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Existence of Weak Solution for a Class of Abstract Coupling System Associated with Stationary Electromagnetic System

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Abstract. We consider the existence of a weak solution for a class of coupling system containing stationary electromagnetic coupling system associated with the Maxwell equations in a multi-connected domain. Mathematically we are concerned with the coupled system containing a p-curl equation and a q-Laplacian equation.

1. Introduction

We consider an electromagnetic field in equilibrium in a bounded domain Ω in $\mathbb{R}^3$ with the boundary Γ. The stationary generalized Maxwell equations is written by

$$\boldsymbol{j} = \operatorname{curl} \boldsymbol{h}, \quad \operatorname{curl} \boldsymbol{e} = \boldsymbol{f}, \quad \operatorname{div} \boldsymbol{h} = 0 \quad \text{in } \Omega,$$

where $\boldsymbol{e}$ is an electric field, $\boldsymbol{h}$ is a magnetic field, $\boldsymbol{j}$ denotes the total current density and $\boldsymbol{f}$ denotes an internal magnetic current. Though $\boldsymbol{f} = \boldsymbol{0}$ in classical Farady's law, in theoretical physics, magnetic monopoles have been postulated by formal consideration (Bossavit [9]), so for mathematical purpose, it is interesting to consider the case $\boldsymbol{f} \neq \boldsymbol{0}$. Here we consider a nonlinear extension of the classical Ohm's law in the form

$$\boldsymbol{e} = \rho \boldsymbol{h},$$

where the resistivity $\rho = \rho(\theta, \boldsymbol{h}, \operatorname{curl} \boldsymbol{h})$ depends on the temperature θ and on the magnetic field $\boldsymbol{h}$. Taking the thermal effect into consideration, we have the equilibrium of energy

$$\operatorname{div} \boldsymbol{q} = \boldsymbol{j} \cdot \boldsymbol{e}, \tag{1.1}$$

where the heat flux $\boldsymbol{q} = -k\nabla\theta$ is given by a nonlinear thermal conductivity

$$k = k(\theta)|\nabla\theta|^{q-2}, \quad q > 1.$$

The right-hand side of (1.1) denotes the Joule heating. We assume that the resistivity is of the form

$$\rho = \nu(\theta)|\operatorname{curl} \boldsymbol{h}|^{p-2}, \quad p > 1.$$

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Then the equations for $\boldsymbol{h}$ and θ is written by

$$\operatorname{curl}\left[\nu(\theta)|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h}\right]=\boldsymbol{f},\quad \operatorname{div}\boldsymbol{h}=0\quad \text{in } \Omega, \tag{1.2a}$$

$$-\operatorname{div}\left[k(\theta)|\nabla\theta|^{q-2}\nabla\theta\right]=\nu(\theta)|\operatorname{curl}\boldsymbol{h}|^{p}\quad \text{in } \Omega. \tag{1.2b}$$

We impose the boundary conditions

$$\boldsymbol{h}\times\boldsymbol{n}=0,\quad \nu(\theta)|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h}\times\boldsymbol{n}=\boldsymbol{0}\quad \text{on } \Gamma \tag{1.3}$$

where $\boldsymbol{n}$ denotes the unit outer normal vector to the boundary Γ.

For the classical solution of the system (1.2a)–(1.2b), we must impose the following compatibility conditions. By (1.2a),

$$\operatorname{div}\boldsymbol{f}=0\quad \text{in } \Omega. \tag{1.4}$$

By (1.2a) and (1.2b), since

$$\boldsymbol{f}\cdot\boldsymbol{n}=\boldsymbol{n}\cdot\operatorname{curl}\big(\nu(\theta)|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h}\big)=\operatorname{Div}\big(\nu(\theta)|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h}\times\boldsymbol{n}\big)=0\quad \text{on } \Gamma,$$

we have

$$\boldsymbol{f}\cdot\boldsymbol{n}=0\quad \text{on } \Gamma, \tag{1.5}$$

where Div denotes the surface divergence. See Mitrea et al. [17].

In Miranda et al. [15], the authors showed the existence of a solution for "weak formulation" of (1.2a)–(1.2b) with boundary condition $\boldsymbol{h}\cdot\boldsymbol{n}=0$ on Γ instead of $\boldsymbol{h}\times\boldsymbol{n}=\boldsymbol{0}$ on Γ in (1.3) in a simply connected domain Ω. Here they call $(\boldsymbol{h},\theta)\in\mathbb{W}^p(\Omega)\times W_0^{1,q}(\Omega)$, where

$$\mathbb{W}^p(\Omega)=\{\boldsymbol{v}\in\boldsymbol{W}^{1,p}(\Omega);\operatorname{div}\boldsymbol{v}=0\text{ in }\Omega,\boldsymbol{v}\cdot\boldsymbol{n}=0\text{ on }\Gamma\}$$

a weak solution if (1.2a)–(1.2b), if $(\boldsymbol{h},\theta)$ satisfies

$$\int_\Omega \nu(\theta)|\operatorname{curl}\boldsymbol{h}|^{p-2}\operatorname{curl}\boldsymbol{h}\cdot\operatorname{curl}\boldsymbol{v}\,dx=\int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx\quad \text{for all } \boldsymbol{v}\in\mathbb{W}^p(\Omega), \tag{1.6a}$$

$$\int_\Omega k(\theta)|\nabla\theta|^{q-2}\nabla\theta\cdot\nabla\xi\,dx=\int_\Omega \nu(\theta)|\operatorname{curl}\boldsymbol{h}|^{p}\xi\,dx\quad \text{for all } \xi\in W_0^{1,q}(\Omega). \tag{1.6b}$$

However, in the case where Ω is multi-connected, since the weak solution $(\boldsymbol{h},\theta)$ does not satisfy (1.2a) in the distribution sense in their weak formulation, $(\boldsymbol{h},\theta)$ is exactly not a weak solution of (1.2a)–(1.2b).

In this paper, we consider a more general system containing (1.2a)–(1.2b) under the boundary conditions (1.3) in a multi-connected domain. Our weak solution of weak formulation of (1.2a)–(1.2b) satisfies the equations in the distribution sense.

The paper is organized as follows. In Section 2, since we allow a domain Ω to be multi-connected, we set the domain appropriately. Moreover, since we consider more general equations than (1.2a)–(1.2b), we must introduce two Carathéodory functions $S(x,s,t)$ and $T(x,s,t)$ on $\Omega\times\mathbb{R}\times[0,\infty)$ and state the structure conditions. We also give a main theorem of this paper. To show the existence of a weak solution, we use the Schauder fixed point theorem. In order to do so, in Section 3, we consider associated minimization problems and consult the properties of the solutions. In Section 4, we show the continuous dependence on given data for the weak solution obtained in Section 3. In Section 5, we consider an approximate problem by truncation. Finally, in Section 6, we prove the main theorem using approximate solution in the preceding section.

2. Preliminaries and the main theorem

In this section, we shall state some preliminaries and give the main theorem with respect to the existence of a weak solution for the generalized system containing the system (1.2a)–(1.2b) with some boundary conditions.

Since we allow that Ω is multi-connected, we assume that Ω has the following conditions as in Amrouche and Seloula [2] (cf. Amrouche and Seloula [1], Dautray and Lions [10] and Girault and Raviart [13]). Let $\Omega\subset\mathbb{R}^3$ be a bounded domain of class $C^{1,1}$ with the boundary $\Gamma=\partial\Omega$ and Ω is locally situated on one side of Γ.

(i) Γ has a finite number of connected components $\Gamma_0,\Gamma_1,\dots,\Gamma_m$ with Γ_0 denoting the boundary of the infinite connected component of $\mathbb{R}^3\setminus\overline{\Omega}$.

(ii) There exist n connected open surfaces Σ_j, $(j=1,\dots,n)$, called cuts, contained in Ω such that

  (a) Σ_j is an open subset of a smooth manifold $\mathcal{M}_j$.

  (b) $\partial\Sigma_j\subset\Gamma$ $(j=1,\dots,n)$ and Σ_j is non-tangential to Γ, where $\partial\Sigma_j$ denotes the boundary of Σ_j.

  (c) $\overline{\Sigma_i}\cap\overline{\Sigma_j}=\emptyset$ $(i\neq j)$.

  (d) The open set $\dot{\Omega}=\Omega\setminus(\bigcup_{i=1}^n\Sigma_i)$ is simply connected and pseudo $C^{1,1}$ class.

The number n is called the first Betti number which is equal to the number of handles of Ω, and m is called the second Betti number which is equal to the number of holes. We say that if $n=0$, then Ω is simply connected, and if $m=0$, then Ω has no holes.

From now on, we use the standard notations $L^p(\Omega)$, $W^{1,p}(\Omega)$ ($m\geq 0$, integer), $W^{s,p}(\Gamma)$ ($s\in\mathbb{R}$) and so on, for the standard L^p and Sobolev spaces of functions. For any Banach space B, we denote $B\times B\times B$ by the boldface character $\boldsymbol{B}$. Hereafter, we use this character to denote vectors and vector valued functions, and we denote the Euclidean inner product of vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ by $\boldsymbol{a}\cdot\boldsymbol{b}$.

For $1<p<\infty$, define two spaces by

$$\mathbb{K}_N^p(\Omega)=\{\boldsymbol{v}\in L^p(\Omega)\mid \operatorname{div}\boldsymbol{v}=0,\operatorname{curl}\boldsymbol{v}=\boldsymbol{0}\text{ in }\Omega,\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}\text{ on }\Gamma\},$$
$$\mathbb{K}_T^p(\Omega)=\{\boldsymbol{v}\in L^p(\Omega)\mid \operatorname{div}\boldsymbol{v}=0,\operatorname{curl}\boldsymbol{v}=\boldsymbol{0}\text{ in }\Omega,\boldsymbol{v}\cdot\boldsymbol{n}=\boldsymbol{0}\text{ on }\Gamma\}.$$

Then it is well known that $\dim\mathbb{K}_N^p(\Omega)=m$ and $\dim\mathbb{K}_T^p(\Omega)=n$. Moreover, define two spaces

$$\mathbb{V}^p(\Omega)=\{\boldsymbol{v}\in W^{1,p}(\Omega)\mid \operatorname{div}\boldsymbol{v}=0\text{ in }\Omega,\boldsymbol{v}\times\boldsymbol{n}=0\text{ on }\Gamma,\\ \langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}=0,i=1,2,\dots,m\},$$

where $\langle\boldsymbol{v}\cdot\boldsymbol{n},1\rangle_{\Gamma_i}$ denotes the duality bracket between $W^{-1/p,p}(\Gamma_i)$ and $W^{1-1/p',p'}(\Gamma_i)$, and p' is the conjugate exponent of p, i.e., $1/p+1/p'=1$, and

$$X_T^p(\Omega)=\{\boldsymbol{v}\in L^p(\Omega)\mid \operatorname{div}\boldsymbol{v}\in L^p(\Omega),\operatorname{curl}\boldsymbol{v}\in L^p(\Omega),\boldsymbol{v}\times\boldsymbol{n}=\boldsymbol{0}\text{ on }\Gamma\}.\tag{2.1}$$

We note that $C_0^\infty(\Omega)\subset X_T^p(\Omega)\subset W^{1,p}(\Omega)$ (cf. [2] for the last inclusion).

Then we have the following (cf. Miranda et al. [16], Aramaki [4]).

Lemma 2.1. *For $1<p<\infty$, the space $\mathbb{V}^p(\Omega)$ is a reflexive, separable Banach space, and the semi-norm $\|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)}:=\|\operatorname{curl}\boldsymbol{v}\|_{L^p(\Omega)}$ is the norm, and it is equivalent to $\|\boldsymbol{v}\|_{W^{1,p}(\Omega)}$.*

From the Sobolev embedding theorem and the trace theorem, we have the following (cf. [16, Remark 1]).

