each $g \in G$, let

$$J_g = \{u \in T \mid ut = \beta_g(t)u \text{ for all } t \in T\}$$

and

$$I_g = \{x \in R \mid xr = \alpha_g(r1_{g^{-1}})x \text{ for all } r \in R\}.$$

Then each J_g (I_g resp.) is a submodule of T (R resp.) over Z (C resp.). In this section, we shall generalize the Kanzaki commutator theorem for Galois extensions to a commutator theorem for partial Galois extensions, and present a relation between $\operatorname{rank}_C(I_g)$ and $\operatorname{rank}_Z(J_g)$ when R is an α-partial Galois algebra.

Lemma 3.1. *If R is an α-partial Galois extension of R^α, then $C_R(R^\alpha) = \bigoplus_{g\in G} J_g 1_R$ as C-modules.*

Proof. Since R is a α-partial Galois extension of R^α, T is a Galois extension of T^β with Galois group G. Hence $C_T(T^\beta) = \bigoplus_{g\in G} J_g$ as Z-modules by Theorem 2.1. Since $C_R(R^\alpha) = C_T(T^\beta)1_R$ and $C = Z1_R$, it follows that $C_R(R^\alpha) = \bigoplus_{g\in G} J_g 1_R$ as C-modules. □

Lemma 3.2. *For each $g \in G$, $J_g 1_R \subseteq I_g \subseteq C_R(R^\alpha) \cap R1_g$.*

Proof. Fix a $g \in G$. Let $u \in J_g$. We have $u1_R \in R$ since R is an ideal of T. For each $r \in R$, $(u1_R)r = (ur)1_R = (\beta_g(r))u1_R = (\beta_g(r)1_R)(u1_R) = \alpha_g(r1_{g^{-1}})(u1_R)$ (see (F2) in Section 2). Hence $u1_R \in I_g$. Next, let $x \in I_g$. Then $x = x1_R = \alpha_g(1_R 1_{g^{-1}})x \in R1_g$; furthermore, if $r \in R^\alpha$, then we have that $xr = \alpha_g(r1_{g^{-1}})x = r1_g x = rx1_g = rx$. Hence $x \in C_R(R^\alpha) \cap R1_g$. □

We are ready to show the generalized Kanzaki commutator theorem for partial Galois extensions.

Theorem 3.3. *If R is a α-partial Galois extension of R^α, then*

$$C_R(R^\alpha) = \bigoplus_{g\in G} I_g \quad \text{as } C\text{-modules.}$$

Proof. By Lemmas 3.1 and 3.2, $C_R(R^\alpha) = \bigoplus_{g\in G} J_g 1_R \subseteq \sum_{g\in G} I_g \subseteq C_R(R^\alpha)$. Therefore, it remains to show that for each g, $I_g \cap \sum_{h\neq g} I_h = (0)$. Fix a g and let $x \in I_g \cap \sum_{h\neq g} I_h$. Then $x \in I_g$ and $x = \sum_{h\neq g} x_h$ for some $x_h \in I_h$ with $h \neq g$. Thus for each $r \in R$, we have

$$xr = \alpha_g(r1_{g^{-1}})x = \beta_g(r)1_R x = \sum_{h\neq g} \beta_g(r)1_R x_h,$$

and on the other hand,

$$xr = \left(\sum_{h\neq g} x_h\right) r = \sum_{h\neq g} \alpha_h(r1_{h^{-1}})x_h = \sum_{h\neq g} \beta_h(r)1_R x_h.$$

Hence for all $r \in R$,

$$0 = \sum_{h\neq g}(\beta_g(r) - \beta_h(r))1_R x_h = \sum_{h\neq g} \beta_g(r - \beta_g^{-1}\beta_h(r))x_h 1_R.$$

Now let $\{a_i, b_i \in R \mid i = 1, 2, \ldots, m\}$ be an α-partial Galois system for R; that is, for each $g \in G$, $\sum_{i=1}^m a_i\alpha_g(b_i 1_{g^{-1}}) = \delta_{1,g}1_R$ or $\sum_{i=1}^m a_i\beta_g(b_i)1_R = \delta_{1,\beta_g}1_R$. By what we just derived above, $\sum_{h\neq g} \beta_g(b_i - \beta_g^{-1}\beta_h(b_i))x_h 1_R = 0$ for each $i = 1, 2, \ldots, m$. Therefore,

$$\begin{aligned}
0 &= \sum_{i=1}^m \beta_g(a_i)\left(\sum_{h\neq g} \beta_g(b_i - \beta_g^{-1}\beta_h(b_i))x_h 1_R\right) \\
&= \sum_{i=1}^m \sum_{h\neq g} \beta_g\left(a_i(b_i - \beta_g^{-1}\beta_h(b_i))\right) x_h 1_R \\
&= \sum_{h\neq g} \beta_g\left(\sum_{i=1}^m a_i\left(b_i - \beta_g^{-1}\beta_h(b_i)\right)\right) x_h 1_R \\
&= \sum_{h\neq g} \beta_g(1_R - 0)x_h 1_R = \beta_g(1_R)1_R\left(\sum_{h\neq g} x_h\right) = 1_g x = x,
\end{aligned}$$

where the second last equality holds by (F1) in Section 2 and the last equality holds because $x \in I_g$ and $I_g \subseteq R1_g$ by Lemma 3.2. We conclude that $C_R(R^\alpha) = \bigoplus_{g\in G} I_g$. □

