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Folding Phenomenon of Major-balance Identities on Restricted Involutions

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Abstract. In this paper we prove refined major-balance identities on the 321-avoiding involutions of length n, respecting the leading element of permutations. The proof is based on sign-reversing involutions on the lattice paths within a $\lfloor n/2 \rfloor \times \lceil n/2 \rceil$ rectangle. Moreover, we prove affirmatively a question about refined major-balance identities on the 123-avoiding involutions, respecting the number of descents.

1. Introduction

Let $\mathfrak{S}_n$ be the set of permutations of $\{1, 2, \ldots, n\}$. A permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$ is called 321-*avoiding* (123-*avoiding*, respectively) if it has no decreasing (increasing, respectively) subsequence of length three. Let $\mathfrak{S}_n(321)$ ($\mathfrak{S}_n(123)$, respectively) be the set of 321-avoiding (123-avoiding, respectively) permutations in $\mathfrak{S}_n$. It is known that $|\mathfrak{S}_n(321)| = |\mathfrak{S}_n(123)| = C_n = \frac{1}{n+1}\binom{2n}{n}$, the nth Catalan number.

1.1. Sign-balance for restricted permutations

The sign-balance of restricted permutations is an interesting theme in enumerative combinatorics. Simion and Schmidt [7] proved the following sign-balance property of $\mathfrak{S}_n(321)$:

$$\sum_{\sigma \in \mathfrak{S}_n(321)} (-1)^{\mathrm{inv}(\sigma)} = \begin{cases} C_{(n-1)/2} & \text{if } n \text{ is odd}, \\ 0 & \text{if } n \text{ is even}, \end{cases}$$

where $\mathrm{inv}(\sigma) = \#\{(\sigma_i, \sigma_j) : \sigma_i > \sigma_j \text{ and } i < j\}$ is the *inversion number* of σ. Adin and Roichman [1] proved a refinement of this result, respecting the position of the last descent (ldes) of σ, i.e., $\mathrm{ldes}(\sigma) = \max\{i : \sigma_i > \sigma_{i+1} \text{ and } 1 \le i \le n-1\}$.

Theorem 1.1 (Adin-Roichman)**.** *For all $n \ge 1$, the following identities hold:*

(i) $\displaystyle\sum_{\sigma \in \mathfrak{S}_{2n+1}(321)} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{ldes}(\sigma)} = \sum_{\sigma \in \mathfrak{S}_n(321)} q^{2\cdot\mathrm{ldes}(\sigma)}$,

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(ii) $$\sum_{\sigma\in\mathfrak{S}_{2n}(321)} (-1)^{\mathrm{inv}(\sigma)} q^{\mathrm{ldes}(\sigma)} = (1-q) \sum_{\sigma\in\mathfrak{S}_{n}(321)} q^{2\cdot\mathrm{ldes}(\sigma)}.$$

Later, Reifegerste [6] proved an analogous refinement for another permutation statistic, the length of the longest increasing subsequence of the permutations. Eu et al. [4] turned to other families of restricted permutations and obtained refined sign-balance results for 321-avoiding alternating permutations, respecting the leading element and the last element of permutations, respectively.

These results share a folding phenomenon that with respect to a certain statistic the sign-balance generating function for restricted permutations of length $2n$ essentially equals the ordinary generating function for the permutations of length n. Eu et al. [5] described the folding phenomenon in the framework

$$\sum_{\pi\in\mathscr{X}_{2n} \text{ or } \mathscr{X}_{2n+1}} (-1)^{\mathrm{stat}_1(\pi)} q^{\mathrm{stat}_2(\pi)} = f(q) \sum_{\pi\in\mathscr{X}_n} q^{2\cdot\mathrm{stat}_2(\pi)},$$

where $\mathscr{X}_n$ is a family of combinatorial objects of size n with statistics stat_1 and stat_2, and $f(q)$ is a rational function. In this paper, we present an instance of such a phenomenon on 321-avoiding involutions.

1.2. Major-balance for 321-avoiding involutions

The *descent set* of σ is defined as $\mathrm{Des}(\sigma) = \{i : \sigma_i > \sigma_{i+1}, 1 \le i \le n-1\}$, and the *descent number* (des) and *major index* (maj) of σ are defined by

$$\mathrm{des}(\sigma) = \sum_{i\in\mathrm{Des}(\sigma)} 1 \quad \text{and} \quad \mathrm{maj}(\sigma) = \sum_{i\in\mathrm{Des}(\sigma)} i.$$

Recall that a permutation σ is called an *involution* if and only if $\sigma^{-1} = \sigma$. Let $\mathscr{I}_n(321)$ ($\mathscr{I}_n(123)$, respectively) be the set of involutions in $\mathfrak{S}_n(321)$ ($\mathfrak{S}_n(123)$, respectively). Simion and Schmidt [7] proved that $|\mathscr{I}_n(321)| = |\mathscr{I}_n(123)| = \binom{n}{\lfloor n/2\rfloor}$. Recently, Eu et al. [5] proved the following refined major-balance result on 321-avoiding involutions.

Theorem 1.2 (Eu-Fu-Pan-Ting)**.** *For all $n \ge 1$, the following identities hold:*

(i) $$\sum_{\sigma\in\mathscr{I}_{4n}(321)} (-1)^{\mathrm{maj}(\sigma)} q^{\mathrm{des}(\sigma)} = \sum_{\sigma\in\mathscr{I}_{2n}(321)} q^{2\cdot\mathrm{des}(\sigma)},$$

(ii) $$\sum_{\sigma\in\mathscr{I}_{4n+2}(321)} (-1)^{\mathrm{maj}(\sigma)} q^{\mathrm{des}(\sigma)} = (1-q) \sum_{\sigma\in\mathscr{I}_{2n}(321)} q^{2\cdot\mathrm{des}(\sigma)},$$

(iii) $$\sum_{\sigma\in\mathscr{I}_{2n+1}(321)} (-1)^{\mathrm{maj}(\sigma)} q^{\mathrm{des}(\sigma)} = \sum_{\sigma\in\mathscr{I}_{n}(321)} q^{2\cdot\mathrm{des}(\sigma)}.$$

Meanwhile, they asked a question about an analogous result for 123-avoiding involutions.

Conjecture 1.3 (Eu-Fu-Pan-Ting)**.** *For all $n \geq 1$, the following identities hold:*

(i) $$\sum_{\sigma \in \mathscr{I}_{4n}(123)} (-1)^{\mathrm{maj}(\sigma)} q^{\mathrm{des}(\sigma)} = q \sum_{\sigma \in \mathscr{I}_{2n}(123)} q^{2\cdot\mathrm{des}(\sigma)},$$

(ii) $$\sum_{\sigma \in \mathscr{I}_{4n+2}(123)} (-1)^{\mathrm{maj}(\sigma)} q^{\mathrm{des}(\sigma)} = (1-q)q^2 \sum_{\sigma \in \mathscr{I}_{2n}(123)} q^{2\cdot\mathrm{des}(\sigma)},$$

(iii) $$\sum_{\sigma \in \mathscr{I}_{2n+1}(123)} (-1)^{\mathrm{maj}(\sigma)} q^{\mathrm{des}(\sigma)} = (-1)^n q^2 \sum_{\sigma \in \mathscr{I}_{n}(123)} q^{2\cdot\mathrm{des}(\sigma)}.$$

1.3. Our work

Recall that the q-binomial coefficients are polynomials defined as

$$\begin{bmatrix} n \\ k \end{bmatrix}_q := \frac{[n]!_q}{[k]!_q [n-k]!_q},$$

where $[n]!_q = [1]_q[2]_q \cdots [n]_q$ and $[i]_q = 1 + q + \cdots + q^{i-1}$ for any positive integer i.

For a permutation $\sigma = \sigma_1 \cdots \sigma_n \in \mathfrak{S}_n$, let $\mathrm{lead}(\sigma)$ denote the first element of σ, i.e., $\mathrm{lead}(\sigma) = \sigma_1$. In addition to answering the above question, one of the main results in this paper is the following enumeration of joint distributions for two statistics of 321-avoiding involutions.

Theorem 1.4. *We have*

(i) $$\sum_{\substack{\sigma \in \mathscr{I}_n(321) \\ \mathrm{des}(\sigma) = k}} q^{\mathrm{maj}(\sigma)} = q^{k^2} \begin{bmatrix} \lceil n/2 \rceil \\ k \end{bmatrix}_q \begin{bmatrix} \lfloor n/2 \rfloor \\ k \end{bmatrix}_q,$$

(ii) $$\sum_{\substack{\sigma \in \mathscr{I}_n(321) \\ \mathrm{lead}(\sigma) = \ell}} q^{\mathrm{maj}(\sigma)} = \sum_{k \geq 0} q^{k^2 + k\ell + \ell - 1} \begin{bmatrix} \lceil n/2 \rceil - 1 \\ k \end{bmatrix}_q \begin{bmatrix} \lfloor n/2 \rfloor - \ell + 1 \\ k \end{bmatrix}_q.$$

The proof of the above theorem involves a specialization of the elementary symmetric functions. We remark that the identity in Theorem 1.4(i) has been proved by Barnabei et al. [2] by a different method.