Lemma 2.2. *There exist positive constants C_r and C_s such that for any $\boldsymbol{v}\in\mathbb{V}^p(\Omega)$,*

$$\|\boldsymbol{v}\|_{L^r(\Omega)}\le C_r\|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)}\quad\textit{with}\begin{cases} r\le 3p/(3-p) & \textit{if } 1<p<3,\\ \textit{any } r<+\infty & \textit{if } p=3,\\ r=+\infty & \textit{if } p>3,\end{cases}$$

$$\|\boldsymbol{v}\|_{L^s(\Gamma)}\le C_s\|\boldsymbol{v}\|_{\mathbb{V}^p(\Omega)}\quad\textit{with}\begin{cases} s\le 2p/(3-p) & \textit{if } 1<p<3,\\ \textit{any } s<+\infty & \textit{if } p=3,\\ s=+\infty & \textit{if } p>3.\end{cases}$$

Here we introduce two Carathéodory functions $S(x,s,t)$ and $T(x,s,t)$ containing $S(x,s,t)=\nu(\theta(x))t^{p/2}$ and $T(x,s,t)=k(\theta(x))t^{q/2}$ as special cases. Assume that $S(x,s,t)$ and $T(x,s,t)$ are two Carathéodory functions on $\Omega\times\mathbb{R}\times[0,\infty)$ satisfying $S(x,s,0)=0$ and $T(x,s,0)=0$ and the following structure conditions. There exist $1<p<\infty$ and

$1 < q < \infty$ (there is no relation between p and q) such that for a.e. $x \in \Omega$ and all $s \in \mathbb{R}$, $S(x,s,t), T(x,s,t) \in C^2((0,\infty))$ as functions of t, and there exist constants $0 < \lambda < \Lambda < \infty$ such that for a.e. $x \in \Omega$, all $s \in \mathbb{R}$ and $t > 0$,

$$\lambda t^{(p-2)/2} \le S_t(x,s,t) \le \Lambda t^{(p-2)/2}, \tag{2.2a}$$

$$\lambda t^{(p-2)/2} \le S_t(x,s,t) + 2tS_{tt}(x,s,t) \le \Lambda t^{(p-2)/2}, \tag{2.2b}$$

$$S_{tt}(x,s,t) < 0 \quad \text{if } 1 < p < 2, \quad \text{and} \quad S_{tt}(x,s,t) \ge 0 \quad \text{if } p \ge 2 \tag{2.2c}$$

and

$$\lambda t^{(q-2)/2} \le T_t(x,s,t) \le \Lambda t^{(q-2)/2}, \tag{2.3a}$$

$$\lambda t^{(q-2)/2} \le T_t(x,s,t) + 2tT_{tt}(x,s,t) \le \Lambda t^{(q-2)/2}, \tag{2.3b}$$

$$T_{tt}(x,s,t) < 0 \quad \text{if } 1 < q < 2, \quad \text{and} \quad T_{tt}(x,s,t) \ge 0 \quad \text{if } q \ge 2. \tag{2.3c}$$

We note that from (2.2a) and (2.3a), we have

$$\begin{aligned} &\frac{2}{p}\lambda t^{p/2} \le S(x,s,t) \le \frac{2}{p}\Lambda t^{p/2} \quad \text{for } t \ge 0,\\ &\frac{2}{q}\lambda t^{q/2} \le T(x,s,t) \le \frac{2}{q}\Lambda t^{q/2} \quad \text{for } t \ge 0. \end{aligned} \tag{2.4}$$

Moreover, from (2.2b) and (2.3b), there exist positive constants C_1 and C_2 depending only on p, λ, Λ and q, λ, Λ, respectively, such that we have

$$|S_{tt}(x,s,t)| \le C_1 t^{(p-4)/2}, \tag{2.5}$$

and similarly,

$$|T_{tt}(x,s,t)| \le C_2 t^{(q-4)/2}.$$

Here and hereafter, for any function $f(x,s,t)$, we denote $f_t = \partial f/\partial t$, $f_{tt} = \partial^2 f/\partial t^2$.

Example 2.3. If we define $S(x,s,t) = \nu(x,s)t^{p/2}$, where $\nu(x,s)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying $0 < \lambda \le \nu(x,s) \le \Lambda < \infty$ for a.e. $x \in \Omega$ and $s \in \mathbb{R}$, then $S(x,s,t)$ satisfies (2.2a)–(2.2c). Similarly, if $T(x,s,t) = \mu(x,s)t^{q/2}$, where $\mu(x,s)$ is a Carathéodory function on $\Omega \times \mathbb{R}$ satisfying $0 < \lambda \le \mu(x,s) \le \Lambda < \infty$, then $T(x,s,t)$ satisfies (2.3a)–(2.3c).

For the proof, see [4] and DiBenedetto [11].

In this paper, we consider the following problem: to find $(\boldsymbol{h}, \theta)$ in an appropriate space such that

$$\operatorname{curl}[S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}] = \boldsymbol{f}, \quad \operatorname{div}\boldsymbol{h} = 0 \quad \text{in } \Omega, \tag{2.6a}$$

$$-\operatorname{div}[T_t(x,\theta,|\nabla\theta|^2)\nabla\theta] = S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)|\operatorname{curl}\boldsymbol{h}|^2 \quad \text{in } \Omega. \tag{2.6b}$$

We impose the boundary conditions as follows.

$$\boldsymbol{h}\times\boldsymbol{n}=\mathbf{0},\quad S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\times\boldsymbol{n}=\mathbf{0}\quad\text{on }\Gamma.$$

In particular, we note that if $\nu(x,s)=\nu(s)$ and $\mu(x,s)=k(s)$ in Example 2.3, then the equations (2.6a)–(2.6b) become (1.2a)–(1.2b).

More precisely, we state the weak formulation of (2.6a)–(2.6b).

For $\boldsymbol{f}\in L^{r'}(\Omega)$ where r is as in Lemma 2.2 and r' is the conjugate exponent of r, i.e., $1/r+1/r'=1$ satisfying (1.4) and (1.5), find $(\boldsymbol{h},\theta)\in\mathbb{V}^p(\Omega)\times W_0^{1,r}(\Omega)$ with $1\vee(q-1)\le r\le q$, where $a\vee b=\max\{a,b\}$ such that

$$\int_\Omega S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\cdot\operatorname{curl}\boldsymbol{v}\,dx=\int_\Omega\boldsymbol{f}\cdot\boldsymbol{v}\,dx\quad\text{for all }\boldsymbol{v}\in X_T^p(\Omega),\tag{2.7a}$$

$$\int_\Omega T_t(x,\theta,|\nabla\theta|^2)\nabla\theta\cdot\nabla\xi\,dx=\int_\Omega S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)|\operatorname{curl}\boldsymbol{h}|^2\xi\,dx\quad\text{for all }\xi\in W_0^{1,\infty}(\Omega).\tag{2.7b}$$

We note that since $C_0^\infty(\Omega)\subset X_T^p(\Omega)$, a weak solution $(\boldsymbol{h},\theta)$ of (2.7a)–(2.7b) is a solution (2.6a)–(2.6b) in the distribution sense. Moreover, in the special case where $S(x,s,t)=\nu(s)t^{p/2}$ and $T(x,s,t)=k(s)t^{q/2}$, equations (2.7a)–(2.7b) become (1.6a)–(1.6b) with $X_T^p(\Omega)$ as the space of test functions instead of $\mathbb{W}^p(\Omega)$.

We are in a position to state the main theorem.

Theorem 2.4. *Assume that Ω and the functions $S(x,s,t)$ and $T(x,s,t)$ satisfy the above conditions with $1<p<\infty$ and $5/3<q<\infty$, respectively. Let $\boldsymbol{f}\in L^{r'}(\Omega)$ satisfy (1.4) and (1.5). Then the problem (2.7a)–(2.7b) has a solution $(\boldsymbol{h},\theta)\in\mathbb{V}^p(\Omega)\times W_0^{1,\overline{r}}(\Omega)$ with $\overline{r}=q$ if $q>3$, and $1<\overline{r}<3(q-1)/2$ if $5/3<q\le 3$.*

In the following, we give some preparations in order to prove this theorem.

Now we give monotonicities of S_t and T_t in the following sense.

Lemma 2.5. *There exists a constant $c>0$ depending only on λ and p such that for any $\boldsymbol{a},\boldsymbol{b}\in\mathbb{R}^3$,*

(i) $$(S_t(x,s,|\boldsymbol{a}|^2)\boldsymbol{a}-S_t(x,s,|\boldsymbol{b}|^2)\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})\ge\begin{cases}c|\boldsymbol{a}-\boldsymbol{b}|^p & \text{if } p\ge 2,\\ c(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}|^2 & \text{if } 1<p<2.\end{cases}$$

In particular,

$$(S_t(x,s,|\boldsymbol{a}|^2)\boldsymbol{a}-S_t(x,s,|\boldsymbol{b}|^2)\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})>0\quad\text{if }\boldsymbol{a}\neq\boldsymbol{b}.$$

(ii) $$(T_t(x,s,|\boldsymbol{a}|^2)\boldsymbol{a}-T_t(x,s,|\boldsymbol{b}|^2)\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})\ge\begin{cases}c|\boldsymbol{a}-\boldsymbol{b}|^q & \text{if } q\ge 2,\\ c(|\boldsymbol{a}|+|\boldsymbol{b}|)^{q-2}|\boldsymbol{a}-\boldsymbol{b}|^2 & \text{if } 1<q<2.\end{cases}$$

In particular,

$$(T_t(x,s,|\boldsymbol{a}|^2)\boldsymbol{a}-T_t(x,s,|\boldsymbol{b}|^2)\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b})>0\quad\text{if }\boldsymbol{a}\neq\boldsymbol{b}.$$

For the proof, see Aramaki [5].

Lemma 2.6. *There exist constants $C_1 > 0$ and $C_2 > 0$ depending only on Λ and p, and Λ and q, respectively, such that for any $\boldsymbol{a}, \boldsymbol{b} \in \mathbb{R}^3$,*

$$\text{(i)}\ \ |S_t(x,s,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,s,|\boldsymbol{b}|^2)\boldsymbol{b}| \le \begin{cases} C_1|\boldsymbol{a}-\boldsymbol{b}|^{p-1} & \text{if } 1<p<2, \\ C_1(|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2}|\boldsymbol{a}-\boldsymbol{b}| & \text{if } p\ge 2. \end{cases}$$

$$\text{(ii)}\ \ |T_t(x,s,|\boldsymbol{a}|^2)\boldsymbol{a} - T_t(x,s,|\boldsymbol{b}|^2)\boldsymbol{b}| \le \begin{cases} C_2|\boldsymbol{a}-\boldsymbol{b}|^{q-1} & \text{if } 1<q\le 2, \\ C_2(|\boldsymbol{a}|+|\boldsymbol{b}|)^{q-2}|\boldsymbol{a}-\boldsymbol{b}| & \text{if } q\ge 2. \end{cases}$$

Proof. It suffices to prove (i). From (2.5) and (2.2a), we have

$$\begin{aligned}
&|S_t(x,s,|\boldsymbol{a}|^2)\boldsymbol{a} - S_t(x,s,|\boldsymbol{b}|^2)\boldsymbol{b}| \\
&= \left| \int_0^1 \frac{d}{d\tau}\left[S_t(x,s,|\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}|^2)(\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b})\right] d\tau \right| \\
&= \int_0^1 \Big| 2S_{tt}(x,s,|\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}|^2)((\boldsymbol{a}-\boldsymbol{b})\cdot(\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}))(\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}) \\
&\qquad + S_t(x,s,|\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}|^2)(\boldsymbol{a}-\boldsymbol{b}) \Big| \, d\tau \\
&\le \int_0^1 \Big[2|S_{tt}(x,s,|\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}|^2)||\boldsymbol{a}-\boldsymbol{b}||\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}|^2 \\
&\qquad + S_t(x,s,|\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}|^2)|\boldsymbol{a}-\boldsymbol{b}| \Big] \, d\tau \\
&\le 2\Lambda \int_0^1 |\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}|^{p-2}|\boldsymbol{a}-\boldsymbol{b}|\, d\tau.
\end{aligned}$$