Corollary 3.4. *If R is an α-partial Galois extension of R^α, then $I_g = J_g 1_R$ for each $g \in G$.*

Proof. By Lemma 3.1 and Theorem 3.3, $C_R(R^\alpha) = \bigoplus_{g\in G} J_g 1_R = \bigoplus_{g\in G} I_g$. But for each $g \in G$, $J_g 1_R \subseteq I_g$ by Lemma 3.2, so $J_g 1_R = I_g$. □

Corollary 3.5. *If R is an α-partial Galois algebra, then $R = \bigoplus_{g\in G} I_g$ as C-modules.*

We next show that R is an α-partial Galois algebra if and only if T is a β-Galois algebra. To do this, we recall some central idempotents of T as defined in [10]. Denote G by $\{\beta_1 = 1, \beta_2, \ldots, \beta_n\}$. Let $e_1 = 1_R$ and $e_i = (1_T - 1_R)(1_T - \beta_2(1_R)) \cdots (1_T - \beta_{i-1}(1_R))\beta_i(1_R)$ for $i = 2, \ldots, n$. Then $e_1, e_2, \ldots, e_n$ are orthogonal central idempotents of T such that $\sum_{i=1}^n e_i = 1_T$. Also in [10], the authors introduced a (left and right) T^β-linear and multiplicative map $\psi\colon T \to T$ defined by $\psi(y) = \sum_{i=1}^n \beta_i(y)e_i$ for each $y \in T$. It was shown in [3, Theorem 1.4] that ψ restricted to R^α is a ring isomorphism from R^α onto T^β with the inverse map given by sending y to $y1_R$ for each $y \in T^\beta$.

Lemma 3.6. *Let e_i, $i = 1, 2, \ldots, n$, be as above. Then $Z = \bigoplus_{i=1}^n \beta_i(C)e_i$ and*

$$\bigcup_{P\subseteq C} \bigcup_{i=1}^{n} \left\{ \beta_i(P)e_i \oplus \sum_{j\neq i} \beta_j(C)e_j \right\},$$

where P runs over all prime ideals of C, is the set of all prime ideals of Z.

Proof. Recall that $C = Z1_R \subseteq Z$. Hence C is an ideal of Z and so is each $\beta_i(C)$. Thus $\bigoplus_i \beta_i(C)e_i$ is an ideal of Z containing $\sum_{i=1}^n \beta_i(1_R)e_i = \sum_{i=1}^n e_i = 1_T$, so $Z = \bigoplus_i \beta_i(C)e_i$. Therefore, for each prime ideal P of C and for each $i = 1, 2, \ldots, n$, $\beta_i(P)e_i \oplus \sum_{j\neq i} \beta_j(C)e_j$ is a prime ideal of Z, and conversely every prime ideal of Z is of this form. □

Lemma 3.7. *$R^\alpha \subseteq C$ if and only if $T^\beta \subseteq Z$.*

Proof. Suppose that $R^\alpha \subseteq C$. Recall the map ψ as defined above. Then

$$T^\beta = \psi(R^\alpha) \subseteq \psi(C) = \left\{ \sum_{i=1}^{n} \beta_i(c)e_i \mid c \in C \right\} \subseteq \sum_{i=1}^{n} \beta_i(C)e_i,$$

which is exactly Z by the preceding lemma. The other direction is easy: if $T^\beta \subseteq Z$, then $R^\alpha = T^\beta 1_R \subseteq Z1_R = C$. □

Proposition 3.8. *R is an α-partial Galois algebra if and only if T is a β-Galois algebra.*

Proof. Recall that R is an α-partial Galois extension of R^α if and only if T is a Galois extension of T^β with Galois group G. Hence the result follows immediately from Lemma 3.7. □

Corollary 3.9. *If R is an α-partial Galois algebra, then I_g and J_g are finitely generated projective modules over C and Z respectively for each $g \in G$.*

Proof. By the preceding proposition, T is a β-Galois algebra. In particular, T is an Azumaya Z-algebra. Hence T is a finitely generated projective Z-module. Furthermore, $T = \bigoplus_{g\in G} J_g$ as Z-modules by the Kanzaki commutator theorem (see Theorem 2.1). Thus each J_g is a finitely generated projective Z-module. Since $I_g = J_g 1_R$ for each $g \in G$ by Corollary 3.4 and $C = Z1_R$, it follows that each I_g is a finitely generated projective C-module. □

Corollary 3.10. *If R is an α-partial Galois algebra, then R is a finitely generated projective C-module.*

Proof. Suppose that R is an α-partial Galois algebra. Then by Corollary 3.5, $R = \bigoplus_{g\in G} I_g$ as C-modules and by Corollary 3.9, each I_g is a finitely generated projective C-module. Hence the result follows. □

Remark 3.11. We can derive the previous two results in reverse order and without going through the process of globalization. Indeed, since R is an α-partial Galois algebra, it follows that R is separable over R^α, which is contained in C. Hence R is an Azumaya C-algebra. In particular, R is a finitely generated projective C-module, and hence so is each I_g by the generalized Kanzaki commutator theorem (see Theorem 3.3) or Corollary 3.5.