The second main result is the following refined major-balance identities on 321-avoiding involutions, respecting the leading element. A curious point is that the right-hand side of the identity in Theorem 1.5(iv) is a combination of the generating functions of leading element for $\mathscr{I}_{2n+1}(321)$ and $\mathscr{I}_{2n}(321)$.

Theorem 1.5. *For all $n \geq 1$, we have*

(i) $$\sum_{\sigma\in\mathscr{I}_{4n}(321)}(-1)^{\mathrm{maj}(\sigma)}q^{\mathrm{lead}(\sigma)}=\frac{1}{q}\sum_{\sigma\in\mathscr{I}_{2n}(321)}q^{2\cdot\mathrm{lead}(\sigma)},$$

(ii) $$\sum_{\sigma\in\mathscr{I}_{4n+2}(321)}(-1)^{\mathrm{maj}(\sigma)}q^{\mathrm{lead}(\sigma)}=\left(\frac{1}{q}-1\right)\sum_{\sigma\in\mathscr{I}_{2n+1}(321)}q^{2\cdot\mathrm{lead}(\sigma)},$$

(iii) $$\sum_{\sigma\in\mathscr{I}_{4n+3}(321)}(-1)^{\mathrm{maj}(\sigma)}q^{\mathrm{lead}(\sigma)}=\left(\frac{2}{q}-1\right)\sum_{\sigma\in\mathscr{I}_{2n+1}(321)}q^{2\cdot\mathrm{lead}(\sigma)},$$

(iv) $$\sum_{\sigma\in\mathscr{I}_{4n+1}(321)}(-1)^{\mathrm{maj}(\sigma)}q^{\mathrm{lead}(\sigma)}=\left(\frac{1}{q}-1\right)\sum_{\sigma\in\mathscr{I}_{2n+1}(321)}q^{2\cdot\mathrm{lead}(\sigma)}+\sum_{\sigma\in\mathscr{I}_{2n}(321)}q^{2\cdot\mathrm{lead}(\sigma)}.$$

2. A bijection between 321-avoiding permutations and grand Dyck paths

Let m, n be positive integers. A *Dyck path* of length $2n$ is a lattice path from $(0,0)$ to (n,n), using *north* step $(0,1)$ and *east* step $(1,0)$, that stays weakly above the line $y=x$. A *partial Dyck path* of length n is a lattice path from $(0,0)$ to the line $x+y=n$ staying weakly above the line $y=x$. Let $\mathscr{P}_n$ be the set of partial Dyck paths of length n. Let $\mathscr{B}(n,m)$ denote the set of lattice paths from $(0,0)$ to (m,n) without restriction. Members of $\mathscr{B}(n,m)$ are called *grand Dyck paths*. Let N and E denote a north step and an east step, respectively.

We shall give combinatorial proofs of Theorems 1.4 and 1.5 on the basis of a bijection between $\mathscr{I}_n(321)$ and $\mathscr{B}(\lfloor n/2\rfloor,\lceil n/2\rceil)$ given by Barnabei et al. [2]. With the partial Dyck paths $\mathscr{P}_n$ being the intermediate stage, the bijection is the composition of two maps $\delta\colon \mathscr{I}_n(321)\to\mathscr{P}_n$ and $\xi\colon \mathscr{P}_n\to\mathscr{B}(\lfloor n/2\rfloor,\lceil n/2\rceil)$. First, we describe the map $\delta\colon \mathscr{I}_n(321)\to\mathscr{P}_n$ given by Deutsch et al. [3].

2.1. The bijection $\delta\colon \mathscr{I}_n(321)\to\mathscr{P}_n$

Given a permutation $\sigma=\sigma_1\cdots\sigma_n\in\mathscr{I}_n(321)$, we associate σ with a path $\tau=\delta(\sigma)=z_1\cdots z_n\in\mathscr{P}_n$, where $z_i=\mathsf{N}$ if $\sigma_i\ge i$ and $z_i=\mathsf{E}$ if $\sigma_i<i$.

To find δ^{-1}, we label the steps of τ from left to right by $1,2,\dots,n$. Proceeding from right to left across τ, couple each E step with the nearest uncoupled N step to its left. Then the cycle structure of the involution $\delta^{-1}(\tau)$ can be determined by taking the labels of a coupled pair as a transposition and an uncoupled N step as a fixed point. Recall that a permutation is an involution if its cycle structure contains no cycle of length greater than two.

Example 2.1. Consider $\sigma=2\,1\,3\,6\,7\,4\,5\,8\,10\,9\,11\in\mathscr{I}_{11}(321)$. The partial Dyck path $\tau=\delta(\sigma)$ is shown on the left-hand side of Figure 2.1. For the inverse map, if we label the steps from left to right by $1, 2, \ldots, 11$ and traverse τ backward, then we obtain the cycle structure of $\sigma = \delta^{-1}(\tau)$, i.e., $(1\,2)(3)(4\,6)(5\,7)(8)(9\,10)(11)$.

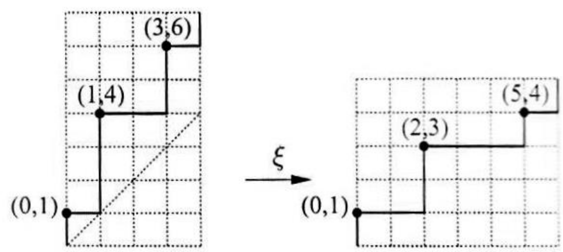

Figure 2.1: An example for the bijection ξ between $\mathscr{P}_{11}$ and $\mathscr{B}(5,6)$

A *peak* at position i of τ is an occurrence $z_i z_{i+1} = \mathsf{NE}$, which is sometimes identified with the point p between z_i and z_{i+1}. Note that the coordinate (x,y) of p satisfies $x+y=i$ and that every descent $\sigma_i > \sigma_{i+1}$ of σ is carried to a peak at position i of τ. We observe that if all fixed points in σ are ignored, each east step of τ must be coupled with a remaining north step to its left. A *valley* of τ is an occurrence of EN. A lattice point with coordinate (x,y) is said to be *even* (*odd*, respectively) if $x+y$ is even (odd, respectively). For convenience, we say that a peak or valley p is *odd* (*even*, respectively) if p is an odd (even, respectively) lattice point. Let $\mathrm{sump}(\tau)$ be the sum of the x-coordinates and y-coordinates of all peaks in τ.

2.2. The bijection $\xi\colon \mathscr{P}_n \to \mathscr{B}(\lfloor n/2\rfloor, \lceil n/2\rceil)$

Let $\tau \in \mathscr{P}_n$ be a partial Dyck path with k more N steps than E steps ($0 \le k \le n$). Match the N steps and E steps that face each other, in the sense that the line segment from the midpoint of N to the midpoint of E has slope 1 and stays below the path τ. Then we construct the path $\xi(\tau) \in \mathscr{B}(\lfloor n/2\rfloor, \lceil n/2\rceil)$ by changing the first $\lceil k/2\rceil$ unmatched N steps into E steps. Note that a peak (x,y) in τ is carried to a peak (x',y') in $\xi(\tau)$ with $x+y = x'+y'$.

Example 2.2. Following Example 2.1, consider the path $\tau = z_1 \cdots z_{11} \in \mathscr{P}_{11}$ shown on the left-hand side of Figure 2.1. Note that $\mathrm{sump}(\tau) = 15$. The unmatched N steps are z_3, z_8 and z_{11}. Then the corresponding path $\xi(\pi) \in \mathscr{B}(5,6)$ is obtained from τ by changing z_3 and z_8 into E steps, shown on the right-hand side of Figure 2.1. Note that the peaks $(1,4)$ and $(3,6)$ of τ are carried to the peaks $(2,3)$ and $(5,4)$ of $\xi(\tau)$, respectively and $\mathrm{sump}(\xi(\tau)) = 15$.

To construct ξ^{-1}, given a grand Dyck path $\pi \in \mathscr{B}(\lfloor n/2\rfloor, \lceil n/2\rceil)$, match the N steps and E steps that face each other in π. Then the path $\xi^{-1}(\pi)$ is recovered from π by changing the remaining unmatched E steps into N steps.

The following properties about the statistics $\mathrm{maj}(\sigma)$ and $\mathrm{lead}(\sigma)$ hold.