When $p \ge 2$, since

$$\int_0^1 |\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}|^{p-2}\, d\tau \le (|\boldsymbol{a}|+|\boldsymbol{b}|)^{p-2},$$

(i) holds. When $1 < p < 2$, if $|\boldsymbol{a}| \ge |\boldsymbol{a}-\boldsymbol{b}|$, we have

$$\begin{aligned}
|\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}|^{p-2} &= |\boldsymbol{a}-(1-\tau)(\boldsymbol{a}-\boldsymbol{b})|^{p-2} \le (|\boldsymbol{a}|-(1-\tau)|\boldsymbol{a}-\boldsymbol{b}|)^{p-2} \\
&\le (|\boldsymbol{a}-\boldsymbol{b}|-(1-\tau)|\boldsymbol{a}-\boldsymbol{b}|)^{p-2} = \tau^{p-2}|\boldsymbol{a}-\boldsymbol{b}|^{p-2}.
\end{aligned}$$

Since $\int_0^1 \tau^{p-2}\, d\tau = 1/(p-1)$, we can see that (i) holds. If $|\boldsymbol{a}| < |\boldsymbol{a}-\boldsymbol{b}|$, then there exists $\tau_* \in (0,1]$ such that $(1-\tau_*)|\boldsymbol{a}-\boldsymbol{b}| = |\boldsymbol{a}|$. Then we have

$$\begin{aligned}
&\int_0^1 |\tau\boldsymbol{a}+(1-\tau)\boldsymbol{b}|^{p-2}\, d\tau|\boldsymbol{a}-\boldsymbol{b}| \\
&\le \int_0^1 \big||\boldsymbol{a}|-(1-\tau)|\boldsymbol{a}-\boldsymbol{b}|\big|^{p-2}\, d\tau|\boldsymbol{a}-\boldsymbol{b}| \\
&= \int_0^{\tau_*} \big||\boldsymbol{a}|-(1-\tau)|\boldsymbol{a}-\boldsymbol{b}|\big|^{p-2}\, d\tau|\boldsymbol{a}-\boldsymbol{b}| + \int_{\tau_*}^1 \big||\boldsymbol{a}|-(1-\tau)|\boldsymbol{a}-\boldsymbol{b}|\big|^{p-2}\, d\tau|\boldsymbol{a}-\boldsymbol{b}|
\end{aligned}$$

$$= -\frac{1}{p-1}\int_0^{\tau_*} \frac{d}{d\tau}\big((1-\tau)|\boldsymbol{a}-\boldsymbol{b}| - |\boldsymbol{a}|\big)^{p-1}\,d\tau + \frac{1}{p-1}\int_{\tau_*}^1 \frac{d}{d\tau}\big(|\boldsymbol{a}| - (1-\tau)|\boldsymbol{a}-\boldsymbol{b}|\big)^{p-1}\,d\tau$$
$$= \frac{1}{p-1}(|\boldsymbol{a}-\boldsymbol{b}| - |\boldsymbol{a}|)^{p-1} + \frac{1}{p-1}|\boldsymbol{a}|^{p-1}$$
$$\le \frac{2}{p-1}|\boldsymbol{a}-\boldsymbol{b}|^{p-1}.$$

Thus (i) holds. □

3. Associated minimization problems

In this section, we consider the minimization problems. Let $S(x,t)$ and $T(x,t)$ be two Carathéodory functions on $\Omega \times [0,\infty)$ satisfying (2.2a)–(2.2c) and (2.3a)–(2.3c) without s-variable, respectively. Then we have the following.

Proposition 3.1. *For given* $\boldsymbol{f} \in L^{r'}(\Omega)$ *satisfying* (1.4) *and* (1.5), *the following minimization problem: to find* $\boldsymbol{h} \in \mathbb{V}^p(\Omega)$ *such that*

$$I[\boldsymbol{h}] = \inf_{\boldsymbol{v}\in\mathbb{V}^p(\Omega)} I[\boldsymbol{v}], \tag{3.1}$$

where

$$I[\boldsymbol{v}] = \frac{1}{2}\int_\Omega S(x, |\operatorname{curl}\boldsymbol{v}|^2)\,dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx$$

has a unique solution $\boldsymbol{h} \in \mathbb{V}^p(\Omega)$. *The minimizer* $\boldsymbol{h}$ *satisfies the following equation*

$$\int_\Omega S_t(x, |\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\cdot\operatorname{curl}\boldsymbol{v}\,dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx \quad \text{for all } \boldsymbol{v} \in X_T^p(\Omega), \tag{3.2}$$

where $X_T^p(\Omega)$ *is defined by* (2.1). *The minimizer* $\boldsymbol{h} \in \mathbb{V}^p(\Omega)$ *is also a unique solution of* (3.2). *Moreover, the solution* $\boldsymbol{h}$ *satisfies the following estimate*

$$\|\boldsymbol{h}\|_{\mathbb{V}^p(\Omega)} \le \lambda^{-1/(p-1)}\|\boldsymbol{f}\|_{L^{r'}(\Omega)}^{1/(p-1)}. \tag{3.3}$$

Proof. First we show that the minimization problem (3.1) has a unique minimizer. If we define $F(x,t) = S(x,t^2)$, it follows from (2.2a) and (2.2b) that

$$F_t(x,t) = 2tS_t(x,t^2) \ge 2\lambda t^{p-1} > 0 \quad \text{for } t>0,$$
$$F_{tt}(x,t) = 2\{S_t(x,t^2) + 2t^2S_{tt}(x,t^2)\} \ge 2\lambda t^{p-2} > 0 \quad \text{for } t>0.$$

Thus I is a proper strictly convex functional on $\mathbb{V}^p(\Omega)$. We show the lower semi-continuity of I. Let $\boldsymbol{v}_j \to \boldsymbol{v}$ in $\mathbb{V}^p(\Omega)$. Since $\operatorname{curl}\boldsymbol{v}_j \to \operatorname{curl}\boldsymbol{v}$ in $L^p(\Omega)$, there exists a subsequence $\{\boldsymbol{v}_{j_k}\}$ of $\{\boldsymbol{v}_j\}$ such that

$$\lim_{k\to\infty}\int_\Omega S(x, |\operatorname{curl}\boldsymbol{v}_{j_k}|^2)\,dx = \liminf_{j\to\infty}\int_\Omega S(x, |\operatorname{curl}\boldsymbol{v}_j|^2)\,dx$$

and $\operatorname{curl} \boldsymbol{v}_{j_k} \to \operatorname{curl} \boldsymbol{v}$ a.e. in Ω. Since $S(x,t)$ is continuous with respect to $t \in [0,\infty)$, $S(x, |\operatorname{curl} \boldsymbol{v}_{j_k}|^2) \to S(x, |\operatorname{curl} \boldsymbol{v}|^2)$ a.e. in Ω. Since $S(x,t) \geq 0$, it follows from the Fatou lemma that

$$\begin{aligned}\int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}|^2)\,dx &\leq \liminf_{k\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}_{j_k}|^2)\,dx \\ &= \lim_{k\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}_{j_k}|^2)\,dx \\ &= \liminf_{j\to\infty} \int_\Omega S(x, |\operatorname{curl} \boldsymbol{v}_j|^2)\,dx.\end{aligned}$$

On the other hand, since using Lemma 2.2 we can easily see that

$$\int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}_j\,dx \to \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\,dx$$

as $j \to \infty$, we have

$$I[\boldsymbol{v}] \leq \liminf_{j\to\infty} I[\boldsymbol{v}_j].$$

Hence I is lower semi-continuous on $\mathbb{V}^p(\Omega)$.

We show that I is coercive on $\mathbb{V}^p(\Omega)$. In fact, from (2.4), Lemma 2.2 and the Hölder inequality, for any $\varepsilon > 0$, we have

$$\begin{aligned}I[\boldsymbol{v}] &\geq \frac{2}{p}\lambda \|\operatorname{curl} \boldsymbol{v}\|^p_{L^p(\Omega)} - \|\boldsymbol{f}\|_{L^{r'}(\Omega)} \|\boldsymbol{v}\|_{L^r(\Omega)} \\ &\geq \frac{2}{p}\lambda \|\boldsymbol{v}\|^p_{\mathbb{V}^p(\Omega)} - C(\varepsilon)\|\boldsymbol{f}\|^{p'}_{L^{r'}(\Omega)} - \varepsilon \|\boldsymbol{v}\|^p_{\mathbb{V}^p(\Omega)}.\end{aligned}$$

If we choose $\varepsilon > 0$ so that $\varepsilon < 2\lambda/p$, we can see that I is coercive. Therefore there exists a unique minimizer (cf. for example, Ekeland and Temam [12, Chapter 2, Proposition 1.2]).

Next we show that if $\boldsymbol{f} \in L^{r'}(\Omega)$ satisfies (1.4) and (1.5), then we claim the following

$$\inf_{\boldsymbol{u}\in\mathbb{V}^p(\Omega)} I[\boldsymbol{u}] = \inf_{\boldsymbol{w}\in X^p_T(\Omega)} I[\boldsymbol{w}]. \tag{3.4}$$

In fact, $\mathbb{V}^p(\Omega) \subset X^p_T(\Omega)$, it is trivial that

$$\inf_{\boldsymbol{u}\in\mathbb{V}^p(\Omega)} I[\boldsymbol{u}] \geq \inf_{\boldsymbol{w}\in X^p_T(\Omega)} I[\boldsymbol{w}].$$

For any $\boldsymbol{u} \in X^p_T(\Omega)$, we consider the following div-curl system

$$\begin{cases} \operatorname{curl} \boldsymbol{v} = \operatorname{curl} \boldsymbol{u} & \text{in } \Omega, \\ \operatorname{div} \boldsymbol{v} = 0 & \text{in } \Omega, \\ \boldsymbol{v} \times \boldsymbol{n} = \boldsymbol{0} & \text{on } \Gamma. \end{cases} \tag{3.5}$$

Since $\operatorname{div}(\operatorname{curl}\boldsymbol{u}) = 0$ in Ω, $\boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{u} = \boldsymbol{n}\cdot\operatorname{curl}\boldsymbol{u}_T = 0$ on Γ, where $\boldsymbol{u}_T = (\boldsymbol{n}\times\boldsymbol{u})\times\boldsymbol{n}$ is the tangent component of $\boldsymbol{u}$ (cf. Monneau [18]), it follows from Aramaki [3, Theorem 3.5] that (3.5) has a solution $\boldsymbol{v}\in W^{1,p}(\Omega)$. Define $\boldsymbol{w} = \boldsymbol{v} - \sum_{k=1}^{m}\langle\boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_k}\boldsymbol{e}_k$, where $\{\boldsymbol{e}_k\}$ is a basis of $\mathbb{K}_N^p(\Omega)$ such that $\langle\boldsymbol{e}_k\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = \delta_{ki}$. We have, for $i = 1,\dots,m$,

$$\langle\boldsymbol{w}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = \langle\boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} - \sum_{k=1}^{m}\langle\boldsymbol{v}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_k}\langle\boldsymbol{e}_k\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0.$$