We end this section by showing that when R is an α-partial Galois algebra, the ranks $\mathrm{rank}_C(I_g)$ and $\mathrm{rank}_Z(J_g)$, $g \in G$, satisfy the following property.

Theorem 3.12. *Let R be an α-partial Galois algebra. Then*

(i) *For each $g \in G$, if $\mathrm{rank}_Z(J_g) = 1$, then $\mathrm{rank}_C(I_g) = 1$.*

(ii) *If $\mathrm{rank}_C(I_g) = 1$ for all $g \in G$, then $\mathrm{rank}_Z(J_g) = 1$ for all $g \in G$.*

Proof. Fix a $g \in G$ and suppose that $\mathrm{rank}_Z(J_g) = 1$. Then $(J_g)_Q \cong Z_Q$ for each prime ideal Q of Z. Take any prime ideal P of C. By Lemma 3.6, $Q = \beta_1(P)e_1 \oplus \sum_{j\neq 1} \beta_j(C)e_j$ is a prime ideal of Z. Since $Q1_R = P$, $Z1_R = C$ and furthermore $I_g = J_g 1_R$ by Corollary 3.4, it follows that $(I_g)_P = (J_g 1_R)_{Q1_R} \cong (Z1_R)_{(Q1_R)} = C_P$. Hence $\mathrm{rank}_C(I_g) = 1$.

Now suppose that $\mathrm{rank}_C(I_g) = 1$ for each $g \in G$. Then $(I_g)_P \cong C_P$ for each $g \in G$ and for any prime ideal P of C, and in particular $J_g \neq (0)$ since $J_g 1_R = I_g$ for each $g \in G$. We firstly show that $(J_g)_Q \neq (0)$ for each $g \in G$ and for each prime ideal Q of Z. Fix a $g \in G$ and a prime ideal Q of Z. We have $J_g = \sum_{j=1}^n J_g e_j$, and by Lemma 3.6, $Q = \beta_k(P)e_k \bigoplus_{t\neq k} \beta_t(C)e_t$ for some $1 \leq k \leq n$ and some prime ideal P of C. Noticing that since $e_k \notin Q$ and $e_j e_k = 0$ for all $j \neq k$, we have $(J_g e_j)_Q = (0)$ for all $j \neq k$. Furthermore, since $e_k = \beta_k(1_R)e_k$, we have $J_g e_k = J_g \beta_k(1_R)e_k = \beta_k(\beta_k^{-1}(J_g)1_R)e_k$. But $\beta_k^{-1}(J_g) = J_{g'}$ for some $g' \in G$ by the proof of Lemma 3.2(1) in [25], so $J_g e_k = \beta_k(J_{g'}1_R)e_k = \beta_k(I_{g'})e_k$. Thus the localization of $J_g e_k$ at Q; that is, the localization of $\beta_k(I_{g'})e_k$ at $\beta_k(P)e_k \bigoplus_{t\neq k} \beta_t(C)e_t$, is isomorphic to $(I_{g'})_P \cong C_P \neq (0)$. Therefore, $(J_g)_Q = \sum_{j=1}^n (J_g e_j)_Q = (J_g e_k)_Q \neq (0)$. This shows that $\text{rank}_{Z_Q}((J_g)_Q) \geq 1$ for each $g \in G$ and for each prime ideal Q of Z. Because T is a β-Galois algebra by Proposition 3.8, $T = \bigoplus_{g\in G} J_g$, $\text{rank}_{T^\beta}(T) = n$ and $\text{rank}_Z(T)$ is defined since T is an Azumaya Z-algebra. Thus we have that

$$n = \text{rank}_{T^\beta}(T) \geq \text{rank}_Z(T) = \text{rank}_{Z_Q}(T_Q) = \sum_{g\in G} \text{rank}_{Z_Q}((J_g)_Q) \geq n$$

for each prime ideal Q of Z. Consequently, $\text{rank}_{Z_Q}((J_g)_Q) = 1$ for each $g \in G$ and for each prime ideal Q of Z. Therefore $\text{rank}_Z(J_g) = 1$ for each $g \in G$. □

Corollary 3.13. *Let R be an α-partial Galois algebra. Then R is a central α-partial Galois algebra if* $\text{rank}_C(I_g) = 1$ *for each* $g \in G$.