Lemma 2.3. *Given a permutation* $\sigma \in \mathscr{I}_n(321)$ *with* $\mathrm{lead}(\sigma) = \ell$, *let* $\tau = \delta(\sigma) \in \mathscr{P}_n$ *and let* $\pi = \xi(\tau) \in \mathscr{B}(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$. *Then the following results hold:*

(i) $\mathrm{maj}(\sigma) = \mathrm{sump}(\tau) = \mathrm{sump}(\pi)$.

(ii) $1 \le \ell \le \lfloor n/2 \rfloor + 1$.

(iii) *The path* π *passes through the points* $(0, \ell - 1)$ *and* $(1, \ell - 1)$.

(iv) *The number of permutations* $\sigma \in \mathscr{I}_n(321)$ *with* $\mathrm{lead}(\sigma) = \ell$ *is* $\binom{n-\ell}{\lceil n/2 \rceil - 1}$.

Proof. Note that every $i \in \mathrm{Des}(\sigma)$ corresponds to a peak (x, y) of τ and a peak (x', y') of π with $x + y = x' + y' = i$. The assertion (i) follows.

Since $\mathrm{lead}(\sigma) = \ell$, $\sigma_\ell = 1$. We observe that either $\ell = 1$ or $\sigma_1 < \cdots < \sigma_{\ell-1}$ if $\ell > 1$ since σ is 321-avoiding. If $\ell > 1$ then the entries $2, \ldots, \ell - 1$ appear to the right of σ_ℓ and hence $2\ell - 2 \le n$. Moreover, by the construction of the maps δ and ξ, we observe that the grand Dyck path π has the prefix $\mathsf{N}^{\ell-1}\mathsf{E}$. The assertions (ii) and (iii) follow.

Note that the number of permutations $\sigma \in \mathscr{I}_n(321)$ with $\mathrm{lead}(\sigma) = \ell$ coincides with the number of lattice paths from the point $(1, \ell - 1)$ to the point $(\lceil n/2 \rceil, \lfloor n/2 \rfloor)$. The assertion (iv) follows. $\square$

3. A combinatorial proof of Theorem 1.4

Define the kth *elementary symmetric function* in n variables

$$e_k = e_k(x_1, x_2, \ldots, x_n) := \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}.$$

Note that $e_0(x_1, \ldots, x_n) = 1$ and $e_k(x_1, \ldots, x_n) = 0$ for $k > n$. Recall the principle specializations of $e_k(x_1, \ldots, x_n)$ (see e.g., [8, Proposition 7.8.3]):

$$e_k(1, 1, \ldots, 1) = \binom{n}{k} \quad \text{and} \quad e_k(1, q, \ldots, q^{n-1}) = q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q.$$

Proof of Theorem 1.4. (i) By the bijection $\xi \circ \delta$ between $\mathscr{I}_n(321)$ and $\mathscr{B}(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$, a permutation $\sigma \in \mathscr{I}_n(321)$ with $\mathrm{des}(\sigma) = k$ is mapped to a grand Dyck path $\pi = \xi(\delta(\sigma))$ with k peaks, say $(x_1, y_1), \ldots, (x_k, y_k)$ with $0 \le x_1 < \cdots < x_k \le \lceil n/2 \rceil - 1$ and $1 \le y_1 <$

$\cdots < y_k \le \lfloor n/2 \rfloor$. Moreover, $\mathrm{maj}(\sigma) = \mathrm{sump}(\pi) = x_1 + \cdots + x_k + y_1 + \cdots + y_k$. Hence

$$\begin{aligned}\sum_{\substack{\sigma\in\mathscr{I}_n(321)\\ \mathrm{des}(\sigma)=k}} q^{\mathrm{maj}(\sigma)} &= \sum_{\substack{0\le x_1<\cdots<x_k\le\lceil n/2\rceil-1\\ 1\le y_1<\cdots<y_k\le\lfloor n/2\rfloor}} q^{x_1+\cdots+x_k}\cdot q^{y_1+\cdots+y_k}\\ &= e_k(1,q,\ldots,q^{\lceil n/2\rceil-1})\cdot e_k(q,q^2,\ldots,q^{\lceil n/2\rceil})\\ &= q^{k^2}\begin{bmatrix}\lceil n/2\rceil\\ k\end{bmatrix}_q\begin{bmatrix}\lfloor n/2\rfloor\\ k\end{bmatrix}_q.\end{aligned}$$

(ii) Let $\sigma \in \mathscr{I}_n(321)$ be a permutation with $\mathrm{lead}(\sigma) = \ell$ and $\mathrm{des}(\sigma) = k$. Then the corresponding grand Dyck path $\pi = \xi(\delta(\sigma))$ can be factorized as $\pi = \mathsf{N}^{\ell-1}\mathsf{E}\mu$. If $\ell = 1$ then the segment μ contains k peaks, say $(x_1, y_1), \ldots, (x_k, y_k)$ with $1 \le x_1 < \cdots < x_k \le \lceil n/2 \rceil - 1$ and $1 \le y_1 < \cdots < y_k \le \lfloor n/2 \rfloor$. Hence

$$\begin{aligned}\sum_{\substack{\sigma\in\mathscr{I}_n(321)\\ \mathrm{des}(\sigma)=k\\ \mathrm{lead}(\sigma)=1}} q^{\mathrm{maj}(\sigma)} &= e_k(q,q^2,\ldots,q^{\lceil n/2\rceil-1})\cdot e_k(q,q^2,\ldots,q^{\lceil n/2\rceil})\\ &= q^{k^2+k}\begin{bmatrix}\lceil n/2\rceil-1\\ k\end{bmatrix}_q\begin{bmatrix}\lfloor n/2\rfloor\\ k\end{bmatrix}_q.\end{aligned}$$

Otherwise $\ell > 1$, and the segment μ contains another $k-1$ peaks, say $(x_1, y_1), \ldots, (x_{k-1}, y_{k-1})$ with $1 \le x_1 < \cdots < x_{k-1} \le \lceil n/2 \rceil - 1$ and $\ell \le y_1 < \cdots < y_{k-1} \le \lfloor n/2 \rfloor$. Hence

$$\begin{aligned}\sum_{\substack{\sigma\in\mathscr{I}_n(321)\\ \mathrm{des}(\sigma)=k\\ \mathrm{lead}(\sigma)=\ell}} q^{\mathrm{maj}(\sigma)} &= q^{\ell-1}e_{k-1}(q,q^2,\ldots,q^{\lceil n/2\rceil-1})e_{k-1}(q^{\ell},q^{\ell+1},\ldots,q^{\lfloor n/2\rfloor})\\ &= q^{(k-1)^2+(k-1)\ell+\ell-1}\begin{bmatrix}\lceil n/2\rceil-1\\ k-1\end{bmatrix}_q\begin{bmatrix}\lfloor n/2\rfloor-\ell+1\\ k-1\end{bmatrix}_q.\end{aligned}$$

The assertion follows. □

We remark that Barnabei et al. [2] proved Theorem 1.4(i) by establishing a bijection between the paths in $\mathscr{B}(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$ and the partitions whose Young diagrams fit inside the $\lfloor n/2 \rfloor \times \lceil n/2 \rceil$-rectangle so that the descent set of $\sigma \in \mathscr{I}_n(321)$ is carried to the hook-decomposition of the mapped partition.

With the result in Theorem 1.4(ii), we give an arithmetic verification of Theorem 1.5 as follows. For positive integers m, n, we have the following facts (i) $[m]_{q=-1} = 0$ if and only if m is even, and (ii) if m, n have the same parity, then

$$\lim_{q\to -1}\frac{[n]_q}{[m]_q} = \begin{cases}\frac{n}{m} & \text{if } m,\, n \text{ are even},\\ 1 & \text{if } m,\, n \text{ are odd}.\end{cases}$$

Making use of the above facts, we observe that

$$\begin{aligned}\begin{bmatrix} n \\ k \end{bmatrix}_{q=-1} &= \lim_{q\to -1} \frac{[n]_q[n-1]_q\cdots[n-k+1]_q}{[1]_q[2]_q\cdots[k]_q} \\ &= \begin{cases} 0 & \text{if } n \text{ is even and } k \text{ is odd,} \\ \binom{\lfloor n/2\rfloor}{\lfloor k/2\rfloor} & \text{otherwise.} \end{cases}\end{aligned} \tag{3.1}$$

Now, we verify the identity in Theorem 1.5(i). By Theorem 1.4(ii), we have the left-hand side

$$\sum_{\substack{\sigma\in\mathscr{I}_{4n}(321)\\ \mathrm{lead}(\sigma)=\ell}} (-1)^{\mathrm{maj}(\sigma)} = \lim_{q\to -1}\sum_{k\ge 0} q^{k^2+k\ell+\ell-1}\begin{bmatrix} 2n-1 \\ k \end{bmatrix}_q \begin{bmatrix} 2n-\ell+1 \\ k \end{bmatrix}_q.$$