Since $\operatorname{div}\boldsymbol{w} = 0$, $\operatorname{curl}\boldsymbol{w} = \operatorname{curl}\boldsymbol{v} = \operatorname{curl}\boldsymbol{u}$ in Ω and $\boldsymbol{w}\times\boldsymbol{n} = \boldsymbol{v}\times\boldsymbol{n} = \boldsymbol{0}$ on Γ, we see that $\boldsymbol{w}\in\mathbb{V}^p(\Omega)$ and $\operatorname{curl}\boldsymbol{w} = \operatorname{curl}\boldsymbol{u}$. Since $\boldsymbol{f}\in L^{r'}(\Omega)$ satisfies (1.4) and (1.5), it holds that $\operatorname{div}\boldsymbol{f} = 0$ in Ω and $\langle\boldsymbol{f}\cdot\boldsymbol{n}, 1\rangle_{\Gamma_i} = 0$ for $i = 0, 1, \dots, m$. Therefore it follows from [2, Lemma 4.1] that there exists $\boldsymbol{g}\in W^{1,r'}(\Omega)$ such that $\boldsymbol{f} = \operatorname{curl}\boldsymbol{g}$ in Ω. By integration by parts,

$$\begin{aligned}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{w}\,dx &= \int_\Omega \operatorname{curl}\boldsymbol{g}\cdot\boldsymbol{w}\,dx\\ &= \int_\Gamma (\boldsymbol{g}\times\boldsymbol{n})\cdot\boldsymbol{w}\,dS + \int_\Omega \boldsymbol{g}\cdot\operatorname{curl}\boldsymbol{w}\,dx\\ &= \int_\Gamma \boldsymbol{g}\cdot(\boldsymbol{n}\times\boldsymbol{w})\,dS + \int_\Omega \boldsymbol{g}\cdot\operatorname{curl}\boldsymbol{u}\,dx\\ &= \int_\Omega \boldsymbol{g}\cdot\operatorname{curl}\boldsymbol{u}\,dx = \int_\Omega \operatorname{curl}\boldsymbol{g}\cdot\boldsymbol{u}\,dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{u}\,dx.\end{aligned}$$

Hence $I[\boldsymbol{w}] = I[\boldsymbol{u}]$. So

$$\inf_{\boldsymbol{w}\in\mathbb{V}^p(\Omega)} I[\boldsymbol{w}] \le I[\boldsymbol{u}] \quad \text{for all } \boldsymbol{u}\in X_T^p(\Omega).$$

Thus we have

$$\inf_{\boldsymbol{w}\in\mathbb{V}^p(\Omega)} I[\boldsymbol{w}] \le \inf_{\boldsymbol{u}\in X_T^p(\Omega)} I[\boldsymbol{u}].$$

Therefore we get (3.4).

Let $\boldsymbol{h}\in\mathbb{V}^p(\Omega)$ be the minimizer of

$$\inf_{\boldsymbol{u}\in\boldsymbol{V}^p(\Omega)} I[\boldsymbol{u}]$$

and $\boldsymbol{v}\in X_T^p(\Omega)$. Then by the Euler-Lagrange equation, we have

$$0 = \frac{d}{d\varepsilon}I[\boldsymbol{h}+\varepsilon\boldsymbol{v}]\Big|_{\varepsilon=0} = \int_\Omega S_t(x, |\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\cdot\operatorname{curl}\boldsymbol{v}\,dx - \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx.$$

Hence $\boldsymbol{h}$ is a solution of (3.2). The uniqueness of the solution for (3.2) follows from the monotonicity of S_t in Lemma 2.5(i).

Finally, we show the estimate (3.3). Taking $\boldsymbol{v} = \boldsymbol{h}$ as a test function of (3.2), it follows from (2.2a), the Hölder inequality and Lemma 2.2 that

$$\lambda\|\operatorname{curl}\boldsymbol{h}\|_{L^p(\Omega)}^p \le \|\boldsymbol{f}\|_{L^{r'}(\Omega)}\|\boldsymbol{h}\|_{\mathbb{V}^p(\Omega)}.$$

This implies (3.3). $\square$

Remark 3.2. (i) Our proof is directly for existence of solution to the equation (3.2). Since the authors of [15] used the result of Lions [14, Theorem 2.1, p. 171], it is necessary to suppose $p > 6/5$. However, by our method, the restriction is unnecessary. Moreover, since $C_0^\infty(\Omega) \subset X_T^p(\Omega)$, $\boldsymbol{h}$ is a solution of

$$\operatorname{curl}[S_t(x, |\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}] = \boldsymbol{f} \quad \text{in } \Omega$$

in the distribution sense.

(ii) In our previous paper Aramaki [6], we showed that if $\boldsymbol{f} \in C^\alpha(\overline{\Omega})$ for some $\alpha \in (0,1)$ satisfies (1.4) and (1.5), then the weak solution $\boldsymbol{h} \in \mathbb{V}^p(\Omega)$ of (3.2) belongs to $C^{1+\beta}(\overline{\Omega})$ for some $\beta \in (0,1)$.

Similarly, taking the Poincaré inequality into consideration, we have the following.

Proposition 3.3. *For a given $k \in W^{-1,q'}(\Omega) = W_0^{1,q}(\Omega)'$, the following equation*

$$\int_\Omega T_t(x, |\nabla\theta|^2)\nabla\theta \cdot \nabla\xi \, dx = \int_\Omega k\xi \, dx \quad \text{for all } \xi \in W_0^{1,q}(\Omega) \tag{3.6}$$

has a unique solution $\theta \in W_0^{1,q}(\Omega)$, where the integral on the right-hand side of (3.6) means the duality of $k \in W^{-1,q'}(\Omega)$ and $\xi \in W_0^{1,q}(\Omega)$. Moreover, there exists a constant $C > 0$ depending only on λ, q and Ω such that

$$\|\theta\|_{W_0^{1,q}(\Omega)} \le C\|k\|_{W^{-1,q'}(\Omega)}^{1/(q-1)}.$$

4. Continuous dependence on known data

In this section, we show the continuous dependence on known data for the weak solution to the equation (3.2).

Assume that $S^{(n)}(x,t)$ and $S(x,t)$ are Carathéodory functions on $\Omega \times [0,\infty)$ satisfying the structure conditions (2.2a)–(2.2c) without s variable and with the same λ and Λ. Let $\boldsymbol{f}_n, \boldsymbol{f} \in L^{r'}(\Omega)$ satisfying (1.4) and (1.5). Let $\boldsymbol{h}_n, \boldsymbol{h} \in \mathbb{V}^p(\Omega)$ be solutions of

$$\int_\Omega S_t^{(n)}(x, |\operatorname{curl}\boldsymbol{h}_n|^2)\operatorname{curl}\boldsymbol{h}_n \cdot \operatorname{curl}\boldsymbol{v} \, dx = \int_\Omega \boldsymbol{f}_n \cdot \boldsymbol{v} \, dx \quad \text{for all } \boldsymbol{v} \in X_T^p(\Omega), \tag{4.1}$$

and (3.2), respectively.

Then we have the following.

Proposition 4.1. *Let $1 < p < \infty$. Assume that $\boldsymbol{f}_n \to \boldsymbol{f}$ in $L^{r'}(\Omega)$ and $S^{(n)}(x,t) \to S(x,t)$ a.e. in $\Omega \times [0,\infty)$. Then $\boldsymbol{h}_n \to \boldsymbol{h}$ in $\mathbb{V}^p(\Omega)$ as $n \to \infty$. More precisely, there exists a constant $C > 0$ independent of n such that*

$$\begin{aligned} \|\boldsymbol{h}_n - \boldsymbol{h}\|_{\mathbb{V}^p(\Omega)}^{p\vee 2} \le C(&\|(S_t^{(n)}(x, |\operatorname{curl}\boldsymbol{h}|^2) - S_t(x, |\operatorname{curl}\boldsymbol{h}|^2))\operatorname{curl}\boldsymbol{h}\|_{L^{p'}(\Omega)}^{p'\wedge 2} \\ &+ \|\boldsymbol{f}_n - \boldsymbol{f}\|_{L^{r'}(\Omega)}^{p'\wedge 2}, \end{aligned} \tag{4.2}$$

where $a \wedge b = \min\{a, b\}$ for any $a, b \in \mathbb{R}$.

Proof. It suffices to prove the estimate (4.2). In fact, from (2.2a),

$$|S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h} - S_t(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h}|^{p'} \le (2\Lambda)^{p'} |\operatorname{curl} \boldsymbol{h}|^p.$$

We note that the right-hand side is an integrable function in Ω which is independent of n. Since $S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h} \to S_t(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h}$ a.e. in Ω, it follows from the Lebesgue dominated theorem that

$$\|S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h} - S_t(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h}\|_{L^{p'}(\Omega)} \to 0$$

as $n \to \infty$. Since $\boldsymbol{f}_n \to \boldsymbol{f}$ in $L^{r'}(\Omega)$, if (4.2) holds, we obtain that $\boldsymbol{h}_n \to \boldsymbol{h}$ in $\mathbb{V}^p(\Omega)$ as $n \to \infty$.

We show the estimate (4.2). If we take $\boldsymbol{v} = \boldsymbol{h}_n - \boldsymbol{h}$ as a test function of (3.2) and (4.1), we have

$$\int_\Omega S_t(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h} \cdot \operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\, dx = \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{h}_n - \boldsymbol{h})\, dx$$

and

$$\int_\Omega S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{h}_n|^2) \operatorname{curl} \boldsymbol{h}_n \cdot \operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\, dx = \int_\Omega \boldsymbol{f}_n \cdot (\boldsymbol{h}_n - \boldsymbol{h})\, dx.$$

Therefore we have

$$\begin{aligned} &\int_\Omega (S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{h}_n|^2) \operatorname{curl} \boldsymbol{h}_n - S_t(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h}) \cdot \operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\, dx \\ &= \int_\Omega (\boldsymbol{f}_n - \boldsymbol{f}) \cdot (\boldsymbol{h}_n - \boldsymbol{h})\, dx, \end{aligned}$$

so

$$\begin{aligned} &\int_\Omega (S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{h}_n|^2) \operatorname{curl} \boldsymbol{h}_n - S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h}) \cdot \operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\, dx \\ &\quad + \int_\Omega (S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h} - S_t(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h}) \cdot \operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\, dx \\ &= \int_\Omega (\boldsymbol{f}_n - \boldsymbol{f}) \cdot (\boldsymbol{h}_n - \boldsymbol{h})\, dx. \end{aligned} \tag{4.3}$$

When $p \ge 2$, using Lemma 2.5(i), we can see that

$$\begin{aligned} &c\|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\|^p_{L^p(\Omega)} \\ &\le \|S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h} - S_t(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h}\|_{L^{p'}(\Omega)} \|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\|_{L^p(\Omega)} \\ &\quad + \|\boldsymbol{f}_n - \boldsymbol{f}\|_{L^{r'}(\Omega)} \|\boldsymbol{h}_n - \boldsymbol{h}\|_{\mathbb{V}^p(\Omega)} \\ &\le C(\varepsilon)\|S_t^{(n)}(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h} - S_t(x, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h}\|^{p'}_{L^{p'}(\Omega)} \\ &\qquad + \varepsilon\|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\|^p_{L^p(\Omega)} + C'(\varepsilon)\|\boldsymbol{f}_n - \boldsymbol{f}\|^{p'}_{L^{r'}(\Omega)} + \varepsilon\|\boldsymbol{h}_n - \boldsymbol{h}\|^p_{\mathbb{V}^p(\Omega)} \end{aligned}$$

for any $\varepsilon > 0$. If we choose $\varepsilon > 0$ so that $2\varepsilon < c$, it follows that there exists a constant $C > 0$ independent of n such that

$$\begin{aligned}&\|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\|^p_{L^p(\Omega)}\\&\le C\left[\|S_t^{(n)}(x, |\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h} - S_t(x, |\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\|^{p'}_{L^{p'}(\Omega)} + \|\boldsymbol{f}_n - \boldsymbol{f}\|^{p'}_{L^{r'}(\Omega)}\right].\end{aligned}$$