Proof. Since R is an α-partial Galois algebra, T is a β-Galois algebra by Proposition 3.8, and thus $T = \bigoplus_{g\in G} J_g$, $\text{rank}_{T^\beta}(T) = n$ and $\text{rank}_Z(T)$ is defined. Suppose that $\text{rank}_C(I_g) = 1$ for each $g \in G$. Then $\text{rank}_Z(J_g) = 1$ for each $g \in G$ by the preceding theorem. Hence $\text{rank}_Z(T) = \sum_{g\in G} \text{rank}_Z(J_g) = n = \text{rank}_{T^\beta}(T)$. It follows that $Z = T^\beta$, and hence $C = Z1_R = T^\beta 1_R = R^\alpha$, as desired. □

4. A structure theorem

In this section, we shall generalize the structure theorem for Galois algebras as described in Theorem 2.2 to a structure theorem for partial Galois algebras. As before, for each $g \in G$, let $I_g = \{x \in R \mid xr = \alpha_g(r1_{g^{-1}})x \text{ for all } r \in R\}$. We firstly derive several properties of these I_g, generalizing those of J_g obtained in [16, 25]. In particular, each I_g determines uniquely a central idempotent of R, from which a Boolean semigroup is then defined. By investigating certain minimal elements of this Boolean semigroup and applying the generalized commutator theorem for partial Galois extensions, we shall obtain a structure theorem for partial Galois algebras without going through the process of globalization.

Lemma 4.1. *Let R be a ring with a partial action α of a group G. Then for any $g, h \in G$,*

(i) $I_g I_h \subseteq I_{gh}$;

(ii) $\alpha_g(I_h 1_{g^{-1}}) = I_{ghg^{-1}} 1_g$;

(iii) $RI_g = I_g R$ *is a two sided ideal of* R.

Proof. Suppose $x \in I_g$, $y \in I_h$ and $r \in R$. First,

$$\alpha_{gh}(r1_{(gh)^{-1}})xy = \alpha_{gh}(r1_{h^{-1}g^{-1}})1_g xy \qquad \text{(since } I_g \subseteq R1_g \text{ by Lemma 3.2)}$$

$$= \alpha_{gh}(r1_{h^{-1}g^{-1}}1_{h^{-1}})xy \qquad \text{(we have applied (F3) in Section 2)}$$
$$= \alpha_g(\alpha_h(r1_{h^{-1}})1_{g^{-1}})xy$$
$$= x\alpha_h(r1_{h^{-1}})y = xyr,$$

so $xy \in I_{gh}$. Hence $I_gI_h \subseteq I_{gh}$. Next,

$$\begin{aligned}
&\alpha_{ghg^{-1}}(r1_{gh^{-1}g^{-1}})\alpha_g(y1_{g^{-1}}) = \alpha_{ghg^{-1}}(r1_{gh^{-1}g^{-1}}1_{gh^{-1}})\alpha_g(y1_{g^{-1}})\\
&= \alpha_g(\alpha_{hg^{-1}}(r1_{gh^{-1}g^{-1}}1_{gh^{-1}}))\alpha_g(y1_{g^{-1}}) = \alpha_g(\alpha_{hg^{-1}}(r1_{gh^{-1}})1_{g^{-1}})\alpha_g(y1_{g^{-1}})\\
&= \alpha_g(\alpha_{hg^{-1}}(r1_{gh^{-1}})y1_h1_{g^{-1}}) \qquad (\text{since } y \in I_h \subseteq R1_h)\\
&= \alpha_g(\alpha_{hg^{-1}}(r1_{gh^{-1}}1_g)y1_{g^{-1}}) = \alpha_g(\alpha_h(\alpha_{g^{-1}}(r1_g)1_{h^{-1}})y1_{g^{-1}})\\
&= \alpha_g(y\alpha_{g^{-1}}(r1_g)1_{g^{-1}}) = \alpha_g(y1_{g^{-1}})\alpha_g(\alpha_{g^{-1}}(r1_g)) = \alpha_g(y1_{g^{-1}})r,
\end{aligned}$$

showing that $\alpha_g(y1_{g^{-1}}) \in I_{ghg^{-1}}$. Hence $\alpha_g(I_h1_{g^{-1}}) \subseteq I_{ghg^{-1}}1_g$, from which we then also have $\alpha_{g^{-1}}(I_{ghg^{-1}}1_g) \subseteq I_h1_{g^{-1}}$; hence $I_{ghg^{-1}}1_g \subseteq \alpha_g(I_h1_{g^{-1}})$. Finally, by definition of I_g, $xr = \alpha_g(r1_{g^{-1}})x \in RI_g$ and $rx = r1_gx = x\alpha_{g^{-1}}(r1_g) \in I_gR$. □

Proposition 4.2. *Let R be an α-partial Galois algebra. Then for any $g, h \in G$,*

(i) $I_gI_h = I_gI_{g^{-1}}I_{gh} = I_hI_{h^{-1}}I_{gh}$;

(ii) $I_gI_{g^{-1}} = I_{g^{-1}}I_g$;

(iii) $I_gI_{g^{-1}}$ *is an ideal of C generated by an idempotent element μ_g of C.*