If ℓ is odd, say $\ell = 2\ell'-1$ then by (3.1) we have

$$\begin{aligned}\sum_{\substack{\sigma\in\mathscr{I}_{4n}(321)\\ \mathrm{lead}(\sigma)=2\ell'-1}} (-1)^{\mathrm{maj}(\sigma)} &= \sum_{k'\ge 0}\binom{n-1}{k'}\binom{n-\ell'+1}{k'} \\ &= \binom{2n-\ell'}{n-1} = |\{\sigma\in\mathscr{I}_{2n}(321) : \mathrm{lead}(\sigma)=\ell'\}|.\end{aligned}$$

If ℓ is even, say $\ell = 2\ell'$, then

$$\begin{aligned}\sum_{\substack{\sigma\in\mathscr{I}_{4n}(321)\\ \mathrm{lead}(\sigma)=2\ell'}} (-1)^{\mathrm{maj}(\sigma)} &= \sum_{k\ge 0}(-1)^{k^2-1}\begin{bmatrix} 2n-1 \\ k \end{bmatrix}_{q=-1}\begin{bmatrix} 2n-2\ell'+1 \\ k \end{bmatrix}_{q=-1} \\ &= \sum_{k=0}^{2n-2\ell'+1}(-1)^{k^2-1}\binom{n-1}{\lfloor k/2\rfloor}\binom{n-\ell'}{\lfloor k/2\rfloor} \\ &= 0.\end{aligned}$$

This agrees with the right-hand side of Theorem 1.5(i). The other identities (ii), (iii) and (iv) of Theorem 1.5 can be verified in a similar manner.

4. A combinatorial proof of Theorem 1.5

For any grand Dyck path $\pi \in \mathscr{B}(n,m)$, we factorize π as $\pi = \mu_0\mu_1\cdots\mu_d$, where each segment μ_{2i} (μ_{2i+1}, respectively) is a maximal sequence of consecutive N steps (E steps, respectively). This is called the *primal factorization* of π. Note that μ_0 is empty if π starts with an east step.

According to the length of μ_0, we partition the set $\mathscr{B}(n,m)$ into subsets $\mathscr{B}_j(n,m)$ for $0\le j\le n$, where $\mathscr{B}_j(n,m)$ consists of the paths passing the points $(0,j)$ and $(1,j)$. By Lemma 2.3(iii), we have the following result.

Lemma 4.1. *For $0 \leq j \leq \lfloor n/2 \rfloor$, the paths in $\mathscr{B}_j(\lfloor n/2 \rfloor, \lceil n/2 \rceil)$ are in one-to-one correspondence with the permutations $\sigma \in \mathscr{I}_n(321)$ with* $\mathrm{lead}(\sigma) = j + 1$.

4.1. The case $\Phi_1 \colon \mathscr{B}(2n, 2n) \to \mathscr{B}(2n, 2n)$

To prove Theorem 1.5(i), we shall establish a sump-parity-reversing involution $\Phi_1 \colon \mathscr{B}(2n, 2n) \to \mathscr{B}(2n, 2n)$ while preserving the initial segment from the beginning to the first east step. Let $\mathscr{F}(2n, 2n) \subseteq \mathscr{B}(2n, 2n)$ be the set of fixed points of the map Φ_1. The set $\mathscr{F}(2n, 2n)$ can be constructed from $\mathscr{B}(n, n)$ as follows.

For each path $\omega \in \mathscr{B}(n, n)$, we form a path $\gamma(\omega)$ by duplicating every step of ω. Then $\gamma(\omega) \in \mathscr{B}(2n, 2n)$. Note that the peaks and valleys of $\gamma(\omega)$ are all even lattice points. Moreover, every path without odd peaks and odd valleys in $\mathscr{B}(2n, 2n)$ can be reduced to a path in $\mathscr{B}(n, n)$ by the reverse operation. For example, for $\omega = \mathsf{NEENEN} \in \mathscr{B}(3, 3)$, the path $\gamma(\omega)$ is shown as Figure 4.1.

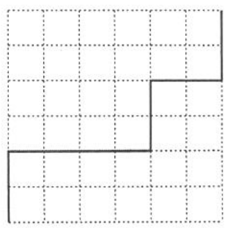

Figure 4.1: Construction of the path $\gamma(\omega)$ for $\omega = \mathsf{NEENEN} \in \mathscr{B}(3, 3)$

The set $\mathscr{F}(2n, 2n)$ is defined by

$$\mathscr{F}(2n, 2n) = \{\gamma(\omega) : \omega \in \mathscr{B}(n, n)\},$$

and for $0 \leq i \leq 2n$, the subset $\mathscr{F}_i(2n, 2n)$ is defined by

$$\mathscr{F}_i(2n, 2n) = \mathscr{F}(2n, 2n) \cap \mathscr{B}_i(2n, 2n).$$

Note that $\mathscr{F}_i(2n, 2n)$ is empty if i is odd. We have the following immediate observation.

Lemma 4.2. *For $0 \leq j \leq n$ and any path $\omega \in \mathscr{B}_j(n, n)$, the following properties hold:*

(i) $\gamma(\omega) \in \mathscr{F}_{2j}(2n, 2n)$ *and* $\mathrm{sump}(\gamma(\omega)) = 2 \cdot \mathrm{sump}(\omega)$.

(ii) $|\mathscr{F}_{2j}(2n, 2n)| = |\mathscr{B}_j(n, n)|$ *and* $|\mathscr{F}_{2j+1}(2n, 2n)| = 0$.

(iii) *The set $\mathscr{F}(2n, 2n)$ consists of all the paths without odd peaks and odd valleys in $\mathscr{B}(2n, 2n)$.*

Now, we construct the involution Φ_1 on $\mathscr{B}(2n, 2n)$.

Algorithm 4.3. *Given a path $\pi \in \mathscr{B}(2n, 2n)$, let $\pi = \mu_0\mu_1 \cdots \mu_d$ be the primal factorization of π. If every segment μ_i contains an even number of steps then $\Phi_1(\pi) = \pi$. Otherwise, find the greatest integer k such that μ_k contains an odd number of steps. The path $\Phi_1(\pi)$ is obtained from π by interchanging the first step of μ_k and the last step of μ_{k-1}.*

Example 4.4. Let π be the path shown on the left-hand side of Figure 4.2, with the primal factorization $\pi = \mu_0\mu_1 \cdots \mu_5$. Then $\mu_4 = \mathsf{NNN}$ is the last segment of odd length. Hence $\Phi_1(\pi)$ is obtained from π by interchanging the first step of μ_4 and the last step of μ_3, shown on the right-hand side of Figure 4.2.

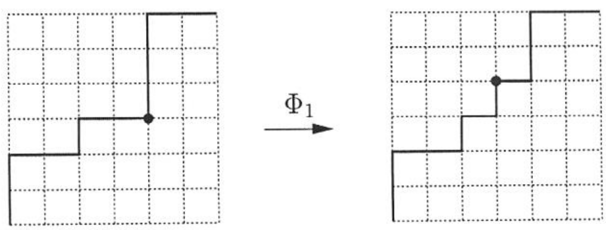

Figure 4.2: An example for sump-parity-reversing involution Φ_1 on grand Dyck paths

.

Lemma 4.5. *For the primal factorization $\pi = \mu_0\mu_1 \cdots \mu_d$ of a path $\pi \in \mathscr{B}(2n, 2n)$, if k is the greatest integer such that μ_k contains an odd number of steps then $k \neq 0$ and $k \neq 1$, i.e., $\Phi_1(\pi)$ preserves the first segment μ_0 of π.*

Proof. The assertion follows from the fact that π has $2n$ east steps and $2n$ north steps. □

Proposition 4.6. *For $0 \leq i \leq 2n$, the map Φ_1 establishes a refined involution on $\mathscr{B}_i(2n, 2n) - \mathscr{F}_i(2n, 2n)$. Furthermore, a path π is carried to a path $\Phi_1(\pi)$ such that* $\operatorname{sump}(\Phi_1(\pi))$ *has the opposite parity of* $\operatorname{sump}(\pi)$.