When $1 < p < 2$, from (4.3) and Lemma 2.5(i) we have

$$\begin{aligned}&c\int_\Omega (|\operatorname{curl}\boldsymbol{h}_n| + |\operatorname{curl}\boldsymbol{h}|)^{p-2}|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})|^2\,dx\\&\le \|S_t^{(n)}(x, |\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h} - S_t(x, |\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\|_{L^{p'}(\Omega)}\|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\|_{L^p(\Omega)}\\&\quad + \|\boldsymbol{f}_n - \boldsymbol{f}\|_{L^{r'}(\Omega)}\|\boldsymbol{h}_n - \boldsymbol{h}\|_{\mathbb{V}^p(\Omega)}.\end{aligned}\tag{4.4}$$

Here if we use the reverse Hölder inequality (cf. Sobolev [19, p. 8]) with $0 < s = p/2 < 1$ and $s' = 2/(p-2)$ (< 0), we have

$$\begin{aligned}&\lambda\int_\Omega (|\operatorname{curl}\boldsymbol{h}_n| + |\operatorname{curl}\boldsymbol{h}|)^{p-2}|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})|^2\,dx\\&\ge \lambda\left(2^{p-1}(\|\operatorname{curl}\boldsymbol{h}_n\|^p_{L^p(\Omega)} + \|\operatorname{curl}\boldsymbol{h}\|^p_{L^p(\Omega)}\right)^{(p-2)/2}\|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\|^2_{L^p(\Omega)}.\end{aligned}$$

From (4.1) with $\boldsymbol{v} = \boldsymbol{h}_n$, using (2.2a), we can see that

$$\lambda\|\operatorname{curl}\boldsymbol{h}_n\|^p_{L^p(\Omega)} \le \|\boldsymbol{f}_n\|_{L^{r'}(\Omega)}\|\boldsymbol{h}_n\|_{\mathbb{V}^p(\Omega)} \le \|\boldsymbol{f}_n\|_{L^{r'}(\Omega)}\|\operatorname{curl}\boldsymbol{h}_n\|_{L^p(\Omega)}.$$

So it follows that $\lambda\|\operatorname{curl}\boldsymbol{h}_n\|^{p-1}_{L^p(\Omega)} \le \|\boldsymbol{f}_n\|_{L^{r'}(\Omega)}$. Since $\boldsymbol{f}_n \to \boldsymbol{f}$ in $L^{r'}(\Omega)$, there exists a constant C_1 independent of n such that

$$\|\operatorname{curl}\boldsymbol{h}_n\|_{L^{r'}(\Omega)} \le C_1.$$

Hence from (4.3), (4.4), Hölder's inequality and Young's inequality, we have

$$\begin{aligned}\|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\|^2_{L^p(\Omega)} &\le C\int_\Omega (|\operatorname{curl}\boldsymbol{h}_n| + |\operatorname{curl}\boldsymbol{h}|)^{p-2}|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})|^2\,dx\\&\le C(\varepsilon)\|S_t^{(n)}(x, |\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h} - S_t(x, |\operatorname{curl}\boldsymbol{h}|)\operatorname{curl}\boldsymbol{h}\|^2_{L^{p'}(\Omega)}\\&\quad + \varepsilon\|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\|^2_{L^p(\Omega)} + C'(\varepsilon)\|\boldsymbol{f}_n - \boldsymbol{f}\|^2_{L^{r'}(\Omega)}\\&\quad + \varepsilon\|\operatorname{curl}(\boldsymbol{h}_n - \boldsymbol{h})\|^2_{L^p(\Omega)}\end{aligned}$$

for any $\varepsilon > 0$. Thus if we choose $\varepsilon > 0$ small enough, we have

$$\begin{aligned}&\|\boldsymbol{h}_n - \boldsymbol{h}\|^2_{\mathbb{V}^p(\Omega)}\\&\le C\left[\|S_t^{(n)}(x, |\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h} - S_t(x, |\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\|^2_{L^{p'}(\Omega)} + \|\boldsymbol{f}_n - \boldsymbol{f}\|^2_{L^{r'}(\Omega)}\right].\end{aligned}$$

This completes the proof. $\square$

5. Weak solution for an approximate problem

In this section, in order to prove Theorem 2.4, we consider weak solution of an approximate problem by truncation.

For $M > 0$, define a continuous function

$$\tau_M(s) = (s \wedge M) \vee (-M) = \begin{cases} -M & \text{if } s \leq -M, \\ s & \text{if } -M < s < M, \\ M & \text{if } s \geq M. \end{cases}$$

We consider the following approximate problem: to find $(\boldsymbol{h}_M, \theta_M) \in \mathbb{V}^p(\Omega) \times W_0^{1,q}(\Omega)$ such that

$$\int_\Omega S_t(x, \theta_M, |\operatorname{curl} \boldsymbol{h}_M|^2) \operatorname{curl} \boldsymbol{h}_M \cdot \operatorname{curl} \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \quad \text{for all } \boldsymbol{v} \in X_T^p(\Omega), \tag{5.1a}$$

$$\begin{aligned} &\int_\Omega T_t(x, \theta_M, |\nabla \theta_M|^2) \nabla \theta_M \cdot \nabla \xi\, dx \\ &= \int_\Omega \tau_M(S_t(x, \theta_M, |\operatorname{curl} \boldsymbol{h}_M|^2) |\operatorname{curl} \boldsymbol{h}_M|^2) \xi\, dx \quad \text{for all } \xi \in W_0^{1,q}(\Omega). \end{aligned} \tag{5.1b}$$

Then we have the following.

Proposition 5.1. *Let $1 < p < \infty$, $1 < q < \infty$ and let $\boldsymbol{f} \in L^{r'}(\Omega)$ satisfy* (1.4) *and* (1.5). *Then the problem* (5.1a)–(5.1b) *has a solution* $(\boldsymbol{h}_M, \theta_M) \in \mathbb{V}^p(\Omega) \times W_0^{1,q}(\Omega)$.

Proof. We shall use the Schauder fixed point theorem. Let $R > 0$ and define a closed convex subset of $L^q(\Omega)$ by

$$D_R = \{\gamma \in L^q(\Omega) \mid \|\gamma\|_{L^q(\Omega)} \leq R\}.$$

Fix $\gamma \in D_R$. We consider the following auxiliary problem: to find $\boldsymbol{h} \in \mathbb{V}^p(\Omega)$ such that

$$\int_\Omega S_t(x, \gamma(x), |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h} \cdot \operatorname{curl} \boldsymbol{v}\, dx = \int_\Omega \boldsymbol{f} \cdot \boldsymbol{v}\, dx \tag{5.2}$$

for all $\boldsymbol{v} \in X_T^p(\Omega)$.

If we put $S(x, t) = S(x, \gamma(x), t)$, the problem is exactly the problem (3.2). Therefore it follows from Proposition 3.1 that (5.2) has a unique solution $\boldsymbol{h} = \boldsymbol{h}(\gamma) \in \mathbb{V}^p(\Omega)$. Taking $\boldsymbol{v} = \boldsymbol{h}(\gamma)$ as a test function of (5.2), we have

$$\begin{aligned} \lambda \int_\Omega |\operatorname{curl} \boldsymbol{h}(\gamma)|^p\, dx &\leq \int_\Omega \boldsymbol{f} \cdot \boldsymbol{h}(\gamma)\, dx \\ &\leq \|\boldsymbol{f}\|_{L^{r'}(\Omega)} \|\boldsymbol{h}(\gamma)\|_{\mathbb{V}^p(\Omega)} \\ &\leq C(\varepsilon) \|\boldsymbol{f}\|_{L^{r'}(\Omega)}^{p'} + \varepsilon \|\operatorname{curl} \boldsymbol{h}(\gamma)\|_{L^p(\Omega)}^p \end{aligned}$$

for any $\varepsilon > 0$. Thus there exists a constant C depending only on λ, p, Ω and $\|\boldsymbol{f}\|_{L^{r'}(\Omega)}$ such that

$$\|\operatorname{curl}\boldsymbol{h}(\gamma)\|_{L^p(\Omega)} \le C. \tag{5.3}$$

Define an operator $S_1\colon D_R\,(\subset L^q(\Omega)) \to \mathbb{V}^p(\Omega)$ by $S_1(\gamma) = \boldsymbol{h}(\gamma)$.

Claim 1. S_1 is continuous.

In fact, let $\gamma_n, \gamma \in D_R$ and $\gamma_n \to \gamma$ in $L^q(\Omega)$ as $n \to \infty$. If we put $S^{(n)}(x,t) = S(x,\gamma_n(x),t)$, then $\boldsymbol{h}_n = \boldsymbol{h}(\gamma_n)$ is a solution of the problem

$$\int_\Omega S_t^{(n)}(x,|\operatorname{curl}\boldsymbol{h}_n|^2)\operatorname{curl}\boldsymbol{h}_n\cdot\operatorname{curl}\boldsymbol{v}\,dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx \quad \text{for all } \boldsymbol{v}\in X_T^p(\Omega).$$

For any subsequence $\{\gamma_{n'}\}$ of $\{\gamma_n\}$, there exists a subsequence $\{\gamma_{n''}\}$ of $\{\gamma_{n'}\}$ such that $\gamma_{n''} \to \gamma$ in $L^q(\Omega)$ and a.e. in Ω. Since $S_t(x,s,t^2)t$ is a Carathéodory function, we can see that

$$S_t^{(n'')}(x,t^2)t \to S_t(x,t^2)t = S_t(x,\gamma(x),t^2)t \quad \text{a.e. in } \Omega\times(0,\infty).$$

By Proposition 4.1, $\boldsymbol{h}_{n''} = \boldsymbol{h}(\gamma_{n''}) \to \boldsymbol{h} = \boldsymbol{h}(\gamma)$ in $\mathbb{V}^p(\Omega)$ as $n'' \to \infty$. Since the limit is unique, we can see that the full sequence converges to $\boldsymbol{h}$ i.e., $\boldsymbol{h}_n = \boldsymbol{h}(\gamma_n) \to \boldsymbol{h} = \boldsymbol{h}(\gamma)$ in $\mathbb{V}^p(\Omega)$ as $n \to \infty$.

Now fix $\gamma \in D_R$ and $S_1(\gamma) = \boldsymbol{h}(\gamma)$. We consider the problem: to find $\theta \in W_0^{1,q}(\Omega)$ such that

$$\int_\Omega T_t(x,\gamma(x),|\nabla\theta|^2)\nabla\theta\cdot\nabla\xi\,dx = \int_\Omega \tau_M(S_t(x,\gamma(x),|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)\xi\,dx \tag{5.4}$$

for all $\xi \in W_0^{1,q}(\Omega)$. Since $\tau_M(S_t(x,\gamma(x),|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma)|^2) \in W_0^{1,q}(\Omega)'$, it follows from Proposition 3.3 that there exists a unique solution $\theta(\gamma) = \theta(\gamma,\boldsymbol{h}(\gamma)) \in W_0^{1,q}(\Omega)$. Using $\xi = \theta(\gamma)$ as a test function of (5.4), we have

$$\lambda\int_\Omega |\nabla\theta(\gamma)|^q\,dx \le \|\tau_M(S_t(x,\gamma(x),|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)\|_{L^{q'}(\Omega)}\|\theta(\gamma)\|_{L^q(\Omega)}.$$

By the Poincaré inequality, there exists a constant $C(\Omega)$ such that

$$\|\theta(\gamma)\|_{L^q(\Omega)} \le C(\Omega)\|\nabla\theta(\gamma)\|_{L^q(\Omega)}.$$

Therefore we have

$$\begin{aligned}\lambda\|\nabla\theta(\gamma)\|_{L^q(\Omega)}^q &\le C(\varepsilon,\Omega)\|\tau_M(S_t(x,\gamma(x),|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)\|_{L^{q'}(\Omega)}^{q'}\\ &\quad + \varepsilon\|\nabla\theta(\gamma)\|_{L^q(\Omega)}^q\end{aligned}$$

for any $\varepsilon > 0$. If we choose $0 < \varepsilon < \lambda$, we can see that

$$\|\theta(\gamma)\|_{W_0^{1,q}(\Omega)} \le C_M, \tag{5.5}$$

where C_M is a constant depending on q, M and Ω. Define $S_2\colon D_R\,(\subset L^q(\Omega)) \to \mathbb{V}^p(\Omega) \to W_0^{1,q}(\Omega)$ by $\gamma \mapsto \boldsymbol{h}(\gamma) \mapsto \theta(\gamma, \boldsymbol{h}(\gamma))$.