Proof. The proof is similar to that of [16, Proposition 2]. Firstly, since R is an α-partial Galois algebra, R is separable over R^α, which is contained in C. Hence R is an Azumaya C-algebra. Thus for each $g \in G$, $I_gR = c_gR$, where $c_g = I_gR \cap C$. Furthermore, by Corollary 3.5 that $R = \bigoplus_{h\in G} I_h$, we have $I_gR = \sum_{h\in G} I_gI_h$ and $c_gR = \bigoplus_{h\in G} c_gI_{gh}$. Since $I_gI_h \subseteq I_{gh}$ by Lemma 4.1(i), it follows that $c_gI_{gh} = I_gI_h$ for any $g, h \in G$, and hence $c_g = I_gI_{g^{-1}}$ by taking $h = g^{-1}$. Similarly, since for each $h \in G$, $c_hR = \bigoplus_{g\in G} c_hI_{gh}$ and $I_hR = RI_h = \sum_{g\in G} I_gI_h$, it follows that $c_hI_{gh} = I_gI_h$ for any $g, h \in G$, and hence by taking $h = g^{-1}$, we obtain $c_{g^{-1}} = I_gI_{g^{-1}}$. At this point, we have derived parts (i) and (ii). In particular, for any $g \in G$, $c_gI_g = I_gC$, so $c_g^2 = c_gI_gI_{g^{-1}} = I_gI_{g^{-1}}C = c_g$. Since I_g is a finitely generated module over C, c_g is a finitely generated idempotent ideal of C, and it follows that c_g is generated by an idempotent element of C. This completes the proof of part (iii). □

Corollary 4.3. *Let R be an α-partial Galois algebra. Then for any $g, h \in G$,*

(i) $R(I_gI_h) = R(I_hI_g)$;

(ii) $RI_g^2 = RI_g$.

Proof. Following the notations in the proof of the preceding proposition, we have $R(I_gI_h) = (RI_g)(RI_h) = (Rc_g)(Rc_h) = (Rc_h)(Rc_g) = (RI_h)(RI_g) = R(I_hI_g)$ and $RI_g^2 = (RI_g)(RI_g) = (Rc_g)(Rc_g) = Rc_g^2 = Rc_g = RI_g$. □

From now on, let R be an α-partial Galois algebra. By Proposition 4.2(iii), for each $g \in G$, there exists (uniquely surely) some idempotent element μ_g of C such that $I_gI_{g^{-1}} = C\mu_g$. Let $(\mathcal{B}, \cdot)$ denote the Boolean semigroup, deleting the zero element of R if exists, generated by those nonzero μ_g under the multiplication of R. For each $\lambda \in (\mathcal{B}, \cdot)$, let $H_\lambda = \{g \in G \mid \lambda\mu_g = \lambda\}$, and for each subset H of G, let $\lambda_H = \prod_{h\in H} \mu_h$. It is obvious that for each element λ in $(\mathcal{B}, \cdot)$, $\lambda = \prod_{h\in H_\lambda} \mu_h = \lambda_{H_\lambda}$ and H_λ is closed under the inverse operation since $\mu_g = \mu_{g^{-1}}$ for each $g \in G$.

Lemma 4.4. *For any nonempty subset H of G, we have*

(i) $R\left(\prod_{h\in H} I_h\right) = R\left(I_{k^{-1}} \prod_{h\in H} I_h\right)$ *for any $k \in H$; in particular, $R\lambda_H = R\left(\prod_{h\in H} I_h\right)$.*

(ii) $\alpha_g(\lambda_H 1_{g^{-1}}) = \lambda_{gHg^{-1}} 1_g$ *for any $g \in G$.*

Proof. By applying Proposition 4.2 and Corollary 4.3, we have

$$\begin{aligned} R\left(\prod_{h\in H} I_h\right) &= R\left(I_gI_k \prod_{h\in H} I_h\right) \quad \text{(for any } g, k \in H) \\ &= R\left(I_kI_{k^{-1}}I_{gk} \prod_{h\in H} I_h\right) = R\left(I_kI_{k^{-1}}^2 I_{gk} \prod_{h\in H} I_h\right) = R\left(I_{k^{-1}} \prod_{h\in H} I_h\right); \end{aligned}$$

hence, $R\lambda_H = R\left(\prod_{h\in H} \mu_h\right) = R\left(\prod_{h\in H} I_{h^{-1}}I_h\right) = R\left(\prod_{h\in H} I_{h^{-1}}\right)\left(\prod_{h\in H} I_h\right)$ $= R\left(\prod_{h\in H} I_h\right)$. Now for any $g \in G$,

$$\begin{aligned} R\alpha_g(\lambda_H 1_{g^{-1}}) &= \alpha_g(R\lambda_H 1_{g^{-1}}) \\ &= \alpha_g\left(R\left(\prod_{h\in H} I_h\right) 1_{g^{-1}}\right) = R\left(\prod_{h\in H} \alpha_g(I_h 1_{g^{-1}})\right) \\ &= R\left(\prod_{h\in H} I_{ghg^{-1}} 1_g\right) \qquad \text{(by Lemma 4.1(ii))} \\ &= R\lambda_{gHg^{-1}} 1_g, \end{aligned}$$

so $\alpha_g(\lambda_H 1_{g^{-1}}) = \lambda_{gHg^{-1}} 1_g$. □

We shall derive a structure theorem for partial Galois algebras via certain minimal elements of $(\mathcal{B}, \cdot)$. To do this, we firstly show that each minimal element has the following property.