Proof. Given a path $\pi \in \mathscr{B}(2n, 2n)$, suppose $\Phi_1(\pi) \neq \pi$. By Lemma 4.2, π contains a peak (or valley) which is an odd lattice point. By Algorithm 4.3, we find the last odd peak (or valley) p. The path $\Phi(\pi)$ is obtained by interchanging the N and E steps adjacent at p. This changes the last odd peak (or valley) of π into the last odd valley (or peak) of $\Phi(\pi)$. Moreover, by Lemma 4.5, Φ_1 is an involution, restricted to each subset $\mathscr{B}_i(2n, 2n) - \mathscr{F}_i(2n, 2n)$. We observe that there is exactly one odd lattice point affected. Hence $\operatorname{sump}(\Phi_1(\pi))$ has the opposite parity of $\operatorname{sump}(\pi)$. □

Proof of Theorem 1.5(i).

$$\begin{aligned}\sum_{\sigma\in\mathscr{I}_{4n}(321)}(-1)^{\mathrm{maj}(\sigma)}q^{\mathrm{lead}(\sigma)} &= \sum_{i=0}^{2n}\left(\sum_{\pi\in\mathscr{B}_i(2n,2n)}(-1)^{\mathrm{sump}(\pi)}\right)q^{i+1}\\ &= \sum_{j=0}^{n}\left(\sum_{\pi\in\mathscr{F}_{2j}(2n,2n)}q^{2j+1}\right)\\ &= \sum_{j=0}^{n}|\mathscr{B}_j(n,n)|q^{2j+1}\\ &= \frac{1}{q}\sum_{\sigma\in\mathscr{I}_{2n}(321)}q^{2\cdot\mathrm{lead}(\sigma)}.\end{aligned}$$ □

4.2. The case $\Phi_2\colon \mathscr{B}(2n+1,2n+1)\to\mathscr{B}(2n+1,2n+1)$

To prove Theorem 1.5(ii), we shall establish a sump-parity-reversing involution $\Phi_2\colon \mathscr{B}_i(2n+1,2n+1)\to\mathscr{B}_i(2n+1,2n+1)$ for $0\le i\le 2n+1$. Let $\mathscr{F}(2n+1,2n+1)\subseteq\mathscr{B}(2n+1,2n+1)$ be the set of fixed points of the map Φ_2. For $0\le i\le 2n+1$, we define

$$\mathscr{F}_i(2n+1,2n+1)=\mathscr{F}(2n+1,2n+1)\cap\mathscr{B}_i(2n+1,2n+1).$$

The set $\mathscr{F}(2n+1,2n+1)$ can be constructed from $\mathscr{B}(n,n+1)$ as follows. For each path $\omega\in\mathscr{B}_j(n,n+1)$ $(0\le j\le n)$, we form a path $\gamma(\omega)$ by duplicating every step of ω. Note that $\gamma(\omega)$ is from $(0,0)$ to $(2n+2,2n)$ with the prefix $\mathsf{N}^{2j}\mathsf{EE}$. Factorize $\gamma(\omega)$ as $\gamma(\omega)=\mathsf{N}^{2j}\mathsf{EE}\beta$. Then we create two paths $\phi_1(\omega),\phi_2(\omega)\in\mathscr{B}(2n+1,2n+1)$ from $\gamma(\omega)$ by

$$\phi_1(\omega)=\mathsf{N}^{2j}\mathsf{EN}\beta,\quad \phi_2(\omega)=\mathsf{N}^{2j+1}\mathsf{E}\beta,$$

i.e., $\phi_1(\omega)$ ($\phi_2(\omega)$, respectively) is obtained from $\gamma(\omega)$ by changing the second (first, respectively) east step into a north step. Note that the segment β of $\phi_1(\omega),\phi_2(\omega)$ goes from $(1,2j+1)$ to $(2n+1,2n+1)$ and that every segment in the primal factorization of β contains an even number of steps. For example, let $\omega=\mathsf{NEENE}\in\mathscr{B}_1(2,3)$. Then the path $\gamma(\omega)$ is shown on the left-hand side of Figure 4.3 and the paths $\phi_1(\omega)$, $\phi_2(\omega)$ are shown on the right-hand side of Figure 4.3.

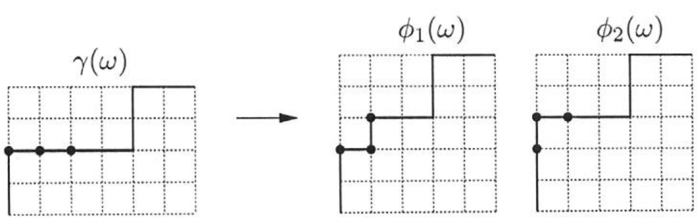

Figure 4.3: Construction of the paths $\phi_1(\omega)$ and $\phi_2(\omega)$ for $\omega=\mathsf{NEENE}$

.

For $0 \leq j \leq n$, define

$$\mathscr{F}_{2j}(2n+1, 2n+1) = \{\phi_1(\omega) : \omega \in \mathscr{B}_j(n, n+1)\},$$
$$\mathscr{F}_{2j+1}(2n+1, 2n+1) = \{\phi_2(\omega) : \omega \in \mathscr{B}_j(n, n+1)\}.$$

We have the following immediate observation.

Lemma 4.7. *For $0 \leq j \leq n$ and any path $\omega \in \mathscr{B}_j(n, n+1)$, the following properties hold:*

(i) *The path $\phi_1(\omega) \in \mathscr{F}_{2j}(2n+1, 2n+1)$ contains a unique odd valley $(1, 2j)$ and no odd peaks.*

(ii) *The path $\phi_2(\omega) \in \mathscr{F}_{2j+1}(2n+1, 2n+1)$ contains a unique odd peak $(0, 2j+1)$ and no odd valleys.*

(iii) $|\mathscr{F}_{2j}(2n+1, 2n+1)| = |\mathscr{F}_{2j+1}(2n+1, 2n+1)| = |\mathscr{B}_j(n, n+1)|$.

It follows that $\mathrm{sump}(\phi_1(\omega))$ is even and $\mathrm{sump}(\phi_2(\omega))$ is odd. Note that every path $\pi \in \mathscr{B}(2n+1, 2n+1)$ contains at least one odd valley or odd peak since π has $2n+1$ east steps and $2n+1$ north steps. In fact, the set $\mathscr{F}(2n+1, 2n+1)$ consists of all the paths in $\mathscr{B}(2n+1, 2n+1)$ either containing a unique odd valley in the line $x = 1$ and no odd peaks, or containing a unique odd peak in the line $x = 0$ and no odd valleys.

Now, we construct the involution Φ_2 on $\mathscr{B}(2n+1, 2n+1)$.

Algorithm 4.8. *Given a path $\pi \in \mathscr{B}(2n+1, 2n+1)$, let $\mu_0\mu_1 \cdots \mu_d$ be the primal factorization of π. According to the parity of the length of μ_0, there are two cases.*

Case 1. μ_0 is odd, say $\mu_0 = \mathsf{N}^{2j+1}$. Find the greatest integer $k \geq 1$ such that μ_k contains an odd number of steps. If $k = 1$ then let $\Phi_2(\pi) = \pi$. Otherwise, the path $\Phi_1(\pi)$ is obtained from π by interchanging the first step of μ_k and the last step of μ_{k-1}.

Case 2. μ_0 is even, say $\mu_0 = \mathsf{N}^{2j}$. If $\mu_1 = \mathsf{E}$ and μ_2 is the only other segment containing an odd number of steps (i.e., μ_t is of even length for all $t \geq 3$) then let $\Phi_2(\pi) = \pi$. Otherwise, find the greatest integer $k \geq 3$ such that μ_k contains an odd number of steps. Then the path $\Phi_2(\pi)$ is obtained from π by interchanging the first step of μ_k and the last step of μ_{k-1}.

It is obvious that the map Φ_2 preserves the segment μ_0, i.e., Φ_2 is a map restricted to each subset $\mathscr{B}_i(2n+1, 2n+1)$ for $0 \leq i \leq 2n+1$.

Example 4.9. Let π be the path shown on the left-hand side of Figure 4.4, with the primal factorization $\pi = \mu_0\mu_1 \cdots \mu_7$. Then $\mu_5 = \mathsf{EEE}$ is the last segment of odd length. Hence $\Phi_2(\pi)$ is obtained from π by interchanging the first step of μ_5 and the last step of μ_4, shown on the right-hand side of Figure 4.4.

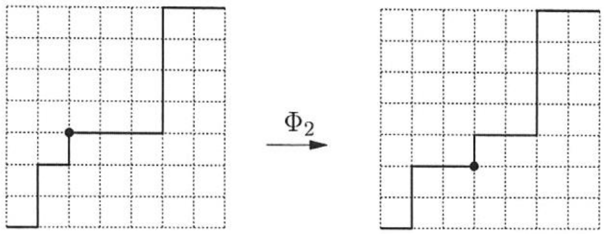

Figure 4.4: An example for the map $\Phi_2\colon \mathscr{B}(7,7) \to \mathscr{B}(7,7)$

.