Claim 2. S_2 is continuous.

In fact, let $\gamma_n, \gamma \in D_R$ and $\gamma_n \to \gamma$ in $L^q(\Omega)$ as $n \to \infty$. From the same arguments of Claim 1, we may assume that $\gamma_n \to \gamma$ a.e. in Ω. Then it follows from Claim 1 that $\boldsymbol{h}(\gamma_n) \to \boldsymbol{h}(\gamma)$ in $\mathbb{V}^p(\Omega)$. We may also assume that $\operatorname{curl} \boldsymbol{h}(\gamma_n) \to \operatorname{curl} \boldsymbol{h}(\gamma)$ in $L^p(\Omega)$ and a.e. in Ω. Taking $\xi = \theta(\gamma_n) - \theta(\gamma)$ as a test function of (5.4), we have

$$\begin{aligned}&\int_\Omega T_t(x, \gamma_n, |\nabla\theta(\gamma_n)|^2)\nabla\theta(\gamma_n) \cdot \nabla(\theta(\gamma_n) - \theta(\gamma))\, dx \\ &= \int_\Omega \tau_M(S_t(x, \gamma_n, |\operatorname{curl} \boldsymbol{h}(\gamma_n)|^2)|\operatorname{curl} \boldsymbol{h}(\gamma_n)|^2)(\theta(\gamma_n) - \theta(\gamma))\, dx\end{aligned}$$

and

$$\begin{aligned}&\int_\Omega T_t(x, \gamma, |\nabla\theta(\gamma)|^2)\nabla\theta(\gamma) \cdot \nabla(\theta(\gamma_n) - \theta(\gamma))\, dx \\ &= \int_\Omega \tau_M(S_t(x, \gamma, |\operatorname{curl} \boldsymbol{h}(\gamma)|^2)|\operatorname{curl} \boldsymbol{h}(\gamma)|^2)(\theta(\gamma_n) - \theta(\gamma))\, dx.\end{aligned}$$

Therefore we have

$$\begin{aligned}&\int_\Omega (T_t(x, \gamma_n, |\nabla\theta(\gamma_n)|^2)\nabla\theta(\gamma_n) - T_t(x, \gamma_n, |\nabla\theta(\gamma)|^2)\nabla\theta(\gamma)) \cdot \nabla(\theta(\gamma_n) - \theta(\gamma))\, dx \\ &\quad + \int_\Omega (T_t(x, \gamma_n, |\nabla\theta(\gamma)|^2)\nabla\theta(\gamma) - T_t(x, \gamma, |\nabla\theta(\gamma)|^2)\nabla\theta(\gamma)) \cdot \nabla(\theta(\gamma_n) - \theta(\gamma))\, dx \\ &= \int_\Omega \Big\{\tau_M(S_t(x, \gamma_n, |\operatorname{curl} \boldsymbol{h}(\gamma_n)|^2)|\operatorname{curl} \boldsymbol{h}(\gamma_n)|^2) \\ &\qquad - \tau_M(S_t(x, \gamma, |\operatorname{curl} \boldsymbol{h}(\gamma)|^2)|\operatorname{curl} \boldsymbol{h}(\gamma)|^2)\Big\}(\theta(\gamma_n) - \theta(\gamma))\, dx.\end{aligned} \tag{5.6}$$

Here we use Lemma 2.5(ii). When $q \geq 2$, we have

$$\begin{aligned}&c\|\nabla(\theta(\gamma_n) - \theta(\gamma))\|_{L^q(\Omega)}^q \\ &\leq \|T_t(x, \gamma_n, |\nabla\theta(\gamma)|^2)\nabla\theta(\gamma) - T_t(x, \gamma, |\nabla\theta(\gamma)|^2)\nabla\theta(\gamma)\|_{L^{q'}(\Omega)}\|\nabla(\theta(\gamma_n) - \theta(\gamma))\|_{L^q(\Omega)} \\ &\quad + \|\tau_M(S_t(x, \gamma_n, |\operatorname{curl} \boldsymbol{h}(\gamma_n)|^2)|\operatorname{curl} \boldsymbol{h}(\gamma_n)|^2) - \tau_M(S_t(x, \gamma, |\operatorname{curl} \boldsymbol{h}(\gamma)|^2)|\operatorname{curl} \boldsymbol{h}(\gamma)|^2)\|_{L^{q'}(\Omega)} \\ &\qquad \times \|\theta(\gamma_n) - \theta(\gamma)\|_{L^q(\Omega)}.\end{aligned}$$

Using Poncaré inequality and Young's inequality, we have

$$\begin{aligned}\|\nabla(\theta(\gamma_n) - \theta(\gamma))\|_{L^q(\Omega)}^q \leq C\Big[&\|T_t(x, \gamma_n, |\nabla\theta(\gamma)|^2)\nabla\theta(\gamma) - T_t(x, \gamma, |\nabla\theta(\gamma)|^2)\nabla\theta(\gamma)\|_{L^{q'}(\Omega)}^{q'} \\ &+ \|\tau_M(S_t(x, \gamma_n, |\operatorname{curl} \boldsymbol{h}(\gamma_n)|^2)|\operatorname{curl} \boldsymbol{h}(\gamma_n)|^2) \\ &\quad - \tau_M(S_t(x, \gamma, |\operatorname{curl} \boldsymbol{h}(\gamma)|^2)|\operatorname{curl} \boldsymbol{h}(\gamma)|^2)\|_{L^{q'}(\Omega)}^{q'}\Big],\end{aligned}$$

where C is a constant independent of n. Since

$$|T_t(x,\gamma_n,|\nabla\theta(\gamma)|^2)\nabla\theta(\gamma)|^{q'} \le \Lambda^{q'}|\nabla\theta|^{(q-1)q'} = \Lambda^{q'}|\nabla\theta|^q$$

and $|\nabla\theta|^q$ is an integrable function in Ω which is independent of n. Since $T_t(x,s,t)$ is a Carathéodory function and $\gamma_n \to \gamma$ a.e. in Ω,

$$T_t(x,\gamma_n,|\nabla\theta(\gamma)|^2)\nabla\theta(\gamma) \to T_t(x,\gamma,|\nabla\theta(\gamma)|^2)\nabla\theta(\gamma) \quad \text{a.e. in } \Omega.$$

Thus by the Lebesgue dominated theorem, we have

$$\lim_{n\to\infty}\|T_t(x,\gamma_n,|\nabla\theta(\gamma)|^2)\nabla\theta(\gamma) - T_t(x,\gamma,|\nabla\theta(\gamma)|^2)\nabla\theta(\gamma)\|^{q'}_{L^{q'}(\Omega)} = 0.$$

Moreover,

$$S_t(x,\gamma_n,|\operatorname{curl}\boldsymbol{h}(\gamma_n)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma_n)|^2 \to S_t(x,\gamma,|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma)|^2 \quad \text{a.e. in } \Omega$$

and τ_M is a continuous function, and $|\tau_M(s)| \le M$. Applying again the Lebesgue theorem,

$$\begin{aligned}\lim_{n\to\infty}\|&\tau_M(S_t(x,\gamma_n,|\operatorname{curl}\boldsymbol{h}(\gamma_n)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma_n)|^2)\\ &- \tau_M(S_t(x,\gamma,|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)\|^{q'}_{L^{q'}(\Omega)} = 0.\end{aligned}$$

Hence $\theta(\gamma_n) = \theta(\gamma_n,\boldsymbol{h}(\gamma_n)) \to \theta(\gamma) = \theta(\gamma,\boldsymbol{h}(\gamma))$ in $W_0^{1,q}(\Omega)$ as $n\to\infty$.

When $1<q<2$, from (5.6) we have

$$\begin{aligned}&c\int_\Omega (|\nabla\theta(\gamma_n)| + |\nabla\theta(\gamma)|)^{q-2}|\nabla(\theta(\gamma_n)-\theta(\gamma))|^2\,dx\\ &\le \|T_t(x,\gamma_n,|\nabla\theta(\gamma)|^2)\nabla\theta(\gamma) - T_t(x,\gamma,|\nabla\theta(\gamma)|^2)\nabla\theta(\gamma)\|_{L^{q'}(\Omega)}\|\nabla(\theta(\gamma_n)-\theta(\gamma))\|_{L^q(\Omega)}\\ &\quad + \|\tau_M(S_t(x,\gamma_n,|\operatorname{curl}\boldsymbol{h}(\gamma_n)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma_n)|^2) - \tau_M(S_t(x,\gamma,|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)\|_{L^{q'}(\Omega)}\\ &\qquad \times \|\theta(\gamma_n)-\theta(\gamma)\|_{L^q(\Omega)}.\end{aligned}$$

If we use the reverse Hölder inequality, we have

$$\begin{aligned}&c\int_\Omega (|\nabla\theta(\gamma_n)| + |\nabla\theta(\gamma)|)^{q-2}|\nabla(\theta(\gamma_n)-\theta(\gamma))|^2\,dx\\ &\ge c\left(2^{q-1}(\|\nabla\theta(\gamma_n)\|^q_{L^q(\Omega)} + \|\nabla\theta(\gamma)\|^q_{L^q(\Omega)})\right)^{(q-2)/2}\|\nabla(\theta(\gamma_n)-\theta(\gamma))\|^2_{L^q(\Omega)}.\end{aligned}$$

Using (5.5), by the arguments similar as the case $q\ge 2$, we have

$$\begin{aligned}\|\nabla(\theta(\gamma_n)-\theta(\gamma))\|^2_{L^q(\Omega)} \le C\Big[&\|T_t(x,\gamma_n,|\nabla\theta(\gamma)|^2)\nabla\theta(\gamma) - T_t(x,\gamma,|\nabla(\gamma)|^2)\nabla\theta(\gamma)\|^2_{L^{q'}(\Omega)}\\ &+ \|\tau_M(S_t(x,\gamma_n,|\operatorname{curl}\boldsymbol{h}(\gamma_n)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma_n)|^2)\\ &\quad - \tau_M(S_t(x,\gamma,|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)|\operatorname{curl}\boldsymbol{h}(\gamma)|^2)\|^2_{L^{q'}(\Omega)}\Big].\end{aligned}$$

Thus we have $\theta(\gamma_n) \to \theta(\gamma)$ in $W_0^{1,q}(\Omega)$ as $n \to \infty$. Hence we have proved that S_2 is continuous.