Proposition 4.5. *If λ is a minimal element of $(\mathcal{B},\cdot)$, then H_λ is a maximal subset of G such that $\prod_{h\in H_\lambda} I_h \neq \{0\}$. Conversely, if H is a maximal subset of G such that $\prod_{h\in H} I_h \neq \{0\}$, then λ_H is a minimal element of $(\mathcal{B},\cdot)$ with $H_{\lambda_H} = H$.*

Proof. Let λ be a minimal element of $(\mathcal{B},\cdot)$ and suppose that $g \in G \setminus H_\lambda$ such that $\left(\prod_{h\in H_\lambda} I_h\right) I_g \neq \{0\}$. Then by Lemma 4.4, $\{0\} \neq R\left(\prod_{h\in H_\lambda} I_h\right) I_g = R\left(\prod_{h\in H_\lambda} I_h\right) I_g I_{g^{-1}} = R\lambda_{H_\lambda}\mu_g = R\lambda\mu_g$. But this means that $\lambda\mu_g$ is an element in $(\mathcal{B},\cdot)$ which is smaller than λ, a contradiction to the minimality of λ.

Suppose that H is a maximal subset of G such that $\prod_{h\in H} I_h \neq \{0\}$. In particular, $\lambda_H \neq 0$ by Lemma 4.4(i). To show that λ_H is minimal, assume that $g \in G \setminus H$ such that $\lambda_H \neq \lambda_H\mu_g \neq 0$. Then $R\lambda_H \neq R\lambda_H\mu_g \neq \{0\}$. It follows that $R\left(\prod_{h\in H} I_h\right) \neq R\left(\prod_{h\in H} I_h\right) I_g I_{g^{-1}} \neq \{0\}$, providing a contradiction to the maximality of H. We conclude that λ_H is a minimal element of $(\mathcal{B},\cdot)$. Finally, we claim that $H_{\lambda_H} = H$. It is obvious that $H \subseteq H_{\lambda_H}$. Assume that $g \in H_{\lambda_H} \setminus H$. Then $\lambda_H\mu_g = \lambda_H$ and $\left(\prod_{h\in H} I_h\right) I_g = \{0\}$. But then $\{0\} \neq R\lambda_H = R\lambda_H\mu_g = R\left(\prod_{h\in H} I_h\right)(I_g I_{g^{-1}}) = \{0\}$, a contradiction. □

In the following proposition, we show that the maximal subset H of G in the preceding proposition is in fact a subgroup of G, and the ideal of R generated by the central idempotent λ_H is a central H-Galois algebra by applying the generalized commutator theorem for partial Galois extensions.

Proposition 4.6. *Suppose that H is a maximal subset of G such that $\prod_{h\in H} I_h \neq \{0\}$. Then H is a subgroup of G and $R\lambda_H$ is a central Galois algebra with Galois group H induced by α.*

Proof. To show that H is a subgroup of G, suppose that $g, k \in H$. Then

$$\begin{aligned} R\left(\prod_{h\in H} I_h\right) &= R\left(\prod_{h\in H} I_h\right) I_g I_k = R\left(\prod_{h\in H} I_h\right) I_g I_{g^{-1}} I_{gk} \\ &= R\left(\prod_{h\in H} I_h\right) I_g I_{g^{-1}} I_{gk} I_{gk} = R\left(\prod_{h\in H} I_h\right) I_{gk}, \end{aligned}$$

which forces that $gk \in H$ by the the maximality of H. Similarly, $R\left(\prod_{h\in H} I_h\right) = R\left(\prod_{h\in H} I_h\right) I_{g^{-1}}$ by Lemma 4.4(i), so $g^{-1} \in H$.

Since H is a subgroup of G and λ_H is a nonzero central idempotent of R, we apply [20, Theorem 4.4] to show that $R\lambda_H$ is a Galois extension with Galois group H induced by α. To do so, recall that $R^{\alpha_H} := \{r \in R \mid \alpha_g(r1_{g^{-1}}) = r1_g \text{ for each } g \in H\}$ and $N(\lambda_H) := \{g \in G \mid \lambda_H 1_g = \lambda_H\}$. We claim that $\lambda_H \in R^{\alpha_H}$ and $H \subseteq N(\lambda_H)$. For any $g \in H$, it follows from Lemma 4.4 that $\alpha_g(\lambda_H 1_{g^{-1}}) = \lambda_{gHg^{-1}} 1_g = \lambda_H 1_g$ and $R\lambda_H 1_g = R\left(\prod_{h\in H} I_h\right) 1_g = R\left(\prod_{h\in H} I_h\right) I_g 1_g = R\left(\prod_{h\in H} I_h\right) I_g = R\lambda_H$, where $I_g 1_g = I_g$