Proposition 4.10. *For $0 \le i \le 2n+1$, the map Φ_2 establishes a refined involution on $\mathscr{B}_i(2n+1, 2n+1) - \mathscr{F}_i(2n+1, 2n+1)$. Moreover, a path π is carried to a path $\Phi_2(\pi)$ such that* $\mathrm{sump}(\Phi_2(\pi))$ *has the opposite parity of* $\mathrm{sump}(\pi)$*.*

Proof. Given a path $\pi \in \mathscr{B}(2n+1, 2n+1)$, suppose $\Phi_2(\pi) \ne \pi$. By Lemma 4.7, π either contains an odd peak (x, y) with $x > 0$ or contains an odd valley (x', y') with $x' > 1$. By Algorithm 4.8, we find the last odd peak (or valley) p and construct the path $\Phi_2(\pi)$ by interchanging the N and E steps adjacent at p. By the same argument as in the proof of Proposition 4.6, the assertion is proved. □

Proof of Theorem 1.5(ii).

$$\begin{aligned}
&\sum_{\sigma\in\mathscr{I}_{4n+2}(321)} (-1)^{\mathrm{maj}(\sigma)} q^{\mathrm{lead}(\sigma)} \\
&= \sum_{i=0}^{2n+1} \left(\sum_{\pi\in\mathscr{B}_i(2n+1,2n+1)} (-1)^{\mathrm{sump}(\pi)} \right) q^{i+1} \\
&= \sum_{j=0}^{n} \left(\sum_{\pi\in\mathscr{F}_{2j}(2n+1,2n+1)} q^{2j+1} - \sum_{\pi\in\mathscr{F}_{2j+1}(2n+1,2n+1)} q^{2j+2} \right) \\
&= \sum_{j=0}^{n} |\mathscr{B}_j(n, n+1)| q^{2j+1} - \sum_{j=0}^{n} |\mathscr{B}_j(n, n+1)| q^{2j+2} \\
&= \left(\frac{1}{q} - 1 \right) \sum_{\sigma\in\mathscr{I}_{2n+1}(321)} q^{2\cdot\mathrm{lead}(\sigma)}.
\end{aligned}$$
□

4.3. The case $\Phi_3\colon \mathscr{B}(2n+1, 2n+2) \to \mathscr{B}(2n+1, 2n+2)$

To prove Theorem 1.5(iii), we shall establish a sump-parity-reversing involution $\Phi_3\colon \mathscr{B}_i(2n+1, 2n+2) \to \mathscr{B}_i(2n+1, 2n+2)$ for $0 \le i \le 2n+1$. Let $\mathscr{F}(2n+1, 2n+2) \subseteq \mathscr{B}(2n+1, 2n+2)$ be the set of fixed points of the map Φ_3. For $0 \le i \le 2n+1$, we define

$$\mathscr{F}_i(2n+1, 2n+2) = \mathscr{F}(2n+1, 2n+2) \cap \mathscr{B}_i(2n+1, 2n+2).$$

The set $\mathscr{F}(2n+1,2n+2)$ can be constructed from $\mathscr{B}(n,n+1)$ as follows. For each path $\omega \in \mathscr{B}_j(n,n+1)$ $(0 \le j \le n)$, we form a path $\gamma(\omega)$ by duplicating every step of ω. Note that $\gamma(\omega)$ is from $(0,0)$ to $(2n+2,2n)$ with the prefix $\mathsf{N}^{2j}\mathsf{EE}$. Factorize $\gamma(\omega)$ as $\gamma(\omega)=\mathsf{N}^{2j}\mathsf{EE}\beta$. Then we create three paths $\psi_0(\omega),\psi_1(\omega),\psi_2(\omega)\in\mathscr{B}(2n+1,2n+2)$ from $\gamma(\omega)$ by

$$\psi_0(\omega)=\gamma(\omega)\mathsf{N},\quad \psi_1(\omega)=\mathsf{N}^{2j}\mathsf{EN}\beta\mathsf{E},\quad \psi_2(\omega)=\mathsf{N}^{2j+1}\mathsf{E}\beta\mathsf{E}.$$

Note that $\psi_0(\omega)$ is obtained from $\gamma(\omega)$ by appending a north step in the end and that $\psi_1(\omega)$ ($\psi_2(\omega)$, respectively) is obtained from $\gamma(\omega)$ inserting a north step after (before, respectively) the first east step and moving the second east step to the end. For example, for $\omega=\mathsf{NEENE}\in\mathscr{B}_1(2,3)$, the paths $\psi_0(\omega)$, $\psi_1(\omega)$ and $\psi_2(\omega)$ are shown in Figure 4.5.

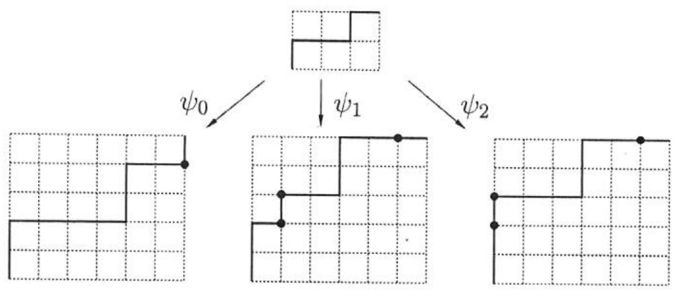

Figure 4.5: Construction of the paths $\psi_0(\omega)$, $\psi_1(\omega)$ and $\psi_2(\omega)$ for $\omega=\mathsf{NEENE}$

We define the refinement of the set $\mathscr{F}(2n+1,2n+2)$. For $0\le j\le n$, let

$$\mathscr{F}_{2j}(2n+1,2n+2)=\{\psi_0(\omega),\psi_1(\omega):\omega\in\mathscr{B}_j(n,n+1)\},$$
$$\mathscr{F}_{2j+1}(2n+1,2n+2)=\{\psi_2(\omega):\omega\in\mathscr{B}_j(n,n+1)\}.$$

We have the following properties of the fixed points $\mathscr{F}(2n+1,2n+2)$.

Lemma 4.11. *For $0\le j\le n$ and any path $\omega\in\mathscr{B}_j(n,n+1)$, the following properties hold:*

(i) *The path $\psi_0(\omega)$ contains no odd peaks and odd valleys.*

(ii) *The path $\psi_1(\omega)$ contains a unique odd valley $(1,2j)$ and no odd peaks.*

(iii) *The path $\psi_2(\omega)$ contains a unique odd peak $(0,2j+1)$ and no odd valleys.*

(iv) $|\mathscr{F}_{2j}(2n+1,2n+2)|=2|\mathscr{B}_j(n,n+1)|$ *and* $|\mathscr{F}_{2j+1}(2n+1,2n+2)|=|\mathscr{B}_j(n,n+1)|$.

It follows that $\mathrm{sump}(\psi_0(\omega)),\mathrm{sump}(\psi_1(\omega))$ are even and $\mathrm{sump}(\psi_2(\omega))$ is odd. In fact, $\mathscr{F}(2n+1,2n+2)$ consists of all the paths in $\mathscr{B}(2n+1,2n+2)$ containing no odd valley (x,y) with $x\ge 2$ and no odd peak (x',y') with $x'\ge 1$.

Now, we construct the involution Φ_3 on $\mathscr{B}(2n+1,2n+2)$.

Algorithm 4.12. *Given a path $\pi \in \mathscr{B}(2n+1, 2n+2)$, let z denote the last step of π. Let π' be the path obtained from π by removing z. We consider the following two cases according to the step z.*

Case 1. *$z = \mathsf{N}$. Then π' goes from $(0,0)$ to $(2n+2, 2n)$. Applying Algorithm* 4.3 *to the primal factorization of π', we determine the path $\Phi_1(\pi')$ associated with π' under the map Φ_1. Then the corresponding path $\Phi_3(\pi)$ is obtained from $\Phi_1(\pi')$ by appending a north step, i.e., $\Phi_3(\pi) = \Phi_1(\pi')\mathsf{N} \in \mathscr{B}(2n+1, 2n+2)$.*

Case 2. *$z = \mathsf{E}$. Then π' goes from $(0,0)$ to $(2n+1, 2n+1)$. Applying Algorithm* 4.8 *to the primal factorization of π', we determine the path $\Phi_2(\pi')$ associated with π' under the map Φ_2. Then the corresponding path $\Phi_3(\pi) \in \mathscr{B}(2n+1, 2n+2)$ is obtained from $\Phi_2(\pi')$ by appending an east step, i.e., $\Phi_3(\pi) = \Phi_2(\pi')\mathsf{E} \in \mathscr{B}(2n+1, 2n+2)$.*

Note that the construction of the map Φ_3 in Case 1 (Case 2, respectively) of Algorithm 4.12 is similar to the construction of Φ_1 by Algorithm 4.3 (Φ_2 by Algorithm 4.8, respectively). The following property of the map Φ_3 can be proved by the same argument as in the proofs of Propositions 4.6 and 4.10.