We note that the inclusion map $W_0^{1,q}(\Omega) \hookrightarrow L^q(\Omega)$ is compact by the Rellich and Kondrachov theorem. For any fixed $M > 0$, if we choose $R > 0$ so that $C_M \le R$, we see that S_2 is a continuous and compact operator from the bounded closed convex set D_R of $L^q(\Omega)$ to D_R. By the Schauder fixed point theorem, S_2 has a fixed point in D_R, i.e., there exists $\theta_M \in W_0^{1,q}(\Omega)$ such that

$$\theta_M = \theta(\theta_M, \boldsymbol{h}(\theta_M)).$$

This implies that $(\boldsymbol{h}_M = \boldsymbol{h}(\theta_M), \theta_M) \in \mathbb{V}^p(\Omega) \times W_0^{1,q}(\Omega)$ is a solution of (5.1a)–(5.1b). $\square$

6. Proof of Theorem 2.4

In this section, we give a proof of Theorem 2.4. When $q > 3$, by Sobolev embedding theorem, $W_0^{1,q}(\Omega) \hookrightarrow C^0(\overline{\Omega})$. In this case, it suffices to repeat the proof of Proposition 5.1 without truncation.

When $5/3 < q \le 3$, we write the solution of (5.1a)–(5.1b) by $(\boldsymbol{h}_M, \theta_M)$. We note that

$$\begin{aligned} \|\tau_M(S_t(x, \theta_M, |\operatorname{curl} \boldsymbol{h}_M|^2)|\operatorname{curl} \boldsymbol{h}_M|^2)\|_{L^1(\Omega)} &\le \Lambda \| |\operatorname{curl} \boldsymbol{h}_M|^p\|_{L^1(\Omega)} \\ &= \Lambda \|\operatorname{curl} \boldsymbol{h}_M\|^p_{L^p(\Omega)} \le C^*, \end{aligned} \tag{6.1}$$

where C^* is a constant independent on M which follows from (5.3). Thus by Boccardo and Gallouët [7, Theorem 1.1 and Lemma 1], for any $1 < \overline{r} < 3(q-1)/2$, $\theta_M \in W_0^{1,\overline{r}}(\Omega)$ and

$$\|\theta_M\|_{W_0^{1,\overline{r}}(\Omega)} \le C_{\overline{r}}, \tag{6.2}$$

where $C_{\overline{r}}$ is a constant independent of M. From (6.1) and (6.2), passing to a subsequence, we may assume that as $M \to \infty$,

$$\begin{aligned} \boldsymbol{h}_M &\to \boldsymbol{h} && \text{strongly in } L^p(\Omega), \\ \operatorname{curl} \boldsymbol{h}_M &\to \operatorname{curl} \boldsymbol{h} && \text{weakly in } L^p(\Omega), \\ \theta_M &\to \theta && \text{strongly in } L^{\overline{r}}(\Omega) \text{ and a.e. in } \Omega, \\ \nabla\theta_M &\to \nabla\theta && \text{weakly in } L^{\overline{r}}(\Omega). \end{aligned}$$

Since

$$\begin{aligned} \|S_t(x, \theta_M, |\operatorname{curl} \boldsymbol{h}_M|^2) \operatorname{curl} \boldsymbol{h}_M\|_{L^{p'}(\Omega)} &\le \Lambda \| |\operatorname{curl} \boldsymbol{h}_M|^{p-1}\|_{L^{p'}(\Omega)} \\ &= \Lambda \|\operatorname{curl} \boldsymbol{h}_M\|^{p-1}_{L^p(\Omega)} \le C, \end{aligned}$$

where C is a constant independent of M which follows from (5.3), we may assume that

$$S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M \to \boldsymbol{\lambda} \quad \text{weakly in } L^{p'}(\Omega).$$

If we show

$$\int_\Omega \boldsymbol{\lambda}\cdot\operatorname{curl}\boldsymbol{v}\,dx = \int_\Omega S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\cdot\operatorname{curl}\boldsymbol{v}\,dx \quad \text{for all } \boldsymbol{v}\in X_T^p(\Omega), \tag{6.3}$$

it follows from (5.1a) that we have

$$\begin{aligned}\int_\Omega S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\cdot\operatorname{curl}\boldsymbol{v}\,dx &= \int_\Omega \boldsymbol{\lambda}\cdot\operatorname{curl}\boldsymbol{v}\,dx\\ &= \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx \quad \text{for all } \boldsymbol{v}\in X_T^p(\Omega).\end{aligned}$$

Thus (2.7a) holds.

We show (6.3). From (5.1a), we have

$$\int_\Omega \boldsymbol{\lambda}\cdot\operatorname{curl}\boldsymbol{v}\,dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{v}\,dx$$

for all $\boldsymbol{v}\in X_T^p(\Omega)$. On the other hand, it follows from (5.1a) with $\boldsymbol{v}=\boldsymbol{h}_M$,

$$\begin{aligned}&\lim_{M\to\infty}\int_\Omega S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M\cdot\operatorname{curl}\boldsymbol{h}_M\,dx\\ &= \lim_{M\to\infty}\int_\Omega \boldsymbol{f}\cdot\boldsymbol{h}_M\,dx = \int_\Omega \boldsymbol{f}\cdot\boldsymbol{h}\,dx = \int_\Omega \boldsymbol{\lambda}\cdot\operatorname{curl}\boldsymbol{h}\,dx.\end{aligned} \tag{6.4}$$

By the monotonicity lemma (see Lemma 2.5(i)), for any $\boldsymbol{v}\in X_T^p(\Omega)$, we have

$$\begin{aligned}&\int_\Omega S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{v})\,dx\\ &\quad-\int_\Omega S_t(x,\theta,|\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v}\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{v})\,dx\\ &= \int_\Omega S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{v})\,dx\\ &\quad-\int_\Omega S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v}\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{v})\,dx\\ &\quad+\int_\Omega (S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{v}|)\operatorname{curl}\boldsymbol{v}-S_t(x,\theta,|\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v})\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{v})\,dx\\ &\ge \int_\Omega (S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v}-S_t(x,\theta,|\operatorname{curl}\boldsymbol{v}|^2)\operatorname{curl}\boldsymbol{v})\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{v})\,dx.\end{aligned} \tag{6.5}$$

Since $S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M \to \boldsymbol{\lambda}$ weakly in $L^{p'}(\Omega)$, taking (6.4) into consideration, we have

$$\int_\Omega S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{v})\,dx \to \int_\Omega \boldsymbol{\lambda}\cdot\operatorname{curl}(\boldsymbol{h}-\boldsymbol{v})\,dx$$

as $M \to \infty$. Therefore the left-hand side of (6.5) converges to

$$\int_\Omega (\boldsymbol{\lambda} - S_t(x, \theta, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v}) \cdot \operatorname{curl}(\boldsymbol{h} - \boldsymbol{v})\, dx.$$

On the other hand, since

$$\begin{aligned}&\left| \int_\Omega (S_t(x, \theta_M, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v} - S_t(x, \theta, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v}) \cdot \operatorname{curl}(\boldsymbol{h}_M - \boldsymbol{v})\, dx \right| \\ &\le \|S_t(x, \theta_M, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v} - S_t(x, \theta, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v}\|_{L^{p'}(\Omega)} \|\operatorname{curl}(\boldsymbol{h}_M - \boldsymbol{v})\|_{L^p(\Omega)}.\end{aligned}$$

Here we note that

$$|S_t(x, \theta_M, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v} - S_t(x, \theta, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v}|^{p'} \le (2\Lambda)^{p'} |\operatorname{curl} \boldsymbol{v}|^p$$

and

$$S_t(x, \theta_M, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v} \to S_t(x, \theta, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v}$$

a.e. in Ω, it follows form the Lebesgue theorem that

$$\|S_t(x, \theta_M, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v} - S_t(x, \theta, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v}\|_{L^{p'}(\Omega)} \to 0 \tag{6.6}$$

as $M \to \infty$. Since $\|\operatorname{curl} \boldsymbol{h}_M\|_{L^p(\Omega)} \le C$, where C is a constant independent of M, the right-hand side of (6.5) converges to zero as $M \to \infty$. Thus we have

$$\int_\Omega (\boldsymbol{\lambda} - S_t(x, \theta, |\operatorname{curl} \boldsymbol{v}|^2) \operatorname{curl} \boldsymbol{v}) \cdot \operatorname{curl}(\boldsymbol{h} - \boldsymbol{v})\, dx \ge 0.$$

If we put $\boldsymbol{v} = \boldsymbol{h} - \alpha \boldsymbol{w}$, $\alpha > 0$, $\boldsymbol{w} \in X_T^p(\Omega)$, then we have

$$\int_\Omega (\boldsymbol{\lambda} - S_t(x, \theta, |\operatorname{curl} \boldsymbol{h} - \alpha \operatorname{curl} \boldsymbol{w})|^2)(\operatorname{curl} \boldsymbol{h} - \alpha \operatorname{curl} \boldsymbol{w}) \cdot \operatorname{curl} \boldsymbol{w}\, dx \ge 0.$$

Letting $\alpha \to +0$, we have

$$\int_\Omega (\boldsymbol{\lambda} - S_t(x, \theta, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h}) \cdot \operatorname{curl} \boldsymbol{w}\, dx \ge 0 \quad \text{for all } \boldsymbol{w} \in X_T^p(\Omega).$$

This implies that

$$\int_\Omega (\boldsymbol{\lambda} - S_t(x, \theta, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h}) \cdot \operatorname{curl} \boldsymbol{w}\, dx = 0 \quad \text{for all } \boldsymbol{w} \in X_T^p(\Omega).$$

Hence (6.3) holds.

From (2.7a) and (5.1a) with $\boldsymbol{v} = \boldsymbol{h}_M - \boldsymbol{h}$, we have

$$\int_\Omega S_t(x, \theta, |\operatorname{curl} \boldsymbol{h}|^2) \operatorname{curl} \boldsymbol{h} \cdot \operatorname{curl}(\boldsymbol{h}_M - \boldsymbol{h})\, dx = \int_\Omega \boldsymbol{f} \cdot (\boldsymbol{h}_M - \boldsymbol{h})\, dx$$

and

$$\int_\Omega S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\,dx=\int_\Omega \boldsymbol{f}\cdot(\boldsymbol{h}_M-\boldsymbol{h})\,dx.$$

Thus we have

$$\begin{aligned}&\int_\Omega(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M-S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\,dx\\ &=\int_\Omega(S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}-S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\,dx.\end{aligned}\tag{6.7}$$

From (6.6) with $\boldsymbol{v}=\boldsymbol{h}$,

$$S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\to S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}$$

in $L^{p'}(\Omega)$, and $\operatorname{curl}\boldsymbol{h}_M\to\operatorname{curl}\boldsymbol{h}$ weakly in $L^p(\Omega)$. Hence the right-hand side of (6.7) converges to zero as $M\to\infty$. We apply the monotonicity condition (see Lemma 2.5(i)) to (6.7). When $p\ge 2$, we have

$$\begin{aligned}&c\int_\Omega|\operatorname{curl}\boldsymbol{h}_M-\operatorname{curl}\boldsymbol{h}|^p\,dx\\ &\le\int_\Omega(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M-S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\,dx\to 0.\end{aligned}$$

When $1<p<2$, we have

$$\begin{aligned}&c\int_\Omega(|\operatorname{curl}\boldsymbol{h}_M|+|\operatorname{curl}\boldsymbol{h}|)^{p-2}|\operatorname{curl}\boldsymbol{h}_M-\operatorname{curl}\boldsymbol{h}|^2\,dx\\ &\le\int_\Omega(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M-S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\,dx.\end{aligned}$$

Applying again the reverse Hölder inequality and (6.1),

$$\begin{aligned}&c\int_\Omega|\operatorname{curl}\boldsymbol{h}_M-\operatorname{curl}\boldsymbol{h}|^2\,dx\\ &\le\int_\Omega(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M-S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\,dx\to 0.\end{aligned}$$

Therefore, passing to a subsequence, we may assume that

$$\operatorname{curl}\boldsymbol{h}_M\to\operatorname{curl}\boldsymbol{h}\quad\text{in } L^p(\Omega)\text{ and a.e. in }\Omega.\tag{6.8}$$