since $I_g \subseteq R1_g$ by Lemma 3.2, so $\lambda_H 1_g = \lambda_H$. We conclude that $R\lambda_H$ is a Galois extension with Galois group H induced by α (see [20, Theorem 4.4]). Next, we apply [13, Theorem 1] to show that the invariant subring of $R\lambda_H$ under H is exactly its center. Notice that since R is an α-partial Galois algebra, R is separable over R^α, and hence so is $R\lambda_H$ over $R^\alpha\lambda_H$, which is contained in both $C\lambda_H$, the center of $R\lambda_H$, and $R^{\alpha_H}\lambda_H = (R\lambda_H)^H$. By Corollary 3.5, $R = \bigoplus_{g\in G} I_g$, so $R\lambda_H = \bigoplus_{g\in G} I_g\lambda_H = \bigoplus_{h\in H} I_h\lambda_H$, where we have applied the maximality of H to derive that for each $g \in G\backslash H$, $RI_g\lambda_H = R\left(\prod_{h\in H} I_h\right) I_g = \{0\}$. On the other hand, for each $h \in H$, let $J_h^{\lambda_H} = \{s \in R\lambda_H \mid st = h(t)s \text{ for all } t \in R\lambda_H\}$. It is easy to see that $J_h^{\lambda_H} = I_h\lambda_H$. Indeed, if $x \in I_h$ and $y \in R$, then $x\lambda_H y\lambda_H = \alpha_h(y1_{h^{-1}})x\lambda_H = \alpha_h(y\lambda_H 1_{h^{-1}})x\lambda_H = h(y\lambda_H)x\lambda_H$; conversely, if $s \in J_h^{\lambda_H}$, then $s = s\lambda_H$ and for each $r \in R$, $sr = sr\lambda_H = h(r\lambda_H)s = \alpha_h(r\lambda_H 1_{h^{-1}})s = \alpha_h(r1_{h^{-1}})\lambda_H s = \alpha_h(r1_{h^{-1}})s$, so $s \in I_h$. Hence $R\lambda_H = \bigoplus_{h\in H} J_h^{\lambda_H}$ and $J_h^{\lambda_H} J_{h^{-1}}^{\lambda_H} = (I_h\lambda_H)(I_{h^{-1}}\lambda_H) = C\mu_h\lambda_H = C\lambda_H$ for each $h \in H$. We conclude from [13, Theorem 1] that $R\lambda_H$ is a central Galois algebra with Galois group H induced by α. □

Corollary 4.7. *Suppose that λ is a minimal element of $(\mathcal{B}, \cdot)$. Then H_λ is a subgroup of G and $R\lambda$ is a central Galois algebra with Galois group H_λ induced by α.*

Proof. This follows immediately from Propositions 4.5 and 4.6 and the fact that $\lambda_{H_\lambda} = \lambda$. □

We next focus on minimal elements of $(\mathcal{B}, \cdot)$ with certain property. For the following results, we define the *length* of an element λ in $(\mathcal{B}, \cdot)$ to be the cardinality of the set H_λ.

Proposition 4.8. *Let $\mathcal{M} = \{\lambda_1, \lambda_2, \ldots, \lambda_r\}$ be the set of minimal elements in $(\mathcal{B}, \cdot)$ with maximum length $\mathcal{L}$. Then $\alpha_g(\mathcal{M}1_{g^{-1}} \setminus \{0\}) = \mathcal{M}1_g \setminus \{0\}$ for each $g \in G$.*

Proof. Let $g \in G$. Suppose $\lambda \in \mathcal{M}$ such that $\lambda 1_{g^{-1}} \neq 0$. By Lemma 4.4(ii), we have $\alpha_g(\lambda 1_{g^{-1}}) = \lambda_{gH_\lambda g^{-1}} 1_g$. If $\lambda_{gH_\lambda g^{-1}}\mu_k$ for some $k \in G$ is smaller than $\lambda_{gH_\lambda g^{-1}}$ in $(\mathcal{B}, \cdot)$, then its length is greater than $|gH_\lambda g^{-1}| = \mathcal{L}$, a contradiction. Thus, $\lambda_{gH_\lambda g^{-1}}$ is a minimal element in $(\mathcal{B}, \cdot)$ with length $\mathcal{L}$; that is $\lambda_{gH_\lambda g^{-1}} \in \mathcal{M}$. We have shown that $\alpha_g(\mathcal{M}1_{g^{-1}} \setminus \{0\}) \subseteq \mathcal{M}1_g \setminus \{0\}$; the other inclusion relation is similar. □

Proposition 4.9. *Suppose that $\lambda_1, \lambda_2, \ldots, \lambda_r$ are as in the preceding proposition. Then $\sum_{i=1}^r \lambda_i$ is an α-invariant element.*

Proof. Suppose $\lambda, \lambda' \in \mathcal{M}$ are distinct such that $\lambda 1_g \neq 0 \neq \lambda' 1_g$, where $g \in G$. Then notice that $\lambda 1_g \neq \lambda' 1_g$, for otherwise, $\lambda 1_g = (\lambda 1_g)^2 = \lambda\lambda' 1_g = 0$, a contradiction. Fix a $g \in G$ and let $N_g^+ = \{i \in \{1, 2, \ldots, r\} \mid \lambda_i 1_g \neq 0\}$ and $N_g^- = \{i \in \{1, 2, \ldots, r\} \mid \lambda_i 1_{g^{-1}} \neq 0\}$. Then by the previous argument and Proposition 4.8, it follows that $\{\lambda_i 1_g \mid i \in N_g^+\} = \{\alpha_g(\lambda_i 1_{g^{-1}}) \mid i \in N_g^-\}$ and this set consists of exactly $|N_g^-| = |N_g^+|$ elements. Hence