Proposition 4.13. *For $0 \le i \le 2n+1$, the map Φ_3 establishes a refined involution on $\mathscr{B}_i(2n+1, 2n+2) - \mathscr{F}_i(2n+1, 2n+2)$. Moreover, a path π is carried to a path $\Phi_3(\pi)$ such that* $\mathrm{sump}(\Phi_3(\pi))$ *has the opposite parity of* $\mathrm{sump}(\pi)$*.*

Proof of Theorem 1.5(iii).

$$\begin{aligned}
&\sum_{\sigma \in \mathscr{I}_{4n+3}(321)} (-1)^{\mathrm{maj}(\sigma)} q^{\mathrm{lead}(\sigma)} \\
&= \sum_{i=0}^{2n+1} \left(\sum_{\pi \in \mathscr{B}_i(2n+1,2n+2)} (-1)^{\mathrm{sump}(\pi)} \right) q^{i+1} \\
&= \sum_{j=0}^{n} \left(\sum_{\pi \in \mathscr{F}_{2j}(2n+1,2n+2)} q^{2j+1} - \sum_{\pi \in \mathscr{F}_{2j+1}(2n+1,2n+2)} q^{2j+2} \right) \\
&= \sum_{j=0}^{n} 2|\mathscr{B}_j(n,n+1)| q^{2j+1} - \sum_{j=0}^{n} |\mathscr{B}_j(n,n+1)| q^{2j+2} \\
&= \left(\frac{2}{q} - 1 \right) \sum_{\sigma \in \mathscr{I}_{2n+1}(321)} q^{2 \cdot \mathrm{lead}(\sigma)}. \qquad \square
\end{aligned}$$

4.4. The case $\Phi_4 \colon \mathscr{B}(2n, 2n+1) \to \mathscr{B}(2n, 2n+1)$

To prove Theorem 1.5(iv), we shall establish a sump-parity-reversing involution $\Phi_4 \colon \mathscr{B}_i(2n, 2n+1) \to \mathscr{B}_i(2n, 2n+1)$ for $0 \le i \le 2n$. Let $\mathscr{F}(2n, 2n+1) \subseteq \mathscr{B}(2n, 2n+1)$ be the set of fixed points of the map Φ_4. For $0 \le i \le 2n$, we define

$$\mathscr{F}_i(2n, 2n+1) = \mathscr{F}(2n, 2n+1) \cap \mathscr{B}_i(2n, 2n+1).$$

The set $\mathscr{F}(2n, 2n+1)$ can be constructed from $\mathscr{B}(n, n+1)$ as follows. For each path $\omega \in \mathscr{B}_j(n, n+1)$ $(0 \le j \le n)$, we form a path $\gamma(\omega)$ by duplicating every step of ω. We consider the following two cases according to the last step z of ω:

- $z = \mathsf{E}$. Then the last two steps of $\gamma(\omega)$ are east steps. Let $\varphi_0(\omega)$ be the path obtained from $\gamma(\omega)$ by removing the last step.
- $z = \mathsf{N}$. Then the last two steps of $\gamma(\omega)$ are north steps. Factorize $\gamma(\omega)$ as $\mathsf{N}^{2j}\mathsf{EE}\beta\mathsf{NN}$ and let $\varphi_1(\omega)$ ($\varphi_2(\omega)$, respectively) be the path obtained from $\gamma(\omega)$ by inserting a north step after (before, respectively) the first east step and then removing the second east step and the last step, i.e.,

$$\varphi_1(\omega) = \mathsf{N}^{2j}\mathsf{EN}\beta\mathsf{N}, \quad \varphi_2(\omega) = \mathsf{N}^{2j+1}\mathsf{E}\beta\mathsf{N}.$$

For example, for $\omega_1 = \mathsf{NEENENE} \in \mathscr{B}_1(3,4)$, the path $\varphi_0(\omega_1)$ is shown as the left-hand side of Figure 4.6. For $\omega_2 = \mathsf{EENENEN} \in \mathscr{B}_0(3,4)$, the paths $\varphi_1(\omega_2)$, $\varphi_2(\omega_2)$ are shown as the right-hand side of Figure 4.6.

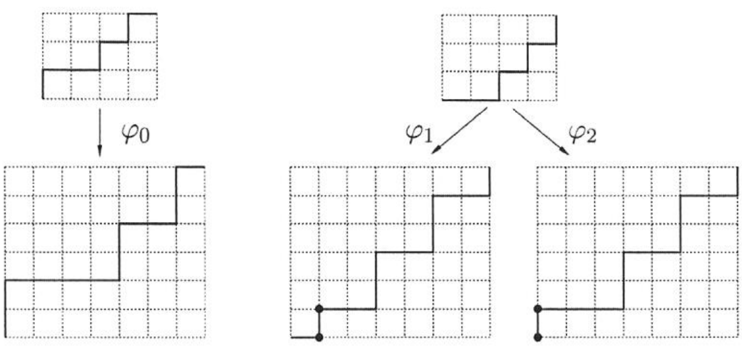

Figure 4.6: Examples of the maps φ_0, φ_1 and φ_2

We define the refinement of the set $\mathscr{F}(2n, 2n+1)$. For $0 \le j \le n$, let $\mathscr{F}_{2j}(2n, 2n+1) = \mathscr{F}^E_{2j}(2n, 2n+1) \cup \mathscr{F}^N_{2j}(2n, 2n+1)$, where

$$\mathscr{F}^E_{2j}(2n, 2n+1) = \{\varphi_0(\omega) : \omega \in \mathscr{B}_j(n, n+1) \text{ ends with an east step}\},$$
$$\mathscr{F}^N_{2j}(2n, 2n+1) = \{\varphi_1(\omega) : \omega \in \mathscr{B}_j(n, n+1) \text{ ends with a north step}\},$$
$$\mathscr{F}_{2j+1}(2n, 2n+1) = \{\varphi_2(\omega) : \omega \in \mathscr{B}_j(n, n+1) \text{ ends with a north step}\}.$$

We have the following properties of the fixed points $\mathscr{F}(2n+1, 2n+2)$.

Lemma 4.14. *For $0 \le j \le n$ and any path $\omega \in \mathscr{B}_j(n, n+1)$, the following properties hold:*

(i) *The path $\varphi_0(\omega)$ contains no odd peaks and odd valleys.*

(ii) *The path $\varphi_1(\omega)$ contains a unique odd valley $(1, 2j)$ and no odd peaks.*

(iii) *The path $\varphi_2(\omega)$ contains a unique odd peak $(0, 2j+1)$ and no odd valleys.*

(iv) $|\mathscr{F}_{2j}(2n+1, 2n+2)| = |\mathscr{B}_j(n, n+1)|$.

(v) $|\mathscr{F}_{2j+1}(2n+1, 2n+2)| = |\mathscr{B}_j(n, n+1)| - |\mathscr{B}_j(n, n)|$.

It follows that $\mathrm{sump}(\varphi_0(\omega))$, $\mathrm{sump}(\varphi_1(\omega))$ are even and $\mathrm{sump}(\varphi_2(\omega))$ is odd. Now, we construct the involution Φ_4 on $\mathscr{B}(2n, 2n+1)$.

Algorithm 4.15. *Given a path $\pi \in \mathscr{B}(2n, 2n+1)$, let z denote the last step of π. Let π' be the path obtained from π by removing z. We consider the following two cases according to the step z.*

Case 1. *$z = \mathsf{E}$. Then π' goes from $(0,0)$ to $(2n, 2n)$. By Algorithm* 4.3*, we determine the path $\Phi_1(\pi') \in \mathscr{B}(2n, 2n)$ associated with π' under the map Φ_1. Then the corresponding path $\Phi_4(\pi)$ is obtained from $\Phi_1(\pi')$ by appending an east step in the end, i.e., $\Phi_4(\pi) = \Phi_1(\pi')\mathsf{E}$.*

Case 2. *$z = \mathsf{N}$. Then π' goes from $(0,0)$ to $(2n+1, 2n-1)$. By the same method as in Algorithm* 4.8*, we determine the path $\Phi_2(\pi') \in \mathscr{B}(2n-1, 2n+1)$ associated with π' under the map Φ_2. Then the corresponding path $\Phi_4(\pi) \in \mathscr{B}(2n, 2n+1)$ is obtained from $\Phi_2(\pi')$ by appending a north step in the end, i.e., $\Phi_4(\pi) = \Phi_2(\pi')\mathsf{N}$.*

The following property of the map Φ_4 can be proved by the same argument as in the proofs of Propositions 4.6 and 4.10 since the construction of Φ_4 in Case 1 (Case 2, respectively) of Algorithm 4.15 is similar to the construction of Φ_1 (Φ_2, respectively).