We show that

$$S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)|\operatorname{curl}\boldsymbol{h}_M|^2\to S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)|\operatorname{curl}\boldsymbol{h}|^2\quad\text{in } L^1(\Omega).\tag{6.9}$$

In fact, we write

$$\begin{aligned}&|S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)|\operatorname{curl}\boldsymbol{h}_M|^2-S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)|\operatorname{curl}\boldsymbol{h}|^2|\\ &\le|S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})|\\ &\quad+|(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M-S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}\boldsymbol{h}|.\end{aligned}$$

Using (6.1) and (6.8),

$$\begin{aligned}
&\|S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M\cdot\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\|_{L^1(\Omega)}\\
&\le \Lambda\|\operatorname{curl}\boldsymbol{h}_M\|_{L^p(\Omega)}^{p-1}\|\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\|_{L^p(\Omega)}\\
&\le C^*\|\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\|_{L^p(\Omega)}\to 0.
\end{aligned}$$

We can write

$$\begin{aligned}
&|(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M-S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}\boldsymbol{h}|\\
&\le |(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M-S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}\boldsymbol{h}|\\
&\quad+|S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)|\operatorname{curl}\boldsymbol{h}|^2-S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)|\operatorname{curl}\boldsymbol{h}|^2.
\end{aligned}$$

We show that the integral of every term on the right-hand side of the above inequality converges to zero as $M\to\infty$. Since

$$S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\to S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\quad\text{in } L^{p'}(\Omega),$$

we can see that

$$\begin{aligned}
&\int_\Omega|(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}-S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}\boldsymbol{h}|\,dx\\
&\le\|S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}-S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\|_{L^{p'}(\Omega)}\|\operatorname{curl}\boldsymbol{h}\|_{L^p(\Omega)}\to 0.
\end{aligned}$$

Using Lemma 2.6(i) and (6.8), if $1<p\le 2$,

$$\begin{aligned}
&\int_\Omega|(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M-S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h})\cdot\operatorname{curl}\boldsymbol{h}|\,dx\\
&\le C\int_\Omega|\operatorname{curl}\boldsymbol{h}_M-\operatorname{curl}\boldsymbol{h}|^{p-1}|\operatorname{curl}\boldsymbol{h}|\,dx\\
&\le C\|\operatorname{curl}\boldsymbol{h}_M-\operatorname{curl}\boldsymbol{h}\|_{L^p(\Omega)}^{p-1}\|\operatorname{curl}\boldsymbol{h}\|_{L^p(\Omega)}\to 0.
\end{aligned}$$

If $p>2$, we have

$$\begin{aligned}
&\|(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)\operatorname{curl}\boldsymbol{h}_M-S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}|^2)\operatorname{curl}\boldsymbol{h}\|_{L^1(\Omega)}\\
&\le C\int_\Omega(|\operatorname{curl}\boldsymbol{h}_M|+|\operatorname{curl}\boldsymbol{h}|)^{p-2}|\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})||\operatorname{curl}\boldsymbol{h}|\,dx\\
&\le C'\int_\Omega(|\operatorname{curl}\boldsymbol{h}_M|^{p-2}|\operatorname{curl}\boldsymbol{h}|+|\operatorname{curl}\boldsymbol{h}|^{p-1})|\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})|\,dx.
\end{aligned}$$

Here

$$\int_\Omega|\operatorname{curl}\boldsymbol{h}|^{p-1}|\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})|\,dx\le\|\operatorname{curl}\boldsymbol{h}\|_{L^p(\Omega)}^{p-1}\|\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\|_{L^p(\Omega)}\to 0,$$

and by Hölder inequality,

$$\begin{aligned}
&\int_\Omega |\operatorname{curl}\boldsymbol{h}||\operatorname{curl}\boldsymbol{h}_M|^{p-2}|\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})|\,dx\\
&\le \|\operatorname{curl}\boldsymbol{h}\|_{L^p(\Omega)}\left(\int_\Omega |\operatorname{curl}\boldsymbol{h}_M|^p\,dx\right)^{p/(p-2)}\|\operatorname{curl}(\boldsymbol{h}_M-\boldsymbol{h})\|_{L^p(\Omega)}\to 0.
\end{aligned}$$

Thus (6.9) holds. Hence we have

$$\tau_M(S_t(x,\theta_M,|\operatorname{curl}\boldsymbol{h}_M|^2)|\operatorname{curl}\boldsymbol{h}_M|^2)\to S_t(x,\theta,|\operatorname{curl}\boldsymbol{h}|^2)|\operatorname{curl}\boldsymbol{h}|^2 \quad \text{in } L^1(\Omega).$$

According to Boccardo and Gallouët [8, Lemma 1], for any $1<\overline{r}<3(q-1)/2$, $\{\theta_M\}$ is compact in $W_0^{1,\overline{r}}(\Omega)$. Passing to a subsequence, we may assume that $\theta_M\to\theta$ in $W_0^{1,\overline{r}}(\Omega)$ and a.e. in Ω. For $1<s<3/2$, we have

$$\begin{aligned}
&|T_t(x,\theta_M,|\nabla\theta_M|^2)\nabla\theta_M - T_t(x,\theta,|\nabla\theta|^2)\nabla\theta|^s\\
&\le 2^{s-1}\Big[|T_t(x,\theta_M,|\nabla\theta_M|^2)\nabla\theta_M - T_t(x,\theta_M,|\nabla\theta|^2)\nabla\theta|^s\\
&\qquad + |T_t(x,\theta_M,|\nabla\theta|^2)\nabla\theta - T_t(x,\theta,|\nabla\theta|^2)\nabla\theta|^s\Big].
\end{aligned}$$

We use Lemma 2.6(ii). We note that $(q-1)s<3(q-1)/2$.

When $5/3<q\le 2$, we have

$$\int_\Omega |T_t(x,\theta_M,|\nabla\theta_M|^2)\nabla\theta_M - T_t(x,\theta_M,|\nabla\theta|^2)\nabla\theta|^s\,dx \le C\int_\Omega |\nabla\theta_M-\nabla\theta|^{(q-1)s}\,dx\to 0$$

as $M\to\infty$.

When $q>2$, since

$$\int_\Omega |\nabla\theta_M|^{(q-1)s}\,dx\le C,$$

where C is a constant independent of M, we have

$$\begin{aligned}
&\int_\Omega |T_t(x,\theta_M,|\nabla\theta_M|^2)\nabla\theta_M - T_t(x,\theta_M,|\nabla\theta|^2)\nabla\theta|^s\,dx\\
&\le C\int_\Omega (|\nabla\theta_M|+|\nabla\theta|)^{(q-2)s}|\nabla\theta_M-\nabla\theta|^s\,dx\\
&\le C\left(\int_\Omega (|\nabla\theta_M|+|\nabla\theta|)^{(q-1)s}\,dx\right)^{(q-2)/(q-1)}\left(\int_\Omega |\nabla\theta_M-\nabla\theta|^{(q-1)s}\,dx\right)^{1/(q-1)}\to 0.
\end{aligned}$$

Using (2.2b), we have

$$|T_t(x,\theta_M,|\nabla\theta|^2)\nabla\theta - T_t(x,\theta,|\nabla\theta|^2)\nabla\theta|^s\le C|\nabla\theta|^{(q-1)s}.$$

Since the right-hand side is an integrable function which is independent of M, and $T_t(x,\theta_M,|\nabla\theta|^2)\nabla\theta\to T_t(x,\theta,|\nabla\theta|^2)\nabla\theta$ a.e. in Ω, it follows from the Lebesgue theorem that

$$\int_\Omega |T_t(x,\theta_M,|\nabla\theta|^2)\nabla\theta - T_t(x,\theta,|\nabla\theta|^2)\nabla\theta|^s\,dx\to 0 \quad \text{as } M\to\infty.$$

Therefore we can see that for any $1 \le s < 3/2$,

$$T_t(x, \theta_M, |\nabla\theta_M|^2)\nabla\theta_M \to T_t(x, \theta, |\nabla\theta|^2)\nabla\theta \quad \text{in } L^s(\Omega) \text{ as } M \to \infty.$$

For any $\xi \in W_0^{1,\infty}(\Omega)$, letting $M \to \infty$ in (5.1b), we get

$$\int_\Omega T_t(x, \theta, |\nabla\theta|^2)\nabla\theta \cdot \nabla\xi \, dx = \int_\Omega S_t(x, \theta, |\operatorname{curl} \boldsymbol{h}|^2)|\operatorname{curl} \boldsymbol{h}|^2 \xi \, dx.$$

This completes the proof of Theorem 2.4.

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References

[1] C. Amrouche and N. E. H. Seloula, *L^p-theory for vector potentials and Sobolev's inequalities for vector fields*, C. R. Math. Acad. Sci. Paris **349** (2011), no. 9-10, 529–534.

[2] ______, *L^p-theory for vector potentials and Sobolev's inequalities for vector fields: Application to the Stokes equations with pressure boundary conditions*, Math. Models Methods Appl. Sci. **23** (2013), no. 1, 37–92.

[3] J. Aramaki, *L^p theory for the div-curl system*, Int. J. Math. Anal. (Ruse) **8** (2014), no. 5-8, 259–271.

[4] ______, *Existence of solutions for a quasilinear parabolic type equation containing p-curl system and applications to blow-up and extinction*, Adv. Differ. Equ. Control Process. **17** (2016), no. 2, 117–158.

[5] ______, *Existence of a solution to a stationary quai-variational inequality in a multi-connected domain*, Submitted.

[6] ______, *Existence and regularity of a weak solution to a class of systems in a multi-connected domain*, Submitted.

[7] L. Boccardo and T. Gallouët, *Nonlinear elliptic and parabolic equations involving measure data*, J. Funct. Anal. **87** (1989), no. 1, 149–169.

[8] ______, *Nonlinear elliptic equations with right-hand side measures*, Comm. Partial Differential Equations **17** (1992), no. 3-4, 641–655.

[9] A. Bossavit, *Computational Electromagnetism: Variational formulations, complementarity, edge elements*, Electromagnetism, Academic Press, San Diego, CA, 1998.

[10] R. Dautray and J.-L. Lions, *Mathematical Analysis and Numerical Methods for Science and Technology, Vol.* 3, Springer-Verlag, New York, 1990.

[11] E. DiBenedetto, *Degenerate Parabolic Equations*, Universitext, Springer-Verlag, New York, 1993.

[12] I. Ekeland and R. Témam, *Convex Analysis and Variational Problems*, Classics in Applied Mathematics **28**, Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 1999.

[13] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier-Stokes Equations: Theory and algorithms*, Springer Series in Computational Mathematics **5**, Springer-Verlag, Berlin, 1986.

[14] J.-L. Lions, *Quelques Méthodes de Résolution des Problèmes aux Limites Non Linéaires*, Dunod, Gauthier-Villars, Paris, 1969.

[15] F. Miranda, J.-F. Rodrigues and L. Santos, *A class of stationary nonlinear Maxwell systems*, Math. Models Methods Appl. Sci. **19** (2009), no. 10, 1883–1905.

[16] ———, *On a p-curl system arising in electromagnetism*, Discrete Contin. Dyn. Syst. Ser. S **5** (2012), no. 3, 605–629.

[17] D. Mitrea, M. Mitrea and J. Pipher, *Vector potential theory on nonsmooth domains in* $\mathbb{R}^3$ *and applications to electromagnetic scattering*, J. Fourier Anal. Appl. **3** (1997), no. 2, 131–192.

[18] R. Monneau, *Quasilinear elliptic system arising in a three-dimensional type-II superconductor for infinite* κ, Nonlinear Anal. **52** (2003), no. 3, 917–930.

[19] S. L. Sobolev, *Applications of Functional Analysis in Mathematical Physics*, Translations of Mathematical Monographs **7**, American Mathematical Society, Providence, R.I., 1963.

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