$\alpha_g((\sum_{i=1}^r \lambda_i)1_{g^{-1}}) = \sum_{i\in N_g^-} \alpha_g(\lambda_i 1_{g^{-1}}) = \sum_{i\in N_g^+} \lambda_i 1_g = (\sum_{i=1}^r \lambda_i)1_g$; that is, $\sum_{i=1}^r \lambda_i$ is α-invariant. □

Corollary 4.10. *Let $\lambda_0 = 1 - \sum_{i=1}^r \lambda_i$. If $\lambda_0 \neq 0$, then $R\lambda_0$ is a partial Galois algebra with the partial action of G induced by α.*

Proof. By the preceding proposition, $\lambda_0 \in R^\alpha$. Since these λ_i, $i = 1, 2, \dots, r$, are minimal elements in $(\mathcal{B}, \cdot)$, they are mutually orthogonal central idempotents of R. Hence $\lambda_0 = 1 - \sum_{i=1}^r \lambda_i$ is also a central idempotent of R. If $\lambda_0 \neq 0$, then by [20, Theorem 4.4], $R\lambda_0$ is a partial Galois extension with the partial action of G induced by α. Since $(R\lambda_0)^\alpha = R^\alpha\lambda_0 \subseteq C\lambda_0 = Z(R\lambda_0)$, the center of $R\lambda_0$, we conclude that $R\lambda_0$ is an α-partial Galois algebra. □

We are now ready to state and prove the generalized structure theorem for partial Galois algebras.

Theorem 4.11. *Suppose that R is an α-partial Galois algebra. Then there are orthogonal central idempotents $e_1, e_2, \dots, e_m$ in R and subgroups $H_1, H_2, \dots, H_m$ of G such that each Re_j is a central Galois algebra with Galois group H_j and $R = \bigoplus_{j=1}^m Re_j$ or $R = \bigoplus_{j=0}^m Re_j$, where $e_0 = 1 - \sum_{j=1}^m e_j$ and $Re_0 = Ce_0$ is a commutative partial Galois algebra with the partial action of G induced by α.*

Proof. Following the previous notations and results, we have $R = \bigoplus_{i=0}^r R\lambda_i$, where each $R\lambda_i$, $1 \le i \le r$, is a central Galois algebra with Galois group H_{λ_i} and $R\lambda_0$, if $\lambda_0 \neq 0$, is an α-partial Galois algebra. Hence if $\lambda_0 = 0$, then we are done; if $\lambda_0 \neq 0$, but $I_g\lambda_0 = \{0\}$ for each $g \neq 1_G$ in G, then the α-partial Galois algebra $R\lambda_0 = \bigoplus_{g\in G} I_g\lambda_0 = I_{1_G}\lambda_0 = C\lambda_0$ is commutative, and hence we are done too. Suppose now that $I_g\lambda_0 \neq \{0\}$ for some $g \neq 1_G$ in G. By applying the preceding result to the α-partial Galois algebra $R\lambda_0$, we obtain some orthogonal central idempotents $\lambda_{01}, \lambda_{02}, \dots, \lambda_{0r_1}$ of $R\lambda_0$ and subgroups $H_{01}, H_{02}, \dots, H_{0r_1}$ of G such that each $R\lambda_0\lambda_{0i}$ is a central Galois algebra with Galois group H_{0i}, and if $\lambda_{00} = \lambda_0 - \sum_{i=1}^{r_1} \lambda_{0i} \neq 0$, then $R\lambda_0\lambda_{00}$ is an α-partial Galois algebra, which is commutative if and only if $I_g\lambda_0\lambda_{00} = \{0\}$ for each $g \neq 1_G$ in G. Hence $R = (\bigoplus_{i=1}^r R\lambda_i) \oplus (\bigoplus_{i=1}^{r_1} R\lambda_{0i}) \oplus R\lambda_0\lambda_{00}$ is a direct sum of central Galois algebras possibly with an α-partial Galois algebra. The idempotents λ_{0i}, $1 \le i \le r_1$, are in fact the minimal elements with maximum length in the Boolean semigroup generated by the nonzero central idempotents of R of the form $\mu_g\lambda_0$, $g \in G$, and $H_{0i} = \{g \in G \mid \lambda_{0i}\mu_g\lambda_0 = \lambda_{0i}\}$ for each $1 \le i \le r_1$. Thus these λ_{0i} and $\lambda_0\lambda_{00}$ are elements of the Boolean algebra generated by μ_g, $g \in G$, which is finite. Therefore if $I_g\lambda_0\lambda_{00} \neq \{0\}$ for some $g \neq 1_G$ in G, then we can continue this process but in finitely many steps, we will stop at a stage where $R = \bigoplus_{j=1}^m Re_j$ or $R = \bigoplus_{j=0}^m Re_j$ for some orthogonal central idempotents e_j such that each Re_j, $1 \le j \le m$, is a central

Galois algebra with Galois group a subgroup of G and Re_0 is a commutative α-partial Galois algebra. □

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