Proposition 4.16. *For $0 \le i \le 2n$, the map Φ_4 establishes a refined involution on $\mathscr{B}_i(2n, 2n+1) - \mathscr{F}_i(2n, 2n+1)$. Moreover, a path π is carried to a path $\Phi_4(\pi)$ such that* $\mathrm{sump}(\Phi_4(\pi))$ *has the opposite parity of* $\mathrm{sump}(\pi)$.

Proof of Theorem 1.5(iv).

$$\begin{aligned}
&\sum_{\sigma \in \mathscr{I}_{4n+1}(321)} (-1)^{\mathrm{maj}(\sigma)} q^{\mathrm{lead}(\sigma)} \\
&= \sum_{i=0}^{2n} \left(\sum_{\pi \in \mathscr{B}_i(2n, 2n+1)} (-1)^{\mathrm{sump}(\pi)} \right) q^{i+1}
\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=0}^{n} \left(\sum_{\pi\in\mathscr{F}_{2j}(2n,2n+1)} q^{2j+1} - \sum_{\pi\in\mathscr{F}_{2j+1}(2n,2n+1)} q^{2j+2} \right) \\
&= \sum_{j=0}^{n} \big(|\mathscr{B}_j(n,n+1)| - |\mathscr{B}_j(n,n+1)|q + |\mathscr{B}_j(n,n)|q\big) q^{2j+1} \\
&= \left(\frac{1}{q} - 1\right) \sum_{\sigma\in\mathscr{I}_{2n+1}(321)} q^{2\cdot \mathrm{lead}(\sigma)} + \sum_{\sigma\in\mathscr{I}_{2n}(321)} q^{2\cdot\mathrm{lead}(\sigma)}. \qquad \square
\end{aligned}
$$

5. Analogous results for 123-avoiding involutions

It is known that by the Robinson-Schensted-Knuth (RSK) algorithm an involution $\sigma \in \mathscr{I}_n(321)$ ($\mathscr{I}_n(123)$, respectively) is associated with a pair (Q, Q) of identical n-cell standard Young tableaux with at most two rows (columns, respectively). Let Q^{T} be the transpose of Q and let σ^{T} be the preimage of the pair $(Q^{\mathrm{T}}, Q^{\mathrm{T}})$ under the RSK correspondence. Then $\sigma \leftrightarrow \sigma^{\mathrm{T}}$ is a bijection between $\mathscr{I}_n(321)$ and $\mathscr{I}_n(123)$.

Lemma 5.1. *We have*

$$\mathrm{Des}(\sigma^{\mathrm{T}}) = \{i : i \notin \mathrm{Des}(\sigma), 1 \le i \le n-1\}.$$

Proof. It is known that a descent $\sigma_i > \sigma_{i+1}$ in σ is translated to the 'descent' of the recording tableau Q that the entry $i{+}1$ is in a row lower than the row of i. For $1 \le i \le n{-}1$, if $i \in \mathrm{Des}(\sigma)$ then i ($i+1$, respectively) is in the first (second, respectively) row in Q. Then $i+1$ is either in the same column as i or in a column to the left of the column of i in Q. Then $i+1$ is not in a lower row than the row of i in Q^{T}. Hence $i \notin \mathrm{Des}(\sigma^{\mathrm{T}})$.

Otherwise, $i \notin \mathrm{Des}(\sigma)$. Then $i+1$ is not in a row lower than the row of i in Q. The element i is in the first row or the second row. In either case, the element $i+1$ is in a column to the right of the column of i. Then $i{+}1$ is in row lower than the row of i in Q^{T}. Hence $i \in \mathrm{Des}(\sigma^{\mathrm{T}})$. $\square$

We obtain the joint distribution of major index and descent number for 123-avoiding involutions.

Corollary 5.2. *We have*

$$\sum_{\substack{\sigma\in\mathscr{I}_n(123)\\ \mathrm{des}(\sigma)=n-1-k}} q^{\mathrm{maj}(\sigma)} = q^{\binom{n}{2}+k^2-nk} \begin{bmatrix} \lceil n/2\rceil \\ k \end{bmatrix}_q \begin{bmatrix} \lfloor n/2\rfloor \\ k \end{bmatrix}_q.$$

Proof. Substituting q^{-1} for q in Theorem 1.4(i), we have

$$\sum_{\substack{\sigma\in\mathscr{I}_n(321)\\ \mathrm{des}(\sigma)=k}} q^{-\mathrm{maj}(\sigma)} = q^{-k^2} \begin{bmatrix} \lceil n/2\rceil \\ k \end{bmatrix}_{q^{-1}} \begin{bmatrix} \lfloor n/2\rfloor \\ k \end{bmatrix}_{q^{-1}} = q^{k^2-kn} \begin{bmatrix} \lceil n/2\rceil \\ k \end{bmatrix}_q \begin{bmatrix} \lfloor n/2\rfloor \\ k \end{bmatrix}_q.$$

By Lemma 5.1, $\operatorname{des}(\sigma) = n-1-\operatorname{des}(\sigma^{\mathrm{T}})$ and $\operatorname{maj}(\sigma) = \binom{n}{2} - \operatorname{maj}(\sigma^{\mathrm{T}})$ for any $\sigma \in \mathscr{I}_n(123)$. Thus we have

$$\sum_{\substack{\sigma\in\mathscr{I}_n(123)\\ \operatorname{des}(\sigma)=n-1-k}} q^{\operatorname{maj}(\sigma)} = \sum_{\substack{\sigma^{\mathrm{T}}\in\mathscr{I}_n(321)\\ \operatorname{des}(\sigma^{\mathrm{T}})=k}} q^{\binom{n}{2}-\operatorname{maj}(\sigma^{\mathrm{T}})} = q^{\binom{n}{2}+k^2-nk} \begin{bmatrix} \lceil n/2\rceil \\ k \end{bmatrix}_q \begin{bmatrix} \lfloor n/2\rfloor \\ k \end{bmatrix}_q,$$

as required. □

With the bijection $\sigma \leftrightarrow \sigma^{\mathrm{T}}$ between $\mathscr{I}_n(321)$ and $\mathscr{I}_n(123)$, we can prove affirmatively Conjecture 1.3. In fact, this result is essentially equivalent to Theorem 1.2.

Theorem 5.3. *For all $n \geq 1$, we have*

(i) $$\sum_{\sigma\in\mathscr{I}_{4n}(123)} (-1)^{\operatorname{maj}(\sigma)} q^{\operatorname{des}(\sigma)} = q \sum_{\sigma\in\mathscr{I}_{2n}(123)} q^{2\cdot\operatorname{des}(\sigma)},$$

(ii) $$\sum_{\sigma\in\mathscr{I}_{4n+2}(123)} (-1)^{\operatorname{maj}(\sigma)} q^{\operatorname{des}(\sigma)} = (1-q)q^2 \sum_{\sigma\in\mathscr{I}_{2n}(123)} q^{2\cdot\operatorname{des}(\sigma)},$$

(iii) $$\sum_{\sigma\in\mathscr{I}_{2n+1}(123)} (-1)^{\operatorname{maj}(\sigma)} q^{\operatorname{des}(\sigma)} = (-1)^n q^2 \sum_{\sigma\in\mathscr{I}_{n}(123)} q^{2\cdot\operatorname{des}(\sigma)}.$$

Proof. Note that $\operatorname{des}(\sigma) = n-1-\operatorname{des}(\sigma^{\mathrm{T}})$ and $\operatorname{maj}(\sigma) = \binom{n}{2} - \operatorname{maj}(\sigma^{\mathrm{T}})$ for any $\sigma \in \mathscr{I}_n(123)$. Making use of the identity in Theorem 1.2(i), we have

$$\begin{aligned}\sum_{\sigma\in\mathscr{I}_{4n}(123)} (-1)^{\operatorname{maj}(\sigma)} q^{\operatorname{des}(\sigma)} &= \sum_{\sigma^{\mathrm{T}}\in\mathscr{I}_{4n}(321)} (-1)^{\binom{4n}{2}-\operatorname{maj}(\sigma^{\mathrm{T}})} q^{4n-1-\operatorname{des}(\sigma^{\mathrm{T}})} \\ &= q \sum_{\sigma^{\mathrm{T}}\in\mathscr{I}_{2n}(321)} q^{2(2n-1-\operatorname{des}(\sigma^{\mathrm{T}}))} \qquad \text{(by Theorem 1.2(i))} \\ &= q \sum_{\sigma\in\mathscr{I}_{2n}(123)} q^{2\cdot\operatorname{des}(\sigma)}.\end{aligned}$$

The assertion (i) follows. Making use of the identities in (ii) and (iii) of Theorem 1.2, the assertions (ii) and (iii) can be proved straightforward in the same manner. □

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